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MODULAR FORMS ON HECKE'S MODULAR GROUPS

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ABSTRACT. Let $H=\{\tau=x+iy:y>0\}$. Let $\lambda>0$, $k>0$, $\gamma=\pm 1$. Let $M(\lambda, k, \gamma)$ denote the set of functions f for which $f(\tau)=\sum_{n=0}^{\infty} a_n e^{2\pi i n\tau/\lambda}$ and $f(-1/\tau)=\gamma(\tau/i)^k f(\tau)$, for all $\tau \in H$. Let $M_0(\lambda, k, \gamma)$ denote the set of $f\in M(\lambda, k. \gamma)$ for which $f(\tau)=O(y^c)$ uniformly for all x as $y\to 0^+$, for some real c. We give a new proof that if $\lambda=2\cos(\pi/q)$ for an integer $q\geqq 3$, then $M(\lambda, k, \gamma)= M_0(\lambda, k, \gamma)$.

Petersson [5, p. 176] and Ogg [4] filled a gap in Hecke's work [2, p. 21] by establishing analytically the theorem below. We present here a short, elementary proof which uses no non-Euclidean geometry.

THEOREM. *Let* $\lambda=2\cos(\pi/q)$ *for an integer* $q\geqq 3$. *Then* $M(\lambda, k, \gamma)= M_0(\lambda, k, \gamma)$.

PROOF. Let $f\in M(\lambda, k, \gamma)$. Let $H_1=\{\tau \in H: |x|\leqq \lambda/2, y\leqq 1\}$. Since $f(\tau)=f(\tau+\lambda)$, it suffices to show that $|y^k f(\tau)|$ is uniformly bounded for all $\tau\in H_1$.

Let $B(\lambda)=\{\tau\in H: |x|<\lambda/2, |\tau|>1\}$ and let $\mathrm{Cl}(B(\lambda))$ denote the closure of $B(\lambda)$. For large y, $|f(\tau)|<|a_0|+1$, and since f is bounded on compact subsets of H, there is a constant A such that $|f(\tau)|\leqq A$ for all $\tau\in \mathrm{Cl}(B(\lambda))$.

Hecke's modular group $G(\lambda)$ is defined to be the group of linear fractional transformations generated by $S_\lambda:\tau\to\tau+\lambda$ and $T:\tau\to -1/\tau$. We shall identify the transformation $\tau\to(a\tau+b)/(c\tau+d)$ with the matrix $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)$. Hecke proved [2, pp. 11–20] that $B(\lambda)$ is a fundamental region (as defined in [3, p. 20]) for $G(\lambda)$. (For an elementary proof, see [1].) Thus for each $\tau\in H$, there exists

$$V_\tau = \begin{pmatrix} a_\tau & b_\tau \\ c_\tau & d_\tau \end{pmatrix} \in G(\lambda)$$

such that $V_\tau\tau\in \mathrm{Cl}(B(\lambda))$.

It can be easily shown that for all $\tau\in H$ and for all $V=\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in G(\lambda)$,

$$|f(\tau)| = |f(V\tau)|\cdot|c\tau+d|^{-k}.$$

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Thus for all $\tau \in H$,

$$|y^k f(\tau)| = y^k\,|f(V_\tau \tau)| \cdot |c_\tau \tau + d_\tau|^{-k} \leqq y^k A \cdot |c_\tau \tau + d_\tau|^{-k}$$
$$= A\,|ic_\tau + (c_\tau x + d_\tau)/y|^{-k}.$$

We shall now show that for all $\tau \in H_1$ and for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G(\lambda)$,

$$|ic + (cx + d)/y|^2 \geqq 1 - \lambda/2.$$

This will show that $|y^k f(\tau)| \leqq A(1-\lambda/2)^{-k/2}$ for all $\tau \in H_1$, which proves our theorem. Fix $\tau \in H_1$ and $V = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G(\lambda)$. Then

$$(cx + d)^2/y^2 + c^2 \geqq (cx + d)^2 + c^2 = c^2(x^2 + 1) + d^2 + 2cdx$$
$$\geqq c^2 + d^2 - \lambda\,|cd| \geqq c^2 + d^2 - (\lambda/2)(c^2 + d_2)$$
$$= (1 - \lambda/2)(c^2 + d^2).$$

It remains to show that $c^2+d^2 \geqq 1$. Suppose that $c^2+d^2<1$. Then $\mathrm{Im}(Vi)= 1/(c^2+d^2)>1$, so i is $G(\lambda)$-equivalent to a point $\tau_1 \in \mathrm{Cl}(B(\lambda))$ such that $\mathrm{Im}(\tau_1)>1$. Thus, by continuity, some point in $B(\lambda)$ close to i is $G(\lambda)$-equivalent to another point in $B(\lambda)$ close to τ_1, contradicting the fact that $B(\lambda)$ is a fundamental region. □

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