# AN ASYMPTOTIC FORMULA FOR EXTENDED EULERIAN NUMBERS

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**1. Introduction.** Fix $\lambda > 1$. Define $d_k(n)$ by $\zeta(s)^k = \sum_{n=1}^{\infty} d_k(n)n^{-s}$, where $\zeta(s)$ is the Riemann zeta function. Note that $d_k(n)$ is a multiplicative function such that $d_k(1) = 1$ and $d_k(p^a) = \binom{a+k-1}{a}$ for $p$ prime, $a \geq 1$. For large Re $(s)$,

$$\frac{\lambda - 1}{\lambda - \zeta(s)} = \frac{\lambda - 1}{\lambda} \cdot \frac{1}{1 - \zeta(s)/\lambda} = \frac{\lambda - 1}{\lambda} \sum_{k=0}^{\infty} \zeta(s)^k \lambda^{-k} = \sum_{n=1}^{\infty} H(n) n^{-s},$$

where $H(n) = H(n, \lambda) = \lambda^{-1}(\lambda - 1) \sum_{k=0}^{\infty} \lambda^{-k} d_k(n)$. The numbers $H(n)$ are the *extended Eulerian numbers*; when $n$ is square-free, $H(n)$ is an *Eulerian number*. Properties of the extended Eulerian numbers may be found in [1].

Let $\Omega = \Omega(n)$ denote (as usual) the total number of prime factors of $n$, e.g. $\Omega(12) = 3$. In this paper we give an asymptotic formula for $H(n)$ as $\Omega \to \infty$. This formula is then used to sharpen some estimates of Hille [2] and to produce various other estimates for $H(n)$.

Hille obtained estimates for certain sums $\sum H(n)$ and therefrom deduced an upper bound and an $\Omega$-result for $H(n)$. He remarked that his upper bound was probably not very sharp when the number of distinct prime factors of $n$ is large. We study the growth of $H(n)$ by estimating the series $\sum_{k=0}^{\infty} \lambda^{-k} d_k(n)$ given above. This direct approach enables us to sharpen Hille's upper bound when $\Omega(n)$ is large and also to improve his $\Omega$-result.

We remark that $H(n)$ grows at least exponentially with $\Omega$; in fact, $H(n) \geq \lambda^{-1}(\lambda/(\lambda - 1))^{\Omega}$. For if $n > 1$,

$$\begin{aligned} H(n) &\geq \lambda^{-1}(\lambda - 1) \sum_{k=0}^{\infty} \lambda^{-k} d_k(2^{\Omega}) = \lambda^{-1}(\lambda - 1) \sum_{k=0}^{\infty} \lambda^{-k} \binom{\Omega + k - 1}{\Omega} \\ &= \lambda^{-2}(\lambda - 1) \sum_{k=0}^{\infty} \binom{\Omega + k}{\Omega} \lambda^{-k} \\ &= \lambda^{-2}(\lambda - 1)(1 - \lambda^{-1})^{-\Omega - 1} = \lambda^{-1}(\lambda/(\lambda - 1))^{\Omega}. \end{aligned}$$

**2. The asymptotic formula for** $H(n)$. For $x \geq 0$ and positive $a_i$, $1 \leq i \leq \nu$, define $f(x) = \lambda^{-x} \prod_{i=1}^{\nu} \binom{a_i + x}{a_i}$ and define

$$H(a_1, \cdots, a_\nu) = \lambda^{-2}(\lambda - 1) \sum_{k=0}^{\infty} \lambda^{-k} f(k). \tag{2.1}$$

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Hence, if $n = \prod_{i=1}^{\nu} p_i^{a_i}$, where the $p_i$ are distinct primes and the $a_i$ are positive integers, it follows that $H(n) = H(a_1, \cdots, a_\nu)$. We emphasize that in general, the $a_i$'s are positive and not necessarily integral.

For each $k \geq 0$ and $a > 0$

$$\binom{a+k}{a} = \prod_{j=1}^{k}\left(1 + \frac{a}{j}\right) < \exp \sum_{j=1}^{k} \frac{a}{j}$$
$$< \exp (a \log (k+1) + a\gamma) = (k+1)^a e^{a\gamma},$$

where $\gamma$ is Euler's constant. Therefore, by (2.1),

$$(2.2) \qquad H(a_1, \cdots, a_\nu) < \lambda^{-2}(\lambda - 1) \sum_{k=0}^{\infty} \lambda^{-k} \prod_{i=1}^{\nu} (k+1)^{a_i} e^{a_i \gamma}$$
$$= e^{\gamma\Omega}\lambda^{-1}(\lambda - 1) \sum_{k=1}^{\infty} \lambda^{-k} k^{\Omega},$$

where $\Omega = \sum_{i=1}^{\nu} a_i$. Thus $H(a_1, \cdots, a_\nu) \to 1$ as $\Omega \to 0$. Consequently, we shall suppose that $\Omega$ is bounded away from 0 in all the estimates of $H(a_1, \cdots, a_\nu)$ throughout this paper.

The main result is the following.

THEOREM 1. *Let $\nu$ be a positive integer and let $a_i > 0$, $1 \leq i \leq \nu$. Suppose that $\Omega = \sum_{i=1}^{\nu} a_i$ is bounded away from 0. Let $c$ be the (unique) positive root of the polynomial $\lambda x^\nu - \prod_{i=1}^{\nu} (x + a_i)$. Then for any constant $F < 1/2$, $H(a_1, \cdots, a_\nu) = \alpha\beta\sqrt{\pi}\,\{1 + O(\Omega^{-F})\}$, where $\alpha = \lambda^{-3/2}(\lambda - 1)e^{-\Omega} \cdot \prod_{i=1}^{\nu} (c + a_i)^{a_i}/\Gamma(a_i + 1)$ and $\beta = \{(1/2c) \sum_{i=1}^{\nu} a_i/(c + a_i)\}^{-\frac{1}{2}}$. The O-constant is independent of $\nu$ and the $a_i$.*

*Proof.* We begin with two lemmas which show that $c$, $\Omega$, and $\beta^2$ have the same order of magnitude.

LEMMA 1. $c \log \lambda < \Omega < c\lambda$.

*Proof.*

$$\lambda = \prod_{i=1}^{\nu} (1 + a_i/c) > \sum_{i=1}^{\nu} a_i/c = \Omega/c$$

and

$$\log \lambda = \sum_{i=1}^{\nu} \log (1 + a_i/c) < \sum_{i=1}^{\nu} a_i/c = \Omega/c. \qquad \text{Q.E.D.}$$

LEMMA 2. *There exist positive constants $K_1$ and $K_2$ such that $K_1\sqrt{c} < \beta < K_2\sqrt{c}$.*

*Proof.* By Lemma 1,

$$\sum_{i=1}^{\nu} a_i/(c + a_i) < \sum_{i=1}^{\nu} a_i/c = \Omega/c < \lambda;$$

hence, $\beta > \sqrt{c}\,(2/\lambda)^{\frac{1}{2}}$. Also by Lemma 1,

$$\sum_{i=1}^{\nu} a_i/(c + a_i) > (c + \Omega)^{-1} \sum_{i=1}^{\nu} a_i = (c + \Omega)^{-1}\Omega$$

$$> (c + c\lambda)^{-1} c \log \lambda = (1 + \lambda)^{-1} \log \lambda;$$

hence, $\beta < \sqrt{c}\,\{(2 + 2\lambda)/\log \lambda\}^{\frac{1}{2}}$. Q.E.D.

Suppose $\Omega$ is restricted to lie in a fixed, finite interval $[x_0, y_0]$, where $x_0 > 0$. Then, by Lemmas 1 and 2, $\beta$ is bounded away from 0. Also, $\alpha$ is bounded away from 0. Moreover, by (2.2), $H(a_1, \cdots, a_\nu) = O(1)$. This proves that Theorem 1 is true when $\Omega \in [x_0, y_0]$. We may consequently assume for the remainder of the proof that $\Omega$ is large.

In the next lemma we obtain a uniform estimate for those terms of the series in (2.1) for which $k$ is close to $c$.

LEMMA 3. *Let $c > 2$ and define $u = k - c$. Fix $E \in (1/2, 1)$ and suppose $|u| \leq c^E$. Then $\lambda^{-2}(\lambda - 1)f(k) = \alpha \exp(-(u/\beta)^2)\{1 + O(\Omega^{3E-2})\}$. The O-constant depends only on $E$ and $\lambda$.*

*Proof.* All O-constants in this proof shall depend only on $E$ and $\lambda$. Write $d_i = c + a_i$, $1 \leq i \leq \nu$. By Stirling's formula and [3; p. 151],

$$\binom{a_i + k}{a_i} = \frac{\Gamma(d_i + u + 1)}{\Gamma(a_i + 1)\Gamma(c + u + 1)} = \Gamma^{-1}(a_i + 1) \exp\{-a_i + x_i + y_i\},$$

where $x_i = (d_i + u + 1/2) \log (d_i + u) - (c + u + 1/2) \log (c + u)$ and

$$y_i = \int_0^\infty \phi(t)\{(t + d_i + u)^{-2} - (t + c + u)^{-2}\}\, dt,$$

where $\phi(t) = \int_0^t ([w] - w + 1/2)\, dw$. Thus

$$f(k) = \lambda^{-k} e^{-\Omega} \prod_{i=1}^{\nu} \Gamma^{-1}(a_i + 1) \exp\left\{\sum_{i=1}^{\nu} x_i + \sum_{i=1}^{\nu} y_i\right\}. \tag{2.3}$$

For convenience we shall at times drop the subscript $i$ from $a_i$ and $d_i$. Observe that $\log (d + u) = \log d + \log (1 + u/d) = \log d + \sum_{m=1}^\infty (-1)^{m+1} u^m/md^m$. Using the analogous formula for $\log (c + u)$, it follows that

$$x_i = a \log d + (c + u + \tfrac{1}{2}) \log\left(\frac{d}{c}\right) + a \sum_{m=1}^{\infty} (-1)^{m+1} u^m/md^m$$
$$+ (c + u + \tfrac{1}{2}) \sum_{m=1}^{\infty} (-1)^{m+1} u^m (d^{-m} - c^{-m})/m. \tag{2.4}$$

Now,

$$a \sum_{m=1}^{\infty} (-1)^{m+1} u^m/md^m = au/d - au^2/2d^2 + O(au^3/c^3). \tag{2.5}$$

Since $d > c$, we have

$$(d^{-m} - c^{-m})/m = (-a/mc^m d^m) \sum_{j=1}^{m} c^{m-j} d^{j-1} = O(ad^{m-1}/c^m d^m) = O(a/c^{m+1}).$$

Therefore, the rightmost series in (2.4) equals

$$(c + u + \tfrac{1}{2})\{-au/cd + a(d + c)u^2/2c^2d^2 + O(au^3/c^4)\}$$
$$= -ua(c + \tfrac{1}{2})/cd + u^2\{-a/cd + a(d + c)(c + \tfrac{1}{2})/2c^2d^2\} + O(au^3/c^3).$$

Combining this expression with the right side of (2.5), we obtain from (2.4), $x_i = a \log d + (c + u + 1/2) \log (d/c) - ua/2cd + u^2\{-a/2cd + a(d + c)/4c^2d^2\} + O(au^3/c^3)$. Observe that $u^3/c^3 = O(c^{3E-3})$, $u/cd = O(u/c^2) = O(c^{3E-3})$, and $u^2(d + c)/c^2d^2 = O(u^2d/c^2d^2) = O(u^2/c^3) = O(c^{3E-3})$. Therefore, $x_i = a_i \log d_i + (c + u + 1/2) \log (d_i/c) - a_iu^2/2cd_i + O(a_ic^{3E-3})$. Thus, by Lemma 1 and the definition of $c$,

$$\sum_{i=1}^{\nu} x_i = \sum_{i=1}^{\nu} a_i \log d_i + (c + u + \tfrac{1}{2}) \log \lambda - (u^2/2c) \sum_{i=1}^{\nu} a_i/d_i + O(\Omega^{3E-2}). \tag{2.6}$$

Since $\phi(t)$ is bounded,

$$y_i = -a_i \int_0^\infty \frac{\phi(t)\{(t + c + u) + (t + d_i + u)\}\, dt}{(t + d_i + u)^2(t + c + u)^2}$$
$$= O\left(a_i \int_0^\infty \frac{dt}{(t + d_i + u)(t + c + u)^2}\right) = O\left(a_i \int_0^\infty (t + c + u)^{-3}\, dt\right)$$
$$= O(a_i/(c + u)^2) = O(a_i/c^2).$$

Thus,

$$\sum_{i=1}^{\nu} y_i = O(\Omega/c^2) = O(\Omega^{3E-2}). \tag{2.7}$$

It follows from (2.6), (2.7), and (2.3) that

$$f(k) = \lambda^{-k}e^{-\Omega}\left(\prod_{i=1}^{\nu} d_i^{a_i}/\Gamma(a_i + 1)\right)\lambda^{k+\frac{1}{2}} \exp(-(u/\beta)^2)\{1 + O(\Omega^{3E-2})\}.$$

Therefore, $\lambda^{-2}(\lambda - 1)f(k) = \alpha \exp(-(u/\beta)^2)\{1 + O(\Omega^{3E-2})\}$. Q.E.D.

We proceed with the proof of Theorem 1. By Lemma 3,

$$\lambda^{-2}(\lambda - 1) \sum_{\substack{k=0 \\ |u| \le c^E}}^{\infty} f(k) = \alpha S\{1 + O(\Omega^{3E-2})\}, \tag{2.8}$$

where

$$S = \sum_{\substack{k=0 \\ |u| \le c^E}}^{\infty} \exp(-(u/\beta)^2).$$

Let $x = \beta^{-1}c^E$. Then

$$J = \int_{-c^E}^{c^E} \exp(-(u/\beta)^2)\, du = 2\beta \int_0^x \exp(-y^2)\, dy$$
$$= \beta\sqrt{\pi} - 2\beta \int_x^\infty \exp(-y^2)\, dy.$$

By Lemma 2, $x \geq c^A$ for some constant $A > 0$. Thus, $J = \beta\sqrt{\pi} + O(\beta e^{-x}) = \beta\sqrt{\pi} + O(\sqrt{c}\exp(-c^A))$. Since $S = J + O(1)$, it follows that $S = \beta\sqrt{\pi} + O(1) = \beta\sqrt{\pi}\,\{1 + O(\Omega^{3E-2})\}$. Hence, by (2.8),

$$\lambda^2(\lambda - 1) \sum_{\substack{k=0 \\ |u| \leq c^E}}^{\infty} f(k) = \alpha\beta\sqrt{\pi}\,\{1 + O(\Omega^{3E-2})\}. \tag{2.9}$$

Let $S_j = \sum_{k \in I_j} f(k)$, $j = 1, 2, 3$, where $I_1 = [0, c - c^E]$, $I_2 = [c + c^E, \Omega^2]$, and $I_3 = [\Omega^2, \infty)$. (Recall that $\Omega$ can be considered large.) It is easily verified that $f(x - 1) - f(x)$ is positive when $x > c$ and negative when $x < c$. Thus $S_1 < cf(c - c^E)$. By Lemma 3 with $u = -c^E$, there exists a constant $A > 0$ such that $f(c - c^E) = O(\alpha \exp(-c^A))$. Therefore,

$$S_1 = O(\alpha c \exp(-c^A)) = O(\alpha). \tag{2.10}$$

Similarly, it follows that

$$S_2 < \Omega^2 f(c + c^E) = O(\alpha). \tag{2.11}$$

Estimating $f(k)$ as in (2.2), we have

$$S_3 = O\Big(e^{\gamma\Omega} \sum_{k=\Omega^2+1}^{\infty} \lambda^{-k} k^{\Omega}\Big) = O\Big(e^{\gamma\Omega} \int_{\Omega^2}^{\infty} \lambda^{-t} t^{\Omega}\, dt\Big)$$
$$= O\Big(e^{\gamma\Omega} \int_{\Omega^2}^{\infty} \lambda^{-t/2}\, dt\Big) = O(e^{\gamma\Omega}\lambda^{-\Omega^2/2}) = O(e^{-\Omega}).$$

Hence, by definition of $\alpha$, $S_3/\alpha = O(\prod_{i=1}^{\nu} \Gamma(a_i + 1)/(c + a_i)^{a_i}) = O(1)$, i.e.,

$$S_3 = O(\alpha). \tag{2.12}$$

By (2.9), (2.10), (2.11), and (2.12), $\lambda^2(\lambda - 1) \sum_{k=0}^{\infty} f(k) = \alpha\beta\sqrt{\pi}\,\{1 + O(\Omega^{3E-2})\}$. By (2.1), $H(a_1, \cdots, a_\nu) = \alpha\beta\sqrt{\pi}\,\{1 + O(\Omega^{3E-2})\}$. Since $E$ can be chosen arbitrarily close to 1/2, the theorem follows. Q.E.D.

**3. Special cases of the asymptotic formula.** The next theorem shows that if each $a_i$ grows slowly relative to $\Omega$, then the asymptotic formula for $H(a_1, \cdots, a_\nu)$ can be considerably simplified. In the simplified formula there is no presence of the variable $c$.

THEOREM 2. *Let $\nu$ be a positive integer and let $a_i > 0$, $1 \leq i \leq \nu$. Let $\Omega = \sum_{i=1}^{\nu} a_i$. Define $M = \max\{a_i : 1 \leq i \leq \nu\}$ and assume $M = o(\Omega^{\frac{1}{3}})$ as $\Omega \to \infty$.*

*Then as* $\Omega \to \infty$,

$$H(a_1, \cdots, a_\nu) \sim \frac{\lambda^{-3/2}(\lambda - 1)\Gamma(\Omega + 1) \exp\{(2\Omega)^{-1}(\log \lambda) \sum_{i=1}^{\nu} a_i^2\}}{(\log \lambda)^{\Omega+1} \prod_{i=1}^{\nu} \Gamma(a_i + 1)}.$$

The proof requires the following lemma.

LEMMA 4. *If* $M = o(\Omega)$, *then*

$$\text{(i)} \qquad \log \lambda = \Omega/c - \sum_{i=1}^{\nu} a_i^2/2c^2 + O(M^2\Omega/c^3)$$

*and*

$$\text{(ii)} \qquad c \sim \Omega/\log \lambda.$$

*Proof.* As $\Omega \to \infty$,

$$\log \lambda = \sum_{i=1}^{\nu} \log(1 + a_i/c) = \sum_{i=1}^{\nu} (a_i/c - a_i^2/2c^2 + O(a_i^3/c^3))$$

$$= \Omega/c - \sum_{i=1}^{\nu} a_i^2/2c^2 + O(M^2\Omega/c^3).$$

This proves (i). Assertion (ii) follows from (i) and the facts that $\sum_{i=1}^{\nu} a_i^2/c^2 = O(M\Omega/c^2) = o(1)$ and $O(M^2\Omega/c^3) = o(1)$. Q.E.D.

We proceed with the proof of Theorem 2. As $\Omega \to \infty$,

$$\beta^{-2} = (1/2c) \sum_{i=1}^{\nu} a_i/(c + a_i) = (1/2c) \sum_{i=1}^{\nu} (a_i/c)(1 + a_i/c)^{-1}$$

$$= (1/2c) \sum_{i=1}^{\nu} (a_i/c + O(a_i^2/c^2)) = \Omega/2c^2 + O(M\Omega/c^3)$$

$$= (\Omega/2c^2)(1 + O(M/c)) \sim \Omega/2c^2.$$

By Lemma 4(ii), $\beta^{-2} \sim (\log^2 \lambda)/2\Omega$ so that $\beta \sim (2\Omega)^{\frac{1}{2}}/\log \lambda$. Thus, by Theorem 1,

$$\text{(3.1)} \qquad H(a_1, \cdots, a_\nu) \sim \lambda^{-3/2}(\lambda - 1)\left(\prod_{i=1}^{\nu} \Gamma(a_i + 1)\right)^{-1} e^{-\Omega}(2\pi\Omega)^{\frac{1}{2}}(\log \lambda)^{-1} \prod_{i=1}^{\nu} (c + a_i)^{a_i}.$$

The logarithm of the rightmost product in (3.1) is

$$L = \sum_{i=1}^{\nu} a_i\{\log c + \log(1 + a_i/c)\} = \Omega \log c + \sum_{i=1}^{\nu} a_i\{a_i/c + O(a_i^2/c^2)\}.$$

Therefore,

$$\text{(3.2)} \qquad L = \Omega \log c + \sum_{i=1}^{\nu} a_i^2/c + O(M^2\Omega/c^2)$$

$$= \Omega \log c + \sum_{i=1}^{\nu} a_i^2/c + o(1).$$

By Lemma 4(i), $c(\log \lambda)/\Omega = 1 - \delta$, where $\delta = \sum_{i=1}^{\nu} a_i^2/2c\Omega + O(M^2/c^2) = O(M/c + M^2/c^2) = o(\Omega^{-\frac{1}{2}})$. Therefore,

$$\Omega \log (c(\log \lambda)/\Omega) = \Omega \log (1 - \delta) = -\Omega\,\delta + O(\Omega\,\delta^2)$$
$$= -\sum_{i=1}^{\nu} a_i^2/2c + o(1)$$

so that $\Omega \log c = \Omega \log (\Omega/\log \lambda) - \sum_{i=1}^{\nu} a_i^2/2c + o(1)$. Substituting this expression in (3.2), we have $L = \Omega \log (\Omega/\log \lambda) + \sum_{i=1}^{\nu} a_i^2/2c + o(1)$. By Lemma 4(i), $\log \lambda = \Omega/c + O(M\Omega/c^2)$ so that

$$(1/2c) \sum_{i=1}^{\nu} a_i^2 = (2\Omega)^{-1}(\log \lambda) \sum_{i=1}^{\nu} a_i^2 + O(M^2\Omega/c^2)$$
$$= (2\Omega)^{-1}(\log \lambda) \sum_{i=1}^{\nu} a_i^2 + o(1).$$

Therefore, $L = \Omega \log (\Omega/\log \lambda) + (2\Omega)^{-1}(\log \lambda) \sum_{i=1}^{\nu} a_i^2 + o(1)$. Exponentiation yields $\prod_{i=1}^{\nu} (c + a_i)^{a_i} \sim \Omega^{\Omega}(\log \lambda)^{-\Omega} \exp \{(2\Omega)^{-1}(\log \lambda) \sum_{i=1}^{\nu} a_i^2\}$. Substituting this expression in (3.1) and using Stirling's formula, we obtain the desired asymptotic formula. Q.E.D.

Corollary 1. *The asymptotic formula in Theorem 2 holds if* $M = o(\nu)$ *and* $\nu M = O(\Omega)$.

*Proof.* If $M = o(\nu)$ and $\nu M = O(\Omega)$, then $M^2 = o(\nu M) = o(\Omega)$ so that $M = o(\Omega^{\frac{1}{2}})$. Q.E.D.

Theorem 3. *Suppose* $a = o(\nu)$ *as* $\nu \to \infty$. *Then* $H(a_1, \cdots, a_\nu) \sim (\lambda - 1)\lambda^{(a-3)/2}\Gamma(\nu a + 1)\Gamma^{-\nu}(a + 1) (\log \lambda)^{-1-\nu a}$ *as* $\nu \to \infty$, *where* $a_1 = \cdots = a_\nu = a$.

*Proof.* This follows immediately from Corollary 1. Q.E.D.

The next theorem is useful, for example, in estimating $H(n^a)$ for large integers $a$, where $n$ is a fixed integer $>1$.

Theorem 4. *Let* $t_1, t_2, \cdots$ *be a fixed sequence of positive numbers. Suppose that* $\nu$ *varies with* $a$ *in such a way that* $\sum_{i=1}^{\nu} 1/t_i = o(a)$ *as* $a \to \infty$. *Then*

$$H(at_1, \cdots, at_\nu)$$
$$\sim (\lambda - 1)\lambda^{-3/2}(2\pi a)^{(1-\nu)/2}q^{1/2}\left(\prod_{i=1}^{\nu} t_i\right)^{-1/2}\left(\sum_{i=1}^{\nu} t_i/(t_i + q)\right)^{-1/2}\left(\prod_{i=1}^{\nu} (1 + q/t_i)^{t_i}\right)^a$$

*as* $a \to \infty$, *where* $q$ *is the (unique) positive root of the polynomial* $\lambda x^\nu = \prod_{i=1}^{\nu} (x + t_i)$.

*Proof.* Put $a_i = at_i$, $1 \le i \le \nu$. By Stirling's formula, $\log \Gamma(a_i + 1) = (a_i + 1/2) \log a_i - a_i + \log (2\pi)^{\frac{1}{2}} + O(1/a_i)$. Thus, by Theorem 1,

$$H(a_1, \cdots, a_\nu) \sim \lambda^{-3/2}(\lambda - 1)(2\pi)^{(1-\nu)/2}\left(\prod_{i=1}^{\nu} a_i\right)^{-1/2} \cdot c^{1/2}\left(\sum_{i=1}^{\nu} a_i/(a_i + c)\right)^{-1/2} \prod_{i=1}^{\nu} (1 + c/a_i)^{a_i} \exp\left\{O\left(\sum_{i=1}^{\nu} a_i^{-1}\right)\right\} \tag{3.3}$$

as $a \to \infty$. Since $\sum_{i=1}^{\nu} a_i^{-1} = o(1)$ by hypothesis, the result follows by replacing $a_i$ and $c$ by $at_i$ and $aq$, respectively, in (3.3). Q.E.D.

COROLLARY 2. *Suppose* $\nu = o(a)$ *as* $a \to \infty$. *Then*

$$H(a_1, \cdots, a_\nu) \sim (2\pi a)^{(1-\nu)/2}\lambda^{-3/2+1/2\nu}(\lambda - 1)\nu^{-1/2}(\lambda^{1/\nu} - 1)^{-1}(\lambda(\lambda^{1/\nu} - 1)^{-\nu})^a$$

*as* $a \to \infty$, *where* $a_1 = \cdots = a_\nu = a$.

*Proof*. Apply Theorem 4 with $t_i = 1$, $1 \le i \le \nu$, and note that $q = (\lambda^{1/\nu} - 1)^{-1}$. Q.E.D.

Corollary 2 shows in particular that for a fixed square-free integer $n > 1$, $H(n^a) \sim K_1 \cdot K_2^a/a^{K_3}$ as the integer $a$ tends to infinity, where $K_1$, $K_2$, and $K_3$ are positive constants. For example, when $\lambda = 2$,

$$H(6^a) \sim \frac{1}{2^{9/4}\pi^{1/2}(\sqrt{2} - 1)} \cdot \frac{(2 + \sqrt{2})^{2a}}{\sqrt{a}}.$$

**4. An upper bound for $H(n)$.** Hille [2; p. 137] proved, in the case $\lambda = 2$, that

$$H(a_1, \cdots, a_\nu) < \frac{1}{\nu}\left(\frac{2^{1/\nu}}{2^{1/\nu} - 1}\right)^{\Omega+1}, \tag{4.1}$$

where $a_1, \cdots, a_\nu$ are positive integers whose sum is $\Omega$. We proceed to give an improvement of (4.1) for large $\Omega$.

LEMMA 5. *Let* $a_i > 0$, $1 \le i \le \nu$. *For fixed* $\Omega = \sum_{i=1}^{\nu} a_i$, $H(a_1, \cdots, a_\nu)$ *attains its maximum when* $a_1 = \cdots = a_\nu$.

*Proof*. Assume $H$ attains its maximum at $(a_1, a_2, \cdots, a_\nu)$, where $\Omega = \sum_{i=1}^{\nu} a_i$ and $a_1 \neq a_2$. We shall obtain a contradiction by showing that $H(a_1, a_2, a_3, \cdots, a_\nu) < H(b, b, a_3, \cdots, a_\nu)$, where $b = (a_1 + a_2)/2$. By the definition of $H(a_1, \cdots, a_\nu)$ given in (2.1), it suffices to prove that for each positive integer $k$,

$$\binom{a_1 + k}{a_1}\binom{a_2 + k}{a_2} < \binom{b + k}{b}^2.$$

By comparing geometric and arithmetic means, we have

$$\binom{a_1 + k}{a_1}\binom{a_2 + k}{a_2} = k!^{-2} \prod_{i=1}^{k} (a_1 + i)(a_2 + i) < k!^{-2} \prod_{i=1}^{k} (b + i)^2 = \binom{b + k}{b}^2.$$

In view of Lemma 5, Hille's inequality (4.1) is equivalent to

$$H(a_1, \cdots, a_\nu) < \frac{1}{\nu}\left(\frac{2^{1/\nu}}{2^{1/\nu} - 1}\right)^{\nu a+1},$$

where $a$ is a positive integer such that $a = a_i$, $1 \le i \le \nu$. If $\nu > 1$ and one of the positive integers $a$ or $\nu$ is large, then Hille's upper bound can be significantly improved, as follows from the following theorem.

THEOREM 5. *Let $a_i = a > 0$, $1 \le i \le \nu$. Then as $a\nu \to \infty$,*

$$H(a_1, \cdots, a_\nu) = O\left\{\left(\frac{\lambda^{1/\nu}}{\lambda^{1/\nu} - 1}\right)^{a\nu} \frac{(a\nu)^{1/2}}{(2\pi a)^{\nu/2}}\right\}.$$

*Proof.* By Theorem 1,

$$H(a_1, \cdots, a_\nu) = O(\alpha\beta). \tag{4.2}$$

By Lemmas 1 and 2,

$$\beta = O(\Omega^{\frac{1}{2}}) = O((\nu a)^{\frac{1}{2}}). \tag{4.3}$$

Since $c = a/(\lambda^{1/\nu} - 1)$,

$$\alpha = O(e^{-\Omega}(c + a)^{a\nu}/\Gamma^\nu(a + 1)) = O\left(e^{-\nu a}\left(\frac{\lambda^{1/\nu}}{\lambda^{1/\nu} - 1}\right)^{a\nu} a^{a\nu}/\Gamma^\nu(a + 1)\right).$$

By Stirling's formula,

$$\alpha = O\left[\frac{e^{-\nu a}\left(\frac{\lambda^{1/\nu}}{\lambda^{1/\nu} - 1}\right)^{a\nu} a^{a\nu}}{e^{-\nu a}a^{a\nu}(2\pi a)^{\nu/2}}\right] = O\left(\left(\frac{\lambda^{1/\nu}}{\lambda^{1/\nu} - 1}\right)^{a\nu}(2\pi a)^{-\nu/2}\right). \tag{4.4}$$

The result now follows from (4.2), (4.3), and (4.4). Q.E.D.

**5. Estimates for $H(n)$ in terms of $n$.** Fix $\nu \ge 1$. Let $\mathcal{S}_\nu$ be a set of $\nu$ distinct primes $p_1, p_2, \cdots, p_\nu$. Let $P_\nu$ denote the set of those integers $n \ge 1$ whose prime factors are all in $\mathcal{S}_\nu$. Put $\zeta(s; P_\nu) = \prod_{p \in \mathcal{S}_\nu} (1 - p^{-s})^{-1}$. Let $\rho_\nu = \rho(P_\nu)$ be the (unique) positive root of the equation $\zeta(s; P_\nu) = \lambda$. As shown in [2; p. 140], there exists a constant $K(P_\nu)$ such that

$$\sum_{\substack{m \le n \\ m \in P_\nu}} H(m) \sim K(P_\nu)n^{\rho_\nu}$$

as $n \to \infty$, provided that $\nu > 1$. Thus if $\nu > 1$, then $H(n) = o(n^{\rho_\nu})$ as $n \to \infty$, $n \in P_\nu$. The next theorem shows that in fact $H(n) = O(n^{\rho_\nu}(\log n)^{(1-\nu)/2})$ for all $n \in P_\nu$. Here, the O-constant depends of course on $P_\nu$ as will all O-constants in this section. Note that this O-estimate clearly holds when $\nu = 1$, since $H(p_1^a) = \lambda^{a-1}/(\lambda - 1)^a$ for $a \ge 1$ [1; p. 669].

THEOREM 6. *Fix $\nu \ge 1$. Then for all $n \in P_\nu$, $H(n) = O(n^{\rho_\nu}(\log n)^{(1-\nu)/2})$.*

For the proof we shall need some notation and a lemma. Let $X_\nu$ be the Cartesian product space $\prod_{i=1}^{\nu} \mathbf{R}^+$, where $\mathbf{R}^+ = (0, \infty)$. Let $\bar{X}_\nu$ denote the closure of $X_\nu$. Define the functions $R_\nu = R_{P_\nu}$ from $X_\nu$ into $\mathbf{R}$ by $R_\nu(a_1, \cdots, a_\nu) = (p_1^{a_i} \cdots p_\nu^{a_\nu})^{-\rho_\nu} \prod_{i=1}^{\nu} (1 + c/a_i)^{a_i}$, where $c$ is defined as in Theorem 1. By the implicit function theorem, $c \in C'$ on $X$, i.e., $\partial c/\partial a_i$ exists and is continuous, $1 \leq i \leq \nu$. Hence $R_\nu \in C'$ on $X$. It is easy to show that $c$ can be extended to a continuous function on $\bar{X}_\nu$ by defining $c$ to be zero at the origin. Thus $R_\nu$ can be extended to a continuous function on $\bar{X}_\nu$ by defining $R_\nu$ to be 1 at the origin and by defining $R_\nu(a_1, \cdots, a_\nu) = (p_1^{a_1} \cdots p_k^{a_k})^{-\rho_\nu} \prod_{i=1}^{k} (1 + c/a_i)^{a_i}$ when $a_1, \cdots, a_k$ are nonzero and $a_{k+1} = \cdots = a_\nu = 0$, $1 \leq k < \nu$. To see that $R_\nu$ is continuous at the origin, for example, observe that as $(a_1, \cdots, a_\nu) \in X$ approaches the origin, $1 < \prod_{i=1}^{\nu} (1 + c/a_i)^{a_i} < \prod_{i=1}^{\nu} e^c = e^{c\nu} \to 1$.

LEMMA 6. *Fix $\nu \geq 1$. For each $(a_1, \cdots, a_\nu) \in \bar{X}_\nu$, $R_\nu(a_1, \cdots, a_\nu) \leq 1$, and equality holds if and only if there exists $a \geq 0$ such that $a_i = a(p_i^{\rho_\nu} - 1)^{-1}$, $1 \leq i \leq \nu$.*

*Proof.* It can be verified by direct calculation that when $a \geq 0$ and $a_i = a(p_i^{\rho_\nu} - 1)^{-1}$, $1 \leq i \leq \nu$ (note that this implies $c = a$), then $R_\nu(a_1, \cdots, a_\nu) = 1$. We proceed to prove the lemma by induction on $\nu$. Writing $P_1 = \{p_1\}$, we have

$$R_1(a_1) = (p_1^{-\rho_1}(1 + c/a_1))^{a_1} = \left\{\frac{\lambda - 1}{\lambda} \cdot \frac{\lambda}{\lambda - 1}\right\}^{a_1} = 1$$

for each $a_1 \in X_1$. Let $\nu \geq 2$ and suppose the lemma is true for all positive integers smaller than $\nu$. Then, except at the origin, $R_\nu < 1$ on the boundary of $X_\nu$. To see this, suppose that for some $k$ such that $1 \leq k < \nu$, $a_1, \cdots, a_k$ are nonzero and $a_{k+1} = \cdots = a_\nu = 0$. As noted in [2; p. 139], $\rho_\nu > \rho_k$ so that $R_\nu(a_1, \cdots, a_\nu) = (p_1^{a_1} \cdots p_k^{a_k})^{-\rho_\nu} \prod_{i=1}^{k} (1 + c/a_i)^{a_i} < R_k(a_1, \cdots, a_k) \leq 1$.

Fix $x > 1$. We wish to maximize $R_\nu$ on $X_\nu$ subject to the constraint $p_1^{a_1} \cdots p_\nu^{a_\nu} = X$. Clearly a maximum is attained; moreover, it must be attained at points in $X_\nu$, since, except at the origin, $R_\nu < 1$ on the boundary of $X_\nu$. It suffices to show that subject to the constraint $x = p_1^{a_1} \cdots p_\nu^{a_\nu}$, $\log \prod_{i=1}^{\nu} (1 + c/a_i)^{a_i}$ attains its maximum value at precisely those points $(a_1, \cdots, a_\nu) \in X$ for which there exists $a \geq 0$ such that $a_i = a(p_i^{\rho_\nu} - 1)^{-1}$, $1 \leq i \leq \nu$. According to Lagrange's method, any point $(a_1, \cdots, a_\nu) \in X_\nu$ at which the maximum is attained must satisfy the following system for some $B$.

$$\text{(5.1)} \quad \frac{\partial}{\partial a_j}\left\{\sum_{i=1}^{\nu} a_i \log (1 + c/a_i) - B\left(\sum_{i=1}^{\nu} a_i \log p_i - \log x\right)\right\} = 0, \ 1 \leq j \leq \nu.$$

Observe that

$$\frac{\partial}{\partial a_j} \sum_{i=1}^{\nu} a_i \log (1 + c/a_i) = \log (1 + c/a_j) + \frac{\partial c}{\partial a_j} \sum_{i=1}^{\nu} \frac{a_i}{c + a_i} - \frac{c}{c + a_j}.$$

Since $\log \lambda = \sum_{i=1}^{\nu} \log (1 + a_i/c)$, it follows by implicit differentiation that

$$\frac{\partial c}{\partial a_j} \sum_{i=1}^{\nu} \frac{a_i}{c + a_i} - \frac{c}{c + a_j} = 0.$$

Thus,

$$\frac{\partial}{\partial a_j} \sum_{i=1}^{\nu} a_i \log (1 + c/a_i) = \log (1 + c/a_j).$$

Thus (5.1) becomes $\log (1 + c/a_j) = B \log p_j$, $1 \leq j \leq \nu$. These equations imply

$$a_j/c = (p_j^{B} - 1)^{-1}, \quad 1 \leq j \leq \nu, \tag{5.2}$$

and

$$\lambda = \prod_{j=1}^{\nu} (1 + a_j/c) = \prod_{j=1}^{\nu} (1 - p_j^{-B})^{-1}. \tag{5.3}$$

By (5.3) and the definition of $\rho_\nu$, $B = \rho_\nu$. Thus (5.2) becomes $a_i = c(p_i^{\rho_\nu} - 1)^{-1}$, $1 \leq i \leq \nu$. Thus the points at which the maximum is attained have the desired form. Q.E.D.

*Proof of Theorem* 6. Let $a_i > 0$, $1 \leq i \leq \nu$. We prove the stronger statement

$$H(a_1, \cdots, a_\nu) = O\left((p_1^{a_1} \cdots p_\nu^{a_\nu})^{\rho_\nu} \left(\sum_{i=1}^{\nu} a_i \log p_i\right)^{(1-\nu)/2}\right)$$

as $\Omega \to \infty$. Since $\sum_{i=1}^{\nu} a_i \log p_i < \Omega \max \{\log p_i : 1 \leq i \leq \nu\}$, it suffices to show, by Theorem 1, that $\alpha = O((p_1^{a_1} \cdots p_\nu^{a_\nu})^{\rho_\nu} \Omega^{-\nu/2})$ or, equivalently,

$$R_\nu(a_1, \cdots, a_\nu) T_\nu(a_1, \cdots, a_\nu) = O(1), \tag{5.4}$$

where $T_\nu(a_1, \cdots, a_\nu) = \Omega^{\nu/2} \prod_{i=1}^{\nu} (a_i/e)^{a_i}/\Gamma(a_i + 1)$. Assume (5.4) is false. Then there exists a sequence of vectors $A = \{(a_{1k}, \cdots, a_{\nu k}) : k \geq 1\}$ such that

$$R_\nu(a_{1k}, \cdots, a_{\nu k}) T_\nu(a_{1k}, \cdots, a_{\nu k}) \to \infty \tag{5.5}$$

as $k \to \infty$. Let $M_k = \max \{a_{ik} : 1 \leq i \leq \nu\}$ and let $b_{ik} = a_{ik}/M_k$. From the definition of $c$ it is easily seen that $R_\nu(a_{1k}, \cdots, a_{\nu k}) = R(b_{1k}, \cdots, b_{\nu k})^{M_k}$. By Lemma 6, $R_\nu(b_{1k}, \cdots, b_{\nu k}) \leq 1$ for each $k$. In fact, $\lim_{k\to\infty} R_\nu(b_{1k}, \cdots, b_{\nu k}) = 1$, as we now show. Assume $R_\nu(b_{1k}, \cdots, b_{\nu k}) \leq \delta < 1$ for infinitely many $k$. Then for these $k$, $R_\nu(a_{1k}, \cdots, a_{\nu k}) \leq \delta^{M_k}$. Now, if $a_i \leq 1$, $(a_i/e)^{a_i}/\Gamma(a_i + 1) = O(1)$. Thus, by Stirling's formula,

$$\begin{aligned} T_\nu(a_1, \cdots, a_\nu) &= O(\Omega^{\nu/2} \prod_{\substack{i=1 \\ a_i > 1}}^{\nu} (2\pi a_i)^{-\frac{1}{2}}) \\ &= O(\Omega^{\nu/2}) = O\{(\nu \max \{a_i : 1 \leq i \leq \nu\})^{\nu/2}\}. \end{aligned}$$

Therefore, $T_\nu(a_{1k}, \cdots, a_{\nu k}) = O(M_k^{\nu/2})$. Thus, for infinitely many $k$, $R_\nu(a_{1k}, \cdots, a_{\nu k}) T_\nu(a_{1k}, \cdots, a_{\nu k}) = O(\delta^{M_k} M_k^{\nu/2}) = O(1)$. This contradicts (5.5), and thus $\lim_{k\to\infty} R_\nu(b_{1k}, \cdots, b_{\nu k}) = 1$.

For each $k$, $|b_{ik}| \leq 1$, $1 \leq i \leq \nu$, and equality holds for at least one $i$. Hence the sequence $B = \{(b_{1k}, \cdots, b_{\nu k}) : k \geq 1\}$ has a cluster point $(b_1, \cdots, b_\nu) \in \bar{X}_\nu$ which is not the origin. We may assume without loss of generality that $(b_1, \cdots, b_\nu)$ is the limit of $B$; otherwise replace $A$ by an appropriate subsequence. By continuity of $R_\nu$, $R_\nu(b_1, \cdots, b_\nu) = 1$. By Lemma 6, there exists an $a > 0$ such that $b_i = a(p_i^{\rho_\nu} - 1)^{-1}$, $1 \leq i \leq \nu$. By Stirling's formula and the definition of $T_\nu$, $T_\nu(a_1, \cdots, a_\nu) < \Omega^{\nu/2} \prod_{i=1}^{\nu} (2\pi a_i)^{-\frac{1}{2}}$. Thus, for each $k$

$$T_\nu(a_{1k}, \cdots, a_{\nu k}) < \left(\sum_{i=1}^{\nu} a_{ik}\right)^{\nu/2} \prod_{i=1}^{\nu} (2\pi a_{ik})^{-\frac{1}{2}}$$
$$= \left(\sum_{i=1}^{\nu} b_{ik}\right)^{\nu/2} \prod_{i=1}^{\nu} (2\pi b_{ik})^{-\frac{1}{2}} .$$

Hence,

$$\limsup_{k\to\infty} T_\nu(a_{1k}, \cdots, a_{\nu k}) \leq \left(\sum_{i=1}^{\nu} b_i\right)^{\nu/2} \prod_{i=1}^{\nu} (2\pi b_i)^{-\frac{1}{2}}$$

so that $T_\nu(a_{1k}, \cdots, a_{\nu k}) = O(1)$. By Lemma 6, $R_\nu(a_{1k}, \cdots, a_{\nu k}) \leq 1$ for each $k$. Hence, $R_\nu(a_{1k}, \cdots, a_{\nu k})T_\nu(a_{1k}, \cdots, a_{\nu k}) = O(1)$, which contradicts (5.5). Q.E.D.

The next theorem shows that the O in Theorem 6 cannot be replaced by o, i.e.,

$$H(n) \neq o(n^{\rho_\nu}(\log n)^{(1-\nu)/2}) \quad \text{as} \quad n \to \infty, n \in P_\nu . \tag{5.6}$$

In the case $\lambda = 2$, Hille [2; p. 141] proved a weaker version of (5.6) in which the exponent $(1 - \nu)/2$ is replaced by $1 - \nu$.

THEOREM 7. *Let* $t_i = (p_i^{\rho_\nu} - 1)^{-1}$ *and let* $e_i = e_i(a) = [at_i] + 1$, $1 \leq i \leq \nu$. *Let* $n_a = p_1^{e_1} \cdots p_\nu^{e_\nu}$. *Then there exists a constant* $C(P_\nu) > 0$ *such that for all large* $a$, $H(n_a) > C(P_\nu)$ $(\log n_a)^{(1-\nu)/2} n_a^{\rho_\nu}$.

*Proof.* By Theorem 4, there exists a constant $C_1(P_\nu)$ such that for all large $a$,

$$\log H(n_a) > \log H(at_1, \cdots, at_\nu)$$
$$> C_1(P_\nu) + \frac{1-\nu}{2} \log a + a \sum_{i=1}^{\nu} t_i \log (1 + q/t_i).$$

By definition of $\rho_\nu$, $\prod_{i=1}^{\nu} (1 + t_i) = \lambda$. Hence, by definition of $q$, $q = 1$. Thus $1 + q/t_i = p_i^{\rho_\nu}$ and

$$\log H(n_a) > C_1(P_\nu) + \frac{1-\nu}{2} \log a + a\rho_\nu \sum_{i=1}^{\nu} t_i \log p_i . \tag{5.7}$$

Now,

$$\log n_a \leq \sum_{i=1}^{\nu} (at_i + 1) \log p_i = a \sum_{i=1}^{\nu} t_i \log p_i + \sum_{i=1}^{\nu} \log p_i .$$

Thus, for large $a$ there exist constants $C_2(P_\nu)$ and $C_3(P_\nu)$ such that $a \sum_{i=1}^{\nu} t_i \log p_i \geq C_2(P_\nu) + \log n_a$ and $\log a \geq C_3(P_\nu) + \log \log n_a$. Therefore, by (5.7), there exists a constant $C_4(P_\nu)$ such that for large $a$, $\log H(n_a) >$

$C_4(P_\nu) + ((1 - \nu)/2) \log \log n_a + \rho_\nu \log n_a$. Exponentiation yields the desired result. Q.E.D.

**6. Estimates for Eulerian numbers in terms of n.** It is well known that the value of the divisor function $d(n) = \sum_{d|n} 1$ comes close to the "maximum order" of $d(n)$, namely, $\exp((\log 2)(\log n)/\log \log n)$, when $n$ is large and square-free. On the other hand, for large square-free $n$, $H(n)$ is far from its maximum order; in fact $H(n) = O(n^{1+\epsilon})$ for any $\epsilon > 0$. To see this, suppose $n$ is the product of $\nu$ primes, $p_1, \cdots, p_\nu$, $\nu \geq 1$. Write $H_\nu = H(n)$. Let $\epsilon > 0$. By Theorem 3,

$$H_\nu \sim (\lambda - 1)\lambda^{-1}\nu! (\log \lambda)^{-\nu-1} \tag{6.1}$$

as $\nu \to \infty$. Therefore, for large $\nu$, $H_\nu < \nu^\nu/(\log \lambda)^\nu < (\nu \log \nu)^\nu < \exp\{(1 + \epsilon/2)\nu \log \nu\}$. Let $2 = q_1 < q_2 < \cdots$ be the primes. It is well known that as $\nu \to \infty$, $\nu \log \nu \sim q_\nu \sim \sum_{i=1}^{\nu} \log q_i$. Since $\sum_{i=1}^{\nu} \log q_i \leq \sum_{i=1}^{\nu} \log p_i = \log n$, it follows that $(1 + \epsilon/2)\nu \log \nu < (1 + \epsilon) \log n$ for large $\nu$. Therefore $H(n) = O(\exp((1 + \epsilon) \log n)) = O(n^{1+\epsilon})$.

**7. Estimates for Eulerian numbers in terms of $\nu$.** In the next theorem we give an estimate for $H_\nu$ which is considerably sharper than (6.1). The proof, however, is not elementary.

THEOREM 8. *Let $\nu \geq 1$. Then $H_\nu = (\lambda - 1)\lambda^{-1}\nu! (\log \lambda)^{-\nu-1} + E_\nu$, where $|E_\nu| < (12\lambda)^{-1}(\lambda - 1)\nu! (\log^2 \lambda + 4\pi^2)^{(1-\nu)/2}$.*

*Proof.* By (2.1) we have

$$H_\nu = (\lambda - 1)\lambda^{-1} \sum_{k=0}^{\infty} \lambda^{-k}k^\nu. \tag{7.1}$$

Now

$$\sum_{k=0}^{\infty} \lambda^{-k}k^\nu = \int_0^\infty \lambda^{-x}x^\nu \, dx + B_\nu, \tag{7.2}$$

where $B_\nu = \int_0^\infty \lambda^{-x}x^\nu d([x] - x + 1/2)$. Integration by parts yields

$$\begin{aligned} B_\nu &= \int_0^\infty ([x] - x + \tfrac{1}{2})\, d(\lambda^{-x}x^\nu) = \int_0^\infty \sum_{n=1}^{\infty} \frac{\sin 2\pi nx}{\pi n}\, d(\lambda^{-x}x^\nu) \\ &= \sum_{n=1}^{\infty} \int_0^\infty \frac{\sin 2\pi nx}{\pi n}\, d(\lambda^{-x}x^\nu) = 2 \sum_{n=1}^{\infty} \int_0^\infty \lambda^{-x}x^\nu \cos 2\pi nx \, dx; \end{aligned}$$

the interchange of integration and summation is justifiable by the Lebesgue convergence theorem. Thus, from (7.1) and (7.2)

$$\begin{aligned} \lambda(\lambda - 1)^{-1}H_\nu &= \sum_{n=-\infty}^{\infty} \int_0^\infty \lambda^{-x}x^\nu \cos 2\pi nx \, dx \\ &= \sum_{n=-\infty}^{\infty} \int_0^\infty \lambda^{-x}x^\nu e^{-2\pi inx} \, dx = \sum_{n=-\infty}^{\infty} \int_0^\infty x^\nu e^{-z} \, dx, \end{aligned}$$

where $z = x(2\pi in + \log \lambda)$. Changing the path of integration, we have $\lambda(\lambda - 1)^{-1}H_\nu = \sum_{n=-\infty}^{\infty} (\log \lambda + 2\pi in)^{-\nu-1} \int_0^\infty z^\nu e^{-z} dz$ so that

$$H_\nu = (\lambda - 1)\lambda^{-1}\nu! \sum_{n=-\infty}^{\infty} (\log \lambda + 2\pi in)^{-\nu-1}. \tag{7.3}$$

*Remark.* Formula (7.3) is actually valid by analytic continuation for all $\lambda$ except 0 and 1, because by [1; p. 671, Formula (2.12)], $H_\nu$ is analytic as a function of $\lambda$ except at $\lambda = 1$.

By (7.3), $E_\nu = 2(\lambda - 1)\lambda^{-1}\nu! \operatorname{Re} \sum_{n=1}^{\infty} (\log \lambda + 2\pi in)^{-(\nu+1)}$. Therefore,

$$\begin{aligned} |E_\nu| &\le 2(\lambda - 1)\lambda^{-1}\nu! \sum_{n=1}^{\infty} (\log^2 \lambda + 4\pi^2 n^2)^{-(\nu+1)/2} \\ &= 2(\lambda - 1)\lambda^{-1}\nu!(\log^2 \lambda + 4\pi^2)^{-(\nu+1)/2} F_\nu , \end{aligned}$$

where

$$F_\nu = \sum_{n=1}^{\infty} \left( \frac{\log^2 \lambda + 4\pi^2}{\log^2 \lambda + 4\pi^2 n^2} \right)^{(\nu+1)/2} < \sum_{n=1}^{\infty} \frac{\log^2 \lambda + 4\pi^2}{4\pi^2 n^2} = \frac{\log^2 \lambda + 4\pi^2}{24}.$$

Therefore, $|E_\nu| < (12\lambda)^{-1}(\lambda - 1)\nu! \, (\log^2 \lambda + 4\pi^2)^{(1-\nu)/2}$. Q.E.D.

## 8. Upper and lower bounds for $H(n)$.

THEOREM 9. *Let $\{a_i : 1 \le i \le \nu\}$ be a set of positive integers, let $\Omega = \sum_{i=1}^{\nu} a_i$, and let $M = \max \{a_i : 1 \le i \le \nu\}$. Then $L < H(a_1, \cdots, a_\nu) < U$, where*

$$L = (\lambda - 1)\lambda^{-1}\Omega!\{(\log \lambda)^{-\Omega-1} - \tfrac{1}{12} (\log^2 \lambda + 4\pi^2)^{(1-\Omega)/2}\} \prod_{i=1}^{\nu} (a_i!)^{-1}$$

*and*

$$U = (\lambda - 1)\lambda^{M-2}\Omega!\{(\log \lambda)^{-\Omega-1} + \tfrac{1}{12} (\log^2 \lambda + 4\pi^2)^{(1-\Omega)/2}\} \prod_{i=1}^{\nu} (a_i!)^{-1}.$$

*Proof.* Let $J = \prod_{i=1}^{\nu} (a_i!)\lambda(\lambda - 1)^{-1}H(a_1, \cdots, a_\nu)$. Then, by (2.1),

$$\begin{aligned} J &= \prod_{i=1}^{\nu} (a_i!) \sum_{k=1}^{\infty} \lambda^{-k} \prod_{i=1}^{\nu} \binom{a_i + k - 1}{a_i} \\ &= \sum_{k=1}^{\infty} \lambda^{-k} \prod_{i=1}^{\nu} (k + a_i - 1) \cdots (k + 1)k. \end{aligned}$$

Thus,

$$\begin{aligned} \sum_{k=1}^{\infty} \lambda^{-k}k^{\Omega} &= \sum_{k=1}^{\infty} \lambda^{-k} \prod_{i=1}^{\nu} k^{a_i} \le J \le \sum_{k=1}^{\infty} \lambda^{-k} \prod_{i=1}^{\nu} (k + M - 1)^{a_i} \\ &= \sum_{k=1}^{\infty} \lambda^{-k}(k + M - 1)^{\Omega} = \lambda^{M-1} \sum_{k=M}^{\infty} \lambda^{-k}k^{\Omega} \le \lambda^{M-1} \sum_{k=1}^{\infty} \lambda^{-k}k^{\Omega}. \end{aligned}$$

Therefore, by (7.1), $\lambda(\lambda - 1)^{-1}H_\Omega \le J \le \lambda(\lambda - 1)^{-1}\lambda^{M-1}H_\Omega$. It follows that $H_\Omega \prod_{i=1}^{\nu} (a_i!)^{-1} \le H(a_1, \cdots, a_\nu) \le \lambda^{M-1}H_\Omega \prod_{i=1}^{\nu} (a_i!)^{-1}$. The desired result now follows from Theorem 8. Q.E.D.

## References

1. L. Carlitz, *Extended Bernoulli and Eulerian numbers*, Duke Math. J., vol. 31(1964), pp. 667–689.
2. E. Hille, *A problem in "Factorisatio Numerorum"*, Acta Arith., vol. 2(1936), pp. 134–144.
3. E. C. Titchmarsh, *The Theory of Functions*, London, Oxford University Press, 1939.

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