SOME MIXED CHARACTER SUM IDENTITIES OF KATZ

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Abstract

A conjecture connected with quantum physics led N. Katz to discover some amazing mixed character sum identities over a field of q elements, where q is a power of a prime $p > 3$. His proof required deep algebro-geometric techniques, and he expressed interest in finding a more straightforward direct proof. Such a proof has been given by Evans and Greene in the case $q \equiv 3 \pmod 4$, and in this paper we give a proof for the remaining case $q \equiv 1 \pmod 4$. Moreover, we show that the identities are valid for all characteristics $p > 2$.

1 Introduction

Let $\mathbb{F}_q$ be a field of q elements, where q is a power of an odd prime p. Throughout this paper, A, B, C, D, χ, λ, ν, μ, ε, ϕ, A_4, A_8 denote complex multiplicative characters on $\mathbb{F}_q^*$, extended to map 0 to 0. Here ε and ϕ always denote the trivial and quadratic characters, respectively, while A_4 denotes a fixed quartic character when $q \equiv 1 \pmod 4$ and A_8 denotes a fixed octic character such that $A_8^2 = A_4$ when $q \equiv 1 \pmod 8$. Define $\delta(A)$ to be 1 or 0 according as A is trivial or not, and let $\delta(j,k)$ denote the Kronecker delta.

For $y \in \mathbb{F}_q$, let $\psi(y)$ denote the additive character

$$\psi(y) := \exp\left(\frac{2\pi i}{p}\left(y^p + y^{p^2} + \cdots + y^q\right)\right).$$

Recall the definitions of the Gauss and Jacobi sums

$$G(A) = \sum_{y \in \mathbb{F}_q} A(y)\psi(y), \quad J(A,B) = \sum_{y \in \mathbb{F}_q} A(y)B(1-y).$$

These sums have the familiar properties

$$G(\varepsilon) = -1, \quad J(\varepsilon,\varepsilon) = q-2,$$

and for nontrivial A,

$$G(A)G(\overline{A}) = A(-1)q, \quad J(A,\overline{A}) = -A(-1), \quad J(\varepsilon, A) = -1.$$

Gauss and Jacobi sums are related by [3, p. 59]

$$J(A,B) = \frac{G(A)G(B)}{G(AB)}, \quad \text{if } AB \neq \varepsilon$$

and

$$J(A,\overline{C}) = \frac{A(-1)G(A)G(\overline{A}C)}{G(C)} = A(-1)J(A,\overline{A}C), \quad \text{if } C \neq \varepsilon.$$

The Hasse–Davenport product relation [3, p. 351] yields

$$A(4)G(A)G(A\phi) = G(A^2)G(\phi). \tag{1.1}$$

As in [8, p. 82], define the hypergeometric ${}_2F_1$ function over $\mathbb{F}_q$ by

$$(1.2) \qquad {}_2F_1\left(\begin{matrix} A, B \\ C \end{matrix} \,\middle|\, x \right) = \frac{\varepsilon(x)}{q} \sum_{y \in \mathbb{F}_q} B(y)\overline{B}C(y-1)\overline{A}(1-xy), \quad x \in \mathbb{F}_q.$$

For $j, k \in \mathbb{F}_q$ and $a \in \mathbb{F}_q^*$, Katz [9, p. 224] defined the mixed exponential sum

$$(1.3) \qquad \begin{aligned} P(j,k) :&= \delta(j,k) + \phi(-1)\delta(j,-k) + \\ &\frac{1}{G(\phi)} \sum_{x \in \mathbb{F}_q^*} \phi(a/x - x)\psi(x(j+k)^2 + (a/x)(j-k)^2). \end{aligned}$$

Note that $P(j,k) = P(k,j)$ and $P(-j,k) = \phi(-1)P(j,k)$. Katz proved an equidistribution conjecture of Wootters [9, p. 226], [1] connected with quantum physics by constructing explicit character sums $V(j)$ [9, pp. 226–229]) for which the identities

$$(1.4) \qquad P(j,k) = V(j)V(k)$$

hold for all $j, k \in \mathbb{F}_q$. (The q-dimensional vector $(V(j))_{j \in \mathbb{F}_q}$ is a minimum uncertainty state, as described by Sussman and Wootters [10].) Katz's proof [9, Theorem 10.2] of the identities (1.4) required the characteristic p to exceed 3, in order to guarantee that various sheaves of ranks 2, 3, and 4 have geometric and arithmetic monodromy groups which are SL(2), SO(3), and SO(4), respectively.

As Katz indicated in [9, p. 223], his proof of (1.4) is quite complex, invoking the theory of Kloosterman sheaves and their rigidity properties, as well as results of Deligne [4] and Beilinson, Bernstein, Deligne [2]. Katz [9, p. 223] wrote, "It would be interesting to find direct proofs of these identities."

The goal of this paper is to respond to Katz's challenge by giving a direct proof of (1.4) (a "character sum proof" not involving algebraic geometry) in the case $q \equiv 1 \pmod 4$. This has the benefit of making the demonstration of his useful identities accessible to a wider audience of mathematicians and physicists. Another advantage of our proof is that it works for all odd characteristics p, including $p = 3$. As a bonus, we obtain some interesting character sum evaluations in terms of Gauss sums; see for example Theorems 2.1, 3.2, and 5.5.

Our method of proof is to show (see Sections 4 and 6) that the Mellin transforms of both sides of (1.4) are equal. A key feature of our proof is the application in Lemma 5.1 of the following hypergeometric ${}_2F_1$ transformation formula over $\mathbb{F}_q$ for $q \equiv 1 \pmod 4$ [5, Theorem 3] :

$$(1.5) \qquad {}_2F_1\left(\begin{array}{c|} D, DA_4 \\ A_4 \end{array} \, z^4 \right) = \overline{D}^4(z-1)\, {}_2F_1\left(\begin{array}{c|} D, D^2\phi \\ D\phi \end{array} \, -\left(\frac{z+1}{z-1}\right)^2 \right),$$

which holds for every character D on $\mathbb{F}_q$ and every $z \in \mathbb{F}_q^*$ with $z \notin \{1, -1\}$. The proof of (1.5) depends on a recently proved finite field analogue [5, Theorem 2], [7, Theorem 17] of a classical quadratic transformation formula of Gauss.

For $q \equiv 1 \pmod 4$, Katz's character sums $V(j)$ are defined in (1.6)–(1.7) below. In the case $q \equiv 3 \pmod 4$, the sums $V(j)$ have a more complex definition, in that they are sums over $\mathbb{F}_{q^2}$ [9, p. 228]. A direct proof of (1.4) for the case $q \equiv 3 \pmod 4$ has been given by Evans and Greene [6].

From here on, let $q \equiv 1 \pmod 4$, so that there exists a primitive fourth root of unity $i \in \mathbb{F}_q$. Thus $\phi(-1) = \phi(i^2) = 1$ and $A_4(-4) = A_4((1+i)^4) = 1$.

For a as in (1.3), define

$$\tau = -\sqrt{qA_4(-a)},$$

where the choice of square root is fixed. For $q \equiv 1 \pmod 4$, the sums $V(j)$ are defined as follows:

$$(1.6) \qquad V(j) := \tau^{-1} \sum_{x \in \mathbb{F}_q^*} A_4(x)\psi(x + aj^4/x), \quad j \in \mathbb{F}_q^*,$$

while for $j = 0$,

$$(1.7) \qquad V(0) := G(A_4)/\tau + \tau/G(A_4).$$

2 Mellin transform of the sums $V(j)$

For any character χ, define the Mellin transform

$$(2.1) \qquad S(\chi) := \sum_{j \in \mathbb{F}_q^*} \chi(j)V(j).$$

The next theorem gives an evaluation of $S(\chi)$ in terms of Gauss sums.

Theorem 2.1. *If χ is not a fourth power, then $S(\chi) = 0$. On the other hand, if $\chi = \nu^4$, then*

$$S(\chi) = \tau^{-1}\overline{\nu}(a) \sum_{m=0}^{3} A_4(a)^{1-m} G(\nu A_4^{m-1}) G(\nu A_4^m). \tag{2.2}$$

Proof. By (1.6),

$$V(j) = V(ji), \quad j \in \mathbb{F}_q^*.$$

Thus $S(\chi) = 0$ when $\chi(i) \neq 1$, i.e., when χ is not a fourth power.

Now set $\chi = \nu^4$ for some character ν, and write $\lambda = \nu^2 \overline{A}_4$. By (1.6),

$$\tau V(j) = \phi(j) \sum_{x \in \mathbb{F}_q^*} A_4(x) \psi(j^2(x + a/x)),$$

so

$$\begin{aligned} \tau S(\chi) &= \sum_{j,x \in \mathbb{F}_q^*} \lambda(j^2) A_4(x) \psi(j^2(x + a/x)) \\ &= \sum_{j,x \in \mathbb{F}_q^*} \lambda(j) A_4(x) \psi(j(x + a/x))(1 + \phi(j)). \end{aligned} \tag{2.3}$$

First suppose that λ^2 is nontrivial. Then for the sums in (2.3), there is no contribution from the terms where $x + a/x = 0$. Thus

$$\tau S(\chi) = G(\lambda) Y(\lambda) + G(\lambda\phi) Y(\lambda\phi), \tag{2.4}$$

where

$$Y(\lambda) := \sum_{x \in \mathbb{F}_q^*} A_4(x) \overline{\lambda}(x + a/x). \tag{2.5}$$

By (2.5),

$$\begin{aligned} Y(\lambda) = Y(\nu^2 \overline{A}_4) &= \sum_{x \in \mathbb{F}_q^*} A_4(x) \overline{\nu}^2 A_4(x + a/x) = \sum_{x \in \mathbb{F}_q^*} \nu(x^2) \overline{\nu}^2 A_4(x^2 + a) \\ &= \sum_{x \in \mathbb{F}_q^*} \nu(x) \overline{\nu}^2 A_4(x + a)(1 + \phi(x)) \\ &= \overline{\nu}(-a)\{A_4(a) J(\nu, \overline{\nu}^2 A_4) + \overline{A}_4(a) J(\nu\phi, \overline{\nu}^2 A_4)\}. \end{aligned}$$

Since $J(B, C) = B(-1)J(B, \overline{BC})$ for all characters B, C, we see that

$$J(\nu, \overline{\nu}^2 A_4) = \nu(-1)J(\nu, \nu\overline{A}_4), \quad J(\nu\phi, \overline{\nu}^2 A_4) = \nu(-1)J(\nu\phi, \nu A_4).$$

Thus

$$Y(\lambda) = \overline{\nu}(a)\{A_4(a)J(\nu, \nu\overline{A}_4) + \overline{A}_4(a)J(\nu\phi, \nu A_4)\}. \tag{2.6}$$

Similarly, we have

$$Y(\lambda\phi) = Y(\nu^2 A_4) = \overline{\nu}(a)\{J(\nu, \nu A_4) + \phi(a)J(\nu\phi, \nu\overline{A}_4)\}. \tag{2.7}$$

Putting (2.6)–(2.7) into (2.4), we easily see that (2.2) holds in the case that λ^2 is nontrivial.

Finally, assume that λ^2 is trivial, so that $\nu^4 = \chi = \phi$. Then $q \equiv 1 \pmod 8$ and ν is an odd power of A_8. By (2.3),

$$\begin{aligned} \tau S(\chi) &= \sum_{j\in\mathbb{F}_q}\sum_{x\in\mathbb{F}_q^*} A_4(x)\psi(j^2(x + a/x)) \\ &= q \sum_{\substack{x\in\mathbb{F}_q^* \\ x+a/x=0}} A_4(x) \; + \; G(\phi)\sum_{x\in\mathbb{F}_q^*} A_4(x)\phi(x + a/x). \end{aligned} \tag{2.8}$$

The first of the two terms on the far right of (2.8) vanishes when $\phi(-a) = -1$, and so this term equals $q(A_8(-a) + A_8^5(-a))$. Thus (2.8) yields

$$\begin{aligned} \tau S(\chi) &= q(A_8(-a) + A_8^5(-a)) + G(\phi)\sum_{x\in\mathbb{F}_q} \overline{A}_8(x)\phi(a + x)(1 + \phi(x)) \\ &= q(A_8(-a) + A_8^5(-a)) + G(\phi)\{A_8^3(-a)J(\overline{A}_8, \phi) + \overline{A}_8(-a)J(A_8^3, \phi)\}. \end{aligned}$$

It follows that

$$\begin{aligned} \tau S(\chi) &= A_8(a)G(A_8)G(\overline{A}_8) + A_8^5(a)G(A_8^3)G(\overline{A}_8^3) \\ &\quad + A_8^3(a)G(\overline{A}_8)G(\overline{A}_8^3) + \overline{A}_8(a)G(A_8)G(A_8^3). \end{aligned} \tag{2.9}$$

No matter which odd power of A_8 is substituted for ν in (2.2), we see that (2.2) matches (2.9). Thus the proof of (2.2) is complete. □

3 Mellin transform of the sums $P(j,0)$

For any character χ, define the Mellin transform

$$T(\chi) := \sum_{j \in \mathbb{F}_q^*} \chi(j) P(j,0). \tag{3.1}$$

Theorem 3.2 below gives an evaluation of $T(\chi)$ in terms of Gauss sums. We will need the following lemma.

Lemma 3.1. *When ν^4 is nontrivial,*

$$ {}_2F_1\left(\begin{matrix} \nu^2, \nu A_4 \\ \nu \overline{A}_4 \end{matrix} \,\middle|\, -1 \right) = \frac{A_4(-1)G(\nu A_4)}{qG(\phi)G(\nu^2)} \{G(\nu)G(A_4) + G(\nu\phi)G(\overline{A}_4)\}.$$

Proof. This follows by first permuting the numerator parameters by means of [8, Corollary 3.21] and then applying a finite field analogue [8, (4.11)] of a classical summation formula of Kummer. □

Theorem 3.2. *If χ is not a fourth power, then $T(\chi) = 0$. On the other hand, if $\chi = \nu^4$, then*

$$\begin{aligned} T(\chi) &= q^{-1} A_4(-1)\{\overline{A}_4(a)G(A_4) + G(\overline{A}_4)\} \times \\ &\times \overline{\nu}(a) \sum_{m=0}^{3} A_4(a)^{1-m} G(\nu A_4^m) G(\nu A_4^{m-1}). \end{aligned} \tag{3.2}$$

Proof. By (3.1) and (1.3),

$$G(\phi)T(\chi) = \sum_{j,x \in \mathbb{F}_q^*} \phi(x - a/x)\chi(j)\psi(j^2(x + a/x)).$$

Therefore $T(\chi) = 0$ unless χ is a square, so suppose that $\chi = (\lambda A_4)^2$ for some character λ. Then

$$\begin{aligned} G(\phi)T(\chi) &= \sum_{j,x \in \mathbb{F}_q^*} \phi(x - a/x)\lambda A_4(j)\psi(j(x + a/x))(1 + \phi(j)) \\ &= U(\lambda) + W(\lambda) + W(\lambda\phi), \end{aligned} \tag{3.3}$$

where

$$U(\lambda) = \sum_{\substack{x \in \mathbb{F}_q^* \\ x + a/x = 0}} \phi(x - a/x) \sum_{j \in \mathbb{F}_q^*} (\lambda A_4(j) + \lambda \overline{A}_4(j)) \tag{3.4}$$

and

$$W(\lambda) = G(\lambda A_4) \sum_{x \in \mathbb{F}_q^*} \phi(x - a/x) \overline{\lambda A}_4 (x + a/x). \tag{3.5}$$

Thus $T(\chi) = 0$ unless $\lambda A_4 = \nu^2$ for some character ν, i.e., unless $\chi = \nu^4$. This proves the first part of Theorem 3.2. For the remainder of this proof, assume that

$$\lambda A_4 = \nu^2, \quad \chi = \nu^4.$$

First consider the case where $\chi = \nu^4$ is nontrivial. Then $U(\lambda) = 0$, since λA_4 and $\lambda \overline{A}_4$ are nontrivial.

We have

$$\begin{aligned} W(\lambda)/G(\nu^2) &= \sum_{x \in \mathbb{F}_q^*} \phi(x - a/x) \overline{\nu}^2 (x + a/x) \\ &= \sum_{x \in \mathbb{F}_q^*} \nu^2 \phi(x) \phi(a - x^2) \overline{\nu}^2 (a + x^2) \\ &= \sum_{x \in \mathbb{F}_q^*} \nu A_4(x) \phi(a - x) \overline{\nu}^2 (a + x)(1 + \phi(x)) \\ &= q \overline{\nu} \overline{A}_4(a) {}_2F_1 \left(\begin{matrix} \nu^2, \nu A_4 \\ \nu \overline{A}_4 \end{matrix} \middle| -1 \right) + q \overline{\nu} A_4(a) {}_2F_1 \left(\begin{matrix} \nu^2, \nu \overline{A}_4 \\ \nu A_4 \end{matrix} \middle| -1 \right). \end{aligned} \tag{3.6}$$

By Lemma 3.1,

$$\begin{aligned} &W(\lambda) = \\ &q^{-1} A_4(-1) G(\nu A_4) G(\phi) \overline{\nu} \overline{A}_4(a) \{ G(\nu) G(A_4) + G(\nu\phi) G(\overline{A}_4) \} \\ &+ q^{-1} A_4(-1) G(\nu \overline{A}_4) G(\phi) \overline{\nu} A_4(a) \{ G(\nu\phi) G(A_4) + G(\nu) G(\overline{A}_4) \}. \end{aligned} \tag{3.7}$$

Similarly,

$$\begin{aligned} &W(\lambda\phi) = \\ &q^{-1} A_4(-1) G(\nu\phi) G(\phi) \overline{\nu}\phi(a) \{ G(\nu A_4) G(A_4) + G(\nu \overline{A}_4) G(\overline{A}_4) \} \\ &+ q^{-1} A_4(-1) G(\nu) G(\phi) \overline{\nu}(a) \{ G(\nu \overline{A}_4) G(A_4) + G(\nu A_4) G(\overline{A}_4) \}. \end{aligned} \tag{3.8}$$

By (3.3), (3.7), and (3.8), we arrive at the desired result (3.2) in the case where $\chi = \nu^4$ is nontrivial.

Finally, suppose that $\chi = \nu^4$ is trivial, so that either $\lambda A_4 = \varepsilon$ or $\lambda A_4 = \phi$. We must show that

$$\sum_{j\in\mathbb{F}_q^*} P(j,0) = q^{-1}A_4(-1)\{\overline{A}_4(a)G(A_4) + G(\overline{A}_4)\} \times \times \sum_{m=0}^{3} A_4(a)^{1-m}G(A_4^m)G(A_4^{m-1}). \tag{3.9}$$

Straightforward computations show that

$$U(\lambda) = (q-1)(A_4(a) + \overline{A}_4(a)) \tag{3.10}$$

and

(3.11)

$$\begin{aligned} W(\lambda) + W(\lambda\phi) =& A_4(a) + \overline{A}_4(a) - A_4(a)J(\overline{A}_4,\phi) - \overline{A}_4(a)J(A_4,\phi) \\ &- 2G(\phi) + \phi(a)J(A_4,\phi)G(\phi) + \phi(a)J(\overline{A}_4,\phi)G(\phi). \end{aligned}$$

The desired result (3.9) now follows from (3.3), (3.10), and (3.11). □

4 Proof of (1.4) when $jk = 0$

We first consider the case where $j = k = 0$. By (1.3),

$$\begin{aligned} G(\phi)(P(0,0) - 2) &= \sum_{x\in\mathbb{F}_q^*} \phi(x)\phi(x^2 - a) = \sum_{u\in\mathbb{F}_q^*} A_4(u)\phi(u-a)(1+\phi(u)) \\ &= 2\ \mathrm{Re}\,\overline{A}_4(a)J(A_4,\phi) = 2\ \mathrm{Re}\,A_4(4/a)J(A_4,A_4), \end{aligned}$$

where the last equality follows from the Hasse–Davenport formula (1.1). Dividing by $G(\phi)$ and using the fact that $A_4(4) = A_4(-1)$, we have

$$P(0,0) = 2 + 2\ \mathrm{Re}\,G(A_4)^2/(qA_4(-a)).$$

It now follows easily from (1.7) that $P(0,0) = V(0)^2$.

To complete the proof of (1.4) for $jk = 0$, it remains to prove that

$$P(j,0) = V(0)V(j), \quad j \in \mathbb{F}_q^*, \tag{4.1}$$

since P is symmetric in its two arguments. By Theorems 2.1 and 3.2, the Mellin transforms of the left and right sides of (4.1) are the same for all characters. By taking inverse Mellin transforms, we see that (4.1) holds, so the proof of (1.4) for $jk = 0$ is complete.

5 Double Mellin Transform of $P(j,k)$

For characters χ_1, χ_2, define the double Mellin transform

$$T = T(\chi_1, \chi_2) := \sum_{j,k \in \mathbb{F}_q^*} \chi_1(j)\chi_2(k)P(j,k). \tag{5.1}$$

Note that $T(\chi_1, \chi_2)$ is symmetric in χ_1, χ_2. In this section we will evaluate T. Theorem 5.4 shows that $T = 0$ when χ_1 and χ_2 are not both fourth powers. Theorem 5.5 evaluates T when χ_1 and χ_2 are both fourth powers.

Since $P(j,k) = P(-j,k)$, we have $T = 0$ unless χ_1 and χ_2 are squares, so we set

$$\chi_i = (\lambda_i A_4)^2 = \lambda_i^2 \phi, \quad i = 1, 2 \tag{5.2}$$

for characters λ_i (which are well-defined up to factors of ϕ).

From the definitions of T and $P(j,k)$, we have

$$G(\phi)\{T - (2q-2)\delta(\lambda_1^2\lambda_2^2)\} = \\ \sum_{j,k,x \in \mathbb{F}_q^*} \lambda_1^2\phi(j)\lambda_2^2\phi(k)\phi(x - a/x)\psi(x(j+k)^2 + a(j-k)^2/x).$$

Replace j by jk to obtain

$$\begin{aligned} &G(\phi)\{T - (2q-2)\delta(\lambda_1^2\lambda_2^2)\} = \\ &\sum_{j,k,x \in \mathbb{F}_q^*} \lambda_1^2\phi(j)\lambda_1^2\lambda_2^2(k)\phi(x - a/x)\psi(k^2\alpha(j,x)) = \\ &\sum_{j,k,x \in \mathbb{F}_q^*} \lambda_1^2\phi(j)\lambda_1\lambda_2(k)\phi(x - a/x)\psi(k\alpha(j,x))(1 + \phi(k)), \end{aligned} \tag{5.3}$$

where

$$\alpha(j,x) := x(j+1)^2 + a(j-1)^2/x. \tag{5.4}$$

Note that $\alpha(j,x)$ cannot vanish when $j = \pm 1$. By (5.3).

$$\begin{aligned} &G(\phi)\{T - (2q-2)\delta(\lambda_1^2\lambda_2^2)\} = \\ &\delta(\lambda_1^2\lambda_2^2)(q-1)H(\lambda_1) + G(\lambda_1\lambda_2)E(\lambda_1,\lambda_2) + G(\lambda_1\lambda_2\phi)E(\lambda_1,\lambda_2\phi), \end{aligned} \tag{5.5}$$

where

$$H(\lambda_1) := \sum_{\substack{j,x\in\mathbb{F}_q^*\\ \alpha(j,x)=0}} \lambda_1^2\phi(j)\phi(x-a/x) \tag{5.6}$$

and

$$E(\lambda_1,\lambda_2) := \sum_{j,x\in\mathbb{F}_q^*} \lambda_1^2\phi(j)\phi(x-a/x)\overline{\lambda_1\lambda_2}(\alpha(j,x)). \tag{5.7}$$

For a character D and $j\in\mathbb{F}_q^*$, define

$$h(D,j) := \sum_{x\in\mathbb{F}_q^*} D(x)\phi(1-x)\overline{D}^2\phi(x(j+1)^2+(j-1)^2). \tag{5.8}$$

By (5.7),

$$E(\lambda_1,\lambda_2) = \sum_{j\in\mathbb{F}_q^*} \chi_1(j)\beta(\lambda_1,\lambda_2,j), \tag{5.9}$$

where

$$\beta(\lambda_1,\lambda_2,j) := \sum_{x\in\mathbb{F}_q^*} \phi(a-x^2)\lambda_1\lambda_2\phi(x)\overline{\lambda_1\lambda_2}(x^2(j+1)^2+a(j-1)^2). \tag{5.10}$$

If $\lambda_1\lambda_2\phi$ is odd, i.e., if $\chi_1\chi_2=\lambda_1^2\lambda_2^2$ is not a fourth power, then we'd have $\delta(\lambda_1^2\lambda_2^2)=0$ and

$$\beta(\lambda_1,\lambda_2,j)=\beta(\lambda_1,\lambda_2\phi,j)=0, \tag{5.11}$$

so that $T=0$ by (5.5). Thus we may assume that

$$\lambda_1\lambda_2\phi=\mu^2 \tag{5.12}$$

for some character μ which is well defined up to factors of A_4. By (5.10),

$$\begin{aligned}\beta(\lambda_1,\lambda_2,j) &= \sum_{x\in\mathbb{F}_q^*} \phi(a-x)\overline{\mu}^2\phi(x(j+1)^2+a(j-1)^2)\{\mu(x)+\mu\phi(x)\}\\ &= \overline{\mu}(a)h(\mu,j)+\overline{\mu}\phi(a)h(\mu\phi,j)\end{aligned} \tag{5.13}$$

and (by replacing λ_2 by $\lambda_2\phi$)

(5.14)
$$\beta(\lambda_1, \lambda_2\phi, j) = \sum_{x\in\mathbb{F}_q^*} \phi(a-x)\overline{\mu}^2(x(j+1)^2 + a(j-1)^2)\{\mu A_4(x) + \mu\overline{A}_4(x)\}$$
$$= \overline{\mu}\overline{A}_4(a)h(\mu A_4, j) + \overline{\mu}A_4(a)h(\mu\overline{A}_4, j).$$

Thus (5.5) is equivalent to

$$G(\phi)\{T - (2q-2)\delta(\mu^4)\} = \delta(\mu^4)(q-1)H(\lambda_1)$$
(5.15)
$$+ \overline{\mu}(a)\sum_{m=0}^{3} G(\mu^2\phi^{m+1})\overline{A}_4^m(a)\sum_{j\in\mathbb{F}_q^*}\chi_1(j)h(\mu A_4^m, j).$$

For $D \in \{\varepsilon, A_4, \overline{A}_4\}$ and $j \in \mathbb{F}_q^*$, we can evaluate $h(D,j)$ directly from definition (5.8), as follows:

(5.16)
$$h(\varepsilon, j) = -2 + q\delta(j^2, -1) + \delta(j^2, 1),$$

(5.17)
$$h(A_4, j) = J(A_4, \phi) - \phi(j^4 - 1),$$

(5.18)
$$h(\overline{A}_4, j) = J(\overline{A}_4, \phi) - \phi(j^4 - 1).$$

For $D \notin \{\varepsilon, A_4, \overline{A}_4\}$, the following lemma expresses $h(D,j)$ in terms of a hypergeometric character sum.

Lemma 5.1. *For $D \notin \{\varepsilon, A_4, \overline{A}_4\}$ and $j \in \mathbb{F}_q^*$,*

(5.19)
$$h(D, j) = \frac{G(D)^2 G(\phi)}{G(D^2\phi)}\, {}_2F_1\left(\begin{array}{c|c} D, DA_4 & \\ A_4 & j^4 \end{array} \right).$$

Proof. First consider the case where $j \neq \pm 1$. By (1.2),

(5.20)
$$h(D, j) = q\overline{D}^4(j-1)\, {}_2F_1\left(\begin{array}{c|c} D^2\phi, D & \\ D\phi & -\left(\frac{j+1}{j-1}\right)^2 \end{array} \right).$$

Since $D \notin \{\varepsilon, A_4, \overline{A}_4\}$, we can permute the two numerator parameters by means of [8, Corollary 3.21] to obtain

$$h(D,j) := \frac{G(D)^2 G(\phi)}{G(D^2\phi)} \overline{D}^4(j-1) {}_2F_1\left(\begin{matrix} D, D^2\phi \\ D\phi \end{matrix} \,\middle|\, -\left(\frac{j+1}{j-1}\right)^2 \right). \tag{5.21}$$

Now (5.19) for $j \neq \pm 1$ follows from (1.5).

Finally, let $j = \pm 1$. By a finite field analogue [8, Theorem 4.9] of Gauss's classical summation formula,

$${}_2F_1\left(\begin{matrix} D, DA_4 \\ A_4 \end{matrix} \,\middle|\, j^4 \right) = \frac{D(-1) G(DA_4) G(\overline{D}^2)}{q G(\overline{D} A_4)}. \tag{5.22}$$

Thus by the Hasse-Davenport formula (1.1) with $A = \overline{D} A_4$,

$${}_2F_1\left(\begin{matrix} D, DA_4 \\ A_4 \end{matrix} \,\middle|\, j^4 \right) = \frac{\overline{D}(4) G(\overline{D}^2)}{G(\overline{D}^2\phi) G(\phi)}. \tag{5.23}$$

From (5.8) with $j = \pm 1$,

$$h(D,j) = \overline{D}(4)^2 J(D,\phi) = \overline{D}(4) G(D)^2 / G(D^2),$$

where the last equality follows from (1.1). Together with (5.23), this completes the proof of (5.19) for $j = \pm 1$. □

Lemma 5.2. *If χ_1 is not a fourth power, then $H(\lambda_1) = 0$. On the other hand, if $\chi_1 = \nu_1^4$, then*

$$H(\lambda_1) = (A_4(a) + \overline{A}_4(a)) \sum_{m=0}^{3} J(\nu_1 A_4^m, \phi). \tag{5.24}$$

Proof. We have

$$\begin{aligned} H(\lambda_1) &= (A_4(-a) + \overline{A}_4(-a)) \sum_{j \neq \pm 1} \chi_1(j) \phi\left(\frac{j-1}{j+1} + \frac{j+1}{j-1}\right) \\ &= (A_4(-a) + \overline{A}_4(-a)) \sum_{j \in \mathbb{F}_q^*} \chi_1(j) \phi(2) \phi(j^2+1) \phi(j^2-1) \\ &= (A_4(a) + \overline{A}_4(a)) \sum_{j \in \mathbb{F}_q^*} \chi_1(j) \phi(j^4 - 1). \end{aligned}$$

If χ_1 is not a fourth power, then replacement of j by ji shows that $H(\lambda_1) = 0$. On the other hand, if $\chi_1 = \nu_1^4$, then

$$H(\lambda_1) = (A_4(a) + \overline{A}_4(a)) \sum_{j \in \mathbb{F}_q^*} \nu_1(j)\phi(1-j) \sum_{m=0}^{3} A_4^m(j),$$

which proves (5.24) □

Lemma 5.3. *If μ^4 is nontrivial and χ_1 is not a fourth power, then $T = 0$.*

Proof. Suppose that μ is not a power of A_4. It follows from Lemma 5.1 that each expression $h(\mu A_4^m, j)$ in (5.15) is unchanged when j is replaced by ji. If χ_1 is not a fourth power, then each sum on j in (5.15) vanishes, and hence T vanishes. □

Theorem 5.4. *$T(\chi_1, \chi_2) = 0$ when the characters χ_1 and χ_2 are not both fourth powers.*

Proof. Since $T(\chi_1, \chi_2)$ is symmetric in the arguments χ_1, χ_2, it suffices to prove that $T = 0$ under the assumption that χ_1 is not a fourth power. In view of Lemma 5.3, we may also assume that μ^4 is trivial, i.e., μA_4^n is trivial for some $n \in \{0, 1, 2, 3\}$. By (5.16)–(5.19), $h(\mu A_4^m, j)$ is unchanged when j is replaced by ji, unless $m = n$. Since $H(\lambda_1) = 0$ by Lemma 5.2, it follows from (5.15) and (5.16) that

$$(5.25)\quad G(\phi)T = G(\phi)(2q-2) + G(\phi) \sum_{j \in \mathbb{F}_q^*} \chi_1(j)\{-2 + q\delta(j^2, -1) + \delta(j^2, 1)\}.$$

Thus

$$(5.26)\qquad T = (2q-2) + q\chi_1(i) + q\chi_1(-i) + \chi_1(1) + \chi_1(-1).$$

Since χ_1 is a square by (5.2),

$$\chi_1(i) = \chi_1(-i) = -1, \quad \chi_1(-1) = \chi_1(1) = 1,$$

so that (5.26) yields the desired result $T = 0$. □

The next theorem evaluates $T(\chi_1, \chi_2)$ when χ_1 and χ_2 are both fourth powers. By (5.2), we may assume that

$$(5.27)\qquad \lambda_i A_4 = \nu_i^2, \ \chi_i = \nu_i^4, \quad i = 1, 2,$$

for characters ν_1 and ν_2 (which are well-defined up to factors of A_4). By (5.12), $(\nu_1\nu_2)^2 = \mu^2$, and we may assume that

$$\mu = \nu_1\nu_2, \tag{5.28}$$

otherwise replace each ν_i with $\nu_i A_4$.

Theorem 5.5. *Suppose that $\chi_i = \nu_i^4$ for $i = 1, 2$. Then*
(5.29)

$$T = \frac{A_4(-a)}{q} \sum_{m=0}^{3} \sum_{n=0}^{3} \overline{\mu}\overline{A}_4^{m+n}(a) G(\nu_1 A_4^{n-1}) G(\nu_1 A_4^n) G(\nu_2 A_4^{m-1}) G(\nu_2 A_4^m).$$

Proof. First assume that μ^4 is nontrivial. By (5.15),

$$G(\phi)T = \sum_{m=0}^{3} G(\mu^2\phi^{m+1}) \overline{\mu}\overline{A}_4^m(a) \sum_{j\in\mathbb{F}_q^*} \chi_1(j) h(\mu A_4^m, j). \tag{5.30}$$

By Lemma 5.1 and (1.2),

$$h(\mu A_4^m, j) = \frac{G(\mu A_4^m)^2 G(\phi)}{qG(\mu^2\phi^{m+1})} \sum_{x\in\mathbb{F}_q^*} \mu A_4^{m+1}(x) \overline{\mu}\overline{A}_4^m(x-1) \overline{\mu}\overline{A}_4^m(1 - xj^4). \tag{5.31}$$

Thus

$$\begin{aligned} \sum_{j\in\mathbb{F}_q^*} \chi_1(j) h(\mu A_4^m, j) &= \frac{G(\mu A_4^m)^2 G(\phi)}{qG(\mu^2\phi^{m+1})} \times \\ \times \sum_{x\in\mathbb{F}_q^*} \mu A_4^{m+1}(x) \overline{\mu}\overline{A}_4^m(x-1) &\sum_{n=0}^{3} \sum_{j\in\mathbb{F}_q^*} \nu_1 A_4^n(j) \overline{\mu}\overline{A}_4^m(1 - xj). \end{aligned} \tag{5.32}$$

Replacing j by j/x on the right side of (5.32), we see from (5.30) that

(5.33)

$$\begin{aligned} T &= \sum_{m=0}^{3} \sum_{n=0}^{3} \frac{\overline{\mu}\overline{A}_4^m(a) G(\mu A_4^m)^2}{q} \sum_{x\in\mathbb{F}_q^*} \mu\overline{\nu_1} A_4^{m-n+1}(x) \overline{\mu}\overline{A}_4^m(x-1) J(\nu_1 A_4^n, \overline{\mu}\overline{A}_4^m) \\ &= \sum_{m=0}^{3} \sum_{n=0}^{3} \frac{\overline{\mu}\overline{A}_4^m(a) G(\mu A_4^m)^2}{q} \mu A_4^m(-1) J(\mu\overline{\nu_1} A_4^{m-n+1}, \overline{\mu}\overline{A}_4^m) J(\nu_1 A_4^n, \overline{\mu}\overline{A}_4^m). \end{aligned}$$

Since μA_4^m is nontrivial for each m, (5.33) yields
(5.34)

$$T = \sum_{m=0}^{3}\sum_{n=0}^{3} \frac{\overline{\mu}\overline{A}_4^m(a)A_4(-1)}{q} G(\nu_1 A_4^{n-1})G(\nu_2 A_4^{m-n})G(\nu_2 A_4^{m+1-n})G(\nu_1 A_4^n),$$

in view of the first equality above (1.1). Replacing m by $m+n-1$ in (5.34), we complete the proof of (5.29) in the case that μ^4 is nontrivial.

Next suppose that μ^4 is trivial, i.e., μ is a power of A_4. By (5.15) and Lemma 5.2,

$$\begin{aligned} G(\phi)T &= G(\phi)(2q-2) + (q-1)(A_4(a)+\overline{A}_4(a))\sum_{m=0}^{3} J(\nu_1 A_4^m, \phi) \\ &\quad - \overline{A}_4(a)\sum_{j\in\mathbb{F}_q^*}\chi_1(j)h(A_4,j) - A_4(a)\sum_{j\in\mathbb{F}_q^*}\chi_1(j)h(\overline{A}_4,j) \\ &\quad + G(\phi)\sum_{j\in\mathbb{F}_q^*}\chi_1(j)h(\varepsilon,j) + \phi(a)G(\phi)\sum_{j\in\mathbb{F}_q^*}\chi_1(j)h(\phi,j). \end{aligned} \tag{5.35}$$

The formula (5.35) can be rewritten as

$$T = \sum_{k=0}^{3} R_k A_4^k(a), \tag{5.36}$$

where

$$R_0 = (2q-2) + \sum_{j\in\mathbb{F}_q^*}\chi_1(j)h(\varepsilon,j), \tag{5.37}$$

$$R_1 = G(\phi)^{-1}\{(q-1)\sum_{m=0}^{3} J(\nu_1 A_4^m,\phi) - \sum_{j\in\mathbb{F}_q^*}\chi_1(j)h(\overline{A}_4,j)\}, \tag{5.38}$$

$$R_3 = G(\phi)^{-1}\{(q-1)\sum_{m=0}^{3} J(\nu_1 A_4^m,\phi) - \sum_{j\in\mathbb{F}_q^*}\chi_1(j)h(A_4,j)\}, \tag{5.39}$$

$$R_2 = \sum_{j\in\mathbb{F}_q^*}\chi_1(j)h(\phi,j). \tag{5.40}$$

By (5.16),

$$R_0 = 4q - (2q-2)\delta(\chi_1). \tag{5.41}$$

By (5.18),

$$R_1 = G(\phi)^{-1}\{q\sum_{m=0}^{3} J(\nu_1 A_4^m, \phi) - \delta(\chi_1)(q-1)J(\overline{A}_4, \phi)\}. \tag{5.42}$$

By (5.17),

$$R_3 = G(\phi)^{-1}\{q\sum_{m=0}^{3} J(\nu_1 A_4^m, \phi) - \delta(\chi_1)(q-1)J(A_4, \phi)\}. \tag{5.43}$$

By Lemma 5.1 and (1.2),

$$R_2 = \sum_{m=0}^{3} J(\overline{\nu_1}\overline{A}_4^{m+1}, \phi)J(\nu_1 A_4^m, \phi). \tag{5.44}$$

A lengthy but straightforward computation now shows that for each k in $\{0,1,2,3\}$,

$$R_k = q^{-1}A_4(-1)\sum G(\nu_1 A_4^{n-1})G(\nu_1 A_4^n)G(\overline{\nu_1}A_4^{m-1})G(\overline{\nu_1}A_4^m), \tag{5.45}$$

where the sum is over all $m, n \in \{0,1,2,3\}$ for which $\overline{A}_4^{m+n-1} = A_4^k$. Putting (5.45) into (5.36), we obtain

(5.46)

$$T = \frac{A_4(-a)}{q}\sum_{m=0}^{3}\sum_{n=0}^{3}\overline{A}_4^{m+n}(a)G(\nu_1 A_4^{n-1})G(\nu_1 A_4^n)G(\overline{\nu_1}A_4^{m-1})G(\overline{\nu_1}A_4^m).$$

Since $\overline{\mu} = A_4^\ell$ for some ℓ, we may substitute $\nu_2 A_4^\ell$ for $\overline{\nu_1}$ in (5.46). Then upon replacing m by $m-\ell$, we complete the proof of (5.29) in the case that μ^4 is trivial. □

6 Proof of Katz's identities (1.4)

The proof for $jk = 0$ was given in Section 4, so we may assume that $jk \neq 0$. Let

$$S(\chi_1, \chi_2) := \sum_{j\in\mathbb{F}_q^*}\sum_{k\in\mathbb{F}_q^*}\chi_1(j)\chi_2(k)V(j)V(k) \tag{6.1}$$

denote the double Mellin transform of $V(j)V(k)$. In the notation of (2.1),

$$S(\chi_1, \chi_2) = S(\chi_1)S(\chi_2). \tag{6.2}$$

If χ_1 and χ_2 are not both fourth powers, then

$$S(\chi_1, \chi_2) = T(\chi_1, \chi_2), \tag{6.3}$$

since both members of (6.3) vanish by Theorems 2.1 and 5.4. On the other hand, if $\chi_1 = \nu_1^4$ and $\chi_2 = \nu_2^4$, then (6.3) holds by Theorems 2.1 and 5.5. Thus the Mellin transforms of the left and right sides of (1.4) are the same for all characters. By taking inverse Mellin transforms, we see that (1.4) holds for $jk \neq 0$, which completes the proof of (1.4).

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