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Gauss and Jacobi sums. (English)

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This fascinating book celebrating the descendants of Gauss's sum $\sum_{n=0}^{k-1} e^{2\pi imn^2/k}$ is obviously a labour of love. Discoveries involving Gauss sums mark some of the most beautiful and path-breaking moments in the history of mathematics – quadratic reciprocity, primes in arithmetic progressions, theta functions and much more. The theme of the book is the classical Gauss sum and related exponential and character sums. These are examined systematically, evaluated explicitly and applied to a wide range of concrete problems in number theory.

Chapter 1 is an account of the basic properties of the Gauss sum

$$\tau_k(\chi) = \sum_{n=1}^{k-1} \chi(n) e^{2\pi in/k},$$

where χ is a Dirichlet character modulo k. It includes Odoni's reduction theorem showing how to evaluate sums with a prime power modulus and many proofs of Gauss's theorem

$$\sum_{n=0}^{p-1} \left(\frac{n}{p}\right) e^{2\pi in/p} = \sum_{n=0}^{p-1} e^{2\pi in^2/p} = \begin{cases} \sqrt{p} & \text{if } p \equiv 1 \bmod 4 \\ i\sqrt{p} & \text{if } p \equiv 3 \bmod 4. \end{cases}$$

Chapters 2 and 3 treat the Jacobi sum

$$J(\chi, \psi) = \sum_n \chi(n)\psi(1-n).$$

This sum is evaluated explicitly for many small orders (the order is the least common multiple of the orders of the characters χ and ψ) and linked to k-th power residuacity and the so-called cyclotomic numbers. For example, for a prime modulus p with a primitive root g and a character χ of order 6 with $\chi(g) = e^{\pi i/3}$, $J(\chi^2, \chi^2) = r_3 + is_3\sqrt{3}$, where r_3 and s_3 are integers such that $r_3^2 + 3s_3^2 = 4p$, $r_3 \equiv -1 \bmod 3$, $s_3 \equiv 0 \bmod 3$ and $3s_3 \equiv (2g^{(p-1)/3} + 1)r_3 \bmod p$.

Chapter 4 takes up the explicit evaluation of Gauss sums of small order. For example, for prime modulus p and a character χ of order 6, $\tau_p(\chi^2)^3 = pJ(\chi^2, \chi^2)$. Kummer calculated Arg $\tau_p(\chi^2)$ for primes up to 100 and conjectured that it had a certain non-uniform distribution. This was finally disproved by *D. R. Heath-Brown* and *S. J. Patterson* [J. Reine Angew. Math. 310, 111-130 (1979; Zbl 0412.10028)] – the distribution is uniform. Another fascinating story surrounds the evaluation of the ambiguous cube root, which was settled by *C. R. Matthews* [Invent. Math. 52, 163-185 (1979; Zbl 0397.10031)]: $\tau_p(\chi^2) = \frac{1}{2}p^{1/3}(r_3 + is_3\sqrt{3})H(\chi^2)$, where r_3 and s_3 are defined above and H is Cassels' product defined initially as a product of values of the Weierstrass elliptic functions. These deep results involving elliptic curves and automorphic functions are beyond the scope of the book but illustrate how the study of Gauss sums is still a thriving area of contemporary research.

Chapters 5, 6 and 7 show how the evaluations of Gauss sums and cyclotomy can be used to determine the existence and non-existence of n-th power residue difference sets for small n, congruences for binomial coefficients such as $r_3 \equiv -\binom{3f}{f} \bmod p$, where $p = 6f + 1$ and r_3 is defined above, and residuacity results such as the fact that 2 is a cubic residue mod p if and only if $2|r_3$. Explicit congruences for binomial coefficients modulo p^2 are derived in Chapter 9 by invoking the Gross-Koblitz formula for the p-adic gamma function. Chapter 10 contains the application to the number of solutions of diagonal equations of the type $\alpha_1 x_1^{k_1} + \cdots + \alpha_n x_n^{k_n} = \alpha$ over finite fields.
Chapter 11 takes up algebraic topics: Stickelberger's congruence for the Gauss sum, the Davenport-Hasse product formula and the Gross-Koblitz formula. These are applied to obtain more subtle congruences and to determine the weight distribution of certain cyclic codes. Chapters 8 and 14 treat higher reciprocity laws by elementary means – the cubic and quartic laws and rational laws, culminating in the Eisenstein reciprocity law for prime powers.
Each chapter is lovingly crafted. Each contains a variety of problems, some routine and some which the authors refer to as 'fairly difficult'. Each chapter also has a historical summary with extensive references. Yet more ideas are described under the heading of research problems. The prerequisites are few; the book could serve as a text for a course in topics in number theory in various ways as well as being a valuable research reference. There are even some hints of the ubiquity of Gauss sums in other fields of mathematics.

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Keywords : Gauss sums; Jacobi sums; cyclotomic numbers; finite fields; difference sets; Jacobsthal sums; power residues; reciprocity laws; congruences for binomial coefficients; diagonal equations over finite fields; Eisenstein sums; Brewer sums; Eisenstein reciprocity law; number of solutions of congruences; generalized Jacobi sums; exponential sums; character sums; Stickelberger's congruence; cyclic codes
Classification :
- ∗11-02 Research monographs (number theory)
- 11L05 Gauss and Kloosterman sums
- 11L10 Complete character sums
- 11R18 Cyclotomic extensions
- 11T71 Algebraic coding theory
- 11-03 Historical (number theory)
- 11T24 Other character sums and Gauss sums
- 11T22 Cyclotomy
- 11D61 Exponential diophantine equations
- 11B65 Binomial coefficients, etc.
- 11A15 Power residues, etc.