Math 120A   Test 1   100 points   February 1, 2013

Directions: Justify ALL answers.
Notation: Let $\mathbb{C}$ denote the field of complex numbers. Let $z \in \mathbb{C}$. As usual, write $z = x + iy = re^{i\theta}$. Let $f : \mathbb{C} \to \mathbb{C}$. As usual, write $f(z) = u + iv$. Here x, y, u, v, r, θ are all real numbers. Write $x = \Re z$, the real part of z.
Points: The first two problems are worth 10 points each, and the others are worth 20 points each.

(1) True or False: As $z \to 0$, the limit of $\left(\frac{e^z}{z} - \frac{1}{z^2}\right)$ equals ∞.

SOLUTION: Subtract the two fractions to get a single fraction with denominator z^2 and numerator $ze^z - 1$. The numerator approaches -1 and the denominator approaches 0, so the fraction approaches infinity. Thus the answer is "True".

(2) Complete each of the following two sentences:
A. $f(z)$ is differentiable at $z = 0$ means that the limit of ....
B. $f(z)$ is analytic at $z = 0$ means that ....

SOLUTION:
A. ... $(f(h) - f(0))/h$ exists in $\mathbb{C}$ as $h \to 0$.
B. ... $f(z)$ is differentiable at every point in some neighborhood of 0.

(3) List, in polar form $re^{i\theta}$, all the solutions to the equation $z^4 + 1 + i = 0$.

SOLUTION: $-1 - i = \sqrt{2}e^{-3i\pi/4}$, so the four 4th roots of $-1 - i$ are

$$2^{1/8}e^{-3i\pi/16 + ik\pi/2}, \quad k = 0, 1, 2, 3.$$

(4) Find every point $z \in \mathbb{C}$ at which the function $f(z) = z(\Re z)^2$ is differentiable. Justify.

SOLUTION: Here $u = x^3$ and $v = yx^2$. Thus $u_x = 3x^2$, $u_y = 0$, $v_x = 2xy$, and $v_y = x^2$. The Cauchy-Riemann equations hold only when $x = 0$, i.e., only for points z on the imaginary axis. The partials are clearly continuous everywhere. Thus $f(z)$ is differentiable at each point on the imaginary axis, but it fails to be differentiable anywhere else. (In particular, there is no point at which $f(z)$ is analytic.)

(5) Let D denote the right half plane, i.e., $D = \{z : \Re z > 0\}$. For $z \in D$, define $f(z) = \ln(r) + i\theta$, with $-\pi/2 < \theta < \pi/2$. Show that $f(z)$ is analytic on D, and find $f'(z)$. *Hint*: $ru_r = v_\theta$.

SOLUTION: Here $u = \ln r$ and $v = \theta$. Thus $u_r = 1/r$, $u_\theta = 0$, $v_r = 0$, and $v_\theta = 1$ on D. The two Cauchy-Riemann equations for polar coordinates are both satisfied on D, and the partials are continuous on D. Thus $f(z)$ is differentiable on D and $f'(z) = e^{-i\theta}(u_r + iv_r) = e^{-i\theta}/r = 1/z$ for each $z \in D$.

(6) Let D denote the right half plane as in the previous problem. If $f'(z) = 0$ for all $z \in D$, prove that $f(z)$ is constant on D. *Hint*: Apply the Mean Value theorem for u and for v.

SOLUTION: It suffices to show that $f(z)$ has the same value at any two points $a+bi$ and $c+di$ in D. Since $f'(z) = 0$, all the partials of u and v are 0 on D. First we move horizontally and note that $f(a + bi) = f(c + bi)$ by the Mean Value Theorem applied to u and v, since $u_x = 0$ and $v_x = 0$. Next we move vertically and note that $f(c+bi) = f(c+di)$ by the Mean Value Theorem applied to u and v, since $u_y = 0$ and $v_y = 0$. Thus $f(a + bi) = f(c + di)$, as desired.