CLAUSEN'S THEOREM AND HYPERGEOMETRIC FUNCTIONS OVER FINITE FIELDS

Ron Evans
Department of Mathematics
University of California at San Diego
La Jolla, CA 92093-0112

and

John Greene
Department of Mathematics and Statistics
University of Minnesota–Duluth
Duluth, MN 55812

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Abstract

We prove a general identity for a ${}_3F_2$ hypergeometric function over a finite field $\mathbb{F}_q$, where q is a power of an odd prime. A special case of this identity was proved by Greene and Stanton in 1986. As an application, we prove a finite field analogue of Clausen's Theorem expressing a ${}_3F_2$ as the square of a ${}_2F_1$. As another application, we evaluate an infinite family of ${}_3F_2(z)$ over $\mathbb{F}_q$ at $z = -1/8$. This extends a result of Ono, who evaluated one of these ${}_3F_2(-1/8)$ in 1998, using elliptic curves.

1 Introduction and main theorems

Let $\mathbb{F}_q$ be a field of q elements, where q is a power of an odd prime p. Throughout this paper, $A, B, C, D, E, R, S, T, M, W, \chi, \psi, \varepsilon, \phi$ will denote complex multiplicative characters on $\mathbb{F}_q^*$, extended to map 0 to 0. The notation ε, ϕ will always be reserved for the trivial and quadratic characters, respectively. Write $\overline{A}$ for the inverse (complex conjugate) of A. For $y \in \mathbb{F}_q$, define the additive character

$$\zeta^y := \exp\left(\frac{2\pi i}{p}\left(y^p + y^{p^2} + \cdots + y^q\right)\right). \tag{1.1}$$

Recall the definitions of the Gauss sum

$$G(A) = \sum_{y \in \mathbb{F}_q} A(y)\zeta^y \tag{1.2}$$

and the Jacobi sum

$$J(A, B) = \sum_{y \in \mathbb{F}_q} A(y)B(1-y). \tag{1.3}$$

Note that

$$G(\varepsilon) = -1, \quad J(\varepsilon, \varepsilon) = q - 2,$$

and for nontrivial A,

$$G(A)G(\overline{A}) = A(-1)q, \quad J(A, \overline{A}) = -A(-1).$$

Gauss and Jacobi sums are related by [5, (1.14)], [2, p. 59]

$$J(A, B) = G(A)G(B)/G(AB), \quad \text{if } AB \neq \varepsilon. \tag{1.4}$$

The Gauss sums satisfy the Hasse–Davenport relation [5, (2.18)], [2, p. 59]

$$A(4)G(A)G(A\phi) = G(A^2)G(\phi). \tag{1.5}$$

For $x \in \mathbb{F}_q$, define the hypergeometric ${}_2F_1$ function over $\mathbb{F}_q$ by [5, p. 82]

$${}_2F_1\left(\begin{matrix} A, B \\ C \end{matrix} \,\middle|\, x\right) = \frac{\varepsilon(x)}{q} \sum_{y \in \mathbb{F}_q} B(y)\overline{B}C(y-1)\overline{A}(1-xy) \tag{1.6}$$

and the hypergeometric ${}_3F_2$ function over $\mathbb{F}_q$ by [5, p. 83]

$$\begin{aligned} &{}_3F_2\left(\begin{matrix} A, B, C \\ D, E \end{matrix} \,\middle|\, x\right) \\ &= \frac{\varepsilon(x)}{q^2} \sum_{y,z \in \mathbb{F}_q} C(y)\overline{C}E(y-1)B(z)\overline{B}D(z-1)\overline{A}(1-xyz). \end{aligned} \tag{1.7}$$

The "binomial coefficient" over $\mathbb{F}_q$ is defined by [5, p. 80]

$$\binom{A}{B} = \frac{B(-1)}{q} J(A, \overline{B}). \tag{1.8}$$

Define the function

$$F(A, B; x) = \frac{q}{q-1} \sum_{\chi} \binom{A\chi^2}{\chi} \binom{A\chi}{B\chi} \chi\left(\frac{x}{4}\right), \quad x \in \mathbb{F}_q, \tag{1.9}$$

and its normalization

$$F^*(A, B; x) = F(A, B; x) + AB(-1)\overline{A}(x/4)/q. \tag{1.10}$$

We will relate the function F^* to a ${}_2F_1$ in both Theorems 1.2 and 1.6 below.

Our main result is the following theorem.

Theorem 1.1. *Let $AB = C^2$ where $C \neq \phi$ and $A, B \notin \{\varepsilon, C\}$. Then for $x \neq 1$,*

$$\begin{aligned} {}_3F_2\left(\begin{matrix} A, B, C\phi \\ C^2, C \end{matrix} \,\middle|\, x\right) &= -\overline{C}(x)\phi(1-x)/q \\ &+ \overline{C}(-4)\overline{C}\phi(1-x) F^*\left(A, C; \frac{x}{x-1}\right) F^*\left(B, C; \frac{x}{x-1}\right). \end{aligned}$$

The proof of Theorem 1.1 is given in Section 2.

The special case $A = B = \phi,\ C = \varepsilon$ of Theorem 1.1 is due to Greene and Stanton [6]. This case was used by Ono [8, Theorem 5], [9] to give explicit determinations of

$$ {}_3F_2\left(\begin{array}{c|} \phi, \phi, \phi \\ \varepsilon, \varepsilon \end{array}\ x \right) $$

for special values of x. For an infinite family of such determinations, see [3].

We proceed to apply Theorem 1.1 to produce a finite field analogue (Theorem 1.5) of Clausen's famous classical identity [1, p. 86]

$$ (1.11) \qquad {}_3F_2\left(\begin{array}{c|} 2c-2s-1,\ 2s,\ c-\frac{1}{2} \\ 2c-1,\ c \end{array}\ x \right) = {}_2F_1\left(\begin{array}{c|} c-s-\frac{1}{2},\ s \\ c \end{array}\ x \right)^2. $$

Formula (1.11) was utilized in de Branges' proof of the Bieberbach conjecture. For further applications of (1.11), consult Askey's Foreword in [4, pp. xiv–xv].

In the special case when the character A is a square, we can relate $F^*(A, C; x)$ to a ${}_2F_1$ as follows.

Theorem 1.2. *Let $R^2 \notin \{\varepsilon, C, C^2\}$. Then*

$$ F^*(R^2, C; x) = R(4)\frac{J(\phi, C\overline{R}^2)}{J(\overline{R}C, \overline{R}\phi)}\ {}_2F_1\left(\begin{array}{c|} R\phi, R \\ C \end{array}\ x \right). $$

Theorem 1.2 is proved in Section 3. Combining Theorems 1.1 and 1.2, we obtain the following result.

Proposition 1.3. *Let $C^2 = R^2S^2$, where $C \neq \phi$ and $R^2, S^2 \notin \{\varepsilon, C\}$. Then for $x \neq 1$,*

$$ \begin{aligned} {}_3F_2\left(\begin{array}{c|} R^2, S^2, C\phi \\ C^2, C \end{array}\ x \right) &= -\overline{C}(x)\phi(1-x)/q \\ &+ \frac{C(-1)\overline{C}\phi(1-x)J(\phi, C\overline{R}^2)J(\phi, C\overline{S}^2)}{J(\overline{R}C, \overline{R}\phi)J(\overline{S}C, \overline{S}\phi)}\, {}_2F_1\left(\begin{array}{c|} R\phi, R \\ C \end{array}\ \frac{x}{x-1} \right) {}_2F_1\left(\begin{array}{c|} S\phi, S \\ C \end{array}\ \frac{x}{x-1} \right). \end{aligned} $$

For $x \neq 1$, there is a transformation formula [5, Thm. 4.4(iv)]

$$ (1.12) \qquad {}_2F_1\left(\begin{array}{c|} R\phi, R \\ C \end{array}\ \frac{x}{x-1} \right) = C(-1)\overline{C}R^2\phi(1-x)\ {}_2F_1\left(\begin{array}{c|} \overline{R}C\phi, \overline{R}C \\ C \end{array}\ \frac{x}{x-1} \right). $$

Using (1.12) in Proposition 1.3, we obtain the following result.

Proposition 1.4. *Let $C = RS$, where $C \neq \phi$ and $R^2, S^2 \notin \{\varepsilon, C\}$. Then for $x \neq 1$,*

$$
{}_3F_2\left(\begin{matrix} R^2, S^2, C\phi \\ C^2, C \end{matrix} \,\middle|\, x\right) = -\overline{C}(x)\phi(1-x)/q + \frac{J(\phi, C\overline{R}^2)J(\phi, C\overline{S}^2)}{J(\overline{R}C, \overline{R}\phi)J(\overline{S}C, \overline{S}\phi)}\overline{S}^2(1-x)\ {}_2F_1\left(\begin{matrix} S\phi, S \\ C \end{matrix} \,\middle|\, \frac{x}{x-1}\right)^2.
$$

For $x \neq 1$, there is another transformation formula [5, Thm 4.4(iii)]

$$
{}_2F_1\left(\begin{matrix} S\phi, S \\ C \end{matrix} \,\middle|\, \frac{x}{x-1}\right) = S(1-x)\ {}_2F_1\left(\begin{matrix} C\overline{S}\phi, S \\ C \end{matrix} \,\middle|\, x\right). \tag{1.13}
$$

Using (1.13) in Proposition 1.4, along with (1.5), we obtain the following direct finite field analogue of Clausen's identity (1.11).

Theorem 1.5. *Let $C \neq \phi$ and $S^2 \notin \{\varepsilon, C, C^2\}$. Then for $x \neq 1$,*

$$
{}_3F_2\left(\begin{matrix} C^2\overline{S}^2, S^2, C\phi \\ C^2, C \end{matrix} \,\middle|\, x\right) = -\overline{C}(x)\phi(1-x)/q + \frac{\overline{C}(4)J(S\overline{C}, S\overline{C})}{J(S,S)}\,{}_2F_1\left(\begin{matrix} C\overline{S}\phi, S \\ C \end{matrix} \,\middle|\, x\right)^2.
$$

Theorem 1.2 relates $F^*(A, C; x)$ to a ${}_2F_1$ when A is a square. We can also relate $F^*(A, C; x)$ to a ${}_2F_1$ when x is a square, as follows.

Theorem 1.6. *Let $C \neq \phi$, $A \neq \varepsilon$, and $u \notin \{0, 1\}$. Then*

$$
F^*(A, C; u^{-2}) = \frac{AC(-1)C\phi(2)A(u)C\overline{A}\phi(1-u)J(A\phi, C\overline{A})}{J(\phi, A\phi)}\,{}_2F_1\left(\begin{matrix} \overline{C}\phi, C\phi \\ C\overline{A}\phi \end{matrix} \,\middle|\, \frac{1-u}{2}\right).
$$

Theorem 1.6 is proved in Section 4, by means of two lemmas relating F^* and ${}_2F_1$ to finite field analogues of Gegenbauer functions.

With $x = 1/(1-u^2)$, use Theorem 1.6 and (4.9) to substitute for the first and second factors F^* in Theorem 1.1, respectively. This yields the following specialization of our main result.

Theorem 1.7. *Let $C \neq \phi$, $A \notin \{\varepsilon, C, C^2\}$, and $u^2 \notin \{0, 1\}$. Then*

$$
{}_3F_2\left(\begin{matrix} A, \overline{A}C^2, C\phi \\ C^2, C \end{matrix} \,\middle|\, \frac{1}{1-u^2}\right) = -\phi(-1)C\phi(1-u^2)/q + \frac{\phi(-1)\overline{A}C^2(1-u)A(1+u)J(A, \overline{A}C^2)}{J(C\phi, C\phi)}\,{}_2F_1\left(\begin{matrix} \overline{C}\phi, C\phi \\ C\overline{A}\phi \end{matrix} \,\middle|\, \frac{1-u}{2}\right)^2.
$$

As an application, we will prove in Section 5 the following evaluation of ${}_3F_2(-1/8)$ for an infinite family of hypergeometric ${}_3F_2$ functions over $\mathbb{F}_q$.

Theorem 1.8. *Suppose that S is a character whose order is not 1, 3, or 4. Then*

$$(1.14)\quad \begin{aligned} &{}_3F_2\left(\begin{matrix} \overline{S}, S^3, S \\ S^2, S\phi \end{matrix} \,\Bigg|\, -\frac{1}{8}\right) \\ &= \begin{cases} -\phi(-1)S(-8)/q, & \text{if } S \text{ is not a square} \\ \phi(-1)S(8)/q + \frac{\phi(-1)S(2)J(\overline{S},S^3)}{q^2 J(S,S)}(J(S,D)^2 + J(S,D\phi)^2), & \text{if } S = D^2. \end{cases} \end{aligned}$$

Formula (1.14) is a direct finite field analogue of the following evaluation [10] of a classical ${}_3F_2$:

$$(1.15)\quad {}_3F_2\left(\begin{matrix} s, 1-s, 3s-1 \\ 2s, s+1/2 \end{matrix} \,\Bigg|\, -\frac{1}{8}\right) = \frac{2^{3s-3}\Gamma(s/2)^2\Gamma(s+1/2)^2}{\pi\Gamma(3s/2)^2}.$$

This classical identity is a consequence of Clausen's Theorem (1.11) and Kummer's Theorem [5, (4.12)]. In Section 5, we show that our identity (1.14) follows analogously from a version of Clausen's Theorem over $\mathbb{F}_q$ (Theorem 1.7) and Kummer's Theorem over $\mathbb{F}_q$ [5, (4.11)].

We remark that it is not difficult to give separate evaluations of the left side of (1.14) in the three exceptional cases where S has order 1, 3, or 4. In the case where S has order 2, i.e., $S = \phi$, Theorem 1.8 reduces to Ono's evaluation of a ${}_3F_2(-1/8)$ in [8, Theorem 6(ii)], [9]. This can be easily seen from the fact [2, Table 3.2.1] that when D is a quartic character on $\mathbb{F}_q$ for a prime $q = x^2 + y^2$ with x odd, then $J(\phi, D)^2 = (x + iy)^2$.

The left side of (1.14) can also be expressed in the form

$$(1.16)\quad S\phi(-8)\ {}_3F_2\left(\begin{matrix} \phi, \overline{S}^2\phi, S^2\phi \\ \overline{S}\phi, S\phi \end{matrix} \,\Bigg|\, -\frac{1}{8}\right);$$

this can be seen by applying [5, Theorem 4.2(i)] with $A = \overline{S}$, $B = S$, $C = S^3$, $D = S\phi$, and $E = S^2$. If we now apply [5, Theorem 4.2(ii)] directly to (1.16), we see that the left side of (1.14) also equals

$$(1.17)\quad S(-8)\phi(-1)\ {}_3F_2\left(\begin{matrix} \phi, S, \overline{S} \\ S^2, \overline{S}^2 \end{matrix} \,\Bigg|\, -8\right).$$

Thus we obtain the following theorem:

Theorem 1.9. *Suppose that S is a character whose order is not 1, 3, or 4. Then*

$$(1.18)\qquad \begin{aligned} &{}_3F_2\left(\begin{matrix}\phi, S, \overline{S}\\ S^2, \overline{S}^2\end{matrix}\;\middle|\; -8\right)\\ &= \begin{cases} -1/q, & \text{if } S \text{ is not a square}\\ 1/q + \frac{\overline{S}(4)J(\overline{S},S^3)}{q^2 J(S,S)}\left(J(S,D)^2 + J(S,D\phi)^2\right), & \text{if } S = D^2. \end{cases}\end{aligned}$$

In the case where $S = \phi$, Theorem 1.9 reduces to Ono's evaluation of a ${}_3F_2(-8)$ in [8, Theorem 6(i)], [9].

We have also evaluated infinite families of ${}_3F_2(-1)$ and ${}_3F_2(1/4)$ over $\mathbb{F}_q$. These more complicated evaluations require further machinery and are thus written up in a separate paper. Note that while Theorem 1.7 covers the argument $z = -1/8$ (via the choice $u = 3$), it cannot be applied to cover $z = -1$ and $z = 1/4$ over all finite fields. We have tried to extend the result of Ono [8, Theorem 6(vii)] by evaluating an infinite family of ${}_3F_2(1/64)$, but our attempts have not been successful.

2 Proof of Theorem 1.1

Let $AB = C^2$ where $C \neq \phi$ and $A, B \notin \{\varepsilon, C\}$. Let $u \neq 1$. The object of this section is to prove

$$(2.1)\qquad \begin{aligned} {}_3F_2\left(\begin{matrix}A, B, C\phi\\ C^2, C\end{matrix}\;\middle|\; u\right) &= -\overline{C}(u)\phi(1-u)/q\\ &+ \overline{C}(-4)\overline{C}\phi(1-u)F^*\left(A, C; \frac{u}{u-1}\right) F^*\left(B, C; \frac{u}{u-1}\right).\end{aligned}$$

Both sides of (2.1) vanish when $u = 0$, so we will assume that $u \notin \{0, 1\}$.

The following proof of (2.1) is best read alongside the paper [5], to which we refer numerous times. We take this opportunity to correct two misprints in [5, p. 94]: the argument 1 is missing on the far right in [5, (4.25)] , and the lower case b should be changed to B in [5, Thm. 4.28].

For a character S on $\mathbb{F}_q$ and an element $y \in \mathbb{F}_q$, define

$$(2.2)\qquad \delta(y) = \begin{cases} 1, & \text{if } y = 0\\ 0, & \text{if } y \neq 0\end{cases}, \qquad \delta(S) = \begin{cases} 1, & \text{if } S = \varepsilon\\ 0, & \text{if } S \neq \varepsilon.\end{cases}$$

Let R, S, T, M, W be characters on $\mathbb{F}_q$, with $R \neq \varepsilon$. By [5, Thm. 4.28], for $t \notin \{0,1\}$,

$$
\begin{aligned}
{}_3F_2\left(\begin{matrix} R, & S, & T \\ & T\overline{R}, & T\overline{S} \end{matrix} \,\middle|\, t\right) &= \frac{(1-q)}{q^2} RT(-1)\delta(S) + \frac{(1-q)}{q^2}\overline{R}(-t)\delta(\overline{RS}T) \\
&+ \frac{1}{q} RST(-1)\delta(1+t) + \frac{1}{q}\binom{S}{RS} ST(-1)T\left(\frac{t-1}{t}\right) \\
&+ ST(-1)\overline{T}(1-t)\frac{q}{q-1}\sum_{\chi}\binom{T\chi^2}{\chi}\binom{T\chi}{\overline{R}T\chi}\binom{\overline{RS}T\chi}{\overline{S}T\chi}\chi\left(\frac{-t}{(1-t)^2}\right).
\end{aligned}
$$

Multiplying both sides by $SMW(-1)\overline{M}(t)MW(1-t)/q$ and the summing over $t \in \mathbb{F}_q$, we obtain

$$
\begin{aligned}
&S(-1)\ {}_4F_3\left(\begin{matrix} R, & S, & T, & \overline{M} \\ & T\overline{R}, & T\overline{S}, & W \end{matrix} \,\middle|\, 1\right) = \\
&\frac{(1-q)}{q^2} RSTW(-1)\binom{MW}{M}\delta(S) + \frac{(1-q)}{q^2} SW(-1)\binom{MW}{MR}\delta(\overline{RS}T) \\
&+ \frac{RTW(-1)MW(2)}{q^2} + \frac{TW(-1)}{q}\binom{S}{RS}\binom{MWT}{W} \\
&+ \frac{q}{q-1}\sum_{\chi}\binom{T\chi^2}{\chi}\binom{T\chi}{\overline{R}T\chi}\binom{\overline{RS}T\chi}{\overline{S}T\chi}\binom{\overline{M}\chi}{\overline{MWT}\chi^2}\chi(-1),
\end{aligned} \tag{2.3}
$$

where the ${}_4F_3$ is defined in [5, Def. 3.10]. Define, for $x \notin \{0,1\}$,

$$
Q(x) = F(A, C; x)F(B, C; x). \tag{2.4}
$$

Then,

(2.5)

$$
\begin{aligned}
Q(x) &= \left(\frac{q}{q-1}\right)^2 \sum_{\chi,\psi}\binom{A\chi^2}{\chi}\binom{A\chi}{C\chi}\binom{B\psi}{C\psi}\binom{B\psi^2}{\psi}\chi\psi\left(\frac{x}{4}\right) \\
&= \left(\frac{q}{q-1}\right)^2 \sum_{\psi}\psi\left(\frac{x}{4}\right)\sum_{\chi}\binom{A\chi^2}{\chi}\binom{A\chi}{C\chi}\binom{B\psi\overline{\chi}}{C\psi\overline{\chi}}\binom{B\psi^2\overline{\chi}^2}{\psi\overline{\chi}} \\
&= C(-1)\frac{q}{q-1}\sum_{\psi}\psi\left(-\frac{x}{4}\right)\left\{\frac{q}{q-1}\sum_{\chi}\binom{A\chi^2}{\chi}\binom{A\chi}{C\chi}\binom{\overline{C\psi}\chi}{\overline{B\psi}\chi}\binom{\overline{\psi}\chi}{\overline{B\psi}^2\chi^2}\chi(-1)\right\}
\end{aligned}
$$

by [6, (2.8)]. By (2.5) and (2.3) with $T=A,\ \ R=A\overline{C},\ \ M=\psi,\ \ S=W=C^2\psi$,

(2.6)
$$Q(x)=Q_1(x)\ +\ C(-1)\frac{q}{q-1}\sum_{\psi}\psi\left(-\frac{x}{4}\right)\left\{\frac{-C\psi(-4)}{q^2}\right.$$
$$\left.-\frac{A\psi(-1)}{q}\binom{C^2\psi}{AC\psi}\binom{AC^2\psi^2}{C^2\psi}+\psi(-1)\ {}_4F_3\left(\begin{matrix}A\overline{C}, & C^2\psi, & A, & \overline{\psi}\\ & C, & \overline{B\psi}, & C^2\psi\end{matrix}\ \middle|\ 1\right)\right\},$$

where

$$Q_1(x)=\frac{1}{q}\overline{C}^2\left(\frac{x}{4}\right)\binom{\overline{C}^2}{\overline{C}^2}+\frac{1}{q}\overline{C}\left(\frac{x}{4}\right)\binom{\varepsilon}{B}.$$

By [5, (2.12)–(2.13)], since $C\neq\phi$,

$$Q_1(x)=\frac{1}{q^2}\overline{C}^2\left(\frac{x}{4}\right)\{-1+(q-1)\delta(C)\}-\frac{1}{q^2}B(-1)\overline{C}\left(\frac{x}{4}\right). \tag{2.7}$$

By [6, (2.6)],

$$\frac{AC(-1)}{q-1}\sum_{\psi}\binom{C^2\psi}{AC\psi}\binom{AC^2\psi^2}{C^2\psi}\psi\left(\frac{x}{4}\right)=\frac{AC(-1)\overline{A}(x/4)}{q}F(B,C;x). \tag{2.8}$$

Since $\sum_{\psi}\psi(x)$ vanishes, it follows from (2.6)–(2.8) that

(2.9)
$$Q(x)=\frac{1}{q^2}\overline{C}^2\left(\frac{x}{4}\right)\{-1+(q-1)\delta(C)\}-\frac{B(-1)}{q^2}\overline{C}\left(\frac{x}{4}\right)$$
$$-\frac{AC(-1)\overline{A}\,(x/4)}{q}F(B,C;x)+\frac{C(-1)q}{q-1}\sum_{\psi}\psi\left(\frac{x}{4}\right){}_4F_3\left(\begin{matrix}A\overline{C},C^2\psi,\overline{\psi},A\\ C,C^2\psi,\overline{B\psi}\end{matrix}\ \middle|\ 1\right).$$

By [5, Thm. 3.15(v)], the degenerate ${}_4F_3$ in (2.9) equals

(2.10)
$${}_4F_3\left(\begin{matrix}A\overline{C},C^2\psi,\overline{\psi},A\\ C,C^2\psi,\overline{B\psi}\end{matrix}\ \middle|\ 1\right)=\binom{\overline{\psi C}}{C\psi}\ {}_3F_2\left(\begin{matrix}A\overline{C},\overline{\psi},A\\ C,\overline{B\psi}\end{matrix}\ \middle|\ 1\right)$$
$$-\frac{1}{q}C\psi(-1)\binom{\overline{BC\psi}}{\overline{C}^2\overline{\psi}}\binom{\overline{B\psi}}{\overline{BC}^2\overline{\psi}^2}+\frac{(q-1)}{q^2}C\psi(-1)\delta(C\psi)\ {}_2F_1\left(\begin{matrix}A\overline{C},A\\ \overline{B\psi}\end{matrix}\ \middle|\ 1\right).$$

By [5, Thm. 4.9], the rightmost term in (2.10) is

$$\frac{q-1}{q^2} A\psi(-1) \binom{A}{C\psi} \delta(C\psi),$$

so the contribution of this term to the right side of (2.9) is

$$\frac{C(-1)q}{q-1}\, \overline{C}\left(\frac{x}{4}\right) \frac{(q-1)}{q^2} A\overline{C}(-1) \binom{A}{\varepsilon} = \frac{-A(-1)\overline{C}\,(x/4)}{q^2}. \tag{2.11}$$

The contribution of the middle term on the right side of (2.10) to the right side of (2.9) is

$$\begin{aligned} -\frac{BC(-1)}{q-1} \sum_{\psi} \psi\left(\frac{x}{4}\right) \binom{BC^2\psi^2}{B\psi} \binom{C^2\psi}{BC\psi} \\ = -\frac{BC(-1)}{q} \overline{B}\left(\frac{x}{4}\right) F(A, C; x). \end{aligned} \tag{2.12}$$

Therefore, by (2.9)–(2.12),

$$\begin{aligned} Q(x) = \frac{1}{q^2} \overline{C}^2 \left(\frac{x}{4}\right) \{-1 + (q-1)\delta(C)\} - \frac{B(-1)}{q^2} \overline{C}\left(\frac{x}{4}\right) \\ - \frac{AC(-1)}{q} \overline{A}\left(\frac{x}{4}\right) F(B, C; x) - \frac{BC(-1)}{q} \overline{B}\left(\frac{x}{4}\right) F(A, C; x) \\ - \frac{A(-1)}{q^2} \overline{C}\left(\frac{x}{4}\right) + Q_2(x), \end{aligned} \tag{2.13}$$

where

$$Q_2(x) := C(-1) \frac{q}{q-1} \sum_{\psi} \psi\left(\frac{x}{4}\right) \binom{\overline{C\psi}}{C\psi} {}_3F_2\left(\begin{matrix} A\overline{C}, \overline{\psi}, A \\ C, \overline{B\psi} \end{matrix} \,\middle|\, 1 \right). \tag{2.14}$$

We proceed to evaluate $Q_2(x)$. By [5, (2.16)],

$$\binom{\overline{C\psi}}{C\psi} = \binom{C\phi\psi}{C\psi} C\psi(-4) + \frac{q-1}{q} \delta(C\psi).$$

Thus (2.14) becomes

$$Q_2(x) = Q_3(x) + Q_4(x), \tag{2.15}$$

where

$$Q_3(x) = C(4)\frac{q}{q-1}\sum_{\psi}\binom{C\phi\psi}{C\psi}\psi(-x)\ {}_3F_2\left(\begin{matrix} A\overline{C}, \overline{\psi}, A \\ C, \overline{B\psi}\end{matrix}\ \middle|\ 1\right) \tag{2.16}$$

and

$$Q_4(x) = \overline{C}\left(\frac{-x}{4}\right){}_3F_2\left(\begin{matrix} A\overline{C}, C, A \\ C, A\overline{C}\end{matrix}\ \middle|\ 1\right). \tag{2.17}$$

By [5, Thm. 3.15(ii) and Cor. 3.16(iii)],

$$\begin{aligned} Q_4(x) &= \overline{C}\left(\frac{-x}{4}\right)B(-1)\binom{C}{B}\binom{B}{C} - \frac{1}{q}\overline{C}\left(\frac{-x}{4}\right){}_2F_1\left(\begin{matrix} A\overline{C}, A \\ A\overline{C}\end{matrix}\ \middle|\ 1\right) \\ &= \frac{1}{q^2}\overline{C}\left(\frac{x}{4}\right)\{q+(1-q)\delta(C)\} + \overline{C}\left(\frac{x}{4}\right)\frac{A(-1)}{q^2}. \end{aligned} \tag{2.18}$$

We now evaluate $Q_3(x)$. By [5, (4.25)],

$$Q_3(x) = C(4)\frac{q}{q-1}\sum_{\psi}\binom{C\phi\psi}{C\psi}\psi(x)\ {}_3F_2\left(\begin{matrix} B, A, \overline{\psi} \\ C^2, C\end{matrix}\ \middle|\ 1\right). \tag{2.19}$$

Thus

(2.20)

$$\begin{aligned} Q_3(x) &= C(4)\frac{q}{q-1}\sum_{\chi}\binom{B\chi}{\chi}\binom{A\chi}{C^2\chi}\frac{q}{q-1}\sum_{\psi}\psi(x)\binom{C\phi\psi}{C\psi}\binom{\chi\overline{\psi}}{\chi C} \\ &= C(-4)\frac{q}{q-1}\sum_{\chi}\binom{B\chi}{\chi}\binom{A\chi}{C^2\chi}\chi(-1)\frac{q}{q-1}\sum_{\psi}\psi(x)\binom{C\phi\psi}{C\psi}\binom{C\psi}{\overline{\chi}\psi} \end{aligned}$$

by [5, (2.6) and (2.8)]. Replacing ψ by $\overline{C}\psi$, we see that

$$Q_3(x) = C\left(\frac{-4}{x}\right)\frac{q}{q-1}\sum_{\chi}\binom{B\chi}{\chi}\binom{A\chi}{C^2\chi}\chi(-1){}_2F_1\left(\begin{matrix} \phi, \varepsilon \\ \overline{C}\overline{\chi}\end{matrix}\ \middle|\ x\right). \tag{2.21}$$

By [5, Cor. 3.16(ii)],

$${}_2F_1\left(\begin{matrix} \phi, \varepsilon \\ \overline{C}\overline{\chi}\end{matrix}\ \middle|\ x\right) = \binom{\overline{C}\overline{\chi}}{\phi\overline{C}\overline{\chi}}\phi(-1)C\chi(x)\overline{C}\overline{\chi}\phi(1-x) - \frac{C\chi(-1)}{q}.$$

Therefore

$$Q_3(x) = -\frac{C(4/x)}{q}\, {}_2F_1\left(\begin{matrix} B, A \\ C^2 \end{matrix}\,\middle|\, 1\right) + Q_5(x), \tag{2.22}$$

where

$$Q_5(x) = C(-4)\overline{C}\phi(1-x)\,{}_3F_2\left(\begin{matrix} B, A, C\phi \\ C^2, C \end{matrix}\,\middle|\, \frac{x}{x-1}\right). \tag{2.23}$$

In view of [5, Thm. 4.9 and (2.12)], the first term on the right of (2.22) equals

$$A(-1)\overline{C}(x/4)/q^2, \tag{2.24}$$

since $A(-1) = B(-1)$. By [5, Thm. 3.20(i)], the (nontrivial) numerator parameters B, A in (2.23) may be interchanged. Thus (2.13) becomes

$$\begin{aligned} Q(x) &= \frac{1}{q^2}\overline{C}^2\left(\frac{x}{4}\right)\{-1 + (q-1)\delta(C)\} - \frac{A(-1)}{q^2}\overline{C}\left(\frac{x}{4}\right) \\ &\quad - \frac{AC(-1)}{q}\overline{A}\left(\frac{x}{4}\right)F(B, C; x) - \frac{BC(-1)}{q}\overline{B}\left(\frac{x}{4}\right)F(A, C; x) \\ &\quad - \frac{A(-1)}{q^2}\overline{C}\left(\frac{x}{4}\right) + \frac{1}{q^2}\overline{C}\left(\frac{x}{4}\right)\{q + (1-q)\delta(C)\} + \frac{A(-1)}{q^2}\overline{C}\left(\frac{x}{4}\right) \\ &\quad + \frac{A(-1)}{q^2}\overline{C}\left(\frac{x}{4}\right) + C(-4)\overline{C}\phi(1-x)\ {}_3F_2\left(\begin{matrix} A, B, C\phi \\ C^2, C \end{matrix}\,\middle|\, \frac{x}{x-1}\right). \end{aligned} \tag{2.25}$$

For $u \notin \{0, 1\}$, take $x = u/(u-1)$ in (2.25), so that $u = x/(x-1)$ and $1-x = 1/(1-u)$. Then (2.25) becomes, in view of definition (1.10),

(2.26)

$$\begin{aligned} {}_3F_2\left(\begin{matrix} A, B, C\phi \\ C^2, C \end{matrix}\,\middle|\, u\right) &= \overline{C}(-4)\overline{C}\phi(1-u)F^*\left(A, C; \frac{u}{u-1}\right)F^*\left(B, C; \frac{u}{u-1}\right) \\ &\quad - \frac{1}{q}\overline{C}(u)\phi(1-u) + \overline{C}(-4)\overline{C}\phi(1-u)\delta(C)\frac{(q-1)}{q^2}\left(C\left(\frac{4u-4}{u}\right) - C^2\left(\frac{4u-4}{u}\right)\right). \end{aligned}$$

The rightmost term in (2.26) vanishes, and so (2.1) is proved.

3 Proof of Theorem 1.2

Let $R^2 \notin \{\varepsilon, C, C^2\}$. Our goal is to prove

$$F^*(R^2, C; x) = R(4)\frac{J(\phi, C\overline{R}^2)}{J(\overline{R}C, \overline{R}\phi)}\, {}_2F_1\left(\begin{matrix} R\phi, R \\ C \end{matrix} \,\middle|\, x \right). \tag{3.1}$$

By definition (1.9) of F,

$$F(R^2, C; x) = \frac{q}{q-1} \sum_{\chi} \binom{R^2\chi^2}{\chi} \binom{R^2\chi}{C\chi} \chi\left(\frac{x}{4}\right).$$

Then from [5, (4.21)],

$$\begin{aligned} F(R^2, C; x) &= \frac{q}{q-1} \sum_{\chi} \binom{R\phi\chi}{\chi} \binom{R\chi}{R^2\chi} \binom{R^2\chi}{C\chi} \binom{\phi}{R\phi}^{-1} R(4)\chi(x) \\ &= \binom{\phi}{R\phi}^{-1} R(4)\ {}_3F_2\left(\begin{matrix} R\phi, R^2, R \\ C, R^2 \end{matrix} \,\middle|\, x \right), \end{aligned} \tag{3.2}$$

where the last equality follows from [5, Def. 3.10]. Thus by [5, Thm.3.15(v)], (3.2) becomes

$$\begin{aligned} &\binom{\phi}{R\phi} \overline{R}(4) F(R^2, C; x) \\ &\qquad = \binom{R\overline{C}}{R^2\overline{C}}\, {}_2F_1\left(\begin{matrix} R\phi, R \\ C \end{matrix} \,\middle|\, x \right) - \frac{C(-1)}{q}\overline{R}^2(x) \binom{\phi\overline{R}}{\overline{R}^2}. \end{aligned} \tag{3.3}$$

By the definition (1.10) of F^*,

$$\binom{\phi}{R\phi} \overline{R}(4) F(R^2, C; x) = \binom{\phi}{R\phi} \overline{R}(4) F^*(R^2, C; x) - R(4) \binom{\phi}{R\phi} \frac{C(-1)}{q}\overline{R}^2(x). \tag{3.4}$$

Applying [5, (2.6)] and then [5, (2.16)] with $A = B = \overline{R}$, we have

$$R(4)\binom{\phi}{R\phi} = \binom{\phi\overline{R}}{\overline{R}^2}.$$

Thus, equating the right sides of (3.3) and (3.4), we obtain

$$\binom{\phi}{R\phi} \overline{R}(4) F^*(R^2, C; x) = \binom{R\overline{C}}{R^2\overline{C}}\, {}_2F_1\left(\begin{matrix} R\phi, R \\ C \end{matrix} \,\middle|\, x \right). \tag{3.5}$$

With the aid of (1.4), we see that (3.5) yields the desired result (3.1).

4 Proof of Theorem 1.6

For $u \in \mathbb{F}_q$, define the function

$$P_R^S(u) = \frac{1}{q} \sum_{t \in \mathbb{F}_q} \overline{R}(t) \overline{S}(1 - 2ut + t^2). \tag{4.1}$$

This is a finite field analogue of the classical Gegenbauer function [7, (5.12.7)]. For the proof of Theorem 1.6, we will need Lemmas 4.1 and 4.2 below, which relate $P_R^S(u)$ to functions ${}_2F_1$ and F^*, respectively.

Lemma 4.1. *Let $u \neq 1$ and $R \notin \{\varepsilon, \overline{S}\phi\}$. Then*

$$P_R^S(u) = \phi(-1)\overline{S}(4) \frac{J(\overline{R}, \overline{S})}{J(\phi, RS)} \, {}_2F_1\left(\begin{matrix} \overline{R}, RS^2 \\ S\phi \end{matrix} \,\middle|\, \frac{1-u}{2} \right). \tag{4.2}$$

Proof. Let $u = 1 - 2v$. Then

$$\begin{aligned} P_R^S(u) &= \frac{1}{q} \sum_{t \neq 1} \overline{R}(t) \overline{S}((1-t)^2 + 4vt) \\ &= \frac{1}{q}\overline{S}(4v) + \frac{1}{q} \sum_t \overline{R}(t) \overline{S}^2(1-t) \overline{S}\left(1 + \frac{4vt}{(1-t)^2}\right). \end{aligned}$$

Applying the finite field analogue [5, (2.10)] of the binomial theorem with $A = S$, we obtain

$$\begin{aligned} P_R^S(u) &= \frac{1}{q}\overline{S}(4v) + \frac{1}{q-1} \sum_\chi \binom{S\chi}{\chi} \chi(-4v) \sum_t \overline{R}\chi(t) \overline{S}^2 \overline{\chi}^2(1-t) \\ &= \frac{1}{q}\overline{S}(4v) + \frac{1}{q-1} \sum_\chi \binom{S\chi}{\chi} \chi(-4v) J(\overline{R}\chi, \overline{S}^2\overline{\chi}^2). \end{aligned} \tag{4.3}$$

Using [5, (2.16)] with $A = \overline{S}\phi\overline{\chi}$ and $B = RS\phi$, we have

$$\begin{aligned} J(\overline{R}\chi, \overline{S}^2\overline{\chi}^2) &= qR\chi(-1) \binom{\overline{S}^2\overline{\chi}^2}{R\overline{\chi}} \\ &= qR\chi(-1) \binom{\phi}{RS\phi}^{-1} \binom{\overline{S}\phi\overline{\chi}}{RS\phi} \binom{\overline{S}\overline{\chi}}{R\overline{\chi}} \overline{S}\overline{\chi}(4). \end{aligned} \tag{4.4}$$

Combining (4.3)–(4.4) and using [5, (2.6)–(2.8)], we have

$$\begin{aligned} P_R^S(u) &= \frac{1}{q}\overline{S}(4v) + \binom{\phi}{RS\phi}^{-1} \overline{S}(4)R\phi(-1)\frac{q}{q-1}\sum_{\chi}\binom{S\chi}{\chi}\binom{RS^2\chi}{S\phi\chi}\binom{\overline{R}\chi}{S\chi}\chi(v) \\ &= \frac{1}{q}\overline{S}(4v) + \binom{\phi}{RS\phi}^{-1} \overline{S}(4)R\phi(-1)\ {}_3F_2\left(\begin{matrix} S, \overline{R}, RS^2 \\ S, S\phi \end{matrix} \middle| v \right). \end{aligned}$$

Thus by [5, Thm. 3.15(iv)],

$$\begin{aligned} P_R^S(u) = \frac{1}{q}\overline{S}(4v) &+ \binom{\phi}{RS\phi}^{-1}\binom{\overline{R}}{S} \overline{S}(4)R\phi(-1)\ {}_2F_1\left(\begin{matrix} \overline{R}, RS^2 \\ S\phi \end{matrix} \middle| v \right) \\ &- \frac{1}{q}RS\phi(-1)\overline{S}(4v)\binom{\phi}{RS\phi}^{-1}\binom{RS}{\phi}. \end{aligned}$$

Since

$$\binom{\phi}{RS\phi} = \binom{\phi}{\overline{RS}} = RS\phi(-1)\binom{RS}{\phi},$$

the first and last terms on the right cancel and the result follows. ☐

Lemma 4.2. *Let $u \neq 0$. Then*

$$P_R^S(u) = R(2u)S(-1)F^*(\overline{R}, \overline{RS}; u^{-2}) \tag{4.5}$$

Proof. Applying [5, (2.10)] (again with $A = S$) to the right side of

$$P_R^S(u) = \frac{1}{q}\sum_t \overline{R}(t)\overline{S}(1 - t(2u - t)),$$

we have

$$P_R^S(u) = \frac{1}{q}\overline{R}(2u) + \frac{1}{q-1}\sum_{\chi}\binom{S\chi}{\chi}\sum_t \overline{R}\chi(t)\chi(2u-t). \tag{4.6}$$

The inner sum in (4.6) equals

$$\overline{R}\chi^2(2u)J(\overline{R}\chi, \chi) = q\overline{R}(2u)\chi(-4u^2)\binom{\overline{R}\chi}{\overline{\chi}}. \tag{4.7}$$

Combining (4.6)–(4.7) and replacing χ by $\overline{\chi}$, we obtain

$$P_R^S(u) = \frac{1}{q}\overline{R}(2u) + \overline{R}(2u)\frac{q}{q-1}\sum_{\chi}\binom{S\overline{\chi}}{\overline{\chi}}\binom{\overline{R}\overline{\chi}}{\chi}\chi\left(\frac{-1}{4u^2}\right).$$

Then from [5, (2.7)–(2.8)],

$$P_R^S(u) = \frac{1}{q}\overline{R}(2u) + \overline{R}(2u)S(-1)\frac{q}{q-1}\sum_{\chi}\binom{\chi}{\overline{S}\chi}\binom{R\chi^2}{\chi}\chi\left(\frac{1}{4u^2}\right).$$

Finally replacing χ by $\overline{R}\chi$, we obtain

$$\begin{aligned}P_R^S(u) &= \frac{1}{q}\overline{R}(2u) + R(2u)S(-1)\frac{q}{q-1}\sum_{\chi}\binom{\overline{R}\chi^2}{\overline{R}\chi}\binom{\overline{R}\chi}{RS\chi}\chi\left(\frac{1}{4u^2}\right)\\ &= R(2u)S(-1)F^*(\overline{R},\overline{RS};u^{-2}),\end{aligned}$$

by [5, (2.6)] and Definition 1.10. □

We proceed to apply Lemmas 4.1 and 4.2 to prove Theorem 1.6. Suppose that $C \neq \phi$, $A \neq \varepsilon$, and $u \notin \{0,1\}$. By (4.2) and (4.5),

$$(4.8)\quad F^*(A,C;u^{-2}) = \left\{\frac{\overline{A}C^2(2)AC(-1)A(u)J(C\overline{A},A\phi)}{J(\phi,A\phi)}\right\}{}_2F_1\left(\begin{matrix}A, A\overline{C}^2\\ \overline{C}A\phi\end{matrix}\,\middle|\,\frac{1-u}{2}\right).$$

First suppose that $u=-1$. Then Theorem 1.6 follows readily from (4.8) and [5, Thm. 4.9]. Thus assume that $u^2 \notin \{0,1\}$.

Since $u \neq -1$, we can apply [5, Thm. 4.4(iv)] to the ${}_2F_1$ in (4.8) to obtain

$$(4.9)\quad F^*(A,C;u^{-2}) = \left\{\frac{\overline{A}C^2(2)AC(-1)A(u)J(C\overline{A},A\phi)}{J(\phi,A\phi)}\right\}C\overline{A}\phi\left(\frac{-1-u}{2}\right)\,{}_2F_1\left(\begin{matrix}\overline{C}\phi, C\phi\\ \overline{C}A\phi\end{matrix}\,\middle|\,\frac{1-u}{2}\right).$$

Again since $u \neq -1$, we can apply [5, Thm. 4.4(i)] to the ${}_2F_1$ in (4.9) to obtain

$$F^*(A,C;u^{-2}) = \left\{\frac{\overline{A}C^2(2)AC(-1)A(u)J(C\overline{A},A\phi)}{J(\phi,A\phi)}\right\}C\overline{A}\phi\left(\frac{-1-u}{2}\right)\overline{C}\phi(-1)\,{}_2F_1\left(\begin{matrix}\overline{C}\phi, C\phi\\ C\overline{A}\phi\end{matrix}\,\middle|\,\frac{1+u}{2}\right).$$

Theorem 1.6 now follows upon replacing u by $-u$.

5 Proof of Theorem 1.8

Let S be a character whose order is not 1, 3, or 4. Then the hypotheses of Theorem 1.7 are satisfied with $A = \overline{S}$, $C = S\phi$, and $u = 3$. With these choices, Theorem 1.7 yields

$$
\tag{5.1}
\begin{aligned}
{}_3F_2\left(\begin{matrix} \overline{S}, S^3, S \\ S^2, S\phi \end{matrix}\,\middle|\, -\frac{1}{8}\right) &= -\phi(-1)S(-8)/q \\
&\quad + \frac{\phi(-1)S(-2)J(\overline{S}, S^3)}{J(S,S)}\, {}_2F_1\left(\begin{matrix} \overline{S}, S \\ S^2 \end{matrix}\,\middle|\, -1\right)^2 .
\end{aligned}
$$

First suppose that S is not a square. Then by [5, (4.11)], the ${}_2F_1$ in (5.1) vanishes, so (1.14) follows in this case.

Finally, suppose that $S = D^2$ for some character D. Then by [5, (4.11)], the ${}_2F_1$ in (5.1) equals

$$S(-1)(J(S,D) + J(S,D\phi))/q,$$

so its square equals

$$(J(S,D)^2 + J(S,D\phi)^2)/q^2 + 2J(S,D)J(S,D\phi)/q^2.$$

It remains to show that

$$2\phi(-1)S(8)/q = \frac{\phi(-1)S(2)J(\overline{S}, S^3)}{J(S,S)}\left(\frac{2J(S,D)J(S,D\phi)}{q^2}\right),$$

or equivalently,

$$qS(4)J(S,S) = J(\overline{S}, S^3)J(S,D)J(S,D\phi), \quad S = D^2.$$

This identity follows easily from (1.4)–(1.5).

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