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ASYMPTOTIC FORMULAS FOR ZERO-BALANCED HYPERGEOMETRIC SERIES*

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Abstract. A hypergeometric series is called s-balanced if the sum of denominator parameters minus the sum of numerator parameters is s. A nonterminating s-balanced hypergeometric series converges at $x=1$ if s is positive. An asymptotic formula for the partial sums of a zero-balanced ${}_3F_2(1)$ is given. A corollary is the behavior of a zero-balanced ${}_3F_2(x)$ as x approaches 1. Some q-analogues are also given.

1. Introduction. For $0<q<1$, define

$$(1.1)\qquad (a)_k=\prod_{j=0}^{k-1}(1-q^ja),\qquad (a)_\infty=\prod_{j=0}^{\infty}(1-q^ja).$$

In the limiting case $q=1$, define

$$(1.2)\qquad (a)_k=\prod_{j=0}^{k-1}(a+j).$$

Let

$$(1.3)\qquad \lambda(x)=x(x)'_\infty/(x)_\infty,$$

where $(x)'_\infty$ denotes the derivative of $(x)_\infty$ with respect to x.

The following two theorems will be proved in §§3 and 4.

THEOREM 1. *If* $abc=de$ *and* $|c|<1$, *then, in the notation of* (1.1),

$$(1.4)\qquad \sum_{k=0}^{\infty}\left\{\frac{(dq^k)_\infty(eq^k)_\infty(q^{k+1})_\infty}{(aq^k)_\infty(bq^k)_\infty(cq^k)_\infty}-\frac{1}{1-q^{k+1}}\right\}=L_q,$$

where

$$(1.5)\qquad L_q=2\lambda(q)-\lambda(a)-\lambda(b)+\sum_{k=1}^{\infty}\frac{(d/c)_k(e/c)_kc^k}{(a)_k(b)_k(1-q^k)};$$

also, as $m\to\infty$,

$$(1.6)\qquad \sum_{k=0}^{m-1}\frac{(a)_k(b)_k(c)_k}{(d)_k(e)_k(q)_k}=\frac{(a)_\infty(b)_\infty(c)_\infty}{(d)_\infty(e)_\infty(q)_\infty}\left\{\sum_{j=0}^{m-1}\frac{1}{1-q^{j+1}}+L_q\right\}+O(q^m),$$

where the implied constant depends on a,b,c,d,e,q *but not on* m.

THEOREM 2. *If* $a+b+c=d+e$ *and* $\mathrm{Re}(c)>0$, *then, in the notation of* (1.2),

$$(1.7)\qquad \sum_{k=0}^{\infty}\left\{\frac{\Gamma(a+k)\Gamma(b+k)\Gamma(c+k)}{\Gamma(d+k)\Gamma(e+k)\Gamma(1+k)}-\frac{1}{k+1}\right\}=L,$$

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where

$$L=-2\gamma-\frac{\Gamma'(a)}{\Gamma(a)}-\frac{\Gamma'(b)}{\Gamma(b)}+\sum_{k=1}^{\infty}\frac{(d-c)_k(e-c)_k}{(a)_k(b)_k k}, \tag{1.8}$$

where γ is Euler's constant; also, as $m\to\infty$,

$$\sum_{k=0}^{m-1}\frac{(a)_k(b)_k(c)_k}{(d)_k(e)_k k!}=\frac{\Gamma(d)\Gamma(e)}{\Gamma(a)\Gamma(b)\Gamma(c)}\{\log m+L+\gamma\}+O\left(\frac{1}{m}\right), \tag{1.9}$$

where the implied constant depends on a,b,c,d,e but not on m.

Theorem 2 gives an asymptotic formula as $m\to\infty$ for the mth partial sums of a zero-balanced hypergeometric series ${}_3F_2\left({abc \atop de}\middle|1\right)$. It would be interesting if such a result could be extended to ${}_4F_3$ series. The special case $c=e$ of (1.9) gives the following known asymptotic formula [4, p. 109, (34)] for partial sums of a zero-balanced hypergeometric series ${}_2F_1\left({ab \atop d}\middle|1\right)$:

$$\sum_{k=0}^{m-1}\frac{(a)_k(b)_k}{(d)_k k!}=\frac{\Gamma(d)}{\Gamma(a)\Gamma(b)}\left\{\log m-\gamma-\frac{\Gamma'(a)}{\Gamma(a)}-\frac{\Gamma'(b)}{\Gamma(b)}\right\}+O\left(\frac{1}{m}\right). \tag{1.10}$$

This paper was motivated by the desire to prove the following theorem, stated (in less precise form) without proof by Ramanujan [6, Entry 24, Cor. 2], [2, Entry 24, Cor. 2]. We are grateful to Bruce Berndt for bringing Ramanujan's result to our attention.

THEOREM 3. *If $a+b+c=d+e$ and* $\mathrm{Re}(c)>0$, *then as $u\to 1$ with* $0<u<1$,

$$\frac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(d)\Gamma(e)}\,{}_3F_2\left(\begin{matrix}a,b,c\\d,e\end{matrix}\middle|u\right)=-\log(1-u)+L+O((1-u)\log(1-u)), \tag{1.11}$$

where L is defined in (1.8).

In §5, we will deduce Theorem 3 from Theorem 2. It is a mystery to us how Ramanujan found the constant term L in the asymptotic expansion (1.11). Because of the inductive nature of our proofs, this paper unfortunately sheds little light on how Ramanujan might have made this remarkable discovery.

Finally, we mention the q-analogue of Theorem 3. If $abc=de$ and $|c|<1$, then as $u\to 1$

$$\frac{(q)_\infty(d)_\infty(e)_\infty}{(a)_\infty(b)_\infty(c)_\infty}\,{}_3\phi_2\left(\begin{matrix}a,b,c\\d,e\end{matrix}\middle|u\right)=g_q(u)+L_q+O((1-u)g_q(u)),$$

where $g_q(u)=\sum_{k=0}^{\infty}u^{k+1}/(1-q^{k+1})$, L_q is defined by (1.5), and ${}_3\phi_2$ is defined at the beginning of §2.

2. Preliminary lemmas. We will use the following notation for q-hypergeometric series:

$${}_3\phi_2\left(\begin{matrix}a,b,c\\d,e\end{matrix}\middle|z\right)=\sum_{k=0}^{\infty}\frac{(a)_k(b)_k(c)_k z^k}{(d)_k(e)_k(q)_k}.$$

Partial sums will be denoted by

$${}_3\phi_2\left(\begin{matrix}a,b,c\\d,e\end{matrix}\middle|z\right)_m=\sum_{k=0}^{m}\frac{(a)_k(b)_k(c)_k z^k}{(d)_k(e)_k(q)_k}.$$

LEMMA 4. *If* $\mathrm{Re}(C)>0$, $S=D+E-A-B-C$, *and* $\mathrm{Re}(S)>0$, *then*

$$ {}_3F_2\left(\begin{matrix}A,B,C\\ D,E\end{matrix}\,\middle|\,1\right)=\frac{\Gamma(D)\Gamma(E)\Gamma(S)}{\Gamma(C)\Gamma(A+S)\Gamma(B+S)}\,{}_3F_2\left(\begin{matrix}D-C,E-C,S\\ A+S,B+S\end{matrix}\,\middle|\,1\right). \tag{2.1}$$

LEMMA 5. *If* $0<q<1$, $|C|<1$, *and* $|DE/ABC|<1$, *then*

$$ {}_3\phi_2\left(\begin{matrix}A,B,C\\ D,E\end{matrix}\,\middle|\,\frac{DE}{ABC}\right)=\frac{(DE/AC)_\infty(C)_\infty(DE/BC)_\infty}{(D)_\infty(E)_\infty(DE/ABC)_\infty}\,{}_3\phi_2\left(\begin{matrix}D/C,E/C,DE/ABC\\ DE/AC,DE/BC\end{matrix}\,\middle|\,C\right). \tag{2.2}$$

Lemma 4 is proved in [1, p. 14]. Lemma 5 is a q-analogue of Lemma 4 whose proof is completely analogous to the proof for Lemma 4; where Gauss's theorem was invoked, one uses instead the q-analogue of Gauss's theorem given in [1, p. 68, (3)].

LEMMA 6. *If* $0<q<1$ *and* D *and* A *are bounded, then, as* $k\to\infty$,

$$\frac{(Dq^k)_\infty}{(Aq^k)_\infty}=1+O(q^k). \tag{2.3}$$

Proof. This follows easily from the q-binomial theorem [1, p. 66, (4)], namely

$$\sum_{j=0}^{\infty}\frac{(a)_j z^j}{(q)_j}=\frac{(az)_\infty}{(z)_\infty},\qquad |z|<1. \tag{2.4}$$

LEMMA 7. *If* d *and* a *are bounded, then as* $z\to\infty$ *with* $\mathrm{Re}(z)>0$,

$$\frac{\Gamma(a+z)}{\Gamma(d+z)}=z^{a-d}(1+O(z^{-1})). \tag{2.5}$$

Proof. This follows from [4, p. 33, (11)].

LEMMA 8. *Fix* $\varepsilon>0$ *and fix a complex number* E. *Let* $\mathrm{Re}(z)\geq\varepsilon$ *and let* k *be a variable positive integer. Then there exists* $N>0$ *such that*

$$\left(1+\frac{z}{k}\right)^E-1=O\left(\frac{z^N}{k}\right), \tag{2.6}$$

where N *and the implied constant are independent of* z *and* k.

Proof. Let $F=\mathrm{Re}(E)$. If $F\geq 0$, then

$$\left(1+\frac{z}{k}\right)^{-E}-1=-\left(1+\frac{z}{k}\right)^{-E}\left(\left(1+\frac{z}{k}\right)^E-1\right)=O\left(\left(1+\frac{z}{k}\right)^E-1\right),$$

so it suffices to consider the case $F\geq 0$. Let $N=F+1$. First suppose that $k\leq|z|$. Then

$$\left|1+\frac{z}{k}\right|^F\leq(1+|z|)^F=O(z^F)=O\left(\frac{z^N}{k}\right).$$

Thus

$$\left(1+\frac{z}{k}\right)^E=O\left(\left(1+\frac{z}{k}\right)^F\right)=O\left(\frac{z^N}{k}\right)$$

and (2.6) follows. Finally suppose that $k>|z|$. Then since $F\geq 0$,

$$\left|\left(1+\frac{z}{k}\right)^{E}-1\right|\leq \sum_{m=1}^{\infty}\left|\binom{E}{m}\right|\left|\frac{z}{k}\right|^{m}\leq\left|\frac{z}{k}\right|\sum_{m=1}^{\infty}\left|\binom{E}{m}\right|=O\left(\frac{z}{k}\right)=O\left(\frac{z^{N}}{k}\right).$$

LEMMA 9. *Fix real $D, D\notin\{0,-1,-2,-3,\cdots\}$. Let k be a variable positive integer. Let* $\mathrm{Re}(z)\geq 0$. *Then in the notation of* (1.2),

$$\frac{(D-z)_k}{(D)_k}=O(e^{2\pi|z|/3}), \tag{2.7}$$

where the implied constant is independent of z and k.

Proof. For some constant $N>0$ independent of z and k,

$$\left|\frac{(D-z)_k}{(D)_k}\right|=\prod_{j=0}^{k-1}\left|\frac{D+j-z}{D+j}\right|=\prod_{j=0}^{k-1}\left|1-\frac{z}{D+j}\right|\ll(1+|z|)^{N}\prod_{\substack{j=0\\ D+j\geq 1}}^{k-1}\left|1-\frac{z}{D+j}\right|.$$

Thus

$$\left|\frac{(D-z)_k}{(D)_k}\right|\ll(1+|z|)^{N}\prod_{\substack{j=0\\ D+j\geq 1}}^{k-1}\left(1-2\,\mathrm{Re}\left(\frac{z}{D+j}\right)+\left|\frac{z}{D+j}\right|^{2}\right)^{1/2}$$

$$\ll(1+|z|)^{N}\prod_{\substack{j=0\\ D+j\geq 1}}^{k-1}\left(1+\left|\frac{z}{D+j}\right|^{2}\right)^{1/2}\ll(1+|z|)^{N}\prod_{m=1}^{\infty}\left(1+\frac{|z|^{2}}{m^{2}}\right)^{1/2}$$

$$=(1+|z|)^{N}\left(\frac{e^{\pi|z|}-e^{-\pi|z|}}{2\pi|z|}\right)^{1/2}\ll(1+|z|)^{N}e^{\pi|z|/2}\ll e^{2\pi|z|/3}.$$

3. Proof of Theorem 1. We begin by proving (1.6) in the case $c=q$. Let $0<t<1$ and let m be a large integer. By the hypothesis $abq=abc=de$,

$$ {}_3\phi_2\left(\begin{matrix}a,b,q\\ d,et\end{matrix}\middle| t\right)_{m-1}=S_1-S_2, \tag{3.1}$$

where

$$S_1={}_3\phi_2\left(\begin{matrix}a,b,q\\ d,et\end{matrix}\middle| t\right) \tag{3.2}$$

and

$$S_2=\frac{(a)_m(b)_m(q)_m t^m}{(d)_m(et)_m(q)_m}\,{}_3\phi_2\left(\begin{matrix}q,bq^m,aq^m\\ dq^m,etq^m\end{matrix}\middle| t\right). \tag{3.3}$$

Apply Lemma 5 with A, B, C, D, E equal to a, b, q, d, et, respectively, to obtain

$$S_1=\frac{(at)_\infty(bt)_\infty(q)_\infty}{(d)_\infty(et)_\infty(t)_\infty}\,{}_3\Phi_2\left(\begin{matrix} d/q, et/q, t\\ at, bt\end{matrix}\,\middle|\, q\right). \tag{3.4}$$

Apply Lemma 5 with A, B, C, D, E equal to $q, bq^m, aq^m, dq^m, etq^m$, respectively, to obtain

$$S_2=\frac{(a)_\infty(qt)_\infty(b)_m(btq^m)_\infty t^m}{(d)_\infty(et)_\infty(t)_\infty}\,{}_3\Phi_2\left(\begin{matrix} d/a, et/a, t\\ qt, btq^m\end{matrix}\,\middle|\, aq^m\right). \tag{3.5}$$

Thus, by (3.1), (3.4), and (3.5),

$$ {}_3\Phi_2\left(\begin{matrix} a, b, q\\ d, et\end{matrix}\,\middle|\, t\right)_{m-1}=R_1(t)+R_2(t)-R_3(t), \tag{3.6}$$

where

$$R_1(t)=\frac{(at)_\infty(bt)_\infty(q)_\infty}{(d)_\infty(et)_\infty(t)_\infty}-\frac{(a)_\infty(qt)_\infty(b)_m(btq^m)_\infty t^m}{(d)_\infty(et)_\infty(t)_\infty}, \tag{3.7}$$

$$R_2(t)=\frac{(at)_\infty(bt)_\infty(q)_\infty}{(d)_\infty(et)_\infty(t)_\infty}\sum_{k=1}^{\infty}\frac{(d/q)_k(et/q)_k(t)_k q^k}{(at)_k(bt)_k(q)_k}, \tag{3.8}$$

and

$$R_3(t)=\frac{(a)_\infty(qt)_\infty(b)_m(btq^m)_\infty t^m}{(d)_\infty(et)_\infty(t)_\infty}\sum_{k=1}^{\infty}\frac{(d/a)_k(et/a)_k(t)_k(aq^m)^k}{(qt)_k(btq^m)_k(q)_k}. \tag{3.9}$$

Taking the limit as $t\to 1$ in (3.6), we obtain

$$ {}_3\Phi_2\left(\begin{matrix} a, b, q\\ d, e\end{matrix}\,\middle|\, 1\right)_{m-1}=R_1+R_2-R_3, \tag{3.10}$$

where

$$R_i=\lim_{t\to 1}R_i(t). \tag{3.11}$$

Now,

$$\begin{aligned} R_1&=\lim_{t\to 1}\frac{1}{(e)_\infty(d)_\infty(1-t)}\left\{\frac{(at)_\infty(bt)_\infty(q)_\infty}{(qt)_\infty}-(a)_\infty(b)_m(btq^m)_\infty t^m\right\}\\ &=\frac{(a)_\infty(b)_\infty}{(d)_\infty(e)_\infty}\{\lambda(q)-\lambda(a)+\lambda(bq^m)-\lambda(b)+m\}. \end{aligned} \tag{3.12}$$

Since

$$\lambda(x)=\sum_{j=0}^{\infty}\frac{-xq^j}{1-xq^j}, \tag{3.13}$$

we have

$$(3.14)\quad \begin{aligned}\lambda(bq^m)-\lambda(b)+m&=\sum_{j=0}^{m-1}\frac{1}{1-bq^j}\\ &=\sum_{j=0}^{m-1}\frac{1}{1-q^{j+1}}+\sum_{j=0}^{m-1}\left\{\frac{-q^{j+1}}{1-q^{j+1}}-\frac{-bq^j}{1-bq^j}\right\}\\ &=\sum_{j=0}^{m-1}\frac{1}{1-q^{j+1}}+\sum_{j=0}^{\infty}\frac{-q^{j+1}}{1-q^{j+1}}-\sum_{j=0}^{\infty}\frac{-bq^j}{1-bq^j}+O(q^m)\\ &=\sum_{j=0}^{m-1}\frac{1}{1-q^{j+1}}+\lambda(q)-\lambda(b)+O(q^m).\end{aligned}$$

By (3.12) and (3.14),

$$(3.15)\qquad R_1=\frac{(a)_\infty(b)_\infty}{(d)_\infty(e)_\infty}\left\{2\lambda(q)-\lambda(a)-\lambda(b)+\sum_{j=0}^{m-1}\frac{1}{1-q^{j+1}}+O(q^m)\right\}.$$

Since

$$(3.16)\qquad \lim_{t\to 1}\frac{(t)_k}{(t)_\infty}=\frac{(q)_{k-1}}{(q)_\infty},$$

we have

$$(3.17)\qquad R_2=\frac{(a)_\infty(b)_\infty}{(d)_\infty(e)_\infty}\sum_{k=1}^{\infty}\frac{(d/q)_k(e/q)_k q^k}{(a)_k(b)_k(1-q^k)}$$

and

$$(3.18)\qquad R_3=\frac{(a)_\infty(b)_\infty}{(d)_\infty(e)_\infty}\sum_{k=1}^{\infty}\frac{(d/a)_k(e/a)_k(aq^m)^k}{(bq^m)_k(q)_k(1-q^k)}=O(q^m).$$

By (3.10), (3.15), (3.17) and (3.18),

(3.19)

$$\begin{aligned}{}_3\phi_2\left(\begin{matrix}a,b,q\\ d,e\end{matrix}\middle|1\right)_{m-1}&=\frac{(a)_\infty(b)_\infty}{(d)_\infty(e)_\infty}\left\{\sum_{j=0}^{m-1}\frac{1}{1-q^{j+1}}+2\lambda(q)\right.\\ &\qquad\left.-\lambda(a)-\lambda(b)+\sum_{k=1}^{\infty}\frac{(d/q)_k(e/q)_k q^k}{(a)_k(b)_k(1-q^k)}\right\}+O(q^m).\end{aligned}$$

This completes the proof of (1.6) in the case $c=q$.

We next prove that (1.6) holds for $c=q^n$ for all positive integers n. Let $c=q^N$ for an integer $N>1$, and assume as induction hypothesis that (1.6) holds with $c=q^n$ for all n such that $1\le n<N$. Since

$$(3.20)\qquad (a)_k=q^k\left(\frac{a}{q}\right)_k+(1-q^k)(a)_{k-1},$$

we have

$$(3.21)\quad {}_3\phi_2\left(\begin{matrix}a,b,c\\d,e\end{matrix}\bigg|1\right)_{m-1}=\frac{(1-d/q)(1-e/q)}{(1-b/q)(1-c/q)}\,{}_3\phi_2\left(\begin{matrix}a,b/q,c/q\\d/q,e/q\end{matrix}\bigg|1\right)_m-\frac{(1-d/q)(1-e/q)}{(1-b/q)(1-c/q)}\,{}_3\phi_2\left(\begin{matrix}a/q,b/q,c/q\\d/q,e/q\end{matrix}\bigg|q\right)_m.$$

Since $a(b/q)(c/q)=(d/q)(e/q)$, the first term on the right of (3.21) can be evaluated by the induction hypothesis.

The last term on the right of (3.21) equals

$$(3.22)\quad \frac{(1-d/q)(1-e/q)}{(1-b/q)(1-c/q)}\,{}_3\phi_2\left(\begin{matrix}a/q,b/q,c/q\\d/q,e/q\end{matrix}\bigg|q\right)+O(q^m),$$

since the ${}_3\phi_2$ in (3.22) converges; by Lemma 5, the first term in (3.22) in turn equals

$$(3.23)\quad \frac{(a)_\infty(b)_\infty(c)_\infty}{(1-b/q)(d)_\infty(e)_\infty(q)_\infty}\sum_{k=1}^{\infty}\frac{(d/c)_k(e/c)_k(c/q)^k}{(a)_k(b)_k}.$$

The relations

$$\frac{1}{(b/q)_k(1-q^k)}-\frac{1}{(b/q)_{k+1}}=\frac{q^k}{(b)_k(1-q^k)}$$

and

$$\frac{1}{1-q^{m+1}}-\lambda(b/q)-\frac{1}{1-b/q}=-\lambda(b)+O(q^m)$$

show that (3.21) and (3.23) imply that (1.6) holds for $c=q^N$. This completes the induction, so (1.6) holds for $c=q^N$ for all positive integers N. Taking the limit as m tends to ∞, we see that (1.4) also holds for all c of the form $c=q^N$.

We next prove that (1.4) holds without the restriction $c=q^N$. Since $q^N\to 0$ as $N\to\infty$, it suffices to show that each member of (1.4) is an analytic function of c on the disk $|c|<1$ for each fixed choice of $a,b,d,$ and q.

Fix $t, 0<t<1$. To show that the right member of (1.4) is analytic in c, it suffices to prove that the series

$$(3.24)\quad \sum_{k=1}^{\infty}\frac{(d/c)_k(ab/d)_k c^k}{(a)_k(b)_k(1-q^k)}$$

converges uniformly in the disk $|c|\le t$. Since $|(d/c)_k c^k|=\Pi_{j=0}^{k-1}|c-dq^j|\ll t_1^k$ for some t_1, $t<t_1<1$, and since $(ab/d)_k/(a)_k(b)_k$ is bounded, the series in (3.24) converges uniformly in the disk $|c|\le t$.

To show that the left member of (1.4) is analytic in c, it suffices to prove that the series

$$(3.25)\quad \sum_{k=0}^{\infty}\left\{\frac{(dq^k)_\infty(q^{k+1})_\infty(abcq^k/d)_\infty}{(aq^k)_\infty(bq^k)_\infty(cq^k)_\infty}-\frac{1}{1-q^{k+1}}\right\}$$

converges uniformly in the disk $|c| \le t$. By Lemma 6, as $k \to \infty$,

$$(3.26) \qquad \frac{(dq^k)_\infty}{(aq^k)_\infty} = 1 + O(q^k), \qquad \frac{(q^{k+1})_\infty}{(bq^k)_\infty} = 1 + O(q^k),$$

$$\frac{(abcq^k/d)_\infty}{(cq^k)_\infty} = 1 + O(q^k).$$

Therefore the summand in (3.25) is $\ll q^k$, so the series in (3.25) converges uniformly in the disk $|c| \le t$. This completes the proof of (1.4).

By (3.26), we see that if the index of summation in (3.25) begins at $k=m$ instead of $k=0$, the resulting series is $O(q^m)$, where the implied constant depends on a,b,c,d,e,q but not on m. Thus (1.6) follows from (1.4).

4. Proof of Theorem 2. By Lemma 7, we see that if the index of summation in (1.7) begins at $k=m$ instead of $k=0$, the resulting series is $O(1/m)$, where the implied constant is independent of m. Since also

$$\sum_{k=0}^{m-1} \frac{1}{k+1} = \log m + \gamma + O\left(\frac{1}{m}\right),$$

(1.9) follows from (1.7). It remains to prove (1.7). If one took $\lim_{q\to 1}$ of each side of (1.4) and then interchanged limits and summations, (1.7) would result. However, since it appears to be a difficult task indeed to justify this interchange of limits and summations, we take a different approach.

The proof in §3 began by showing that (1.4) holds for each c of the form $c=q^n$, where n is a positive integer. Mimicking this proof with $q=1$, we can deduce that (1.9) holds for $c=1$, as follows. In place of (3.1), write, for $\varepsilon > 0$,

$${}_3F_2\left(\begin{matrix} a,b,1 \\ d,e+\varepsilon \end{matrix}\middle| 1\right)_{m-1} = H_1 - H_2,$$

where

$$H_1 = {}_3F_2\left(\begin{matrix} a,b,1 \\ d,e+\varepsilon \end{matrix}\middle| 1\right)$$

and

$$H_2 = \frac{(a)_m (b)_m}{(d)_m (e+\varepsilon)_m} \, {}_3F_2\left(\begin{matrix} 1,b+m,a+m \\ d+m,e+\varepsilon+m \end{matrix}\middle| 1\right).$$

Apply Lemma 4 to get analogues of (3.4) and (3.5) for H_1 and H_2. Let $\varepsilon \to 0$ to obtain the analogue of (3.10) of the form

$$(4.1) \qquad {}_3F_2\left(\begin{matrix} a,b,1 \\ d,e \end{matrix}\middle| 1\right)_{m-1} = G_1 + G_2 - G_3.$$

The analogue of (3.12) is

$$G_1 = \frac{\Gamma(d)\Gamma(e)}{\Gamma(a)\Gamma(b)} \left(\frac{\Gamma'(1)}{\Gamma(1)} - \frac{\Gamma'(a)}{\Gamma(a)} - \frac{\Gamma'(b)}{\Gamma(b)} + \frac{\Gamma'(b+m)}{\Gamma(b+m)} \right).$$

Since [4, p. 33, (8)]

$$\frac{\Gamma'(b+m)}{\Gamma(b+m)}=\log m+O\left(\frac{1}{m}\right),$$

we obtain the following analogue of (3.15):

$$G_1=\frac{\Gamma(d)\Gamma(e)}{\Gamma(a)\Gamma(b)}\left(-\gamma-\frac{\Gamma'(a)}{\Gamma(a)}-\frac{\Gamma'(b)}{\Gamma(b)}+\log m\right)+O\left(\frac{1}{m}\right). \tag{4.2}$$

Apply Lemma 7 to obtain the following analogues of (3.17) and (3.18):

$$G_2=\frac{\Gamma(d)\Gamma(e)}{\Gamma(a)\Gamma(b)}\sum_{k=1}^{\infty}\frac{(d-1)_k(e-1)_k}{(a)_k(b)_k k} \tag{4.3}$$

and

$$G_3=\frac{\Gamma(d)\Gamma(e)}{\Gamma(a)\Gamma(b)}\sum_{k=1}^{\infty}\frac{(d-a)_k(e-a)_k}{(b+m)_k(1)_k k}=O\left(\frac{1}{m}\right). \tag{4.4}$$

Combining (4.1)–(4.4), we deduce that (1.9) holds for $c=1$.

An induction argument analogous to that following (3.19) shows that (1.9) holds for each positive integer c. Taking the limit as m tends to ∞, we see that (1.7) also holds for each positive integer c.

To prove that (1.7) holds for all c with $\operatorname{Re}(c)>0$, it suffices by Carlson's theorem [1, p. 39] to prove that, *for fixed* a, b, d *and fixed* $\varepsilon>0$, both sides of (1.7) are analytic in c and equal to $O(e^{2\pi|c|/3})$ for $\operatorname{Re}(c)\geq\varepsilon$.

Write $D=\operatorname{Re}(d-\varepsilon)$, adjusting ε if necessary so that $D\notin\{0,-1,-2,-3,\cdots\}$. Write $z=c+D-d$, so in the notation of (1.2),

$$S:=\sum_{k=1}^{\infty}\frac{(d-c)_k(e-c)_k}{(a)_k(b)_k k}=\sum_{k=1}^{\infty}A_k\frac{(D-z)_k}{(D)_k}$$

with

$$A_k=\frac{(a+b-d)_k(D)_k}{(a)_k(b)_k k}.$$

By Lemma 7, $A_k=O(k^{-1-\varepsilon})$. By Lemma 9, $(D-z)_k/(D)_k=O(e^{2\pi|z|/3})$. Thus S is analytic in z and equals $O(e^{2\pi|z|/3})$ for $\operatorname{Re}(z)\geq 0$. It follows that S is analytic in c and equal to $O(e^{2\pi|c|/3})$ for $\operatorname{Re}(c)\geq\varepsilon$.

It remains to prove that

$$T:=\sum_{k=1}^{\infty}\left\{\frac{\Gamma(a+k)\Gamma(b+k)\Gamma(c+k)}{\Gamma(1+k)\Gamma(d+k)\Gamma(a+b-d+c+k)}-\frac{1}{k+1}\right\}$$

is analytic in c and equal to $O(e^{2\pi|c|/3})$ for $\operatorname{Re}(c)\geq\varepsilon$. Let $E=d-a-b$. By Lemma 7,

$$\begin{aligned}T&=\sum_{k=1}^{\infty}\left\{k^{-E-1}(c+k)^{E}(1+k^{-1}O(1))-\frac{1}{k+1}\right\}\\&=O(1)+\sum_{k=1}^{\infty}k^{-1}\left\{\left(1+\frac{c}{k}\right)^{E}-1\right\}\{1+k^{-1}O(1)\},\end{aligned}$$

where the expressions $O(1)$ are bounded analytic functions of c for $\operatorname{Re}(c)\geq\varepsilon$. By Lemma 8, $(1+c/k)^E-1=O(c^N/k)$, so T is analytic in c and equals $O(c^N)$ for $\operatorname{Re}(c)\geq\varepsilon$.

5. Proof of Theorem 3. Define

$$f(k)=\frac{\Gamma(a+k)\Gamma(b+k)\Gamma(c+k)}{\Gamma(d+k)\Gamma(e+k)\Gamma(1+k)},$$

and $V=\sum_{k=0}^{\infty}f(k)u^k+\log(1-u)-L$, where L is defined in (1.8). We must show that as $u\to1$,

$$V=O((1-u)\log(1-u)).$$

By (1.7),

$$V=\sum_{k=0}^{\infty}\left(f(k)-\frac{1}{k+1}\right)(u^k-1)+\sum_{k=0}^{\infty}\frac{u^k-u^{k+1}}{k+1}.$$

The last sum is $(u-1)/u\log(1-u)=O((1-u)\log(1-u))$ as $u\to1$. Finally, by Lemma 7,

$$\begin{aligned}\sum_{k=1}^{\infty}\left|\left(f(k)-\frac{1}{k+1}\right)(u^k-1)\right| &\ll\sum_{k=1}^{\infty}\frac{1-u^k}{k^2}\\ &=(1-u)\sum_{k=1}^{\infty}k^{-2}\sum_{n=0}^{k-1}u^n=(1-u)\sum_{n=0}^{\infty}u^n\sum_{k=n+1}^{\infty}k^{-2}\\ &<(1-u)\left\{\frac{\pi^2}{6}+\sum_{n=1}^{\infty}\frac{u^n}{n}\right\}=O((1-u)\log(1-u)).\end{aligned}$$

6. Concluding remarks. The series

$${}_3F_2\left(\begin{matrix}a,b,c\\ d,e\end{matrix}\middle|1\right)$$

converges for $\operatorname{Re}(e+d-a-b-c)>0$. Theorem 2 gives information of the divergence at the boundary $a+b+c=d+e$. We have not investigated related problems, such as $a+b+c=d+e+1$.

Bailey and Darling have given transformations for truncated 1-balanced ${}_3F_2$'s [1, p. 94–95]. We were unable to use similar techniques to derive Theorem 2. There may be similar results for special truncated very well poised ${}_6F_5$'s.

The special case $c=e$ of Theorem 3 gives an asymptotic expansion of a zero-balanced ${}_2F_1(x)$ as $x\to1$. This is equivalent to (1.10). This result is easy to obtain in the following way. The point $x=1$ is a regular singular point of the differential equation for ${}_2F_1(x)$. There are two independent solutions (u_1 and u_2) near $x=1$. If the ${}_2F_1$ is zero-balanced, one solution is logarithmic. The precise definitions of u_1 and u_2 and the constants c_1 and c_2 such that ${}_2F_1(x)=c_1u_1+c_2u_2$, are given in [3, eq. 2.10 (14)]. The asymptotic formula follows immediately.

For the ${}_3F_2(x)$ case, Norlund [5] has explicitly given three independent solutions $(u_1,u_2,$ and $u_3)$ near $x=1$. (The authors would like to thank Dennis Hejhal for pointing this out.) Again the zero-balancing condition gives a logarithmic solution. So an expansion of the form of Theorem 3 is guaranteed. However, the constant L is not given. One would need to find the constants c_1, c_2, and c_3 such that ${}_3F_2(x)=c_1u_1+c_2u_2+c_3u_3$. This is not an easy task.

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