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A FAMILY OF POLYNOMIALS WITH CONCYCLIC ZEROS. II

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ABSTRACT. Let $\lambda_1, \ldots, \lambda_J$ be nonzero real numbers. Expand

$$E(z) = \prod(-1 + \exp \lambda_j z),$$

rewrite products of exponentials as single exponentials, and replace every $\exp(az)$ by its approximation $(1 + an^{-1}z)^n$, where $n \geq J$. The resulting polynomial has all zeros on the (possibly infinite) circle of radius $|r|$ centered at $-r$, where $r = n / \sum \lambda_j$.

1. Introduction. Our purpose is to establish Conjecture [1] of [**S2**]. For positive integers n let P_n be the linear mapping from the exponential polynomials over **C** to the polynomials over **C** that replaces $\exp(az)$ by

$$\left(1 + \frac{az}{n}\right)^n \tag{1.1}$$

but is otherwise the identity. For example,

$$P_n\{(e^{5z} - 1)(e^z - 1)\} = \left(1 + \frac{6z}{n}\right)^n - \left(1 + \frac{5z}{n}\right)^n - \left(1 + \frac{z}{n}\right)^n + 1.$$

Thus P_∞ applied to any exponential polynomial $E(z)$ would be the identity. Next, a set of points in the complex plane is said to be concyclic if each of its points lies on the same circle, or on the same line.

The above-mentioned conjecture is now the

THEOREM. *Assume $n \geq J$. Let the λ_j, for $1 \leq j \leq J$, be nonzero real numbers. Then the zeros of $P_n E(z)$, where*

$$E(z) = \prod_{j=1}^{J} (e^{\lambda_j z} - 1), \tag{1.2}$$

are concyclic. In fact, they all lie on $C(r)$, the circle of radius $|r|$ centered at $-r$, where

$$r = n \Big/ \sum_{j=1}^{J} \lambda_j. \tag{1.3}$$

If $\sum \lambda_j = 0$, this means the zeros are purely imaginary.

The condition $n \geq J$ is needed to insure that $P_n E(z)$ is not identically zero. The fact that $P_n E(z)$ is identically zero if and only if $J > n$ is established in the course of the proof (see formula (3.5)).

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Our proof uses a theorem of N. Obrechkoff [**O**] and seems quite different from the approach used in [**S1**] where a theorem of A. Cohn [**C**] was used to obtain partial results. However, Cohn's theorem can be used to obtain a "q-analogue" of these [**S3**].

We remark that the present method of proof also establishes the result for

$$E(z) = \sum_{j=1}^{J} (e^{\lambda_j z + ib_j} - 1) \tag{1.4}$$

where $b_1, \ldots, b_J$ are any real numbers. For other results related to zeros of exponential polynomials see [**DeB, I, L-S**] and the references of [**S1**].

2. A theorem of Obrechkoff. For complex α and a fixed real h let $T(\alpha) = T_h(\alpha)$ be the operator on the set of all polynomials that is defined by

$$T(\alpha)g(z) = g(z+h) + \alpha g(z-h). \tag{2.1}$$

In [**O**, pp. 95–97] Obrechkoff showed (his angular parameter ϕ may be set equal to zero with no loss of generality) that if α lies on the unit circle U (i.e. $|\alpha| = 1$) and the zeros of $g(z)$ lie in a vertical strip S, then the zeros of $T(\alpha)g(z)$ lie in the same strip S.

Now define an operator Δ_i by

$$\Delta_i g(z) = g(z + \lambda_i) - g(z). \tag{2.2}$$

LEMMA. *If all zeros of the nonconstant polynomial $g(z)$ lie on $\operatorname{Re} z = \sigma$, then all zeros of $\Delta_i g(z)$ lie on*

$$\operatorname{Re} z = \sigma - (\lambda_i/2). \tag{2.3}$$

PROOF. Let $s = z + (\lambda_i/2)$. Then

$$\Delta_i g(z) = g(s + (\lambda_i/2)) - g(s - (\lambda_i/2)). \tag{2.4}$$

Since all zeros of $g(s)$ have real part σ, the result follows from the case $\alpha = -1$ of Obrechkoff's theorem.

COROLLARY. *All zeros of*

$$\Delta_1 \Delta_2 \cdots \Delta_J z^n = 0 \tag{2.5}$$

lie on

$$\operatorname{Re} z = -\left(\sum_{j=1}^{J} \lambda_j\right) \Big/ 2. \tag{2.6}$$

PROOF. Apply the above lemma J times with $\sigma = 0$.

The above lemma can also be deduced from a lemma in [**T**, p. 238].

3. Proof of the theorem. Clearly

$$R_n(z) := P_n E(z) = 1 - \sum_j \left(1 + \frac{\lambda_j z}{n}\right)^n + \sum_{i<j} \left(1 + \frac{(\lambda_i + \lambda_j) z}{n}\right)^n - \cdots. \tag{3.1}$$

Set $w = n/z$. Thus

$$w^n R_n(n/w) = w^n - \sum_j (w+\lambda_j)^n + \sum_{i<j}(w+\lambda_i+\lambda_j)^n - \cdots \tag{3.2}$$

where the signs alternate and (consider w^n as the 0th sum on the right) the kth sum has the form

$$\sum (w + \lambda(j_1) + \cdots + \lambda(j_k))^n \tag{3.3}$$

where $\lambda(j) = \lambda_j$ and the sum is over all k-tuples

$$j_1 < \cdots < j_k. \tag{3.4}$$

It is now clear that

$$w^n R_n(n/w) = \Delta_1 \Delta_2 \cdots \Delta_J w^n, \tag{3.5}$$

and that the right-hand side is not identically zero unless $J > n$. By the previous corollary,

$$R_n(z) = 0 \tag{3.6}$$

implies

$$\operatorname{Re} w = -\left(\sum \lambda_j\right) \Big/ 2 \tag{3.7}$$

so $z = n/w$ lies on a circle through the origin that is symmetric with respect to the real axis, and cuts the real axis at

$$x_0 = -2n \Big/ \sum \lambda_j. \tag{3.8}$$

Hence z must lie on a circle of radius $|r|$ and center $-r$ where

$$r = n \Big/ \sum_{j=1}^{J} \lambda_j. \tag{3.9}$$

If r is infinite, the above argument shows that the zeros all lie on the imaginary axis. This completes the proof.

REMARK. Define the operators Δ and B_J by

$$\Delta F(n) = F(n+1) - F(n) \tag{3.10}$$

and

$$B_J F(n) = J!^{-1} \Delta^J F(n) \tag{3.11}$$

for any number theoretic function $F = F(n)$. Thus,

$$B_J F(n) = \frac{1}{J!} \sum_{k=0}^{J} (-1)^{J-k} \binom{J}{k} F(n+k). \tag{3.12}$$

If $F = F_u(n)$ is the function n^u, where u is a nonnegative integer, then it is well known that $B_J F_u(0)$ is a Stirling number of the second kind, and

$$B_J F_u(0) = \delta(J, u), \qquad 0 \le u \le J, \tag{3.13}$$

where $\delta(i,j)$ is the Kronecker delta. Professor Graydon Bell of the University of Northern Arizona has pointed out to us that

$$B_J P_n e^z = \sum_{k=0}^{J} \frac{z^k}{k!}. \tag{3.14}$$

Hence the operator B_J provides a link between the two most common approximations to e^z. Formula (3.14) can be deduced from (3.13). Also, it can be inverted to yield

$$\left(1+\frac{z}{n}\right)^n = \sum_{k=0}^{n} kn^{-k-1}\frac{n!}{(n-k)!}\sum_{j=0}^{k}\frac{z^j}{j!}. \tag{3.15}$$

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