FINITE FIELDS AND THEIR APPLICATIONS 1, 376–394 (1995)

Spherical Functions for Finite Upper Half Planes with Characteristic 2

RONALD EVANS

Department of Mathematics, University of California at San Diego, 9500 Gilman Drive, La Jolla, California 92093-0112
E-mail: revans@euclid.ucsd.edu

Communicated by Neal Koblitz

Received September 1, 1994

Let $GF(q)$ be a field of q elements. Let G denote the group of matrices $M(x, y) = \binom{y\ x}{0\ 1}$ over $GF(q)$ with $y \neq 0$. Fix an irreducible polynomial $u^2 + tu + n \in GF(q)[u]$. For each $a \in GF(q)$, let X_a be the graph whose vertices are the $q^2 - q$ elements of G, with two vertices $M(x, y)$, $M(v, w)$ joined by an edge iff $(x - v)^2 + t(x - v)(y - w) + n(y - w)^2 = wya$. The graphs X_a with $a \notin \{0, t^2 - 4n\}$ are $(q + 1)$-regular connected graphs. These X_a are known to be Ramanujan graphs for odd q, largely by the work of Katz, Soto-Andrade, and Terras. Using representation theory for $GL(2, q)$ and recent character sum estimates of Katz, we show that these X_a are Ramanujan graphs for even q as well. This settles the beautiful conjecture of Terras that all $(q + 1)$-regular finite upper half plane graphs X_a are Ramanujan graphs. Further, let $L(G)$ denote the Hilbert space of functions mapping G into the complex numbers, with inner product $(f, h) = \sum_{g \in G} f(g)\overline{h(g)}$. We give $q^2 - q$ character sums which comprise a common basis in $L(G)$ of mutually orthogonal eigenfunctions for the ajacency operators of all the connected graphs X_a. © 1995 Academic Press, Inc.

1. INTRODUCTION

Let $GF(q)$ be the field of q elements. Let $\mathcal{G}$ denote the general linear group $GL(2, q)$ of 2×2 nonsingular matrices over $GF(q)$. Then $|\mathcal{G}| = (q^2 - q)(q^2 - 1)$. Fix a zero $\theta \in GF(q^2)$ of an irreducible polynomial

$$u^2 + tu + n \in GF(q)[u]. \tag{1.1}$$

1071-5797/95 $12.00

The set of points

$$z = x + y\theta \in GF(q^2), \qquad \text{with } x \in GF(q),\ y \in GF(q)^*,$$

is called the finite upper half plane over $GF(q)$. Write

$$N(z) = (x + y\theta)(x + y\theta^q) = x^2 + txy + ny^2 \tag{1.2}$$

and

$$\mathrm{Im}(z) = y. \tag{1.3}$$

For $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathscr{G}$, we have the action

$$gz = (az + b)/(cz + d) \tag{1.4}$$

on the finite upper half plane. Define a "pseudo-distance" $\Delta: \mathscr{G} \times \mathscr{G} \to GF(q)$ by

$$\Delta(g_1, g_2) = N(z_1 - z_2)/(\mathrm{Im}(z_1)\mathrm{Im}(z_2)), \tag{1.5}$$

where $z_i = g_i\theta$. It follows from (1.5) that

$$\Delta(gg_1, gg_2) = \Delta(g_1, g_2), \qquad g, g_1, g_2 \in \mathscr{G}. \tag{1.6}$$

Let G be the subgroup of $\mathscr{G}$ defined by

$$G = \left\{\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}: x \in GF(q),\ y \in GF(q)^*\right\}. \tag{1.7}$$

Then $|G| = q^2 - q$, and $\{g\theta : g \in G\}$ is the finite upper half plane. The point θ in the finite upper half plane corresponds to the identity $I = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ in G.

For $a \in GF(q)$, let X_a be the graph whose vertices are the $q^2 - q$ elements of G, with two vertices $g_1, g_2 \in G$ joined by an edge iff $\Delta(g_1, g_2) = a$. If the irreducible polynomial in (1.1) were changed, so that θ were replaced by say $c + d\theta$, where $c, d \in GF(q)$, then the new graph X_a would be isomorphic to the old graph X_{a/d^2}. The graphs X_a with $a \notin \{0, t^2 - 4n\}$ are $(q + 1)$-regular connected graphs [1].

Let A_a denote the adjacency matrix of the graph X_a. (Thus A_a is a $|G| \times |G|$ symmetric matrix whose entries are 0's and 1's.) Due to the work of Katz [4], Soto-Andrade [10], and Terras (see [11]), it is known that for *odd* q and $a \notin \{0, t^2 - 4n\}$, the adjacency matrix A_a has the simple eigenvalue $(q + 1)$ with all remaining eigenvalues $\leq 2\sqrt{q}$ in absolute value, so that X_a is a Ramanujan graph [11, p. 126; 7].

Let $L(G)$ denote the Hilbert space of functions mapping G into $\mathbb{C}$ with inner product

$$(f, h) = \sum_{g \in G} f(g)\, \overline{h(g)}. \tag{1.8}$$

For odd q, we have produced [3] a basis of $q^2 - q$ mutually orthogonal eigenfunctions in $L(G)$ for all of the adjacency operators A_a with $a \notin \{0, t^2 - 4n\}$. See also Kuang [6]. The purpose of this paper is to obtain analogous results on the eigenvalues and eigenfunctions for *even* q. The proof that the $(q + 1)$-regular graphs X_a are Ramanujan graphs is substantially different for even q than for odd q. Nevertheless, some of our arguments are the same for all q, and for these we will simply refer the reader to the arguments for odd q found in [3].

In Section 2, we discuss the regularity of the graphs X_a. In Section 3, we develop the necessary results on spherical functions which enable us to construct our eigenvalues and eigenfunctions in Sections 4 and 5, respectively.

For the rest of this paper, let q be even. The polynomial $u^2 + tu + n \in GF(q)[u]$ is irreducible iff $\mathrm{Tr}(n/t^2) = 1$, where Tr denotes the trace from $G(q)$ to $GF(2)$. (To see this, replace u by ut and note that $\mathrm{Tr}(n) = 0$ iff $n = v^2 + v$ for some $v \in GF(q)$, by Hilbert's Theorem 90.) Without loss of generality, we specify the polynomial in (1.1) by taking

$$t = 1, \ \mathrm{Tr}(n) = 1. \tag{1.9}$$

Fix multiplicative characters

$$\chi \text{ on } GF(q)^* \text{ of order } q - 1, \tag{1.10}$$

and

$$\omega \text{ on } GF(q^2)^* \text{ of order } q^2 - 1. \tag{1.11}$$

Let ψ denote the additive character on $GF(q)$ defined by

$$\psi(x) = (-1)^{\mathrm{Tr}(x)}, \qquad x \in GF(q). \tag{1.12}$$

For $g = \left(\begin{smallmatrix} y & x \\ 0 & 1 \end{smallmatrix}\right) \in G$ and $b \in GF(q)$, define the K-Bessel character sum

$$K_{j,b}(g) = \psi(-bx)\chi^j(y) \sum_{u \in GF(q)} \psi(bu)\overline{\chi}^j(u^2 + uy + ny^2), \tag{1.13}$$

and for $t \in GF(q)^*$, define the double character sum

$$H_{t,j,b}(g) = \frac{-\psi(-bx)}{q+1} \sum_{u \in GF(q)}^{*} \psi(bu) \sum_{\substack{\alpha \in GF(q^2) \\ N(\alpha)=1, \alpha \neq 1}} \omega^{j}(\alpha)\psi\left(\frac{b\alpha}{\alpha^2+1}\left(\frac{t}{y} + \frac{u}{y} + 1 + \frac{nu^2}{ty} + \frac{nu}{t} + \frac{n^2y}{t}\right)\right), \tag{1.14}$$

where $N(\alpha) = \alpha^{q+1}$ is the norm of α and the asterisk on the summation in (1.14) indicates that if $y = t/n$, then the terms for $u = 0$ and $u = y$ are to be multiplied by $q + 1$. For each pair b, j with $b \in GF(q)^*$ and $1 \leq j \leq q/2$, choose $t \in GF(q)^*$ so that $H_{t,j,b}$ is not identically zero on G; this is shown to be possible by Theorem 5.2.

The adjacency operator A_a can be viewed as a linear transformation of $L(G)$ into itself (or similarly $L(\mathscr{G})$ into itself) defined by

$$(A_a f)(g) = \sum_{s \in S_a} f(gs) \qquad (g \in G, f \in L(G)), \tag{1.15}$$

where

$$S_a = \{s \in G : \Delta(s, I) = a\}, \tag{1.16}$$

since $\Delta(gs, g) = a$ by (1.16) and (1.6).

One of our results is that the $(q^2 - q)/2$ functions

$$H_{t,j,b} \qquad (b \in GF(q)^*, 1 \leq j \leq q/2) \tag{1.17}$$

(chosen above to be nonzero), together with the $(q^2 - q)/2$ functions

$$K_{j,b} \qquad (b = 0, 0 \leq j < q - 1), \tag{1.18}$$

$$K_{j,b} \qquad (b \in GF(q)^*, 1 \leq j \leq (q-2)/2), \tag{1.19}$$

comprise a common basis in $L(G)$ of mutually orthogonal (nonzero) eigenfunctions for all the adjacency operators A_a with $a \neq 0, 1$. See Theorems 5.1–5.4.

Theorems 4.2 and 5.1 below show that the eigenfunction $K_{j,b}$ for A_a corresponds to the eigenvalue $q + 1$ if $j = b = 0$, and to the (real) eigenvalue

$$-\sum_{\substack{v \in GF(q) \\ v \neq 1}} \chi^{j}(v)\psi\left(\frac{av}{v^2+1}\right) \tag{1.20}$$

for $1 \leq j < q - 1$, independent of the choice of $b \in GF(q)$. Theorems 4.1 and 5.1 below show that the eigenfunction $H_{t,j,b}$ $(1 \leq j \leq q/2)$ corresponds to the (real) eigenvalue

$$- \sum_{\substack{\alpha \in GF(q^2) \\ N(\alpha)=1, \alpha \neq 1}} \omega^j(\alpha)\psi\left(\frac{a\alpha}{\alpha^2 + 1}\right), \tag{1.21}$$

independent of the choice of $b \in GF(q)^*$. (Note that $\psi(a\alpha/(\alpha^2 + 1))$ makes sense since $\alpha/(\alpha^2 + 1) = 1/(\alpha + \alpha^q) \in GF(q)$.) The eigenvalues in (1.20)–(1.21) involve *additive* characters; this was not the case for *odd* q (see [3, (1.18), (1.19)].

Viewing the $q^2 - q$ eigenfunctions in (1.17)–(1.19) as mutually orthogonal column vectors u_i each of length $q^2 - q = |G|$, and normalizing so that the inner products (u_i, u_i) all equal 1, we see that the matrix U with columns u_i gives simultaneous unitary diagonalizations

$$U^*A_aU = \Lambda_a, \qquad a \neq 0, 1, \tag{1.22}$$

where Λ_a is the diagonal matrix with the eigenvalue $q + 1$ occurring once, the eigenvalue (1.21) occurring $(q - 1)$ times for each j such that $1 \leq j \leq q/2$, and the eigenvalue (1.20) occurring $(q + 1)$ times for each j such that $1 \leq j \leq (q - 2)/2$.

It will be shown in the next section that the graphs X_a are $(q + 1)$-regular graphs iff $a \neq 0, 1$. To verify that the graphs X_a are Ramanujan for $a \neq 0, 1$, one must show that the character sums in (1.20)–(1.21) are bounded in absolute value by $2\sqrt{q}$. This estimate has recently been achieved in the work of Katz [5], thus completing the establishment of the fact that the $(q + 1)$-regular graphs X_a are all Ramanujan graphs, for both even and odd characteristic. We remark that related graphs defined over rings instead of fields are not necessarily Ramanujan graphs; see [2]. For another approach to the results of Katz, see [7A].

2. Regularity of the Graphs X_a

Let $g \in G$, $a \in GF(q)$. We have seen that the set of neighbors of g in X_a (i.e., vertices connected by an edge to g) is $\{gs : s \in S_a\}$, where S_a is defined in (1.16). Note that

$$S_a = \left\{\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} : x^2 + x(y - 1) + n(y - 1)^2 + ay = 0\right\}, \tag{2.1}$$

and

$$\bar{S}_a = \{s^{-1} : s \in S_a\}. \tag{2.2}$$

It follows from (1.9) and (2.1) that

$$S_0 = \{I\}. \tag{2.3}$$

Moreover, since for $y \neq 1$,

$$\begin{aligned}\operatorname{Tr}\left(\frac{n(y-1)^2 + ay}{(y-1)^2}\right) &= 1 + \operatorname{Tr}\left(\frac{ay}{(y-1)^2}\right) \\ &= 1 + \operatorname{Tr}\left(\frac{ay}{y-1} + a\left(\frac{y}{y-1}\right)^2\right),\end{aligned} \tag{2.4}$$

and since the members of (2.4) equal 1 when $a = 1$, we have

$$S_1 = \left\{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right\}. \tag{2.5}$$

Thus X_a is not $(q + 1)$-regular when $a = 0$ or $a = 1$. Assume throughout the rest of this paper that $a \neq 0, 1$. Then X_a is $(q + 1)$-regular [1]. We give a short new proof of this fact in the following theorem.

THEOREM 2.1. *For $a \neq 0, 1$, the graph X_a is $(q + 1)$-regular.*

Proof. We must show that

$$|S_a| = q + 1, \qquad a \neq 0, 1. \tag{2.6}$$

The quadratic equation

$$p_y(x) := x^2 + x(y - 1) + n(y - 1)^2 + ya = 0 \tag{2.7}$$

has the solution $y = 1$, $x = \sqrt{a}$. For each fixed $y \neq 1$ in $GF(q)^*$ for which $p_y(x)$ is reducible over $GF(q)$, there are two distinct solutions $x \in GF(q)$, so we must show that $p_y(x)$ is reducible for $q/2$ values of $y \neq 0, 1$. Equivalently, we must show that the left side of (2.4) vanishes for $q/2$ values $y \neq 0, 1$. Let $v = y/(y - 1)$. By (2.4), it suffices to show that $\operatorname{Tr}(a(v + v^2)) = 1$ for $q/2$ values of v. Since $w = v + v^2$ runs twice through the elements of trace 0 as v runs through $GF(q)$, it suffices to show that there are exactly $q/4$ values of $w \in GF(q)$ such that $\operatorname{Tr}(aw) = 1$ and

$\mathrm{Tr}(w) = 0$. The number of such w is counted by the character sum

$$\frac{1}{4} \sum_{w \in GF(q)} (1 - \psi(aw))(1 + \psi(w)),$$

which clearly equals $q/4$ since $a \neq 0, 1$. ■

For $a \neq 0, 1$, the graph X_a has diameter ≤ 4; i.e., every $g \in G$ has the form $s_1 s_2 s_3 s_4$ with $s_i \in S_a \cup \{I\}$. In particular, X_a is connected for $a \neq 0, 1$. For a proof, see [1].

3. Spherical Functions

Let K be the subgroup of the general linear group $\mathcal{G}$ defined by

$$K = \{g \in \mathcal{G} : g\theta = \theta\}. \tag{3.1}$$

Thus K is the cyclic, non-normal subgroup

$$K = \left\{ \begin{pmatrix} c & dn \\ d & c + d \end{pmatrix} : c, d \in GF(q),\ c \text{ and } d \text{ not both } 0 \right\} \tag{3.2}$$

isomorphic to $GF(q^2)^*$. As $|K| = q^2 - 1$, we have $|\mathcal{G}/K| = q^2 - q = |G|$. Each coset $gK \in \mathcal{G}/K$ has a unique representative in G, i.e.,

$$gK = \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} K \tag{3.3}$$

for a unique $\left(\begin{smallmatrix} y & x \\ 0 & 1 \end{smallmatrix}\right) \in G$. Define the function $\chi : \mathcal{G} \to \mathbb{C}$ by

$$\chi(g) = \chi(y), \tag{3.4}$$

where g and y are as in (3.3) and $\chi(y)$ is defined by (1.10). For $g \in \mathcal{G}$, define

$$S(g) = S_b, \qquad \text{with } b = \Delta(g, I), \tag{3.5}$$

where S_b is defined by (1.16). Just as in [3, (2.5)], we have

$$S(g) = KgK \cap G, \tag{3.6}$$

and the intersections with G of the q double cosets in $K\backslash\mathcal{G}/K$ are precisely the q sets S_b, $b \in GF(q)$.

Given a complex irreducible representation π of $\mathcal{G}$, let λ_π denote its character $\mathrm{Tr}(\pi)$. Define

$$h_\pi(g) = \frac{1}{|K|} \sum_{k \in K} \lambda_\pi(kg), \quad g \in \mathcal{G}. \tag{3.7}$$

It is easily seen that h_π is both left and right K-invariant, so we may write

$$h_\pi(g) = h_\pi(KgK) = h_\pi(S(g)), \qquad g \in \mathcal{G}. \tag{3.8}$$

Assume from now on that π is chosen such that

$$h_\pi(I) = 1. \tag{3.9}$$

The functions $h_\pi \in L(\mathcal{G})$ satisfying (3.7) and (3.9) are the zonal spherical functions. The representations π for which (3.9) hold are characterized in the following theorem.

THEOREM 3.1. *The q nonequivalent complex irreducible representations π of $\mathcal{G}$ for which $h_\pi(I) = 1$ are (in the notation of* [9, p. 70])

$$\pi = \rho(\bar{\omega}^{(q-1)j}), \qquad 1 \le j \le q/2 \tag{3.10}$$

(the $q/2$ "cuspidal" representations each having dimension $q - 1$);

$$\pi = \chi^j, \qquad j = 0 \tag{3.11}$$

(the trivial representation χ^0 of dimension 1); and

$$\pi = \rho(\chi^j, \bar{\chi}^j), \qquad 1 \le j \le (q-2)/2 \tag{3.12}$$

(the $(q-2)/2$ representations each having dimension $(q+1)$).

Proof. We consider first the *cuspidal* representations π. By [9, p. 41], the irreducible cuspidal representations $\rho(\nu)$ on $\mathcal{G}$ are indexed by the $(q^2 - q)/2$ characters

$$\nu = \bar{\omega}^i, \qquad 1 \le i \le (q^2 - 2)/2,\ (q+1) \nmid i. \tag{3.13}$$

Let $\pi = \rho(\nu)$ with ν as in (3.13). By (3.2), K consists of the $(q-1)$ matrices $\begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}$, $c \in GF(q)^*$, with eigenvalues c, c, together with the $q^2 - q$ matrices $\begin{pmatrix} c & dn \\ d & c+d \end{pmatrix}$, $c, d \in GF(q)$, $d \neq 0$, with eigenvalues $c + d\theta$, $c + d + d\theta$. Hence by [9, pp. 15, 70] and (3.7), we have

$$|K|h_\pi(I) = (q-1)\sum_{c\in GF(q)^*} \nu(c) - 2\sum_{\substack{c\in GF(q)\\ d\in GF(q)^*}} \nu(c+d\theta)$$

$$= (q+1)\sum_{c\in GF(q)} \nu(c) - 2\sum_{z\in GF(q^2)} \nu(z) = (q+1)\sum_{c\in GF(q)} \nu(c)$$

$$= \begin{cases} |K|, & \text{if } \nu = \bar{\omega}^{j(q-1)},\ 1 \le j \le q/2, \\ 0, & \text{otherwise.} \end{cases}$$

Thus $h_\pi(I) = 1$ iff π is as in (3.10).

Next we consider those π of dimension 1. By [9, pp. 1, 27], these are the $(q-1)$ representations

$$\pi(g) = \chi^j(\det g), \qquad 0 \le j < q-1, \tag{3.14}$$

where $g \in \mathcal{G}$. With π as in (3.14),

$$|K|h_\pi(I) = \sum_{c,d\in GF(q)} \chi^j(c^2 + cd + d^2 n)$$

$$= \sum_{c\in GF(q)} \chi^j(c^2) + \sum_{d\in GF(q)^*} \chi^j(d^2) \sum_{c\in GF(q)} \chi^j(c^2 + c + n)$$

$$= \begin{cases} |K|, & \text{if } j = 0 \\ 0, & \text{if } 0 < j < q-1. \end{cases}$$

Thus $h_\pi(I) = 1$ iff π is as in (3.11).

Next we consider those π of dimension $(q+1)$. By [9, pp. 1, 27], these are the $(q-1)(q-2)/2$ representations of the type

$$\pi = \rho(\chi^j, x^k), \qquad 0 \le j < k < q-1. \tag{3.15}$$

With π as in (3.15), we have

$$|K|h_\pi(I) = (q+1)\sum_{c\in GF(q)} \chi^{j+k}(c) = \begin{cases} |K|, & \text{if } k = q-1-j, \\ 0, & \text{otherwise.} \end{cases}$$

Thus $h_\pi(I) = 1$ iff π is as in (3.12).

Finally, we consider those π of dimension q. By [9, pp. 1, 27], these are the $(q-1)$ representations of type

$$\rho(\chi^j, \chi^j), \qquad 0 \le j < q-1. \tag{3.16}$$

With π as in (3.16), we have

$$|K|h_\pi(I) = q \sum_{c\in GF(q)} \chi^{2j}(c) - \sum_{\substack{c,d\in GF(q)\\ d\neq 0}} \chi^j(c^2+cd+nd^2) = 0.$$

Thus $h_\pi(I)$ never equals 1 for π of dimension q. ■

Just as in [3, Thm. 2.1], we have:

THEOREM 3.2. *For $a \neq 0$, 1 and π as in Theorem 3.1, the function $h_\pi \in L(\mathcal{G})$ is an eigenfunction of A_a with eigenvalue $(q+1)h_\pi(S_a)$.*

We know that $h_\pi(S_0) = 1$ by (3.9) and (2.3). The next theorem evaluates $h_\pi(S_1)$.

THEOREM 3.3. *We have*

$$h_\pi(S_1) = \begin{cases} -1, & \text{for cuspidal } \pi \text{ as in (3.10)},\\ 1, & \text{for noncuspidal } \pi \text{ as in (3.11)–(3.12)}. \end{cases} \tag{3.17}$$

Proof. By (3.7)–(3.8) and (2.5),

$$|K|h_\pi(S_1) = \sum_{k\in K} \lambda_\pi(kg),$$

with $g = \left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right) \in S_1$. Thus, by (3.2),

$$|K|h_\pi(S_1) = \sum_{\substack{c,d\in GF(q)\\ c,d \text{ not both } 0}} \lambda_\pi\left(\begin{pmatrix} c & c+dn\\ d & c\end{pmatrix}\right). \tag{3.18}$$

The eigenvalues of $\left(\begin{smallmatrix}c & c+dn\\ d & c\end{smallmatrix}\right)$ both equal $\sqrt{c^2+cd+nd^2} \in GF(q)^*$. Since $\omega^{q-1}(x) = 1$ for all $x \in GF(q)^*$, it follows from [9, p. 70] that each summand on the right of (3.18) equals -1 for cuspidal π, whereas each summand equals 1 for noncuspidal π. Thus the result follows. ■

For cuspidal π as in (3.10), write

$$C_j = h_\pi, \qquad 1 \le j \le q/2, \tag{3.19}$$

and for noncuspidal π as in (3.11)–(3.12), write

$$B_j = h_\pi, \qquad 0 \le j \le (q-2)/2. \tag{3.20}$$

These spherical functions B_j, C_j will be calculated as explicit character sums in the next section. As in [3, Thm. 2.2], the spherical functions B_j, C_j are mutually orthogonal on G. Also, as in [3, Cor. 2.3], if U, V are any two distinct members of the set of q spherical functions $\{B_j, C_j\}$, then $U(S_a) \ne V(S_a)$ for some $a \ne 0, 1$.

4. Spherical Functions As Character Sums

In Theorem 4.1, we express the cuspidal spherical functions C_j on $\mathcal{G}$ as character sums. In Theorem 4.2, we similarly express noncuspidal spherical functions B_j.

THEOREM 4.1. *For $g \in \mathcal{G}$ and for cuspidal π as in* (3.10), *we have*

$$C_j(g) = \frac{-1}{|S(g)|} \sum_{\substack{\alpha \in GF(q^2) \\ N(\alpha)=1, \alpha \ne 1}} \omega^j(\alpha)\psi\left(\frac{\alpha\Delta(g, I)}{\alpha^2+1}\right), \qquad 1 \le j \le q/2. \tag{4.1}$$

Proof. Note that

$$g := \begin{pmatrix} 1 & \sqrt{a} \\ 0 & 1 \end{pmatrix} \in S_a, \qquad a \in GF(q), \tag{4.2}$$

since $\Delta(g, I) = a$. By (3.8), it suffices to prove (4.1) for g as in (4.2). If $a = 0$, both sides of (4.1) equal 1. Next let $a = 1$. For α as in (4.1), we have

$$\operatorname{Tr}\left(\frac{\alpha}{\alpha^2+1}\right) = 1, \tag{4.3}$$

since the polynomial $(u-\alpha)(u-\alpha^q) \in GF(q)[u]$ is irreducible. Thus the factor $\psi\ (\alpha/\alpha^2 + 1))$ in (4.1) always equals -1, so the right side of (4.1) equals -1. Thus (4.1) follows from (3.17).

It remains to show that for g as in (4.2) with $a \ne 0, 1$, we have

$$C_j(g) = \frac{-1}{q+1} \sum_{\substack{\alpha \in GF(q^2) \\ N(\alpha)=1, \alpha \ne 1}} \omega^j(\alpha)\psi\left(\frac{a\alpha}{\alpha^2+1}\right), \qquad 1 \le j \le q/2. \tag{4.4}$$

By (3.2) and (3.7), we must prove, for $a \neq 0, 1$, that

$$\sum_{\substack{c,d \in GF(q) \\ c,d \text{ not both } 0}} \lambda_\pi \left(\begin{pmatrix} c & c\sqrt{a} + dn \\ d & d\sqrt{a} + d + c \end{pmatrix} \right) = (1 - q) \sum_{\substack{\alpha \in GF(q^2) \\ N(\alpha)=1, \alpha \neq 1}} \omega^j(\alpha) \psi \left(\frac{a\alpha}{\alpha^2 + 1} \right). \tag{4.5}$$

By [9, p. 70], the terms on the left of (4.5) with $d = 0$ contribute

$$M_0 := \sum_{c \in GF(q)^*} -\bar{\omega}^{j(q-1)}(c) = 1 - q. \tag{4.6}$$

Let M_1 denote the sum of those terms on the left of (4.5) with $d \neq 0$. The matrix in (4.5) has trace

$$T = d(1 + \sqrt{a}) \neq 0, \tag{4.7}$$

and determinant

$$N = c^2 + cd + d^2 n, \tag{4.8}$$

so its eigenvalues are the zeros of the polynomial

$$u^2 + Tu + N \in GF(q)[u]. \tag{4.9}$$

From (4.7)–(4.8),

$$c^2 + \frac{cT}{1 + \sqrt{a}} + \frac{nT^2}{1 + a} = N. \tag{4.10}$$

Instead of summing over all $c, d \in GF(q)$ with $d \neq 0$ in (4.5) to evaluate M_1, we instead sum twice over all variables $N, T \in GF(q)$ with $T \neq 0$ and

$$\operatorname{Tr} \left(\frac{N(a + 1)}{T^2} \right) = 1. \tag{4.11}$$

This is because for fixed N, T with $T \neq 0$, Eq. (4.10) has two distinct solutions $c \in GF(q)$ when (4.11) holds and no solutions $c \in GF(q)$ otherwise. By [9, p. 70], there is no contribution to the left sum in (4.5) when the two eigenvalues of the matrix are distinct elements of $GF(q)$. Hence we sum twice over all $N, T \in GF(q)$ with $T \neq 0$ for which (4.11) holds and for which

$$\mathrm{Tr}(N/T^2) = 1, \tag{4.12}$$

since the polynomial in (4.9) is irreducible over $GF(q)$ iff (4.12) holds. Thus, by [9, p. 70],

$$M_1 = \sum_{\substack{N,T\in GF(q),T\neq 0\\ \mathrm{Tr}(N/T^2)=1,\mathrm{Tr}(Na/T^2)=0}} -2(\bar{\omega}^{j(q-1)}(\beta) + \bar{\omega}^{j(q-1)}(\beta^q)), \tag{4.13}$$

where $\beta = \beta(N, T) \in GF(q^2) - GF(q)$ is a zero of the irreducible polynomial in (4.9). Since $N = \beta^{q+1}$, $T = \beta + \beta^q$, and $N/T^2 = 1/(\beta^{1-q} + \beta^{q-1})$, (4.13) becomes

$$M_1 = \sum_{\substack{\beta\in GF(q^2)-GF(q)\\ \mathrm{Tr}(1/(\beta^{1-q}+\beta^{q-1}))=1,\mathrm{Tr}(a/(\beta^{1-q}+\beta^{q-1}))=0}} -2\omega^j(\beta^{q-1}). \tag{4.14}$$

Thus,

$$M_1 = 2(1-q) \sum_{\substack{\alpha\in GF(q^2)\\ N(\alpha)=1,\alpha\neq 1\\ \mathrm{Tr}(\alpha/(\alpha^2+1))=1,\mathrm{Tr}(a\alpha/(\alpha^2+1))=0}} \omega^j(\alpha), \tag{4.15}$$

since β^{q-1} runs through each α exactly $(q-1)$ times. In view of (4.3), the first trace restriction in (4.15) is superfluous, and (4.15) becomes

$$M_1 = (1-q) \sum_{\substack{\alpha\in GF(q^2)\\ N(\alpha)=1,\alpha\neq 1}} \omega^j(\alpha) \left\{1 + \psi\left(\frac{a\alpha}{\alpha^2+1}\right)\right\}. \tag{4.16}$$

Adding (4.6) and (4.16), we obtain the desired result (4.5). ■

It is clear that

$$B_0 \equiv 1. \tag{4.17}$$

We conclude this section by evaluating the remaining spherical functions B_j.

THEOREM 4.2. *For $g \in \mathcal{G}$ and for noncuspidal π as in* (3.12), *we have*

$$B_j(g) = \frac{-1}{|S(g)|} \sum_{\substack{v \in GF(q) \\ v \neq 1}} \chi^j(v)\psi\left(\frac{v\Delta(g, I)}{v^2 + 1}\right), \qquad 1 \leq j \leq \frac{(q-2)}{2}. \quad (4.18)$$

Proof. We may assume g is as in (4.2). If $a = 0$, both sides of (4.18) equal 1. Next let $a = 1$. For $v \in GF(q)$, $v \neq 1$,

$$\frac{v}{v^2+1} = \frac{v}{v+1} + \left(\frac{v}{v+1}\right)^2, \quad (4.19)$$

so

$$\operatorname{Tr}\left(\frac{v}{v^2+1}\right) = 0, \qquad \psi\left(\frac{v}{v^2+1}\right) = 1. \quad (4.20)$$

Thus the right side of (4.18) equals 1, so (4.18) follows by (3.17).

It remains to prove that for $a \neq 0, 1$,

$$\sum_{\substack{c, d \in GF(q) \\ c, d \text{ not both } 0}} \lambda_\pi\left(\begin{pmatrix} c & c\sqrt{a} + dn \\ d & d\sqrt{a} + d + c \end{pmatrix}\right) = (1-q) \sum_{\substack{v \in GF(q) \\ v \neq 1}} \chi^j(v)\psi\left(\frac{av}{v^2+1}\right). \quad (4.21)$$

By [9, p. 70], the terms on the left of (4.21) with $d = 0$ contribute

$$N_0 := \sum_{c \in GF(q)^*} 1 = q - 1. \quad (4.22)$$

Let N_1 denote the sum of those terms on the left of (4.21) with $d \neq 0$. By [9, p. 70], there is no contribution to the left sum in (4.21) when the two eigenvalues of the matrix are distinct elements of $GF(q^2) - GF(q)$. Thus, as in the proof of Theorem 4.1, instead of summing over all $c, d \in GF(q)$ with $d \neq 0$ in (4.21) to evaluate N_1, we sum twice over all variables $N, T \in GF(q)$ with $T \neq 0$ and

$$\operatorname{Tr}\left(\frac{N(a+1)}{T^2}\right) = 1, \qquad \operatorname{Tr}\left(\frac{N}{T^2}\right) = 0. \quad (4.23)$$

Therefore, by [9, p. 70],

$$N_1 = \sum_{\substack{N,T\in GF(q),T\neq 0\\ \mathrm{Tr}(Na/T^2)=1,\mathrm{Tr}(N/T^2)=0}} 2\left(\chi^j\left(\frac{r}{r+T}\right) + \chi^j\left(\frac{r+T}{r}\right)\right), \tag{4.24}$$

where $r = r(N, T) \in GF(q)^*$ is a zero of the reducible polynomial

$$x^2 + Tx + N \in GF(q)[x] \tag{4.25}$$

(so the other zero is $r + T$). By (4.24),

$$N_1 = \sum_{\substack{T,r\in GF(q)^*\\ \mathrm{Tr}(a(r^2+rT)/T^2)=1}} 2\chi^j\left(\frac{r}{r+T}\right). \tag{4.26}$$

Replace the variable r by rT to obtain

$$N_1 = 2\sum_{\substack{r,T\in GF(q)^*\\ \mathrm{Tr}(ar+ar^2)=1}} \chi^j\left(\frac{r}{r+1}\right). \tag{4.27}$$

Make the change of variables $v = r/(r + 1)$, so $r = v/(v + 1)$. Then

$$N_1 = 2(q-1)\sum_{\substack{v\in GF(q),v\neq 1\\ \mathrm{Tr}(av/(v^2+1))=1}} \chi^j(v) = (q-1)\sum_{\substack{v\in GF(q)\\ v\neq 1}} \chi^j(v)\left(1 - \psi\left(\frac{av}{v^2+1}\right)\right). \tag{4.28}$$

Adding (4.22) and (4.28), we obtain the desired result (4.21). ■

5. EIGENFUNCTIONS OF A_a

In this section, we show that the $q^2 - q$ functions in (1.17)–(1.19) comprise a common basis in $L(G)$ of mutually orthogonal (nonzero) eigenfunctions for all the adjacency operators A_a, $a \neq 0, 1$.

THEOREM 5.1. *Let $a \in GF(q)$, $a \neq 0, 1$. The sums $K_{j,b}$, $H_{t,j,b}$ in* (1.13), (1.14) *are eigenfunctions of A_a on G with eigenvalues $(q + 1)B_j(S_a)$, $(q + 1)C_j(S_a)$, respectively.*

Proof. By (1.14) and (4.1), for all $g \in G$ and all $t \in GF(q)^*$, we have

$$H_{t,j,b}(g) = \sum_{u \in GF(q)} \psi(bu)C_j(g_u g), \tag{5.1}$$

where $g_u = \begin{pmatrix} 0 & t \\ 1 & u \end{pmatrix} \in \mathcal{G}$. By (1.13), for all $g \in G$,

$$K_{j,b}(g) = \sum_{u \in GF(q)} \psi(bu)\chi^j(g_u g), \tag{5.2}$$

where $g_u = \begin{pmatrix} 0 & 1 \\ 1 & u \end{pmatrix} \in \mathcal{G}$. The proof is now completed with the use of Theorem 3.2, in a way completely analogous to the proof of [3, Thm. 3.1]. ■

It remains to show that the eigenfunctions in (1.17)–(1.19) form a mutually orthogonal basis for $L(G)$. We begin by showing in Theorem 5.2 that the functions in (1.17) may indeed be taken to be nonzero on G.

THEOREM 5.2. *Let $b \in GF(q)^*$, $1 \leq j \leq q/2$. Then there exists $t \in GF(q)^*$ for which $H_{t,j,b}$ is not identically zero on G.*

Proof. If $H_{t,j,b}$ vanishes on G for every $t \in GF(q)^*$, then replacing y by yt and u by ut in (1.14), we see that $H_{1,j,b}$ vanishes on G for *every* $b \in GF(q)^*$. By summing in (5.1) over all $b \in GF(q)^*$ with $t = 1$ and $g = \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}$, it would follow that

$$\sum_{u \in GF(q)} C_j\left(\begin{pmatrix} 0 & 1 \\ y & uy \end{pmatrix}\right) = qC_j\left(\begin{pmatrix} 0 & 1 \\ y & 0 \end{pmatrix}\right), \tag{5.3}$$

for each $y \in GF(q)^*$. Fix $y = 1/n$. Then

$$\Delta\left(\begin{pmatrix} 0 & 1 \\ y & uy \end{pmatrix}, I\right) = (u + 1)^2. \tag{5.4}$$

Therefore, the terms for $u = 1$ and $u = 0$ in (5.3) are 1 and -1, respectively, by (3.9) and (3.17). Thus (5.3) becomes

$$L := \sum_{u \neq 0,1} C_j\left(\begin{pmatrix} 0 & 1 \\ y & uy \end{pmatrix}\right) = -q. \tag{5.5}$$

We will obtain a contradiction by showing that $L = 0$.

By (4.1) and (5.4),

$$\begin{aligned} L &= \frac{-1}{q+1} \sum_{u \neq 0,1} \sum_{\substack{N(\alpha)=1 \\ \alpha \neq 1}} \omega^j(\alpha) \psi\left(\frac{u^2\alpha}{\alpha^2+1}\right) \\ &= \frac{-1}{q+1} \sum_{\substack{N(\alpha)=1 \\ \alpha \neq 1}} \omega^j(\alpha) \sum_{u \in GF(q)} \psi\left(\frac{u^2\alpha}{\alpha^2+1}\right), \end{aligned} \tag{5.6}$$

in view of (4.3). The last sum on u in (5.6) vanishes, so $L = 0$. ∎

We now show that the functions $K_{j,b}$ in (1.18)–(1.19) are nonzero on G. If $j = 0 = b$, this is clear, for then $K_{j,b} \equiv q$ by (1.13). The remaining cases are addressed in the following theorem.

THEOREM 5.3. *If $1 \leq j < q - 1$ and $b \in GF(q)$, then $K_{j,b}$ is not identically zero on G.*

Proof. By (5.2), we have for all $b \in GF(q)$ and all $g = \left(\begin{smallmatrix} y & x \\ 0 & 1 \end{smallmatrix}\right) \in G$,

$$\psi(bx)K_{j,b}(g) = \sum_u \psi(bu)\chi^j\left(\begin{pmatrix} 0 & 1 \\ y & u \end{pmatrix}\right). \tag{5.7}$$

Suppose for the purpose of contradiction that $K_{j,b}$ is identically zero on G. If $b = 0$, then by (5.7),

$$\sum_u \chi^j\left(\begin{pmatrix} 0 & 1 \\ y & u \end{pmatrix}\right) = 0 \qquad \text{for all } y \in GF(q)^*. \tag{5.8}$$

On the other hand, if $b \neq 0$, then $K_{j,b}$ is identically zero on G for *every* $b \in GF(q)^*$ by (1.13), so by summing over $b \in GF(q)^*$ in (5.7), we obtain

$$\sum_{u \in GF(q)} \chi^j\left(\begin{pmatrix} 0 & 1 \\ y & u \end{pmatrix}\right) = q\chi^j\left(\begin{pmatrix} 0 & 1 \\ y & 0 \end{pmatrix}\right) \qquad \text{for all } y \neq 0. \tag{5.9}$$

The left side of (5.9) is $K_{j,0}(g)$ by (5.7), so we deduce from (5.8)–(5.9) that

$$|K_{j,0}(g)| \in \{0, q\} \qquad \text{for all } y \neq 0. \tag{5.10}$$

Fix $y = 1$. Then by (1.13),

$$|K_{j,0}(g)| = \left|\sum_{u \in GF(q)} \chi^j(u^2 + u + n)\right|. \tag{5.11}$$

Let N denote the norm map from $GF(q^2)$ to $GF(q)$. Then $\lambda = \chi^j \circ N$ is a nontrivial character on $GF(q^2)$, since $0 < j < q - 1$. Let E denote the sum on the right side of (5.11). Then

$$E = \sum_{u \in GF(q)} \lambda(u + \theta). \tag{5.12}$$

Since the argument of λ in (5.12) runs through all of the elements in $GF(q^2)$ whose trace down to $GF(q)$ equals 1, E is an Eisenstein sum. By [8, p. 264],

$$E = G_2(\lambda)/G_1(\lambda), \tag{5.13}$$

where G_1 and G_2 are Gauss sums defined by

$$G_1(\lambda) = \sum_{u \in GF(q)} \lambda(u)\psi(u), \tag{5.14}$$

$$G_2(\lambda) = \sum_{z \in GF(q^2)} \lambda(z)\Psi(z); \tag{5.15}$$

here $\Psi(z)$ is the additive character on $GF(q^2)$ defined by

$$\Psi(z) = (-1)^{\mathrm{Tr}(z)}.$$

The Gauss sums in (5.14), (5.15) are nontrivial and thus have absolute values $\sqrt{q}$, q, respectively; see [8, p. 193]. Thus $|E| = \sqrt{q}$, which contradicts (5.10). ∎

The following theorem completes the proof that the eigenfunctions (1.17)–(1.19) form a mutually orthogonal basis for $L(G)$. The proof is completely analogous to that of [3, Thm. 3.4].

THEOREM 5.4. *The* $q^2 - q$ *(nonzero) eigenfunctions* $K_{j,b}$, $H_{i,j,b}$ *in* (1.17)–(1.19) *are pairwise orthogonal on* G.

REFERENCES

1. J. Angel, Finite upper half planes over finite fields, *Finite Fields Appl.*
2. J. Angel, B. Shook, A. Terras, and C. Trimble, Graph spectra for finite upper half planes over rings, *Linear Algebra Appl.*, to appear.
3. R. Evans, Character sums as orthogonal eigenfunctions of adjacency operators for Cayley graphs, *in* "Contemporary Mathematics," Vol. 168, Amer. Math. Soc., Providence, RI, 1994.

4. N. Katz, Estimates for Soto-Andrade sums, *J. Reine Angew. Math.* **438** (1993), 143–161.

5. N. Katz, A note on exponential sums, *Finite Fields Appl.* **1** (1995), 395–398.

6. J. Kuang, A. natural orthogonal basis of eigenfunctions of the Hecke algebra acting on Cayley graphs, *Proc. Amer. Math. Soc.*, to appear.

7. W. Li, Character sums and abelian Ramanujan graphs, *J. Number Theory* **41** (1992), 199–217.

7a. W. Li, Some character sum estimates, preprint.

8. H. Niederreiter and R. Lidl, "Finite Fields," Addison-Wesley, Reading, 1983.

9. I. Piatetski-Shapiro, Complex representations of GL(2, K) for finite fields K, *in* "Contemporary Mathematics," Vol. 16, Amer. Math. Soc. Providence, RI, 1983.

10. J. Soto-Andrade, Geometrical Gel'fand models, tensor quotients, and Weil representations, *in* "Proceedings of Symposia in Pure Mathematics," Vol 47, pp. 305–316, Amer. Math. Soc., Providence, RI, 1987.

11. A. Terras, Are finite upper half plane graphs Ramanujan? DIMACS Ser. Discrete Math. Theoret. Comp. Sci. **10** (1993), 125–142.