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Half Gauss Sums

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1. Introduction

Suppose that χ is a real, primitive character modulo k, where $k>1$ is odd. Since the quadratic Gauss sum $G(\chi)=\sum_{n=1}^{k-1}\chi(n)e^{2\pi in/k}$ has the values [15, p. 256]

$$G(\chi)=\begin{cases}\sqrt{k}, & \text{if } \chi \text{ is even},\\ i\sqrt{k}, & \text{if } \chi \text{ is odd},\end{cases} \tag{1.1}$$

if follows that

$$\sum_{n=1}^{k-1}\chi(n)\cos(2\pi n/k)=2\sum_{n=1}^{(k-1)/2}\chi(n)\cos(2\pi n/k)=\sqrt{k}, \qquad \text{if } \chi \text{ is even},$$

and

$$\sum_{n=1}^{k-1}\chi(n)\sin(2\pi n/k)=2\sum_{n=1}^{(k-1)/2}\chi(n)\sin(2\pi n/k)=\sqrt{k}, \qquad \text{if } \chi \text{ is odd}.$$

Lehmer [12] has made the following interesting conjectures for the signs of "half Gauss sums":

$$\chi(2)\sum_{n=1}^{(k-1)/2}\chi(n)\cos(2\pi n/k)>0, \qquad \text{if } \chi \text{ is odd}, \tag{1.2}$$

and

$$\chi(2)\sum_{n=1}^{(k-1)/2}\chi(n)\sin(2\pi n/k)<0, \qquad \text{if } \chi \text{ is even}. \tag{1.3}$$

Lehmer also has formulated some conjectures on the signs of half Gauss sums like those above but with $2\pi nr/k$ in place of $2\pi n/k$, for certain integers r. For example,

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for $p \equiv 1 \pmod 4$, she has conjectured that

$$\sum_{n=1}^{(p-1)/2} \left(\frac{n}{p}\right) \sin(4\pi n/p) > 0,$$

where $\left(\frac{n}{p}\right)$ denotes the Legendre symbol.

The primary purpose of this paper is to prove some general theorems from which the conjectures of Lehmer follow as corollaries. In Sect. 2, we establish some representations for trigonometric character sums. These formulas are employed in Sect. 3 to determine the signs of half Gauss sums. We conclude this paper by showing how some of the theorems of Sect. 2 can be reformulated to yield some classical identities for trigonometric character sums.

2. Representation Theorems

Throughout the sequel, χ is a primitive character (mod k) with $k > 1$, and α is a real number. Define

$$G(\alpha, \chi) = \sum_{n=1}^{k-1} \chi(n) e^{2\pi i \alpha n/k}.$$

Thus, if α is an integer relatively prime to k, then $G(\alpha, \chi)$ is a Gauss sum (mod k). Note that $G(1, \chi) = G(\chi)$.

Theorem 1. *If α is not an integer, then*

$$G(\alpha, \chi) = \frac{1 - e^{2\pi i \alpha}}{2\pi i} G(\chi) \sum_{j=1}^{\infty} \bar{\chi}(j) \left\{ \frac{1}{j - \alpha} - \frac{\chi(-1)}{j + \alpha} \right\}. \tag{2.1}$$

Proof. Since χ is primitive, $\chi(n) = G(\chi)\overline{G(n, \chi)}/k$ when n is an integer [1, p. 171]. Thus,

$$\begin{aligned} G(\alpha, \chi) &= \frac{G(\chi)}{k} \sum_{n=0}^{k-1} \overline{G(n, \chi)} e^{2\pi i \alpha n/k} \\ &= \frac{G(\chi)}{k} \sum_{n=0}^{k-1} \sum_{m=1}^{k-1} \bar{\chi}(m) e^{2\pi i(\alpha - m)n/k} \\ &= \frac{G(\chi)}{k} \sum_{m=1}^{k-1} \bar{\chi}(m) \left\{ \frac{1 - e^{2\pi i \alpha}}{1 - e^{2\pi i(\alpha - m)/k}} \right\} \\ &= \frac{(1 - e^{2\pi i \alpha}) G(\chi)}{2k} \sum_{m=1}^{k-1} \bar{\chi}(m) \{1 - i \cot(\pi(m - \alpha)/k)\} \\ &= \frac{(1 - e^{2\pi i \alpha}) G(\chi)}{2ik} \sum_{m=1}^{k-1} \bar{\chi}(m) \cot(\pi(m - \alpha)/k). \end{aligned} \tag{2.2}$$

Since

$$\cot(\pi x) = \lim_{N \to \infty} \frac{1}{\pi} \sum_{n=-N}^{N} \frac{1}{n + x},$$

when x is not an integer, we deduce that

$$G(\alpha,\chi)=\lim_{N\to\infty}\frac{(1-e^{2\pi i\alpha})G(\chi)}{2\pi i}\sum_{n=-N}^{N}\sum_{m=1}^{k-1}\frac{\bar{\chi}(m)}{kn+m-\alpha}$$

$$=\lim_{N\to\infty}\frac{(1-e^{2\pi i\alpha})G(\chi)}{2\pi i}\sum_{j=-kN+1}^{kN+k-1}\frac{\bar{\chi}(j)}{j-\alpha},$$

from which the theorem now follows. QED

Theorem 1, in a slightly different form, was established by Hamburger [10], by a different method. For a more general result and some related remarks, see [6, pp. 171–173].

As usual, let $L(s,\chi)=\sum_{n=1}^{\infty}\chi(n)n^{-s}$, $\operatorname{Re}s>0$, denote a Dirichlet L-function. The following result is well known; see, for example, [2; 5, Example 2] and [8, (6.24)].

Corollary 2. *Let r be a natural number. Then*

$$\sum_{n=1}^{k-1}\chi(n)n^r=-r!k^rG(\chi)\sum_{j=1}^{r}\{1+\chi(-1)(-1)^j\}\frac{L(j,\bar{\chi})}{(r-j+1)!(2\pi i)^j}.$$

Proof. Using Leibniz's rule, differentiate both sides of (2.1) r times with respect to α. Upon letting α tend to 0, we deduce the desired result. QED

If α is not an integer, define

$$R(\alpha,\chi)=\sum_{j=1}^{\infty}\frac{\chi(j)}{j^2-\alpha^2}\quad\text{and}\quad S(\alpha,\chi)=\sum_{j=1}^{\infty}\frac{j\chi(j)}{j^2-\alpha^2}.$$

Thus, the series in (2.1) equals $2\alpha R(\alpha,\bar{\chi})$ or $2S(\alpha,\bar{\chi})$ according as χ is even or odd.

Corollary 3. *If α is not an integer, then*

$$\sum_{n=1}^{k-1}\chi(n)\cos(2\pi\alpha n/k)=\begin{cases}-\dfrac{\alpha}{\pi}\sin(2\pi\alpha)G(\chi)R(\alpha,\bar{\chi}), & \textit{if } \chi \textit{ is even},\\[2ex] \dfrac{1}{\pi i}(1-\cos(2\pi\alpha))G(\chi)S(\alpha,\bar{\chi}), & \textit{if } \chi \textit{ is odd},\end{cases}$$

and

$$\sum_{n=1}^{k-1}\chi(n)\sin(2\pi\alpha n/k)=\begin{cases}-\dfrac{\alpha}{\pi}(1-\cos(2\pi\alpha))G(\chi)R(\alpha,\bar{\chi}), & \textit{if } \chi \textit{ is even},\\[2ex] -\dfrac{1}{\pi i}\sin(2\pi\alpha)G(\chi)S(\alpha,\bar{\chi}), & \textit{if } \chi \textit{ is odd}.\end{cases}$$

Proof. Since the sums on the left sides above are $\frac{1}{2}\{G(\alpha,\chi)+G(-\alpha,\chi)\}$ and $\frac{1}{2i}\{G(\alpha,\chi)-G(-\alpha,\chi)\}$, respectively, the corollary follows immediately from Theorem 1. QED

Corollary 4. *If α is not an integer and χ is real, then*

$$G(\alpha,\chi)=\begin{cases} -\dfrac{\alpha\sqrt{k}}{\pi}\{\sin(2\pi\alpha)+i(1-\cos(2\pi\alpha))\}R(\alpha,\chi), & \text{if } \chi \text{ is even},\\ \dfrac{\sqrt{k}}{\pi}\{1-\cos(2\pi\alpha)-i\sin(2\pi\alpha)\}S(\alpha,\chi), & \text{if } \chi \text{ is odd}. \end{cases} \tag{2.3}$$

Proof. The result follows easily from Theorem 1 and (1.1). QED

If β is not an odd integer, we define

$$T(\beta,\chi)=\sum_{\substack{j=1\\ j\text{ odd}}}^{\infty}\frac{\chi(j)}{j^2-\beta^2} \quad\text{and}\quad U(\beta,\chi)=\sum_{\substack{j=1\\ j\text{ odd}}}^{\infty}\frac{j\chi(j)}{j^2-\beta^2}.$$

The series R, S, T, and U are connected as shown in the following two lemmas. In the proofs,

$$\sum_{j=-\infty}^{\infty} \text{ denotes } \lim_{N\to\infty}\sum_{j=-N}^{N}.$$

Lemma 5. *Let k be odd and assume that β is not an odd integer. Then*

$$R\left(\frac{k+\beta}{2},\chi\right)=\frac{4\bar{\chi}(2)\beta}{k+\beta}T(\beta,\chi), \qquad \text{if } \chi \text{ is even},$$

and

$$S\left(\frac{k+\beta}{2},\chi\right)=2\bar{\chi}(2)U(\beta,\chi), \qquad \text{if } \chi \text{ is odd}.$$

Proof. First, suppose that χ is even. Then

$$\begin{aligned} R\left(\frac{k+\beta}{2},\chi\right)&=\frac{1}{k+\beta}\sum_{j=-\infty}^{\infty}\frac{\chi(j)}{j-(k+\beta)/2}=\frac{2\bar{\chi}(2)}{k+\beta}\sum_{\substack{j=-\infty\\ j\text{ even}}}^{\infty}\frac{\chi(j)}{j-k-\beta}\\ &=\frac{2\bar{\chi}(2)}{k+\beta}\sum_{\substack{j=-\infty\\ j\text{ even}}}^{\infty}\frac{\chi(j-k)}{j-k-\beta}=\frac{2\bar{\chi}(2)}{k+\beta}\sum_{\substack{j=-\infty\\ j\text{ odd}}}^{\infty}\frac{\chi(j)}{j-\beta}\\ &=\frac{4\beta\bar{\chi}(2)}{k+\beta}T(\beta,\chi). \end{aligned}$$

The proof of the second part of Lemma 5 follows along the same lines. QED

Lemma 6. *Let k be even and assume that α is not an integer. Then*

$$R(\alpha+k/2,\chi)=-\frac{2\alpha}{2\alpha+k}R(\alpha,\chi)=-\frac{2\alpha}{2\alpha+k}T(\alpha,\chi), \qquad \text{if } \chi \text{ is even},$$

and

$$S(\alpha+k/2,\chi)=-S(\alpha,\chi)=-U(\alpha,\chi), \qquad \text{if } \chi \text{ is odd}.$$

Proof. Note that $k/2$ is even, for otherwise the conductor of χ would divide $k/2$, which contradicts the fact that χ is primitive. Moreover, $\chi(j-k/2)=-\chi(j)$. For if $\chi(j-k/2)=\chi(j)$ for some odd integer j, then

$$1=\bar{\chi}(j)\chi(j)=\bar{\chi}(j)\chi(j-k/2)=\chi(j^{-1}j-k/2)=\chi(1-k/2),$$

which again contradicts the primitivity of χ.

First, suppose that χ is even. Then

$$\begin{aligned}R(\alpha+k/2,\chi)&=\frac{1}{2\alpha+k}\sum_{j=-\infty}^{\infty}\frac{\chi(j)}{j-\alpha-k/2}=-\frac{1}{2\alpha+k}\sum_{j=-\infty}^{\infty}\frac{\chi(j-k/2)}{j-\alpha-k/2}\\&=-\frac{1}{2\alpha+k}\sum_{j=-\infty}^{\infty}\frac{\chi(j)}{j-\alpha}=-\frac{2\alpha}{2\alpha+k}R(\alpha,\chi).\end{aligned}$$

The proof of the second part of Lemma 6 is analogous. QED

Theorem 7. *Let k be odd and assume that β is not an odd integer. Then*

$$\sum_{n=1}^{k-1}(-1)^n\chi(n)\cos(\pi\beta n/k)=\begin{cases}\dfrac{2}{\pi}\beta\chi(2)\sin(\pi\beta)G(\chi)T(\beta,\bar{\chi}), & \textit{if } \chi \textit{ is even},\\[2mm] \dfrac{2}{\pi i}\chi(2)(1+\cos(\pi\beta))G(\chi)U(\beta,\bar{\chi}), & \textit{if } \chi \textit{ is odd},\end{cases}$$

and

$$\sum_{n=1}^{k-1}(-1)^n\chi(n)\sin(\pi\beta n/k)=\begin{cases}-\dfrac{2}{\pi}\beta\chi(2)(1+\cos(\pi\beta))G(\chi)T(\beta,\bar{\chi}), & \textit{if } \chi \textit{ is even},\\[2mm] \dfrac{2}{\pi i}\chi(2)\sin(\pi\beta)G(\chi)U(\beta,\bar{\chi}), & \textit{if } \chi \textit{ is odd}.\end{cases}$$

Proof. In Corollary 3, replace α by $(k+\beta)/2$. Applying Lemma 5, we complete the proof. QED

Theorem 8. *Let k be even and assume that α is not an integer. Then*

$$\sum_{n=1}^{k-1}(-1)^n\chi(n)\cos(2\pi\alpha n/k)=\begin{cases}\dfrac{\alpha}{\pi}\sin(2\pi\alpha)G(\chi)R(\alpha,\bar{\chi}), & \textit{if } \chi \textit{ is even},\\[2mm] -\dfrac{1}{\pi i}(1-\cos(2\pi\alpha))G(\chi)S(\alpha,\bar{\chi}), & \textit{if } \chi \textit{ is odd},\end{cases}$$

and

$$\sum_{n=1}^{k-1}(-1)^n\chi(n)\sin(2\pi\alpha n/k)=\begin{cases}\dfrac{\alpha}{\pi}(1-\cos(2\pi\alpha))G(\chi)R(\alpha,\bar{\chi}), & \textit{if } \chi \textit{ is even},\\[2mm] \dfrac{1}{\pi i}\sin(2\pi\alpha)G(\chi)S(\alpha,\bar{\chi}), & \textit{if } \chi \textit{ is odd}.\end{cases}$$

Proof. In Corollary 3, replace α by $\alpha+k/2$ and then apply Lemma 6 to complete the proof. QED

3. The Signs of Half Gauss Sums

Theorem 12 and Corollaries 15 and 17 below were conjectured by Lehmer. Our goal in this section is to prove these and some generalizations.

Lemma 9. *Let χ be real and assume that $0<\alpha<1$. Then*

$$R(\alpha,\chi)>\frac{1}{1-\alpha^2}-\frac{3}{4} \quad \textit{and} \quad S(\alpha,\chi)>\alpha^2\left(\frac{1}{1-\alpha^2}-\frac{1}{4}\right).$$

In particular, $R(\alpha,\chi)$ and $S(\alpha,\chi)$ are positive.

Proof. First,

$$\begin{aligned} R(\alpha,\chi) &> \frac{1}{1-\alpha^2}-\sum_{j=2}^{\infty}\frac{1}{j^2-\alpha^2} \\ &> \frac{1}{1-\alpha^2}-\sum_{j=2}^{\infty}\frac{1}{j^2-1}=\frac{1}{1-\alpha^2}-\frac{3}{4}. \end{aligned}$$

Secondly,

$$\begin{aligned} S(\alpha,\chi) &= \sum_{j=1}^{\infty}\frac{\chi(j)}{j}+\alpha^2\sum_{j=1}^{\infty}\frac{\chi(j)}{j(j^2-\alpha^2)} \\ &> L(1,\chi)+\alpha^2\left(\frac{1}{1-\alpha^2}-\sum_{j=2}^{\infty}\frac{1}{j(j^2-1)}\right) \\ &= L(1,\chi)+\alpha^2\left(\frac{1}{1-\alpha^2}-\frac{1}{4}\right) \\ &\geqq \alpha^2\left(\frac{1}{1-\alpha^2}-\frac{1}{4}\right), \end{aligned}$$

since it is well known that $L(1,\chi)\geqq 0$ [7, p. 267]. QED

Theorem 10. *Let χ be real and assume that $0<\alpha<1$. Then*

$$\sum_{n=1}^{k-1}\chi(n)\cos(2\pi\alpha n/k)\begin{cases} <0, & \text{if } \chi \text{ is even and } \alpha<\tfrac{1}{2}, \\ =0, & \text{if } \chi \text{ is even and } \alpha=\tfrac{1}{2}, \\ >0, & \text{otherwise}, \end{cases}$$

and

$$\sum_{n=1}^{k-1}\chi(n)\sin(2\pi\alpha n/k)\begin{cases} >0, & \text{if } \chi \text{ is odd and } \alpha>\tfrac{1}{2}, \\ =0, & \text{if } \chi \text{ is odd and } \alpha=\tfrac{1}{2}, \\ <0, & \text{otherwise}. \end{cases}$$

Proof. Taking the real and imaginary parts on both sides of (2.3) in Corollary 4 and using Lemma 9, we immediately deduce the two results above, respectively. QED

Clearly, analogues of Theorem 10 can be readily established for the sums in Theorems 7 and 8.

The next lemma expresses half Gauss sums in terms of the sums in Theorem 10.

Lemma 11. *Let χ be real, k odd, and r a natural number. Then*

$$\sum_{n=1}^{(k-1)/2} \chi(n)\cos(2\pi rn/k) = \begin{cases} \frac{1}{2}\chi(2) \sum_{m=1}^{k-1} \chi(m)\cos(\pi rm/k), & \text{if } \chi(-1)=(-1)^r, \\ \frac{1}{2}\chi(2) \sum_{m=1}^{k-1} (-1)^m \chi(m)\cos(\pi rm/k), & \text{if } \chi(-1)=(-1)^{r+1}, \end{cases}$$

and

$$\sum_{n=1}^{(k-1)/2} \chi(n)\sin(2\pi rn/k) = \begin{cases} \frac{1}{2}\chi(2) \sum_{m=1}^{k-1} \chi(m)\sin(\pi rm/k), & \text{if } \chi(-1)=(-1)^{r+1}, \\ \frac{1}{2}\chi(2) \sum_{m=1}^{k-1} (-1)^m \chi(m)\sin(\pi rm/k), & \text{if } \chi(-1)=(-1)^r. \end{cases}$$

Proof. We have

$$\chi(2) \sum_{n=1}^{(k-1)/2} \chi(n)e^{2\pi irn/k} = \sum_{\substack{m=1 \\ m \text{ even}}}^{k} \chi(m)e^{\pi irm/k} = \sum_{\substack{m=1 \\ m \text{ odd}}}^{k} \chi(k-m)e^{\pi ir(k-m)/k}$$

$$= \chi(-1)(-1)^r \sum_{\substack{m=1 \\ m \text{ odd}}}^{k} \chi(m)e^{-\pi irm/k}.$$

Now twice the first sum above is equal to the second sum plus the fourth sum above. The lemma now follows by considering the various cases. QED

Theorem 12. *Let χ be real and k odd. Then*

$$\chi(2) \sum_{n=1}^{(k-1)/2} \chi(n)\cos(2\pi n/k) > 0, \quad \text{if } \chi \text{ is odd},$$

and

$$\chi(2) \sum_{n=1}^{(k-1)/2} \chi(n)\sin(2\pi n/k) < 0, \quad \text{if } \chi \text{ is even}.$$

Proof. Apply Lemma 11 with $r=1$ and then Theorem 10 with $\alpha=1/2$. The theorem now follows. QED

Theorem 12 establishes the conjectures (1.2) and (1.3) of Lehmer. The special case of Theorem 12 when k is prime is proposed in [13].

Theorem 13. *Let k be odd, and let χ be real and even. Assume that r is odd and >1. Suppose that $\chi(n)=1$ for $1 \leqq n \leqq (r-1)/2$ but that $\chi((r+1)/2) \neq 1$. Then*

$$\chi(2) \sum_{n=1}^{(k-1)/2} \chi(n)\sin(2\pi rn/k) > 0.$$

Proof. Using successively Lemma 11 and Corollary 3, we find that

$$\chi(2)\sum_{n=1}^{(k-1)/2}\chi(n)\sin(2\pi rn/k)=\tfrac{1}{2}\sum_{m=1}^{k-1}\chi(m)\sin(\pi rm/k)$$
$$=-\frac{r\sqrt{k}}{2\pi}R(r/2,\chi)\,.$$

It remains to show that $R(r/2,\chi)<0$. Now,

$$-\tfrac{1}{4}R(r/2,\chi)=\sum_{j=1}^{\infty}\frac{\chi(j)}{r^2-(2j)^2}$$
$$=\sum_{j=1}^{(r-1)/2}\frac{1}{r^2-(2j)^2}+\frac{\chi((r+1)/2)}{r^2-(r+1)^2}+\sum_{j=(r+3)/2}^{\infty}\frac{\chi(j)}{r^2-(2j)^2}$$
$$\geqq\sum_{j=1}^{(r-1)/2}\frac{1}{r^2-(2j)^2}+\sum_{j=(r+3)/2}^{\infty}\frac{1}{r^2-(2j)^2}$$
$$=\sum_{j=1}^{\infty}\frac{1}{r^2-(2j)^2}+\frac{1}{2r+1}$$
$$=-\frac{1}{2r^2}+\frac{1}{2r+1}>0\,.\quad \text{QED}$$

Corollary 14. *Let p be a prime with* $p\equiv 5\pmod 8$. *Then*

$$\sum_{n=1}^{(p-1)/2}\left(\frac{n}{p}\right)\sin(6\pi n/p)<0\,.$$

Corollary 15. *Let p be a prime with* $p\equiv 17\pmod{24}$. *Then*

$$\sum_{n=1}^{(p-1)/2}\left(\frac{n}{p}\right)\sin(10\pi n/p)>0\,.$$

Theorem 16. *Let k be odd, and let* χ *be real and even. Suppose that r is positive and even. Assume that* $\chi(n)=1$ *for all odd natural numbers* $n<r$. *Then*

$$\sum_{n=1}^{(k-1)/2}\chi(n)\sin(2\pi rn/k)>0\,.$$

Proof. By applying Lemma 11 and Theorem 7, we find that

$$\sum_{n=1}^{(k-1)/2}\chi(n)\sin(2\pi rn/k)=\frac{1}{2}\chi(2)\sum_{m=1}^{k-1}(-1)^m\chi(m)\sin(\pi rm/k)$$
$$=-\frac{2r\sqrt{k}}{\pi}T(r,\chi)\,.$$

It remains to show that $T(r,\chi)<0$. Now,

$$-T(r,\chi)=\sum_{\substack{j=1\\ j\,\mathrm{odd}}}^{\infty}\frac{\chi(j)}{r^2-j^2}$$

$$=\sum_{\substack{j=1\\ j\,\mathrm{odd}}}^{r-1}\frac{1}{r^2-j^2}+\sum_{\substack{j=r+1\\ j\,\mathrm{odd}}}^{\infty}\frac{\chi(j)}{r^2-j^2}$$

$$>\sum_{\substack{j=1\\ j\,\mathrm{odd}}}^{\infty}\frac{1}{r^2-j^2}=0. \quad \mathrm{QED}$$

Corollary 17. *Let p be a prime with* $p\equiv 1 \pmod 4$. *Then*

$$\sum_{n=1}^{(p-1)/2}\left(\frac{n}{p}\right)\sin(4\pi n/p)>0.$$

Corollary 18. *For primes* $p\equiv 1 \pmod{12}$, *we have*

$$\sum_{n=1}^{(p-1)/2}\left(\frac{n}{p}\right)\sin(8\pi n/p)>0.$$

Corollary 19. *For primes* $p\equiv 1$ *or* $49 \pmod{60}$, *we have*

$$\sum_{n=1}^{(p-1)/2}\left(\frac{n}{p}\right)\sin(12\pi n/p)>0.$$

4. Identities for Trigonometric Character Sums

Most of the theorems in Sect. 2 involve the infinite series $R(\alpha,\chi)$ and $S(\alpha,\chi)$. By using (2.1) and (2.2), we can express $R(\alpha,\chi)$ and $S(\alpha,\chi)$ as finite trigonometric character sums:

$$R(\alpha,\chi)=\frac{\pi}{2\alpha k}\sum_{m=1}^{k-1}\chi(m)\cot(\pi(m-\alpha)/k), \qquad \text{if } \chi \text{ is even}, \tag{4.1}$$

and

$$S(\alpha,\chi)=\frac{\pi}{2k}\sum_{m=1}^{k-1}\chi(m)\cot(\pi(m-\alpha)/k), \qquad \text{if } \chi \text{ is odd}, \tag{4.2}$$

where α is not an integer. These identities can be reformulated. If χ is even, then

$$R(\alpha,\chi)=-\frac{\pi}{2\alpha k}\sin(2\pi\alpha/k)\sum_{m=1}^{k-1}\frac{\chi(m)}{\cos(2\pi m/k)-\cos(2\pi\alpha/k)}; \tag{4.3}$$

if χ is odd, then

$$S(\alpha,\chi)=-\frac{\pi}{2k}\sum_{m=1}^{k-1}\frac{\chi(m)\sin(2\pi m/k)}{\cos(2\pi m/k)-\cos(2\pi\alpha/k)}. \tag{4.4}$$

We proceed to prove only (4.3), as the proof of (4.4) is analogous.

Suppose that χ is even. In view of (4.1), it suffices to show that

$$-\sum_{m=1}^{k-1}\chi(m)\cot(\pi(m-\alpha)/k)=\sum_{m=1}^{k-1}\frac{\chi(m)\sin(2\pi\alpha/k)}{\cos(2\pi m/k)-\cos(2\pi\alpha/k)}. \tag{4.5}$$

Choosing $A=\pi(m+\alpha)/k$ and $B=\pi(m-\alpha)/k$ in the easily proved identity

$$\cot A-\cot B=\frac{2\sin(A-B)}{\cos(A+B)-\cos(A-B)},$$

we see that the sum of the m-th and $(k-m)$-th terms on each side of (4.5) are equal. This proves (4.5) and thus (4.3).

In [16], Schemmel proved a version of Corollary 3 in which $R(\alpha,\chi)$ and $S(\alpha,\chi)$ are replaced by the right sides of (4.3) and (4.4), respectively. We shall indicate additional identities which can be obtained from (4.1)–(4.4) and results in Sect. 2.

Making use of (4.1) and (4.2) in the proof of Corollary 2, we can express $\sum_{n=1}^{k-1}\chi(n)n^r$ as a linear combination of the sums $\sum_{m=1}^{k-1}\bar{\chi}(m)\cot^j(\pi m/k)$, $1\leqq j\leqq r$. For such identities in a more general setting, see [4, Sect. 4]. The cases $r=1,2$ yield the well-known identities

$$\sum_{n=1}^{k-1}\chi(n)n=\frac{iG(\chi)}{2}\sum_{m=1}^{k-1}\bar{\chi}(m)\cot(\pi m/k), \tag{4.6}$$

when χ is odd, and

$$\sum_{n=1}^{k-1}\chi(n)n^2=\frac{G(\chi)}{2}\sum_{m=1}^{k-1}\bar{\chi}(m)\csc^2(\pi m/k), \tag{4.7}$$

when χ is even. Lebesgue [11] first established (4.6). For references to other proofs, see [9, Chap. 6] and [6, p. 156]. See also a paper of Lerch [14] for a thorough discussion of identities like (4.6) and (4.7).

Let k be odd and χ be odd. From (4.2) and Corollary 3 with $\alpha=k/2$, we find that

$$2\chi(2)\sum_{n=1}^{(k-1)/2}\chi(n)=\sum_{n=1}^{k-1}\chi(n)(-1)^n=\frac{iG(\chi)}{k}\sum_{m=1}^{k-1}\bar{\chi}(m)\tan(\pi m/k).$$

Again, let k be odd. From (4.3), (4.4), and Corollary 3 with $\alpha=k/4$, we deduce that

$$\chi(4)\sum_{n=1}^{(k-1)/2}\chi(n)=(-1)^{(k+1)/2}\sum_{\substack{n=1\\ n\,\mathrm{odd}}}^{k-1}\chi(n)(-1)^{(n-1)/2}=\frac{iG(\chi)}{2k}\sum_{m=1}^{k-1}\bar{\chi}(m)\tan(2\pi m/k),$$

when χ is odd, and

$$(-1)^{(k-1)/2}\chi(4)\left\{\sum_{1\leqq n<k/4}\chi(n)-\sum_{k/4<n<k/2}\chi(n)\right\}$$
$$=\sum_{\substack{n=1\\ n\,\mathrm{odd}}}^{k-1}\chi(n)(-1)^{(n-1)/2}=\frac{G(\chi)}{2k}\sum_{m=1}^{k-1}\bar{\chi}(m)\sec(2\pi m/k),$$

when χ is even.

Finally, we mention that by expressing $\cos^r(2\pi\alpha n/k)$ as a linear combination of the terms $\cos(2\pi\alpha jn/k)$, $0 \leq j \leq r$, we can apply Corollary 3 to obtain identities for the sums $\sum_{n=1}^{k-1} \chi(n)\cos^r(2\pi\alpha n/k)$ for any natural number r. In particular, the elegant identities in [3, p. 34] can be proved in this way.

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