Canad. Math. Bull. Vol. **21** (4), 1978

AN ASYMPTOTIC FORMULA FOR A SUM OF PRODUCTS OF POWERS

BY
RONALD J. EVANS

1. Introduction. Fix an integer $r \geq 2$ and positive numbers $b_1, \ldots, b_r$. Write $\sigma = b_1 + \cdots + b_r$. Let $t \in \mathbb{Z}$, $k \in \mathbb{N}$. In this note we evaluate the constant A (when it exists) for which

$$(1) \qquad k^{1-\sigma-r} \sum j_1^{b_1} \cdots j_r^{b_r} \to A \qquad (k \to \infty),$$

where the sum is over all vectors

$$(2) \qquad (j_1, \ldots, j_r) \in \mathbb{N}^r, \quad \text{with} \quad j_1 + \cdots + j_r \equiv t (\text{mod } k) \quad \text{and} \quad 1 \leq j_i \leq k.$$

We also obtain upper and lower bounds for the sum in (1).

If t is allowed to vary with k, one cannot generally expect an asymptotic constant A to exist. However, if t is so restricted that t/k approaches a limit α as $k \to \infty$, then A does exist and we evaluate it in terms of Bernoulli polynomials $B_\nu(\alpha)$. In the case $t = 0$, $b_1 = \cdots = b_r = 1$, our formula (1) reduces essentially to that in [2].

2. **Notation.** If not otherwise indicated, a summation $\sum$ is over the vectors in (2). The Bernoulli polynomials $B_\nu(x)$ are defined by

$$\frac{we^{wx}}{e^w - 1} = \sum_{\nu=0}^{\infty} \frac{B_\nu(x)}{\nu!} w^\nu \quad (0 < |w| < 2\pi).$$

For $\nu \geq 2$, $0 \leq x \leq 1$, these polynomials have the following Fourier expansions [1, p. 267]:

$$B_\nu(x) = -\frac{2\nu!}{(2\pi i)^\nu} \sum_{j=1}^{\infty} \frac{\cos 2\pi jx}{j^\nu}, \quad \text{if} \quad 2 \mid \nu$$

and

$$B_\nu(x) = -\frac{2i\nu!}{(2\pi i)^\nu} \sum_{j=1}^{\infty} \frac{\sin 2\pi jx}{j^\nu}, \quad \text{if} \quad 2 \nmid \nu.$$

For $b > 0$, $j \in \mathbb{Z}$, define

$$(3) \qquad C(b, j) = \int_0^1 x^b e^{2\pi ijx}\, dx.$$

Received by the editors August 11, 1977.

For non-zero j, integration by parts yields

(4) $$C(b, j)=\frac{1}{2\pi ij}-\frac{b}{2\pi ij}\int_0^1 x^{b-1}e^{2\pi ijx}\,dx.$$

It follows that

(5) $$|C(b, j)|\leq 1/\pi\,|j|.$$

In the case $b\in\mathbb{N}$, repeated use of (4) shows that

(6) $$C(b, j)=P_b(-1/2\pi ij),$$

where

(7) $$P_b(x)=-\sum_{m=0}^{b-1}\frac{b!}{(b-m)!}x^{m+1}.$$

For $b_1,\ldots, b_r\in\mathbb{N}$, denote the polynomial $\prod_{i=1}^r P_{b_i}(x)$ by $\sum_{\nu=r}^{\sigma} e_\nu x^\nu$.
Define

$$J=\left\{j\in\mathbb{Z}: 1\leq|j|\leq\left[\frac{k-1}{2}\right]\right\}.$$

3. **Upper and lower bounds.** THEOREM 1. *Fix* $r\geq 2$ *and positive numbers* $b_1,\ldots, b_r$. *Let* $t\in\mathbb{Z}$, $k\in\mathbb{N}$. *Then as* $k\to\infty$,

$$M_r\leq\liminf k^{1-\sigma-r}\sum j_1^{b_1}\cdots j_r^{b_r}\leq\limsup k^{1-\sigma-r}\sum j_1^{b_1}\cdots j_r^{b_r}\leq N_r,$$

where

$$M_r=\frac{\Gamma(b_1+1)\Gamma(b_2+1)}{\Gamma(b_1+b_2+2)}\prod_{i=3}^r(b_i+1)^{-1}$$

and

$$N_r=(b_1+b_2+1)^{-1}\prod_{i=3}^r(b_i+1)^{-1}.$$

Proof. First suppose that $r=2$. Let $L(m)$ denote the least positive residue of $m(\text{mod } k)$. Then

$$\sum j_1^{b_1}j_2^{b_2}=\sum_{j=1}^k j^{b_1}L^{b_2}(t-j).$$

Since the sequence $L(t-1)$, $L(t-2),\ldots, L(t-k)$ is a permutation of $1, 2,\ldots, k$, it follows that

(8) $$\sum_{j=1}^k j^{b_1}(k+1-j)^{b_2}\leq\sum j_1^{b_1}j_2^{b_2}\leq\sum_{j=1}^k j^{b_1+b_2}.$$

As $k\to\infty$, the rightmost member of (8) is asymptotic to $k^{b_1+b_2+1}N_2$ and the leftmost is asymptotic to

$$\int_0^{k+1} x^{b_1}(k+1-x)^{b_2}\, dx \sim k^{b_1+b_2+1}M_2.$$

Therefore, the result follows from (8) in the case $r=2$.

Now let $r>2$ and suppose that the theorem holds for $r-1$ in place of r. We have

$$\sum j_1^{b_1}\cdots j_r^{b_r} = \sum_{j=1}^{k} j^{b_r} \sum^{*} j_1^{b_1}\cdots j_{r-1}^{b_{r-1}}$$

where the sum $\sum^*$ is over all vectors $(j_1,\ldots,j_{r-1})\in\mathbb{N}^{r-1}$ with $j_1+\cdots+j_{r-1}\equiv t-j \pmod k$ and $1\le j_i\le k$. Applying the induction hypothesis to $\sum^*$ and using the fact that

$$\sum_{j=1}^{k} j^{b_r} \sim \frac{k^{b_r+1}}{b_r+1},$$

the theorem follows. QED

In case (1) holds, Theorem 1 says that $M_r\le A\le N_r$. It is proved in [2] that $A=2^{-r}-B_r(0)/r!$ in the case $t=0$, $b_1=\cdots=b_r=1$. This example shows that in general M_r cannot be replaced by the larger number $\prod_{i=1}^r(b_i+1)^{-1}$, nor can N_r be replaced by the smaller number $\prod_{i=1}^r(b_i+1)^{-1}$.

4. **Lemmas.** LEMMA 2. *Let $g_u(x) (1\le u\le r)$ be complex valued functions, and set*

$$f_u(x)=\sum_{n=1}^{k} g_u(n)x^n.$$

Write

$$F(x)=\prod_{u=1}^{r} f_u(x).$$

Then

$$\sum g_1(j_1)\cdots g_r(j_r) = k^{-1}\sum_{j=1}^{k} e^{-2\pi ijt/k}F(e^{2\pi ij/k}).$$

Proof. We may assume $0\le t<k$. Write

$$F(x)=\sum_{m=0}^{kr} c_m x^m \quad\text{and}\quad a_s=\sum_{\substack{0\le m\le kr\\ m\equiv s \pmod k}} c_m \qquad (0\le s<k).$$

Let $V=\{e^{2\pi ij/k}: j=1,2,\ldots,k\}$. For each $v\in V$,

$$v^{-t}F(v)=v^{-t}\sum_{m=0}^{kr} c_m v^m = \sum_{s=0}^{k-1} a_s v^{s-t}.$$

Since

$$\sum_{v\in V} v^{s-t} = \begin{cases} 0, & \text{if } s\neq t \\ k, & \text{if } s=t, \end{cases}$$

we have

$$\sum_{v\in V} v^{-t}F(v) = ka_t = k\sum g_1(j_1)\cdots g_r(j_r). \qquad \text{QED}$$

We remark that if $t=1$ and the g_u are taken to be primitive characters (mod k) such that $g_1\cdots g_u$ is non-principal, then Lemma 2 yields the well known formula for r-fold Jacobi sums in terms of Gauss sums; see [4, p. 100].

LEMMA 3. *For each $b\in\mathbb{N}$ and $j\in J$,*

$$\sum_{n=1}^{k} n^b e^{2\pi i n j/k} = k^{b+1}C(b,j)+0(k^b),$$

where the implied constant depends only on b.

Proof. Curiously, the result does not seem to be readily deducible from the Euler–Maclaurin summation formula, so we utilize complex analysis. Taking bth derivatives in the identity

$$\sum_{n=1}^{k} e^{zn} = (e^{kz}-1)(1-e^{-z})^{-1},$$

we have

$$\sum_{n=1}^{k} n^b e^{zn} = \sum_{m=0}^{b}\binom{b}{m}(e^{kz}-1)^{(b-m)}\left(\frac{1}{1-e^{-z}}\right)^{(m)}.$$

Restrict z to the annulus $0<|z|<\pi$. We have the Laurent expansion

$$(1-e^{-z})^{-1} = z^{-1}+d_0+d_1z+d_2z^2+\cdots$$

Hence

$$\left(\frac{1}{1-e^{-z}}\right)^{(m)} = (-1)^m m!\, z^{-m-1}+0(1),$$

where the implied constant depends only on m. Hence

$$\sum_{n=1}^{k} n^b e^{zn} = (e^{kz}-1)\left(\frac{1}{1-e^{-z}}\right)^{(b)} + \sum_{m=0}^{b-1}\binom{b}{m}k^{b-m}e^{kz}(-m!\,(z)^{-m-1}+0(1)).$$

Setting $z=2\pi ij/k$ with $j\in J$, we have the desired result, in view of (6).

LEMMA 4. *Assume that $t/k\to\alpha$ as $k\to\infty$. Then for each $\nu\geq 2$,*

$$\lim_{k\to\infty}\sum_{j=1}^{[(k-1)/2]}\frac{e^{2\pi ijt/k}}{(2\pi ij)^\nu} = \sum_{j=1}^{\infty}\frac{e^{2\pi ij\alpha}}{(2\pi ij)^\nu}$$

Proof. Put $N=(t/k-\alpha)^{-1}$ (if $t/k=\alpha$, $N=\infty$). Let $k\to\infty$. Then $N\to\infty$ and

$$\sum_{j=1}^{[(k-1)/2]} \frac{e^{2\pi ijt/k}}{(2\pi ij)^{\nu}} = \sum_{j=1}^{N} \frac{e^{2\pi ijt/k}}{(2\pi ij)^{\nu}} + o(1) = \sum_{j=1}^{N} \frac{\exp(2\pi ij\alpha + 0(j/N))}{(2\pi ij)^{\nu}} + o(1)$$

$$= \sum_{j=1}^{N} \frac{e^{2\pi ij\alpha}}{(2\pi ij)^{\nu}} + \sum_{j=1}^{N} 0\left(\frac{1}{jN}\right) + o(1) \to \sum_{j=1}^{\infty} \frac{e^{2\pi ij\alpha}}{(2\pi ij)^{\nu}}. \qquad \text{QED}$$

5. **Asymptotic formula.** THEOREM 5. *Fix $r\geq 2$ and $b_1,\ldots,b_r\in\mathbb{N}$. Let t and k be integers such that $0\leq t<k$ and $t/k\to\alpha$ as $k\to\infty$. Then as $k\to\infty$,*

$$\sum j_1^{b_1}\cdots j_r^{b_r} \sim Ak^{\sigma+r-1},$$

where

$$A=\prod_{i=1}^{r}(b_i+1)^{-1}-\sum_{\nu=r}^{\sigma}\frac{e_\nu}{\nu!}B_\nu(\alpha).$$

Proof. By Lemma 2,

$$\sum j_1^{b_1}\cdots j_r^{b_r}=k^{-1}\sum_{j=1}^{k}e^{-2\pi ij\,t/k}\prod_{u=1}^{r}\sum_{n=1}^{k}n^{b_u}e^{2\pi ij\,n/k}.$$

The rightmost sum is asymptotic to $k^{b_u+1}/(b_u+1)$ when $j=k$, and it is equal to $0(k^{b_u})$ when $j=k/2$. Thus,

$$\sum j_1^{b_1}\cdots j_r^{b_r}\sim k^{\sigma+r-1}\prod_{i=1}^{r}(b_i+1)^{-1}+k^{\sigma+r-1}H,$$

where

$$H=k^{-\sigma-r}\sum_{j\in J}e^{-2\pi ijt/k}\prod_{u=1}^{r}\sum_{n=1}^{k}n^{b_u}e^{2\pi inj/k}.$$

It remains to show that

$$H\to-\sum_{\nu=r}^{\sigma}\frac{e_\nu}{\nu!}B_\nu(\alpha) \quad \text{as} \quad k\to\infty.$$

By Lemma 3 and (5),

$$H=k^{-\sigma-r}\sum_{j\in J}e^{-2\pi ijt/k}\left\{k^{\sigma+r}\prod_{u=1}^{r}C(b_u,j)+0\left(\frac{k^{\sigma+r-1}}{j^{r-1}}\right)\right\}$$

where the implied constant depends only on $b_1,\ldots,b_r$. Thus,

$$H=\left\{1+0\left(\frac{\log k}{k}\right)\right\}\sum_{j\in J}e^{-2\pi ijt/k}\prod_{u=1}^{r}C(b_u,j).$$

Therefore, by (6) and (7),

$$H=\left\{1+0\left(\frac{\log k}{k}\right)\right\}\sum_{j\in J}e^{-2\pi ijt/k}\sum_{\nu=r}^{\sigma}e_\nu\left(\frac{-1}{2\pi ij}\right)^\nu$$

$$=\left\{1+0\left(\frac{\log k}{k}\right)\right\}\sum_{\nu=r}^{\sigma}e_\nu\cdot\sum_{j=1}^{[(k-1)/2]}\frac{e^{2\pi ijt/k}+(-1)^\nu e^{-2\pi ijt/k}}{(2\pi ij)^\nu}.$$

By Lemma 4,

$$H\to\sum_{\nu=r}^{\sigma}e_\nu\sum_{j=1}^{\infty}\frac{e^{2\pi ij\alpha}+(-1)^\nu e^{-2\pi ij\alpha}}{(2\pi ij)^\nu}=-\sum_{\nu=r}^{\sigma}\frac{e_\nu}{\nu!}B_\nu(\alpha).$$

COROLLARY 6. *Under the hypotheses of Theorem 5,*

$$\sum j_1\cdots j_r\sim k^{2r-1}(2^{-r}+(-1)^{r+1}B_r(\alpha)/r!)$$

and

$$\sum(j_1\cdots j_r)^2\sim k^{3r-1}\left(3^{-r}+(-1)^{r+1}\sum_{n=0}^{r}\binom{r}{n}2^nB_{n+r}(\alpha)/(n+r)!\right).$$

COROLLARY 7. *Fix $r\geq 2$. Let $k\in\mathbb{N}$ tend to ∞. Then*

$$\sum_{\substack{1\leq j_i\leq 2k\\(j_1+\cdots+j_r)/k\ \text{odd}}} j_1\cdots j_r\sim(2k)^{2r-1}(2^{-r}+(-1)^{r+1}B_r(\tfrac{1}{2})/r!).$$

For $h_1,\ldots,h_r\in\mathbb{N}$, let $A=A(h_1,\ldots,h_r)$ be as in Theorem 5. Let $A'(h_1,\ldots,h_r)$ be obtained from $A(h_1,\ldots,h_r)$ by replacing α by $1-\alpha$. For $b_1,\ldots,b_r>0$, define

$$B=B(b_1,\ldots,b_r)=\sum_{h_1=0}^{\infty}\cdots\sum_{h_r=0}^{\infty}(-1)^{h_1+\cdots+h_r}\binom{b_1}{h_1}\cdots\binom{b_r}{h_r}A'(h_1,\ldots,h_r).$$

Since $|A'(h_1,\ldots,h_r)|<1$ by Theorem 1 and since

$$\sum_{h=0}^{\infty}\binom{b}{h}$$

converges absolutely for $b>0$ [3, p. 90], the r-fold series for B converges absolutely.

The following theorem extends Theorem 5.

THEOREM 8. *Fix $r\geq 2$ and $b_1,\ldots,b_r>0$. Let $t\in\mathbb{Z}$, $k\in\mathbb{N}$, $0\leq t<k$, and $t/k\to\alpha$ as $k\to\infty$. Then as $k\to\infty$,*

$$\sum j_1^{b_1}\cdots j_r^{b_r}\sim Bk^{\sigma+r-1}.$$

Proof. Let $\sum'$ be obtained from $\sum$ by replacing t by $-t$. Then

$$\sum j_1^{b_1}\cdots j_r^{b_r}=\sum{}'(k-j_1)^{b_1}\cdots(k-j_r)^{b_r}$$

$$=k^{\sigma+r-1}\sum_{h_1=0}^{\infty}\cdots\sum_{h_r=0}^{\infty}(-1)^{h_1+\cdots+h_r}\binom{b_1}{h_1}\cdots\binom{b_r}{h_r}k^{1-r-(h_1+\cdots+h_r)}\sum{}'j_1^{h_1}\cdots j_r^{h_r}.$$

We have

$$\sum j_1^{b_1}\cdots j_r^{b_r}=k^{\sigma+r-1}B+k^{\sigma+r-1}\sum_{h_1=0}^{\infty}\cdots\sum_{h_r=0}^{\infty}(-1)^{h_1+\cdots+h_r}\binom{b_1}{h_1}\cdots\binom{b_r}{h_r}\theta, \tag{9}$$

where

$$\theta=\theta(k,t,h_1,\ldots,h_r)=k^{1-r-(h_1+\cdots+h_r)}\sum{}'j_1^{h_r}\cdots j_r^{h_r}-A'(h_1,\ldots,h_r).$$

By Theorem 1, $|\theta|$ is bounded by an absolute constant. Since also $\theta\to 0$ as $k\to\infty$, it follows that the r-fold series in (9) approaches 0 as $k\to\infty$. Thus (9) yields the desired result.

References

1. T. M. Apostol, *Introduction to analytic number theory*, Springer-Verlag, N.Y., 1976.
2. B. C. Berndt, Problem 226, *Canadian Math. Bull.* **19** (1976), 250–251.
3. E. Hewitt and K. Stromberg, *Real and abstract analysis*, Springer-Verlag, N.Y., 1965.
4. K. Ireland and M. Rosen, *Elements of number theory*, Bogden and Quigley, Tarrytown-on-Hudson, 1972.

Department of Mathematics
University of California at San Diego
La Jolla, California 92093