FREE PRODUCTS OF TWO REAL CYCLIC MATRIX GROUPS

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1. Introduction. We exhibit a large class K^* of real 2×2 matrices of determinant ± 1 such that, for nearly all A and B in K^*, the group generated by A and B^t (the transpose of B) is the free product of the cyclic groups $\langle A\rangle$ and $\langle B^t\rangle$. It is shown that K^* contains all matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ of determinant ± 1 with integer entries satisfying $|b|>|a|, |c|, |d|$. This gives a generalization of a theorem of Goldberg and Newman [2]. We also prove related results concerning the dominance of b and the discreteness of the free products $\langle A\rangle * \langle B^t\rangle$.

The matrices A will be identified with linear fractional transformations on $\mathbb{R}^*$ (the extended reals), except in §5.

2. Definitions and notation.

(1) A matrix M is *unimodular* if $\det M = \pm 1$.

(2) A will always denote the real unimodular matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

(3) $\mathbb{Z}$ denotes the integers.

(4) An entry of A is called *dominant* if its absolute value is larger than that of each other entry.

(5) $I=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $T=\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $g=2^{-\frac{1}{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$.

(6) Γ denotes the interval $(-1, 1)$.

(7) $\Delta=\mathbb{R}^*-[-1,1]$.

(8) If C is a 2×2 matrix and S is a set of 2×2 matrices, then $S^C=\{B^C : B\in S\}$, where $B^C=CBC^{-1}$.

(9) $A=\begin{bmatrix} + & + \\ - & - \end{bmatrix}$ means that either the matrix A or $-A$ has the indicated sign pattern, i.e. $a, b\geqq 0$, $c, d\leqq 0$ or $a, b\leqq 0$, $c, d\geqq 0$.

(10) A real linear fractional transformation is called *minimal* if it has a matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ of determinant 1 which satisfies the conditions $c>0$, $a+d=2\cos(\pi/q)(q\in\mathbb{Z}, q\geqq 2)$. If a transformation A has period $q\geqq 2$ and $\det A=1$, then $\langle A\rangle$ has a unique minimal member of period q which can be found as follows. Write $|\operatorname{tr} A|=2\cos(\pi p/q)$, where $(p,q)=1$, $q\geqq 2$. Choose r such that $rp\equiv 1 (\operatorname{mod} q)$. Then either A^r or A^{-r} is minimal.

(11) $J=\{A : |a+b|\geqq|c+d|, |a-b|\geqq|c-d| \text{ and } |b|>|a|\}$.

(12) $K_1=\{A\in J : |\operatorname{tr} A|\geqq 2, \det A=1\}$.

(13) $K_2=\{A : A^r\in J \text{ and } A^r \text{ is minimal, for some } r\}$.

It will be shown in Lemma 7 that $K_2 \subset J$. On the other hand, not every $A \in J$ of determinant 1 and finite period is in K_2. For example, if $\lambda = 2\cos(\pi/5)$, we have

$$A = \begin{bmatrix} 0 & \lambda \\ 1-\lambda & \lambda-1 \end{bmatrix} \in J,\ A^2 \notin J, \text{ and } A^2 \text{ is minimal.}$$

(14) $K_3 = \{A \in J : \det A = -1, \operatorname{tr} A = 0\}$. Observe that K_3 consists of all $A \in J$ of determinant -1 with finite period.

(15) $K_4 = \{A \in J : \det A = -1, A^2 \in J\}$.

(16) $K = K_1 \cup K_2 \cup K_3 \cup K_4$.

3. Free products of transformations.

THEOREM 1. *Let $A, B \in K$, $C = B^t$. Then $\langle A, C\rangle = \langle A\rangle * \langle C\rangle$ if and only if, for every pair r, s satisfying $s^2 - r^2 = 1$, we have* $\{A, C\} \not\subset \left\{\begin{bmatrix} r & s \\ -s & -r \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right\}$.

Before proving this theorem, we prove Proposition 1. The following lemmas lead up to Proposition 1.

LEMMA 1. *$A \in J$ if and only if $A(\Gamma) \subset \Delta$.*

Proof. Let $A \in J$; then $|A(1)| \geqq 1$, $|A(-1)| \geqq 1$, and $A(x)$ does not vanish for $x \in \Gamma$. Moreover, $A(x)$ is monotone on the intervals $(-\infty, -d/c)$ and $(-d/c, \infty)$, since $d/dx(A(x)) = \det A/(cx+d)^2$. It is thus readily seen that $\min\{|A(x)| : x \in [-1, 1]\}$ is attained at $x = 1$ or $x = -1$. Thus $A(\Gamma) \subset \Delta$. The converse is readily verified.

LEMMA 2. *$A \in J$ if and only if $A^{-1} \in J$.*

Proof. This follows from Lemma 1.

LEMMA 3. *Let $\det A = -1$ (so that the fixed points of A are in $\mathbb{R}^*$). Then $A \notin J$ if and only if there is a fixed point of A in Γ.*

Proof. Suppose that $A \notin J$. Then the graph of $A(x)$ must intersect the open square whose vertices are $(1, 1)$, $(1, -1)$, $(-1, 1)$, and $(-1, -1)$. Since $A(x)$ is monotone decreasing on $(-\infty, -d/c)$ and on $(-d/c, \infty)$, the graph of $A(x)$ must intersect the line $y = x$ inside the square. Thus A has a fixed point in Γ. The converse is obvious.

LEMMA 4. *Let $\det A = -1$. Then, if $A^2 \in J$, $A \in J$.*

Proof. If $A \notin J$, then, by Lemma 3, there exists $x \in \Gamma$ such that $A(x) = x$. Thus $A^2(x) = x$, so that $A^2 \notin J$.

LEMMA 5. *Let $A \in K_1$. Then $A^n \in J$ for all $n > 0$.*

Proof. By Lemma 1, the fixed points of A must lie in $\mathbb{R}^* - \Gamma$. For any $x \in \Gamma$, the sequence $x, Ax, A^2x, \ldots$ converges to one of these fixed points in that cyclic order on $\mathbb{R}^*$. Thus, for all $n > 0$, $A^n(x) \in \Delta$, and so $A^n(\Gamma) \subset \Delta$.

LEMMA 6. *Let $A \in K_4$. Then $A^n \in J$ for all $n > 0$.*

Proof. An easy calculation shows that $A^2 \in K_1$. By Lemma 5, $A^{2n} \in J$ for all $n > 0$. By Lemma 4, $A^n \in J$ for all $n > 0$.

LEMMA 7. *Let $A \in K_2$. Then $A^n \in J$ for all n such that $A^n \neq I$.*

Proof. Let $B \in \langle A \rangle$ be minimal of period q, so that $B \in J$. Fix $x \in \Gamma$. The points $x, Bx, B^2x, \ldots, B^{q-1}x$ occur in that cyclic order on $\mathbb{R}^*$. If one of these points other than x were in $[-1, 1]$, then either $Bx \in [-1, 1]$ or $B^{-1}x \in [-1, 1]$. This is impossible since $B \in J$. Thus $\{Bx, B^2x, \ldots, B^{q-1}x\} \subset \Delta$. Therefore, for all n such that $A^n \neq I$, we have $A^n(x) \in \Delta$, and so $A^n(\Gamma) \subset \Delta$.

LEMMA 8. *Let $A \in K_3$. Then $A^n \in J$ for all n such that $A^n \notin I$.*

Proof. Since each $A \in K_3$ is an involution, the assertion is obvious.

PROPOSITION 1. *If $A \in K$, then $A^n \in J$ for all n such that $A^n \neq I$.*

Proof. This follows from Lemmas 2, 5, 6, 7 and 8.

Proof of Theorem 1. Suppose that $B^m \neq I$. By Proposition 1, $B^{-m} \in J$. Thus

$$C^m(\Delta) = TB^{-m}T(\Delta) \subset TB^{-m}(\Gamma) \subset T(\Delta) = \Gamma.$$

Thus, by the Lemma in [4, p. 161], $\langle A, C \rangle = \langle A \rangle * \langle C \rangle$ unless A and C are involutions such that $(AC)^n = I\,(n > 0)$. Suppose that the latter event occurs. We must show that

$$\{A, C\} \subset \left\{ \begin{bmatrix} r & s \\ -s & -r \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$$

for some pair r, s satisfying $s^2 - r^2 = 1$. Let $E = \{-1, 1\}$. Assume that $C(E) \neq E$. Then there exists $e \in E$ such that $C(e) \in \Gamma$, so that $AC(e) \in \Delta$. By induction, $e = (AC)^n(e) \in \Delta$, a contradiction. Thus $C(E) = E$. Since $(CA)^n = I$, similar reasoning shows that $A(E) = E$. Therefore

$$A^g(g(E)) = C^g(g(E)) = g(E) = \{0, \infty\}.$$

It follows that A^g and C^g each have one of the forms $\begin{bmatrix} 0 & u \\ -1/u & 0 \end{bmatrix}$ or $\begin{bmatrix} v & 0 \\ 0 & -1/v \end{bmatrix}$. (The forms

$$\begin{vmatrix} 0 & u \\ 1/u & 0 \end{vmatrix} \text{ and } \begin{vmatrix} v & 0 \\ 0 & 1/v \end{vmatrix} \text{ are ruled out because } \begin{vmatrix} 0 & u \\ 1/u & 0 \end{vmatrix}^{g^{-1}} \text{ and } \begin{vmatrix} v & 0 \\ 0 & 1/v \end{vmatrix}^{g^{-1}}$$

are not in J, by definition of J.) The latter form is an involution only if $v = 1$. Suppose that A^g and C^g both have the former form, say

$$A^g = \begin{vmatrix} 0 & u \\ -1/u & 0 \end{vmatrix}, \quad C^g = \begin{vmatrix} 0 & w \\ -1/w & 0 \end{vmatrix}.$$

Then $A^g = C^g$, since otherwise $(AC)^g$ has infinite period. Therefore we conclude that, for some u,

$$\{A^g, C^g\} \subset \left\{ \begin{vmatrix} 0 & u \\ -1/u & 0 \end{vmatrix}, \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} \right\},$$

i.e.

$$\{A, C\} \subset \left\{ \begin{vmatrix} r & s \\ -s & -r \end{vmatrix}, \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \right\}$$

for some pair r, s satisfying $s^2 - r^2 = 1$.

4. Discreteness. The free products $\langle A \rangle * \langle C \rangle$ in Theorem 1 are, in fact, discrete. We shall prove this now in the special case in which $\det A = \det C = 1$; we prove the result in full generality in a paper to be submitted later. First we establish some propositions.

If we could find a larger class $K' \supset K$ for which Proposition 1 held, we would be able to improve Theorem 1. The next result (the converse of Proposition 1) shows that no such K' exists.

PROPOSITION 2. *Let $A \neq I$ satisfy $A^n \in J$ for all n such that $A^n \neq I$. Then $A \in K$.*

Proof. First suppose that $\det A = 1$. If $|\operatorname{tr} A| \geqq 2$, then $A \in K_1$. Suppose that $|\operatorname{tr} A| < 2$. If A had infinite period, then $\{A^n(0) : n = 1, 2, \dots\}$ would be dense in $\mathbb{R}$; so there would exist an $n > 0$ such that $A^n \notin J$, a contradiction. Thus $|\operatorname{tr} A| = 2\cos(\pi p/q)$, with $(p, q) = 1$, $q \geqq 2$. Since the power of A that is minimal is in J by hypothesis, $A \in K_2$.

Now suppose that $\det A = -1$. If A has finite period, then $A \in K_3$. If A has infinite period, then, since $A^2 \in J$ by hypothesis, $A \in K_4$.

LEMMA 9. *$A \in J^g$ if and only if $A = \begin{bmatrix} + & + \\ - & - \end{bmatrix}$.*

Proof. Suppose that $A \in J^g$. Then

$$A[(0, \infty)] = A[g(\Gamma)] \subset g(\Delta) = (-\infty, 0).$$

It follows that $\{A(0), A^{-1}(0), A(\infty)\} \subset [-\infty, 0]$. This shows that $A = \begin{bmatrix} + & + \\ - & - \end{bmatrix}$.

Conversely, if $A = \begin{bmatrix} + & + \\ - & - \end{bmatrix}$, then $A[(0, \infty)] \subset (-\infty, 0)$, so that $A \in J^g$.

PROPOSITION 3. *Let $A \neq I$. Then $A \in K^g$ if and only if $A^n = \begin{bmatrix} + & + \\ - & - \end{bmatrix}$ for all n such that $A^n \neq I$.*

Proof. This follows from Propositions 1 and 2 and Lemma 9.

The next theorem implies that $\langle A, B^t \rangle$ is the discrete free product $\langle A \rangle * \langle B^t \rangle$ for all $A, B \in K$ of determinant 1. Another consequence is that all the real groups investigated by Lyndon and Ullman in [4] are discrete.

THEOREM 2. *Let Γ_0 be an open interval in $\mathbb{R}^*$ and let $\bar{\Gamma}_0$ be its closure. Let $\Delta_0 = \mathbb{R}^* - \bar{\Gamma}_0$. Suppose that A and C are real 2×2 matrices of determinant* 1 *satisfying the conditions*

(1) $A^n(\Gamma_0) \subset \Delta_0$ *for all n such that $A^n \neq I$, and*
(2) $C^n(\Delta_0) \subset \Gamma_0$ *for all n such that $C^n \neq I$.*

Then $\langle A, C\rangle$ is the discrete free product $\langle A\rangle * \langle C\rangle$.

Proof. By conjugating A and C, we may assume without loss of generality that $\Gamma = \Gamma_0$ and $\Delta = \Delta_0$. Let $B = C^t$. Since $B = TC^{-1}T$, we have $B^n(\Gamma) \subset \Delta$ for all n such that $B^n \neq I$. Thus, by Proposition 2, $A, B \in K$. By Proposition 3, we have $A^g, B^g = \begin{bmatrix} + & + \\ - & - \end{bmatrix}$. Define $A_1 = (A^g)^j$ where j is chosen as follows. If $A \in K_1$, choose $j \in \{1, -1\}$ so that A_1 has a matrix whose upper entries are $\leqq 0$ and whose trace is $\geqq 2$. If $A \in K_2$, choose j so that A_1 has a matrix whose upper entries are $\leqq 0$ and whose trace is $2\cos(\pi/q)$ with $q \geqq 2$, $q \in \mathbb{Z}$. Define B_1 analogously. It is readily seen that $\langle A_1, B_1^t\rangle$ is the discrete free product $\langle A_1\rangle * \langle B_1^t\rangle$ if and only if $\langle A^g, (B^g)^t\rangle$ is the discrete free product $\langle A^g\rangle * \langle (B^g)^t\rangle$. Since $(B^g)^t = (B^t)^g$, it suffices to show that $\langle A_1, B_1^t\rangle$ is the discrete free product $\langle A_1\rangle * \langle B_1^t\rangle$. This follows immediately from Newman's theorem [6, p. 159]. (For a proof of Newman's theorem, see [7, p. 212].) This completes the proof.

The next theorem shows that, if A and C satisfy certain conditions given in [7, p. 210], one can always find an interval Γ_0 for which the hypotheses of Theorem 2 hold.

THEOREM 3. *Let A and C be real 2×2 matrices of determinant* 1, *neither of which is elliptic of infinite period. If A has infinite period, let A_1 be the matrix for A satisfying* $\operatorname{tr} A_1 \geqq 2$; *if A has finite period, let A_1 be the matrix for the minimal transformation in $\langle A\rangle$ satisfying* $\operatorname{tr} A_1 = 2\cos(\pi/q)$ *with $q \geqq 2$, $q \in \mathbb{Z}$. Define C_1 analogously. Suppose that $A_1 \neq -C_1$ and* $\operatorname{tr}(A_1^{-1}C_1) \leqq -2$. *Then A and C satisfy the conditions of Theorem 2 for some Γ_0.*

Proof. View A_1 and C_1 as transformations. It suffices to prove the conclusion with A and C replaced by A_1 and C_1, respectively. As shown in [7, pp. 210–211], we may assume, by conjugation, that $A_1 = \begin{bmatrix} 0 & -\rho \\ 1/\rho & \lambda \end{bmatrix}$ and $C_1 = \begin{bmatrix} 0 & -\rho_1 \\ 1/\rho_1 & \lambda_1 \end{bmatrix}$ with $\rho\rho_1 < 0$. Suppose, without loss of generality, that $\rho > 0$. Letting $B_1 = C_1^t$, we have $A_1, B_1 = \begin{bmatrix} + & + \\ - & - \end{bmatrix}$. By Lemma 9, $A_1, B_1 \in J^g$, so that $A_1, B_1 \in K^g$ by definition of K. The result now follows from Proposition 1.

5. Free products of matrices. In this section, unless otherwise specified, we interpret matrices as elements of the real unimodular 2×2 matrix group G rather than the group $\bar{G}$ of real linear fractional transformations. We define $\bar{A}$ in $\bar{G}$ as the image of $A \in G$ under the natural homomorphism $G \to \bar{G}$. Define $K^* = \{A : \bar{A} \in K\}$.

THEOREM 4. *Let $A, B \in K^*$, $C = B^t$. Then $\langle A, C\rangle = \langle A\rangle * \langle C\rangle$ if any only if*

$$\{A, B\} \not\subset \left\{\pm\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right\}$$

and neither A nor B has even period $\geqq 4$.

Proof. Suppose that A or C, say A, has period $2n$ ($n \geqq 2$). Then $A^n = -I$. Consequently, $A^nCA^nC^{-1} = I$, so that $\langle A, C\rangle \neq \langle A\rangle * \langle C\rangle$. Conversely, suppose that

$$\{A, B\} \not\subset \left\{\pm\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right\}$$

and neither A nor B has even period $\geqq 4$. Then it follows from Theorem 1 that $\langle \bar{A}, \bar{C}\rangle = \langle \bar{A}\rangle * \langle \bar{C}\rangle$. Assume that a reduced word $\ldots A^nC^m \ldots$ in $\langle A, C\rangle$ equals I. Then $\ldots \bar{A}^n\bar{C}^m \ldots$ equals $\bar{I}$, which is impossible because $-I \notin \langle A\rangle$, $-I \notin \langle C\rangle$. Thus $\langle A, C\rangle = \langle A\rangle * \langle C\rangle$. This completes the proof.

Let L^* be the set of unimodular matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with integer entries, infinite period, and b dominant. Let $L = \{\bar{A} : A \in L^*\}$. Goldberg and Newman [2] proved that, for all $A, B \in L^*$, $\langle A, B^t\rangle$ is free. The next theorem shows that this result is a special case of Theorem 4.

THEOREM 5. $L^* \subset K^*$.

Proof. We must show that $L \subset K$. Let $A \in L$. As is mentioned in [2, p. 446], $|b-a| \geqq |d-c|$. If the same reasoning is applied to $\begin{bmatrix} a & -b \\ -c & d \end{bmatrix} \in L$, we obtain $|a+b| \geqq |c+d|$. Hence $A \in J$. This proves, incidentally, that $L \subset J$.

Suppose that $\det A = 1$. Since A has infinite period, $|\operatorname{tr} A| \geqq 2$. Thus $A \in K_1$. Now suppose that $\det A = -1$. It remains to show that $A^2 \in L$, for then $A^2 \in J$ and consequently $A \in K_4$. Since A has infinite period, $t = \operatorname{tr} A \neq 0$. Observe that $A^2 = tA + I = t\begin{bmatrix} a+t^{-1} & b \\ c & d+t^{-1} \end{bmatrix}$. We may assume that $|a+t^{-1}| \geqq |d+t^{-1}|$, because there is no loss of generality in replacing A^2 by its inverse, since $L = \{A^{-1} : A \in L\}$. It remains to show that $|b| > |a+t^{-1}|$. Clearly, $|a+t^{-1}| \leqq |a|+1 \leqq |b|$. Assume that $|a+t^{-1}| = |b|$. Then $t = \operatorname{sgn}(a)$ and

$$A = \begin{bmatrix} a & \pm(1+|a|) \\ c & -a+\operatorname{sgn}(a) \end{bmatrix}$$

so that $\pm c = -1 + a^2/(1+|a|)$. Since $c \in \mathbb{Z}$, we must have $a = 0$. Therefore $|b| = 1$, which contradicts the fact that b is dominant in A.

6. Dominance. For each A, write $A^n = \begin{bmatrix} a_n & b_n \\ c_n & d_n \end{bmatrix}$. In [2] it is proved that, if $A \in L$, then b_n is dominant in A^n for all $n \neq 0$. The next theorem generalizes this result. We first prove one lemma.

LEMMA 10. *Let $A \in K$ and suppose that $(A^t)^n \in J$ for some n. Then*

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \textit{or} \quad A = \begin{bmatrix} a & b \\ -b & -a \end{bmatrix}.$$

Proof. Let $B = A^n$. Since $B^t \in J$, $B \neq I$. Thus $B \in K$, by Propositions 1 and 2. By Lemma 9, $B^g, (B^g)^t = \begin{bmatrix} + & + \\ - & - \end{bmatrix}$. Thus $B^g = \begin{bmatrix} 0 & u \\ -1/u & 0 \end{bmatrix}$ or $\begin{bmatrix} v & 0 \\ 0 & -1/v \end{bmatrix}$ for some u, v. If B^g has the latter form, then $(B^g)^2 = \begin{vmatrix} + & + \\ + & + \end{vmatrix} \notin J^g$. Hence $B \notin K_4$; so we must have $B \in K_3$ and consequently $v = 1$. We have thus shown that $B^g = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ or $\begin{bmatrix} 0 & u \\ -1/u & 0 \end{bmatrix}$, i.e., $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ or $\begin{bmatrix} r & s \\ -s & -r \end{bmatrix}$ for some r, s. In either case A has even period $2m$. If $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, then $\det A = -1$, so that $A \in K_3$. Then $A = B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, the desired result. Suppose therefore that $B = \begin{bmatrix} r & s \\ -s & -r \end{bmatrix}$. Let $M \in \langle A \rangle$ be minimal. The sequence $1, M(1), \ldots, M^{2m-1}(1)$ occurs in $\mathbb{R}^*$ in that cyclic order and each term lies outside of Γ by Lemma 7. However, -1 must be in the sequence because $B(1) = -1$ and B is a power of M. This is possible only if $-1 \in \{M(1), M^{-1}(1)\}$. Thus $B = M$ or $B = M^{-1}$, so that $M^2 = I$. Therefore $A^2 = I$ and $A = B$, the desired result.

THEOREM 6. *Let $A \in K$. Then b_n is dominant in A^n for all n such that $A^n \neq I$, unless* $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ *or* $A = \begin{bmatrix} a & b \\ -b & -a \end{bmatrix}$.

Proof. Let $B = A^n \neq I$. Suppose without loss of generality that $b_n > 0$. Since $B(0)$ and $B^{-1}(0)$ are not in Γ, by Proposition 1, $b_n > |a_n|, |d_n|$. Assume that b_n is not dominant in B. Then we have $|c_n| \geqq b_n > |a_n|, |d_n|$. Since $B \in J$, we have

(1) $b_n + a_n \geqq (c_n + d_n)s$, and
(2) $b_n - a_n \geqq (c_n - d_n)s$,

where $s = \operatorname{sgn}(c_n)$. Adding, we have $b_n \geqq |c_n|$. Thus $b_n = |c_n|$ and equality must hold in (1) and (2). It follows that $B = \begin{bmatrix} a_n & b_n \\ sb_n & sa_n \end{bmatrix}$. Hence $B^t \in J$. By Lemma 10, $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ or $A = \begin{bmatrix} a & b \\ -b & -a \end{bmatrix}$, the desired result.

7. Comments on the literature. In [1], Brenner showed that $A = \begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ m & 1 \end{bmatrix}$ generate a free group if $|m| \geqq 2$. Brenner asked if there were any algebraic $m \in (0, 2)$ for which $\langle A, B \rangle$ is free ($\langle A, B \rangle$ is easily seen to be free for transcendental m). In fact, Brenner's work answers his own question. For (as pointed out in [5]), it follows immediately that $\langle A, B\rangle$ is free when m has an algebraic conjugate of absolute value $\geqq 2$. Since each $m\in S=\{4\cos\pi\theta : \theta \text{ rational}, \theta\in(\frac{1}{3}, \frac{1}{2})\}$ has a conjugate of absolute value $\geqq 2$, we have a dense set of algebraic $m\in(0, 2)$ for which $\langle A, B\rangle$ is free. Thus Knapp [3, p. 304] was incorrect when he claimed (in effect) that $\langle A, B\rangle$ is free for no value of $m\in(0, 2)$.

In [5, p. 1399], it is claimed that $\begin{bmatrix}1 & u\\ 0 & 1\end{bmatrix}$ and $\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$ generate a discontinuous group (on the upper half-plane) when $u=2\cos\pi\alpha$, with α rational. The condition "α rational" should be replaced by the condition "$\alpha = 1/q$, with $q\in\mathbb{Z}^+$".

In [4, p. 165], the description of a minimal transformation is rather ambiguous, since, if $|\operatorname{tr} A|$ is maximal, so is $|\operatorname{tr} A^{-1}|$. With our definition of minimal in §2, the ambiguity is eliminated and the theorems in [4] involving minimal transformations are correct. In particular, Purzitsky's counterexample [7, p. 214] does not apply because the transformation $\begin{bmatrix}0 & 1\\ -1 & 1\end{bmatrix}$ is not minimal.

Purzitsky's other counterexample [7, p. 213] is incorrect, since $(3+\sqrt{5})/2 > (5-\sqrt{21})/2$.

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