Pfaffians of Toeplitz payoff matrices

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Abstract

The purpose of this paper is to evaluate the Pfaffians of certain Toeplitz payoff matrices associated with integer choice matrix games.

1 Introduction

For $w \in \mathbb{R}$, let $M(n, k, w)$ denote the $n \times n$ skew-symmetric Toeplitz matrix whose k superdiagonals in the upper right corner have all entries $-w$, and whose remaining $n - k - 1$ superdiagonals have all entries 1. For example, $M(6, 3, w)$ is the matrix

$$\begin{bmatrix} 0 & 1 & 1 & -w & -w & -w \\ -1 & 0 & 1 & 1 & -w & -w \\ -1 & -1 & 0 & 1 & 1 & -w \\ w & -1 & -1 & 0 & 1 & 1 \\ w & w & -1 & -1 & 0 & 1 \\ w & w & w & -1 & -1 & 0 \end{bmatrix}.$$

In earlier work [4], the Pfaffians of (scalar multiples of) $M(2n, 2n-2, w)$ were evaluated. Our main result (Theorem 2.4) evaluates the Pfaffians of the matrices $M(2n, m, w)$ for all $n \geq 1$ with $n \geq m \geq 0$. Up to sign, these Pfaffians turn out to be the polynomials $F_m(w)$ defined in (2.1). We remark that $F_m(w)$ is the difference of two Chebyshev polynomials of the second kind, namely

$$F_m(w) = U_m((w+1)/2) - U_{m-1}((w+1)/2).$$

A connection between Chebyshev polynomials of the second kind and Pfaffians of some skew-symmetric Toeplitz band matrices may be found in [2]. For a paper involving Pfaffians of general skew-symmetric Toeplitz matrices, see [5].

The matrices $M(n, k, w)$ arise in connection with the following integer choice matrix game. Rowan secretly chooses an integer $i \in \{1, 2, \ldots, n\}$ and his opponent Calum secretly chooses an integer $j \in \{1, 2, \ldots, n\}$. If $i = j$, the game is a draw. If $i \neq j$, the player with the smallest integer wins one dollar from his opponent, unless $|i - j| \geq n - k$, in which case the player with the smallest integer pays w dollars to his opponent. The matrix $M(n, k, w)$ is the payoff matrix for Rowan, when identifying Rowan's i with the i-th row and Calum's j with the j-th column.

Let $\boldsymbol{p} = (p_1, p_2, \ldots, p_n)$ be a vector of probabilities p_i with $p_1 + \cdots + p_n = 1$ and $p_i \geq 0$. If $\boldsymbol{p} \in \ker M(n, k, w)$, then $\boldsymbol{p}$ is an optimal strategy for this payoff matrix. If further $\ker M(n, k, w) = \mathbb{R}\boldsymbol{p}$ and each $p_i > 0$, then the optimal strategy $\boldsymbol{p}$ is unique [4, Prop. 2.1]. To simplify future notation, we will identify an optimal strategy $\boldsymbol{p}$ with any positive scalar multiple of $\boldsymbol{p}$. An example of an optimal strategy for the payoff matrix $M(6,3,w)$ with $w = 2\cos(\pi/7) - 1$ is $\boldsymbol{p} = (w^2, w, 1 - w^2, w, w^2, 0)$. That optimal strategy is not unique, as another is $(w^2 + w - 1, 1, w, w, 1, w^2 + w - 1)$.

From here on, we will assume that $n \geq 1$ and $n \geq m \geq 0$. In Lemma 2.1, we prove that the dimension of $\ker M(2n+1, m, w)$ is 1 for each $w \in \mathbb{R}$. The special case $m = n$ was already proved in [3, Lemma 1]. It turns out that $\ker M(2n+1, m, w)$ is generated by a vector whose entries up to sign are all Pfaffians of matrices of the form $M(2k, k, w)$ with $k \leq m$.

In Lemma 2.3, we apply Lemma 2.1 to evaluate the determinant of $M(2n, m, w)$. This determines the Pfaffian up to sign. Finally, we determine the sign in Theorem 2.4.

2 Pfaffians

As in [3], define monic polynomials $F_m := F_m(w)$ of degree m by

$$F_0 = 1, \; F_1 = w, \; F_m = (w+1)F_{m-1} - F_{m-2}, \quad \text{for } m > 1. \tag{2.1}$$

It is convenient to also set $F_{-1} = 1$. For example, $F_{-1} = F_0 = 1$, $F_1 = w$, $F_2 = w^2 + w - 1$, $F_3 = w^3 + 2w^2 - w - 1$, $F_4 = w^4 + 3w^3 - 3w$, and $F_5 = w^5 + 4w^4 + 2w^3 - 5w^2 - 2w + 1$. As noted in [3], for each $m \geq 1$, the m zeros of F_m are

$$2\cos(h\pi/(2m+1)) - 1, \quad h = 1, 3, 5, \ldots, 2m-1. \tag{2.2}$$

In particular, the zeros of F_m are distinct and they lie in the interval $(-3, 1)$.

For each $m \geq 0$, define the sequences

$$S_m = \begin{cases} F_{m-1}, F_{m-3}, F_{m-5}, \ldots F_0, & \text{for odd } m \\ F_{m-1}, F_{m-3}, F_{m-5}, \ldots F_1, & \text{for even } m, \end{cases}$$

and

$$T_m = \begin{cases} F_1, F_3, F_5, \ldots F_{m-2}, & \text{for odd } m \\ F_0, F_2, F_4, \ldots F_{m-2}, & \text{for even } m, \end{cases}$$

and let S'_m and T'_m denote the respective sequences written in reverse order. Interpret S_m and T_m as empty sequences when $m = 0$. Note that the combined sequence S_m, T_m consists of the m distinct terms $F_0, F_1, \ldots, F_{m-1}$. Define the alternating sequence $V_{n,m}$ of length $2n+1-2m$ by

$$V_{n,m} = F_m, -F_m, F_m, -F_m, \ldots, F_m.$$

Define the column vector $Y_{n,m}$ of length $2n+1$ by

$$Y_{n,m} = Y_{n,m}(w) = (S_m, T_m, V_{n,m}, T'_m, S'_m).$$

For example, when $n = m$ (in which case the sequence $V_{n,m}$ consists of the single term F_m),

$$Y_{0,0} = (F_0),\ Y_{1,1} = (F_0, F_1, F_0),\ Y_{2,2} = (F_1, F_0, F_2, F_0, F_1),$$

$$Y_{3,3} = (F_2, F_0, F_1, F_3, F_1, F_0, F_2),$$
$$Y_{4,4} = (F_3, F_1, F_0, F_2, F_4, F_2, F_0, F_1, F_3),$$
$$Y_{5,5} = (F_4, F_2, F_0, F_1, F_3, F_5, F_3, F_1, F_0, F_2, F_4).$$

The following lemma shows that the matrix $M(2n+1, m, w)$ has a nullspace of dimension 1 generated by the vector $Y_{n,m}$.

Lemma 2.1. *For each $w \in \mathbb{R}$ and all $n \geq 1$ with $n \geq m \geq 0$, we have*

$$\ker M(2n+1, m, w) = \mathbb{R}Y_{n,m}(w).$$

Proof. Our proof follows closely the proof in [3, Lemma 1]. We first show that $Y_{n,m}$ is in the nullspace of $M(2n+1, m, w)$ by showing that

$$M_i Y_{n,m} = 0 \quad \text{for} \quad 1 \leq i \leq 2n+1, \tag{2.3}$$

where M_i denotes the i-th row of $M(2n+1, m, w)$. For $2 \leq i \leq 2n+1$, the vector $M_i - M_{i-1}$ either has exactly three nonzero entries or else has exactly two nonzero entries which are consecutive and equal to -1. In the first case, the three term recurrence in (2.1) shows that

$$(M_i - M_{i-1})Y_{n,m} = 0. \tag{2.4}$$

In the second case, the two consecutive -1's occur in the central portion of length $2n+1-2m$ in the vector $M_i - M_{i-1}$. Since the signs in the sequence $V_{n,m}$ are alternating, it follows that (2.4) holds in this second case as well. The vector M_{n+1} begins with n entries -1, followed by 0, ending with n

entries 1. Thus $M_{n+1}Y_{n,m} = 0$. Together with (2.4), this completes the proof of (2.3).

It remains to prove that $M(2n+1, m, w)$ has nullity 1. The case $m = 0$ follows immediately from the special case $\nu = -1$ of [3, Lemma 1]. Thus we may stipulate that $m \geq 1$. Assume for the purpose of contradiction that for some real w, there exists m taken minimal for which the nullity of $M(2n+1, m, w)$ exceeds 1. Since skew-symmetric matrices have even rank, the nullity must be at least 3. Thus there is a nonzero vector C in the nullspace of $M(2n+1, m, w)$ whose middle and last entries are both 0. Let $\bar{C}$ be the vector of length $2n-1$ obtained from C by deleting the middle and last entries. The matrix $M(2n-1, m-1, w)$ is obtained upon deleting the middle and last rows and columns from $M(2n+1, m, w)$. Therefore $M(2n-1, m-1, w)\bar{C} = 0$. Replacing C by a scalar multiple of C if necessary, we may assume that $\bar{C} = Y_{n-1,m-1}$, in view of the minimality of m. Thus $C = (\underline{\quad\quad}, 0, \underline{\quad\quad}, 0)$, where the first blank is filled with the first n entries of $Y_{n-1,m-1}$ and the second blank is filled with the remaining $n-1$ entries of $Y_{n-1,m-1}$. We have $M_{n+1}C = 0$, and everything cancels in the dot product on the left except for the middle entry $(-1)^{n-m}F_{m-1}$ of $Y_{n-1,m-1}$. Thus $F_{m-1} = 0$. Since also $M_{2n+1}C = 0$, we have

$$(w-1)(F_0 + F_1 + \cdots + F_{m-1}) + F_{m-1} = 0.$$

The left side above equals F_m, as is easily seen by induction. Thus $F_m = 0$, and since $F_{m-1} = 0$ as well, we obtain a contradiction to (2.1). $\square$

We now mention an application of Lemma 2.1 to matrix games. First suppose that $n = m$. By (2.2), all the entries of $Y_{n,m}(w)$ are positive when $w > 2\cos(\pi/(2m+1) - 1$. For every such w, it follows from Lemma 2.1 that $Y_{n,m}(w)$ is the unique optimal strategy for the payoff matrix $M(2n+1, m, w)$. Now suppose that $n > m$. Then it is no longer true that $Y_{n,m}(w)$ is a strategy for $M(2n+1, m, w)$ when $w > 2\cos(\pi/(2m+1) - 1$, because positivity fails due to the alternating signs in the sequence $V_{n,m}$. On the other hand, when $w = 2\cos(\pi/(2m+1) - 1$, then all entries in $Y_{n,m}(w)$ are nonnegative, so that $Y_{n,m}(w)$ is indeed an optimal strategy for $M(2n+1, m, w)$, although not unique. For example, take $n = 4$, $m = 3$, and $w = 2\cos(\pi/7) - 1$. Then

$$Y_{4,3}(w) = (w^2 + w - 1, 1, w, 0, 0, 0, w, 1, w^2 + w - 1)$$

is an optimal strategy, and so is

$$(0, w+1, w^2 + 2w, 0, 0, 0, 1, w^2 + 2w, w + 1).$$

We will need the following simple lemma for the proof of Lemma 2.3.

Lemma 2.2. *Let $E(k)$ denote the $k \times k$ skew-symmetric matrix with all entries above the diagonal equal to 1. Then the determinant $d(k)$ of $E(k)$ is 1 or 0 according as k is even or odd.*

Proof. Denote the j-th row of $E(k)$ by E_j. In the matrix $E(k)$, first replace E_j by $E_j - E_{j+1}$ for each $j = 1, 2, \ldots, k-1$. Each of the first $k-1$ rows now has two nonzero entries, which are consecutive and equal to 1. To evaluate $d(k)$, expand along the first column to see that

$$d(k) = d(k-1) + (-1)^k.$$

The result now follows by induction. □

The next lemma shows that up to sign, the Pfaffian of $M(2n, m, w)$ is $F_m(w)$. The sign will be determined in Theorem 2.4.

Lemma 2.3. *Let $D(w)$ denote the determinant of $M(2n, m, w)$. For each $w \in \mathbb{R}$ and all $n \geq 1$ with $n \geq m \geq 0$, we have $D(w) = F_m(w)^2$.*

Proof. Let $\bar{Y}_{n,m}$ be the vector of length $2n$ obtained from $Y_{n,m}$ by deleting the middle entry $(-1)^{n-m} F_m$. One obtains the matrix $M(2n, m, w)$ upon deleting the middle row and column from $M(2n+1, m, w)$. Thus by Lemma 2.1,

$$M(2n, m, w)\bar{Y}_{n,m}(w) = Z_{n,m}(w), \tag{2.5}$$

where $Z_{n,m}(w)$ denotes the vector of length $2n$ each of whose first n entries is $(-1)^{n-m+1}F_m(w)$ and each of whose last n entries is $(-1)^{n-m}F_m(w)$. Let w' be any one of the m distinct zeros of $F_m(w)$. By (2.5),

$$M(2n, m, w')\bar{Y}_{n,m}(w') = Z_{n,m}(w') = 0.$$

Thus the nonzero vector $\bar{Y}_{n,m}(w')$ is in the nullspace of $M(2n, m, w')$, so the determinant of $M(2n, m, w')$ must be 0. Since the polynomial $D(w)$ vanishes at each zero of $F_m(w)$, it must be divisible by $F_m(w)$, and hence by $F_m(w)^2$. The polynomial $D(w)$ has degree $2m$, so $D(w) = \alpha F_m(w)^2$, where α is the leading coefficient of $D(w)$. It remains to prove that $\alpha = 1$. From the Leibniz formula for the determinant, it can be seen that α equals the determinant of the $(2n-2m) \times (2n-2m)$ submatrix situated in the center of the matrix $M(2n, m, w)$. This submatrix is $E(2n-2m)$, so by Lemma 2.2, $\alpha = d(2n-2m) = 1$. □

Theorem 2.4. *Write $M = M(2n, m, w)$ and let $Pf(M)$ denote its Pfaffian. For each $w \in \mathbb{R}$ and all $n \geq 1$ with $n \geq m \geq 0$, we have*

$$Pf(M) = (-1)^{[(m+1)/2]} F_m(w).$$

Proof. Let β denote the leading coefficient of the polynomial $Pf(M)$. In view of Lemma 2.3, it suffices to show that

$$\beta = (-1)^{[(m+1)/2]}. \tag{2.6}$$

From [1, eq. 1.4], (2.6) holds when $n = 1$, so assume that $n > 1$. For $2 \leq j \leq 2n$, let $M^{(j)}$ denote the $(2n-2) \times (2n-2)$ matrix obtained by removing the first and j-th rows and columns. From [1, eq. 1.5],

$$Pf(M) = \sum_{j=2}^{2n} (-1)^j a_j Pf(M^{(j)}), \tag{2.7}$$

where a_j is the j-th entry in the first row of M.

When $m = 0$, it follows by induction on n that the right side of (2.7) is the alternating sum $1 - 1 + 1 - \cdots + 1$, so that $Pf(M) = 1$, proving (2.6). Thus assume that $m \geq 1$. The only term on the right side of (2.7) that contributes to the leading coefficient of $Pf(M)$ is the term for $j = 2n+1-m$, and for this j, $a_j = -w$. By induction on m, we may assume that for this j, $Pf(M^{(j)}) = (-1)^{[m/2]}$. Therefore the right side of (2.7) contributes $(-1)^{m+[m/2]}$ to the leading coefficient of $Pf(M)$. Since $(-1)^{m+[m/2]} = (-1)^{[(m+1)/2]}$, the proof of (2.6) is complete. □

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