

# SEVENTH POWER MOMENTS OF KLOOSTERMAN SUMS

BY

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ABSTRACT

Evaluations of the  $n$ -th power moments  $S_n$  of Kloosterman sums are known only for  $n \leq 6$ . We present here substantial evidence for an evaluation of  $S_7$  in terms of Hecke eigenvalues for a weight 3 newform on  $\Gamma_0(525)$  with quartic nebentypus of conductor 105. We also prove some congruences modulo 3, 5 and 7 for the closely related quantity  $T_7$ , where  $T_n$  is a sum of traces of  $n$ -th symmetric powers of the Kloosterman sheaf.

## 1. Introduction

For an odd prime  $p$ , let  $\mathbb{F}_p$  denote a field of  $p$  elements, and write  $\zeta_p = \exp(2\pi i/p)$ . Consider the Kloosterman sums

$$(1.1) \quad K(a) = \sum_{x=1}^{p-1} \zeta_p^{x+a/x}, \quad a \in \mathbb{F}_p,$$

and their  $n$ -th power moments

$$(1.2) \quad S_n = \sum_{a=0}^{p-1} K(a)^n, \quad n \in \mathbb{N}.$$

It is well-known [5, §4.4] that

$$(1.3) \quad S_1 = 0, \quad S_2 = p^2 - p, \quad S_3 = \left(\frac{p}{3}\right)p^2 + 2p, \quad S_4 = 2p^3 - 3p^2 - 3p.$$

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The work in [8], [9] shows that  $S_5$  can be expressed in terms of the  $p$ -th eigenvalue for a weight 3 newform on  $\Gamma_0(15)$ . The work in [4] shows that  $S_6$  can be expressed in terms of the  $p$ -th eigenvalue for a weight 4 newform on  $\Gamma_0(6)$ . See also [1].

In Conjecture 1.1 below, we propose an evaluation of  $S_7$  in terms of the  $p$ -th eigenvalue for a weight 3 newform on  $\Gamma_0(525)$ . This conjecture is based on substantial numerical evidence.

Write

$$(1.4) \quad K(a) = -g(a) - h(a), \quad a \neq 0,$$

where  $g(a)$ ,  $h(a)$  are the two Frobenius eigenvalues for the Kloosterman sheaf at  $a$ , given by

$$(1.5) \quad g(a) = p^{1/2} \exp(i\theta_p(a)), \quad h(a) = p^{1/2} \exp(-i\theta_p(a)),$$

with  $\theta_p(a) \in [0, \pi]$ . (In fact,  $\theta_p(a) \in (0, \pi)$ ; see [2, Theorem 6.1].) By (1.2) and (1.4),

$$(1.6) \quad S_n = (-1)^n + (-1)^n \sum_{a=1}^{p-1} (g(a) + h(a))^n.$$

As noted in [5, p. 63], one should study the “more natural” related expressions

$$(1.7) \quad T_n = \sum_{a=1}^{p-1} (g(a)^n + g(a)^{n-1}h(a) + \cdots + h(a)^n).$$

The summand in (1.7) is the trace of the  $n$ -th symmetric power of the Kloosterman sheaf at  $a$ , and equals

$$(1.8) \quad p^{n/2} U_n(2 \cos \theta_p(a)),$$

where  $U_n$  is the  $n$ -th monic Chebyshev polynomial of the second kind. We have the bound [3, Theorem 0.2], [6]

$$(1.9) \quad |1 + T_n| \leq \left[ \frac{n-1}{2} \right] p^{(n+1)/2}, \quad \text{if } p > n > 0,$$

whose proof is based on Deligne’s theory of exponential sums for varieties over  $\mathbb{F}_p$ . (A slightly weaker bound which holds for all  $p > 2$  is given in [5, Theorem 4.6].)

The expressions  $S_n$  and  $T_n$  are related by the formula

$$(1.10) \quad (-1)^n S_n - 1 = \sum_{k=0}^{\lfloor n/2 \rfloor} \left\{ \binom{n}{k} - \binom{n}{k-1} \right\} p^k T_{n-2k}.$$

In [1, (1.11)], it is proved that  $S_n$  is an integer multiple of  $p$  satisfying

$$(1.11) \quad S_n \equiv p(n-1)(-1)^{n-1} \pmod{p^2}.$$

From (1.10)–(1.11), it follows by induction that

$$(1.12) \quad T_n \equiv -1 \pmod{p^2}, \quad n > 0.$$

By (1.3) and (1.10), we have

$$(1.13) \quad T_0 + 1 = p, \quad T_1 + 1 = T_2 + 1 = 0, \quad T_3 + 1 = -\left(\frac{p}{3}\right)p^2, \quad T_4 + 1 = -p^2.$$

By [1, (1.8)], we have for  $p > 5$ ,

$$(1.14) \quad a_p := \frac{-1 - T_5}{p^2} = \begin{cases} 2p - 12u^2, & \text{if } p = 3u^2 + 5v^2 \\ 4x^2 - 2p, & \text{if } p = x^2 + 15y^2 \\ 0, & \text{if } \left(\frac{p}{15}\right) = -1. \end{cases}$$

Define

$$(1.15) \quad c_p := (-1 - T_7)/p^2.$$

By (1.12),  $a_p$  and  $c_p$  are integers, and by (1.9), we have

$$(1.16) \quad |a_p| \leq 2p, \quad |c_p| \leq 3p^2.$$

Putting  $n = 7$  in (1.10) yields

$$(1.17) \quad S_7 = p^2 c_p + 6p^3 a_p + 14\left(\frac{p}{3}\right)p^4 + 14p^3 + 14p^2 + 6p.$$

Hence by (1.16),

$$(1.18) \quad |S_7| \leq 29p^4 + 14p^3 + 14p^2 + 6p.$$

In view of (1.14) and (1.17), an evaluation of  $c_p$  would yield an evaluation of  $S_7$ . Hence we focus on  $c_p$  in Conjecture 1.1 below, and in the sequel.

Let  $\chi_5$  denote the quartic Dirichlet character (mod 5) defined by  $\chi_5(2) = -i$ , and let  $\psi$  denote the quartic character of conductor 105 defined by

$$(1.19) \quad \psi(d) = \left(\frac{d}{21}\right)\chi_5(d), \quad d \in \mathbb{Z}.$$

CONJECTURE 1.1: For  $p > 7$ ,

$$(1.20) \quad c_p = \left(\frac{p}{105}\right)(-p^2 + b(p)^2 \bar{\psi}(p)) = \left(\frac{p}{105}\right)(-p^2 + |b(p)|^2),$$

where  $b(p)$  is the  $p$ -th Hecke eigenvalue for a weight 3 newform  $f$  on  $\Gamma_0(525)$  with nebentypus  $\psi$  and eigenfield  $\mathbb{Q}(i, \sqrt{6}, \sqrt{14})$ .

In Section 2, we motivate Conjecture 1.1 and discuss the evidence for it. In Section 3, we examine the integers  $c_p$  modulo 3, 5, and 7, proving in the process some observations of Katz [7]. Section 4, the Appendix, records a Sage [10] session which exhibits numerical evidence for Conjecture 1.1.

## 2. Motivation and evidence for Conjecture 1.1

The following conjecture has been verified for each of the 396 primes  $p$  in the interval  $7 < p \leq 2741$ .

CONJECTURE 2.1: Let  $p > 7$ , and define the signature  $\alpha_p := ((\frac{p}{3}), (\frac{p}{5}), (\frac{p}{7}))$ . Then

$$(2.1) \quad \left(\frac{p}{105}\right)c_p + p^2 = x(p)^2$$

for a nonnegative number  $x(p)$  of the form:

$$\begin{aligned} &2m\sqrt{7} \text{ with } m \equiv \pm 1 \pmod{10}, \quad 3 \nmid m, \quad \text{if } \alpha_p = (1, -1, -1); \\ &4m\sqrt{3} \text{ with } m \equiv \pm 1 \pmod{10}, \quad \text{if } \alpha_p = (-1, -1, 1); \\ &2m\sqrt{42} \text{ with } m \equiv \pm 1 \pmod{5}, \quad \text{if } \alpha_p = (1, -1, 1); \\ &6m\sqrt{2} \text{ with } m \equiv \pm 2 \pmod{5}, \quad \text{if } \alpha_p = (-1, -1, -1); \\ &2m \text{ with } m \equiv \pm(3 - 2\chi_5(p)) \pmod{10}, \quad 3 \nmid m, \quad \text{if } \alpha_p = (1, 1, 1); \\ &4m\sqrt{21} \text{ with } m \equiv \pm(1 + \chi_5(p)) \pmod{5}, \quad \text{if } \alpha_p = (-1, 1, -1); \\ &2m\sqrt{6} \text{ with } m \equiv \pm(2 - 2\chi_5(p)) \pmod{5}, \quad \text{if } \alpha_p = (1, 1, -1); \\ &6m\sqrt{14} \text{ with } m \equiv \pm(2 - 2\chi_5(p)) \pmod{5}, \quad \text{if } \alpha_p = (-1, 1, 1) \end{aligned}$$

where  $m$  is a positive integer.

The values of  $x(p)$  for  $7 < p < 100$  are given in Table 2.1 below.

Motivated by our Conjecture 2.1, Katz [7] proposed the following scenario. For  $p > 7$ , the number  $c_p/p^2$  (which lies in  $[-3, 3]$  by (1.16)) is the trace of  $\text{Frob}_p$  in a representation towards  $\text{O}(3)$  (the orthogonal group with respect to

$p$	11	13	17	19	23	29	31
$x(p)$	0	$2\sqrt{7}$	$18\sqrt{2}$	$8\sqrt{6}$	$4\sqrt{3}$	$6\sqrt{14}$	$10\sqrt{6}$

  

$p$	37	41	43	47	53	59	61
$x(p)$	$2\sqrt{42}$	$12\sqrt{21}$	$8\sqrt{42}$	$12\sqrt{2}$	$36\sqrt{3}$	$20\sqrt{21}$	$30\sqrt{6}$

  

$p$	67	71	73	79	83	89	97
$x(p)$	$12\sqrt{42}$	$30\sqrt{14}$	$38\sqrt{7}$	50	$78\sqrt{2}$	$20\sqrt{21}$	$38\sqrt{7}$

Table 1

a trace form). This  $\text{Frob}_p$  has determinant  $(\frac{p}{105})$ , so  $(\frac{p}{105})c_p/p^2$  is the trace of  $\text{Frob}_p$  in a representation towards  $\text{SO}(3)$ . For some Dirichlet character  $\chi$ , this representation is  $\bar{\chi}(p) \otimes \text{Sym}^2(V)$  for a 2-dimensional representation  $V$ , where  $\text{Frob}_p$  in  $V$  has eigenvalues  $\alpha, \beta$  with  $|\alpha| = |\beta| = 1$  and  $\alpha\beta = \chi(p)$ . After equating traces, we obtain

$$\chi(p) \left(\frac{p}{105}\right) c_p/p^2 = \chi(p) + \alpha^2 + \beta^2,$$

so

$$\chi(p) \left\{ \left(\frac{p}{105}\right) c_p + p^2 \right\} = p^2(\alpha + \beta)^2.$$

Define  $b(p) := p(\alpha + \beta)$ , so that  $|b(p)| \leq 2p$  and  $b(p)/p$  is the trace of  $\text{Frob}_p$  in  $V$ . In the notation of (2.1), it follows that

$$(2.2) \quad \chi(p)x(p)^2 = b(p)^2, \quad p > 7.$$

Assuming the validity of Katz’s scenario, we hoped to find a Dirichlet character  $\chi$ , a level  $N$ , and a weight 3 newform

$$(2.3) \quad f(z) = \sum_{m=1}^{\infty} \hat{f}(m)e^{2\pi imz}, \quad \hat{f}(p) = b(p)$$

on  $\Gamma_0(N)$  with nebentypus  $\chi$  such that  $x(p) = |b(p)|$  for  $p > 7$ . The equality  $x(p) = |b(p)|$  is equivalent to (2.2), by [5, (6.57)]. Our search for  $N, \chi, f$  culminated with the discovery of a weight 3 newform (2.3) on  $\Gamma_0(525)$  with nebentypus  $\psi$  and eigenfield  $\mathbb{Q}(i, \sqrt{6}, \sqrt{14})$  such that  $x(p) = |b(p)|$  for  $7 < p < 100$ . Equivalently,

$$\psi(p)x(p)^2 = b(p)^2, \quad 7 < p < 100,$$

which is powerful evidence that (1.20) in fact holds for all  $p > 7$ .

We proceed to describe how this newform  $f$  of level 525 was discovered. While browsing William Stein's *Modular Forms Explorer* found in his *Modular Forms Database* [11], we had encountered a weight 3 newform  $g(z)$  on  $\Gamma_0(168)$  with quadratic nebentypus of conductor 168 and eigenfield  $\mathbb{Q}(i\sqrt{2}, \sqrt{3}, i\sqrt{7})$ . For each  $p$  with  $7 < p < 100$ ,  $|\widehat{g}(p)|$  appeared to be an integer multiple of one of  $\sqrt{1}, \sqrt{2}, \sqrt{3}, \sqrt{6}, \sqrt{7}, \sqrt{14}, \sqrt{21}, \sqrt{42}$ , just as was the case for  $x(p)$  (cf. Table 2.1). Moreover, analogous to the situation in Conjecture 2.1, the particular choice of square root occurring in  $|\widehat{g}(p)|$  seemed to be completely determined by the signature  $((\frac{p}{3}), (\frac{p}{7}), (\frac{-8}{p}))$ . The product of the conductors of the three quadratic characters in this signature is  $3 \cdot 7 \cdot 8 = 168$ , which equals the conductor of the nebentypus of  $g$ . It seemed reasonable to guess by analogy that the product of the conductors of the three quadratic characters in  $\alpha_p$ , namely  $3 \cdot 5 \cdot 7 = 105$ , should be the conductor of the nebentypus  $\chi$  of the newform  $f$  that we were seeking. Since  $f$  has odd weight,  $\chi$  is odd. The simplest odd character of conductor 105 is the quartic character  $\psi$  defined in (1.19). Thus we took  $\chi = \psi$  as a first guess, and the evidence strongly suggests that this was the right choice.

As a first guess for the level  $N$ , we took  $N = 105$ , hoping that the level would equal the conductor of the nebentypus as was the case for the newform  $g$  on  $\Gamma_0(168)$ . However, for newforms  $f$  on  $\Gamma_0(105)$ , there were already small primes  $p > 7$  for which  $|\widehat{f}(p)|$  failed to equal  $x(p)$ . Our next guess was that the level equals 105 times a small prime factor. The levels  $2 \cdot 105$  and  $3 \cdot 105$  each failed, but the level  $5 \cdot 105 = 525$  provided a happy ending. Indeed the Sage session in the Appendix shows the existence of a weight 3 newform  $f$  on  $\Gamma_0(525)$  with nebentypus  $\psi$  and eigenfield  $\mathbb{Q}(i, \sqrt{6}, \sqrt{14})$  such that  $|\widehat{f}(p)| = x(p)$  for all  $p$  with  $7 < p < 100$ . As was noted above, this is powerful evidence for Conjecture 1.1.

### 3. Congruences for $c_p$

Let  $p > 7$ . It follows from [1, Theorem 2.1] that  $S_7 \equiv -(\frac{p}{105}) \pmod{4}$ . Thus, by (1.17),

$$(3.1) \quad 2 \nmid c_p.$$

Katz [7] observed that numerical evidence moreover suggests

$$(3.2) \quad 5 \nmid c_p$$

and

$$(3.3) \quad 7 \nmid c_p.$$

On the other hand, we conjecture that for every prime  $q \notin \{2, 5, 7\}$ , one has  $q \mid c_p$  for infinitely many primes  $p$ .

In Theorem 3.1, we prove (3.3). In Theorem 3.2, we give an evaluation of  $c_p \pmod{5}$  which in particular proves (3.2). In Theorem 3.3, we evaluate  $c_p \pmod{3}$ .

Our proofs will make use of the simple fact that

$$(3.4) \quad n \mid S_n, \quad \text{for prime } n.$$

To justify (3.4), note that

$$S_n = \sum_{a=0}^{p-1} \left( \sum_{x=1}^{p-1} \zeta_p^{x+a/x} \right)^n \equiv \sum_{x=1}^{p-1} \sum_{a=0}^{p-1} \zeta_p^{n(x+a/x)} \equiv 0 \pmod{n}.$$

**THEOREM 3.1:** *For each  $p > 7$ , we have  $7 \nmid c_p$ .*

*Proof.* By (1.17),

$$p^2 c_p \equiv S_7 + p^3 a_p + p \pmod{7}.$$

Since  $7 \mid S_7$  by (3.4),

$$(3.5) \quad p^2 c_p \equiv \left(\frac{p}{7}\right) a_p + p \pmod{7}.$$

It remains to prove that

$$(3.6) \quad \left(\frac{p}{7}\right) a_p + p \not\equiv 0 \pmod{7}.$$

We may assume that  $a_p \neq 0$ , since otherwise (3.6) is clear. By (1.14), either

$$(3.7) \quad a_p = 10v^2 - 6u^2 \quad \text{with} \quad p = 3u^2 + 5v^2.$$

or

$$(3.8) \quad a_p = 2x^2 - 30y^2 \quad \text{with} \quad p = x^2 + 15y^2.$$

In the case (3.7),

$$\left(\frac{p}{7}\right) a_p + p = u^2 \left(3 - 6\left(\frac{p}{7}\right)\right) + v^2 \left(5 + 10\left(\frac{p}{7}\right)\right) \not\equiv 0 \pmod{7},$$

since  $(-3 + 6\left(\frac{p}{7}\right))(5 + 10\left(\frac{p}{7}\right))$  is a nonsquare  $\pmod{7}$ . In the case (3.8),

$$\left(\frac{p}{7}\right) a_p + p = x^2 \left(1 + 2\left(\frac{p}{7}\right)\right) + y^2 \left(15 - 30\left(\frac{p}{7}\right)\right) \not\equiv 0 \pmod{7},$$

since  $(1 + 2(\frac{p}{7}))(-15 + 30(\frac{p}{7}))$  is a nonsquare (mod 7). ■

THEOREM 3.2: For  $p > 7$ ,

$$c_p \equiv p + p\left(\frac{p}{5}\right) + \left(\frac{p}{21}\right) \pmod{5}.$$

In particular,  $5 \nmid c_p$ .

*Proof.* All congruences in this proof are modulo 5. By (1.17),

$$p^2 c_p \equiv S_7 - p^3 a_p + \left(\frac{p}{3}\right)p^4 + p^3 + p^2 - p.$$

Since  $p^2 \equiv (\frac{p}{5})$ , we have

$$(3.9) \quad c_p \equiv \left(\frac{p}{5}\right)S_7 - pa_p + \left(\frac{p}{15}\right) + p + 1 - \left(\frac{p}{5}\right)p.$$

It remains to prove

$$(3.10) \quad a_p \equiv \left(\frac{p}{3}\right)p + \left(\frac{p}{5}\right)p$$

and

$$(3.11) \quad S_7 \equiv 2p + \left(\frac{p}{105}\right),$$

since the theorem follows from (3.9)–(3.11).

By (1.10) and (1.13),

$$(3.12) \quad p^2 a_p = S_5 - 4p^3 \left(\frac{p}{3}\right) - 5p^2 - 4p.$$

Thus

$$(3.13) \quad a_p \equiv \left(\frac{p}{5}\right)S_5 + \left(\frac{p}{3}\right)p + \left(\frac{p}{5}\right)p.$$

This proves (3.10), since  $5 \mid S_5$  by (3.4).



To prove (3.11), observe that

$$\begin{aligned}
 S_7 &= \sum_{a=0}^{p-1} K(a)^7 \equiv \sum_{a=0}^{p-1} K(a)^2 \sum_{x=1}^{p-1} \zeta_p^{5(x+a/x)} = \sum_{a=0}^{p-1} K(a)^2 K(25a) \\
 &= \sum_{a=0}^{p-1} \sum_{x,y,z \neq 0} \zeta_p^{x+y+z+a(\frac{1}{x}+\frac{1}{y}+\frac{25}{z})} \\
 &= p \sum_{\substack{x,y,z \neq 0 \\ \frac{1}{x}+\frac{1}{y}+\frac{25}{z}=0}} \zeta_p^{x+y+z} \\
 &= p \sum_{\substack{x,y \neq 0 \\ x+y \neq 0}} \zeta_p^{x+y-25xy/(x+y)}.
 \end{aligned}$$

With the change of variables

$$r = x + y, \quad s = xy,$$

this becomes

$$S_7 \equiv p \sum_{r,s \neq 0} \zeta_p^{r-25s/r} \left\{ 1 + \left( \frac{r^2 - 4s}{p} \right) \right\} = p \sum_{r,s \neq 0} \zeta_p^{r(1-25s)} \left\{ 1 + \left( \frac{1 - 4s}{p} \right) \right\},$$

where in the last step we replaced  $s$  by  $sr^2$ . Replacing  $s$  by  $(1-s)/4$ , we obtain

$$\begin{aligned}
 S_7 &\equiv p \sum_{r \neq 0, s \neq 1} \zeta_p^{r(\frac{-21}{4} + \frac{25s}{4})} \left\{ 1 + \left( \frac{s}{p} \right) \right\} \\
 &= 2p - p^2 + p \sum_{r,s} \zeta_p^{r(\frac{-21}{4} + \frac{25s}{4})} \left\{ 1 + \left( \frac{s}{p} \right) \right\} \\
 &= 2p - p^2 + p^2 \left\{ 1 + \left( \frac{21}{p} \right) \right\} = 2p + p^2 \left( \frac{p}{21} \right) \equiv 2p + \left( \frac{p}{105} \right).
 \end{aligned}$$

This completes the proof of (3.11). ■

**THEOREM 3.3:** For  $p > 7$ ,

$$c_p \equiv 1 + \left( \frac{p}{3} \right) + \left( \frac{p}{35} \right) \pmod{3}.$$

In particular,  $3 \mid c_p$  if and only if  $\left( \frac{p}{3} \right) = \left( \frac{p}{35} \right) = 1$ .

*Proof.* By (1.17),

$$c_p \equiv S_7 + \left( \frac{p}{3} \right) + p + 1 \pmod{3}.$$

Thus it remains to show that

$$(3.14) \quad S_7 \equiv \left(\frac{p}{35}\right) - p \pmod{3}.$$

We have

$$\begin{aligned} S_7 &\equiv \sum_{a=0}^{p-1} K(a) \left(\sum_{x=1}^{p-1} \zeta_p^{3(x+a/x)}\right)^2 = \sum_{a=0}^{p-1} K(a)K(9a)^2 \\ &= \sum_{a=0}^{p-1} \sum_{x,y,z \neq 0} \zeta_p^{x+y+z+a(\frac{9}{x}+\frac{9}{y}+\frac{1}{z})} \pmod{3}. \end{aligned}$$

The rest of the proof of (3.14) proceeds as in the proof of (3.11). ■

### 4. Appendix

The Sage session below shows the existence of a weight 3 newform  $f$  on  $\Gamma_0(525)$  with nebentypus  $\psi$  and eigenfield  $\mathbb{Q}(i, \sqrt{6}, \sqrt{14})$ , such that the  $p$ -th Fourier coefficients  $b(p)$  of  $f$  satisfy (1.20) for  $7 < p < 100$ .

The session begins by setting  $\mathbf{G}$  equal to the group of 16 Dirichlet characters modulo 525 of order dividing 4. The elements of  $\mathbf{G}$  are placed into a list  $\mathbf{X}$ , whose last element  $\mathbf{Y} = \mathbf{X}[15]$  equals the quartic character  $\psi$  of conductor 105 defined in (1.19).

Let  $\mathbf{M}$  denote a modular symbols space of level 525, weight 3, with character  $\psi$ . This is a vector space of dimension 160 over  $\mathbb{Q}(i)$ . It has a “cuspidal subspace”  $\mathbf{S}$  of dimension 148, and  $\mathbf{S}$  in turn has a “new subspace”  $\mathbf{N}$  of dimension 92. The space  $\mathbf{N}$  is decomposed into 10 further subspaces, each invariant under Hecke operators, and  $\mathbf{D}$  denotes a sorted list of these 10 subspaces. For more information about these spaces, see the Sage documentation at [10].

Our desired eigenfunction  $f$  lies in the fifth invariant subspace  $\mathbf{D}[4]$ , and  $\mathbf{f}$  gives the first 97 terms of its  $q$ -expansion. Finally, `parent(f)` tells us that the Fourier coefficients of our eigenfunction all lie in the eigenfield  $\mathbb{Q}(\mathbf{zeta4}, \mathbf{a})$ , where  $\mathbf{zeta4} = i$  and  $\mathbf{a}$  is a zero of

$$x^4 + (4i + 4)x^3 + 20ix^2 + (24i - 24)x - 120.$$

We may take  $\mathbf{a} = \sqrt{7}z^7 - z(\sqrt{2} + \sqrt{3})$ , where  $z = \exp(2\pi i/8)$ . Then the eigenfield is easily seen to be  $\mathbb{Q}(i, \sqrt{6}, \sqrt{14})$ . Simplifying the  $q$ -expansion  $\mathbf{f}$ , we obtain the Fourier coefficients of  $f$  corresponding to primes  $7 < p < 100$  given in Table 4.1.

$p$	11	13	17	19	23	29	31
$b(p)$	0	$2\sqrt{7}z^7$	$18\sqrt{2}z^3$	$8\sqrt{6}$	$4\sqrt{3}z^3$	$6\sqrt{14}$	$10\sqrt{6}z^2$

$p$	37	41	43	47	53	59	61
$b(p)$	$2\sqrt{42}z^3$	$12\sqrt{21}z^4$	$8\sqrt{42}z^5$	$12\sqrt{2}z^7$	$36\sqrt{3}z^7$	$20\sqrt{21}z^2$	$30\sqrt{6}z^2$

$p$	67	71	73	79	83	89	97
$b(p)$	$12\sqrt{42}z^3$	$30\sqrt{14}z^6$	$38\sqrt{7}z^3$	$50z^2$	$78\sqrt{2}z$	$20\sqrt{21}z^6$	$38\sqrt{7}z$

Comparison of Tables 2.1 and 4.1 shows that  $x(p) = |b(p)|$ , and so (1.20) holds for  $7 < p < 100$ .

### SAGE SESSION

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| Sage Version 4.1, Release Date: 2009-07-09           |
| Type notebook() for the GUI, and license() for information. |
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sage: G=DirichletGroup(525,CyclotomicField(4));X=G.list();
sage: Y=X[15];Y;Y.conductor();Y.order()
[-1, -zeta4, -1]
105
4

sage: M=ModularSymbols(Y,3,sign=1);M
Modular Symbols space of dimension 160 and level 525, weight 3, character
[-1, -zeta4, -1], sign 1, over Cyclotomic Field of order 4 and degree 2

sage: S=M.cuspidal_subspace();S
Modular Symbols subspace of dimension 148 of Modular Symbols space of
dimension 160 and level 525, weight 3, character [-1, -zeta4, -1], sign 1,
over Cyclotomic Field of order 4 and degree 2

sage: N=S.new_subspace();N
Modular Symbols subspace of dimension 92 of Modular Symbols space of
dimension 160 and level 525, weight 3, character [-1, -zeta4, -1], sign 1,
over Cyclotomic Field of order 4 and degree 2

sage: D=N.decomposition();D
[
Modular Symbols subspace of dimension 2 of Modular Symbols space of
dimension 160 and level 525, weight 3, character [-1, -zeta4, -1], sign 1,
over Cyclotomic Field of order 4 and degree 2,
Modular Symbols subspace of dimension 2 of Modular Symbols space of
dimension 160 and level 525, weight 3, character [-1, -zeta4, -1], sign 1,
over Cyclotomic Field of order 4 and degree 2,

```

Modular Symbols subspace of dimension 4 of Modular Symbols space of dimension 160 and level 525, weight 3, character  $[-1, -\zeta_4, -1]$ , sign 1, over Cyclotomic Field of order 4 and degree 2,  
 Modular Symbols subspace of dimension 4 of Modular Symbols space of dimension 160 and level 525, weight 3, character  $[-1, -\zeta_4, -1]$ , sign 1, over Cyclotomic Field of order 4 and degree 2,  
 Modular Symbols subspace of dimension 4 of Modular Symbols space of dimension 160 and level 525, weight 3, character  $[-1, -\zeta_4, -1]$ , sign 1, over Cyclotomic Field of order 4 and degree 2,  
 Modular Symbols subspace of dimension 4 of Modular Symbols space of dimension 160 and level 525, weight 3, character  $[-1, -\zeta_4, -1]$ , sign 1, over Cyclotomic Field of order 4 and degree 2,  
 Modular Symbols subspace of dimension 8 of Modular Symbols space of dimension 160 and level 525, weight 3, character  $[-1, -\zeta_4, -1]$ , sign 1, over Cyclotomic Field of order 4 and degree 2,  
 Modular Symbols subspace of dimension 8 of Modular Symbols space of dimension 160 and level 525, weight 3, character  $[-1, -\zeta_4, -1]$ , sign 1, over Cyclotomic Field of order 4 and degree 2,  
 Modular Symbols subspace of dimension 16 of Modular Symbols space of dimension 160 and level 525, weight 3, character  $[-1, -\zeta_4, -1]$ , sign 1, over Cyclotomic Field of order 4 and degree 2,  
 Modular Symbols subspace of dimension 40 of Modular Symbols space of dimension 160 and level 525, weight 3, character  $[-1, -\zeta_4, -1]$ , sign 1, over Cyclotomic Field of order 4 and degree 2  
 ]

```

sage: f=D[4].q_eigenform(98,"a");f
q + (1/20*zeta4*a^3 + (3/20*zeta4 - 3/20)*a^2 - 1/5*a)*q^2 + (-1/20*zeta4*a^3 + (-3/20*zeta4 + 3/20)*a^2 + 6/5*a)*q^3 - zeta4*q^4 + ((-1/20*zeta4 + 1/20)*a^3 + 4/5*a^2 + (6/5*zeta4 + 6/5)*a + 3*zeta4)*q^6 + (-1/20*a^3 + (-13/20*zeta4 + 7/20)*a^2 + (-6/5*zeta4 + 2)*a + 2*zeta4 + 4)*q^7 + (1/4*a^3 + (3/4*zeta4 + 3/4)*a^2 + zeta4*a)*q^8 + ((1/10*zeta4 - 1/10)*a^3 - 3/5*a^2 + (-12/5*zeta4 - 12/5)*a - 9*zeta4)*q^9 + (-1/20*a^3 + (-3/20*zeta4 - 3/20)*a^2 - 6/5*zeta4*a)*q^12 + (-1/10*zeta4*a^3 + (-3/10*zeta4 + 3/10)*a^2 + 12/5*a + 2*zeta4 + 2)*q^13 + ((-3/20*zeta4 + 3/20)*a^3 + (-1/2*zeta4 + 9/10)*a^2 + (13/5*zeta4 + 23/5)*a + 6*zeta4 + 3)*q^14 + 11*q^16 + (18*zeta4 - 18)*q^17 + (1/4*a^3 + (-1/4*zeta4 - 1/4)*a^2 - 3*zeta4*a - 6*zeta4 + 6)*q^18 + ((-2/5*zeta4 - 2/5)*a^3 - 12/5*zeta4*a^2 + (-8/5*zeta4 + 8/5)*a)*q^19 + ((-3/10*zeta4 + 2/5)*a^3 + (3/10*zeta4 + 11/10)*a^2 + (3/5*zeta4 - 9/5)*a - 6*zeta4 - 9)*q^21 + (1/5*a^3 + (3/5*zeta4 + 3/5)*a^2 + 4/5*zeta4*a)*q^23 + ((-1/4*zeta4 - 1/4)*a^3 - 4*zeta4*a^2 + (-6*zeta4 + 6)*a + 15)*q^24 + (a^2 + (2*zeta4 + 2)*a + 6*zeta4)*q^26 + (-1/20*a^3 + (-3/20*zeta4 - 3/20)*a^2 - 6/5*zeta4*a + 18*zeta4 - 18)*q^27 + (1/20*zeta4*a^3 + (-7/20*zeta4 - 13/20)*a^2 + (-2*zeta4 - 6/5)*a - 4*zeta4 + 2)*q^28 + ((-3/10*zeta4 + 3/10)*a^3 + 9/5*a^2 + (36/5*zeta4 + 36/5)*a + 12*zeta4)*q^29 + ((-1/2*zeta4 + 1/2)*a^3 + 3*a^2 + (2*zeta4 + 2)*a)*q^31 + (-9/20*zeta4*a^3 + (-27/20*zeta4 + 27/20)*a^2 + 9/5*a)*q^32 + ((-9/10*zeta4 - 9/10)*a^3 - 27/5*zeta4*a^2 + (-18/5*zeta4 + 18/5)*a)*q^34 + ((1/10*zeta4 + 1/10)*a^3 + 3/5*zeta4*a^2 + (12/5*zeta4 - 12/5)*a - 9)*q^36 + ((-zeta4 + 1)*a^2 + 4*a + 6*zeta4 + 6)*q^37 + (-24*zeta4 - 24)*q^38 + ((1/10*zeta4 - 1/10)*a^3 - 3/5*a^2 + (-12/5*zeta4 - 12/5)*a - 18*zeta4)*q^39 + (6*a^2 + (12*zeta4 + 12)*a + 36*zeta4)*q^41 + ((-7/20*zeta4 - 3/10)*a^3 + (-29/20*zeta4 - 7/20)*a^2 + (-36/5*zeta4 - 3/5)*a - 30*zeta4 + 24)*q^42 + ((4*zeta4 + 4)*a^2 + 16*zeta4*a + 24*zeta4 - 24)*q^43 + 12*q^46 + (-12*zeta4 + 12)*q^47 + (-11/20*zeta4*a^3 + (-33/20*zeta4 + 33/20)*a^2 + 66/5*a)*q^48 + ((-7/10*zeta4 - 7/10)*a^3 - 21/5*zeta4*a^2 + (-14/5*zeta4 + 14/5)*a - 35*zeta4)*q^49 + ((9/10*zeta4 + 9/10)*a^3 + 27/5*zeta4*a^2 + (108/5*zeta4 - 108/5)*a)*q^51 + (-1/10*a^3 + (-3/10*zeta4 - 3/10)*a^2 - 12/5*zeta4*a - 2*zeta4 + 2)*q^52 + (-9/5*a^3 +
  
```

$$\begin{aligned}
 &(-27/5*\zeta_4 - 27/5)*a^2 - 36/5*\zeta_4*a)*q^{53} + ((-19/20*\zeta_4 - 19/20)*a^3 - \\
 &31/5*\zeta_4*a^2 + (-24/5*\zeta_4 + 24/5)*a + 3)*q^{54} + ((-3/4*\zeta_4 - 3/4)*a^3 + \\
 &(-9/2*\zeta_4 - 5/2)*a^2 + (-23*\zeta_4 + 13)*a - 15*\zeta_4 + 30)*q^{56} + (4/5*\zeta_4*a^3 + \\
 &32/5*\zeta_4 - 32/5)*a^2 - 96/5*a - 24*\zeta_4 - 24)*q^{57} + ((3*\zeta_4 + 3)*a^2 + \\
 &12*\zeta_4*a + 18*\zeta_4 - 18)*q^{58} + (-10*\zeta_4*a^2 + (-20*\zeta_4 + 20)*a + 60)*q^{59} + \\
 &((-3/2*\zeta_4 + 3/2)*a^3 + 9*a^2 + (6*\zeta_4 + 6)*a)*q^{61} + (-30*\zeta_4 + 30)*q^{62} + \\
 &((1/4*\zeta_4 - 7/5)*a^3 + (-119/20*\zeta_4 - 149/20)*a^2 + (-78/5*\zeta_4 - 6)*a - \\
 &6*\zeta_4 + 24)*q^{63} - 71*\zeta_4*q^{64} + ((-6*\zeta_4 + 6)*a^2 + 24*a + 36*\zeta_4 + 36)*q^{67} + \\
 &(18*\zeta_4 + 18)*q^{68} + ((-1/5*\zeta_4 - 1/5)*a^3 - 16/5*\zeta_4*a^2 + (-24/5*\zeta_4 + \\
 &24/5)*a + 12)*q^{69} + ((-3/2*\zeta_4 - 3/2)*a^3 - 9*\zeta_4*a^2 + (-36*\zeta_4 + 36)*a + \\
 &60)*q^{71} + (-5/4*\zeta_4*a^3 + (5/4*\zeta_4 - 5/4)*a^2 - 15*a - 30*\zeta_4 - 30)*q^{72} + \\
 &(19/10*\zeta_4*a^3 + (57/10*\zeta_4 - 57/10)*a^2 - 228/5*a - 38*\zeta_4 - 38)*q^{73} + \\
 &((-3/10*\zeta_4 + 3/10)*a^3 + 9/5*a^2 + (36/5*\zeta_4 + 36/5)*a + 12*\zeta_4)*q^{74} + \\
 &((2/5*\zeta_4 - 2/5)*a^3 - 12/5*a^2 + (-8/5*\zeta_4 - 8/5)*a)*q^{76} + (7/10*a^3 + \\
 &(11/10*\zeta_4 + 11/10)*a^2 - 6/5*\zeta_4*a - 6*\zeta_4 + 6)*q^{78} + 50*\zeta_4*q^{79} + \\
 &((\zeta_4 + 1)*a^3 + 6*\zeta_4*a^2 + (24*\zeta_4 - 24)*a - 9)*q^{81} + (9/5*a^3 + (27/5*\zeta_4 + \\
 &27/5)*a^2 + 216/5*\zeta_4*a + 36*\zeta_4 - 36)*q^{82} + (78*\zeta_4 + 78)*q^{83} + ((-2/5*\zeta_4 - \\
 &3/10)*a^3 + (-11/10*\zeta_4 + 3/10)*a^2 + (9/5*\zeta_4 + 3/5)*a + 9*\zeta_4 - 6)*q^{84} + \\
 &((6/5*\zeta_4 + 6/5)*a^3 + 36/5*\zeta_4*a^2 + (144/5*\zeta_4 - 144/5)*a - 48)*q^{86} + \\
 &(-3/5*a^3 + (-9/5*\zeta_4 - 9/5)*a^2 - 72/5*\zeta_4*a - 54*\zeta_4 + 54)*q^{87} + \\
 &(10*\zeta_4*a^2 + (20*\zeta_4 - 20)*a - 60)*q^{89} + ((-7/10*\zeta_4 + 7/10)*a^3 + 21/5*a^2 + \\
 &(14/5*\zeta_4 + 14/5)*a - 14)*q^{91} + (-1/5*\zeta_4*a^3 + (-3/5*\zeta_4 + 3/5)*a^2 + \\
 &4/5*a)*q^{92} + (-a^3 + (-8*\zeta_4 - 8)*a^2 - 24*\zeta_4*a - 30*\zeta_4 + 30)*q^{93} + \\
 &((3/5*\zeta_4 + 3/5)*a^3 + 18/5*\zeta_4*a^2 + (12/5*\zeta_4 - 12/5)*a)*q^{94} + ((9/20*\zeta_4 - \\
 &9/20)*a^3 - 36/5*a^2 + (-54/5*\zeta_4 - 54/5)*a - 27*\zeta_4)*q^{96} + (19/10*a^3 + \\
 &(57/10*\zeta_4 + 57/10)*a^2 + 228/5*\zeta_4*a + 38*\zeta_4 - 38)*q^{97} + 0(q^{98})
 \end{aligned}$$

```

sage: parent(f)
Power Series Ring in q over Number Field in a with defining polynomial
x^4 + (4*zeta4 + 4)*x^3 + 20*zeta4*x^2 + (24*zeta4 - 24)*x - 120 over its base field

```

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