$\boxed{\text{NC hierarchy}}$, "Nick's class" – named by Steve Cook

NC^i = {languages L recognizable by circuits of polynomial size and depth $O(\log^i n)$ w/ 2-input $\wedge, \vee, \neg$}

$i \geq 1$

Uniform NC^1 = Alternating logarithmic time

(uniform) $NC^1 \subseteq L \subseteq NL \subseteq NC^2 \subseteq NC^3 \subseteq \cdots \subseteq NC \subseteq P$

$\bigcup NC^i = NC$

(Circuit model for polylogarithmic parallel time)

$\boxed{\text{AC-hierarchy}}$; "A" = alternating.

AC^i = {languages L recognizable by unbounded fanin circuits of polynomial size and depth $O(\log^i n)$}

AC^0 = constant depth polynomial size circuits.

(uniform) $AC^0 \subseteq$ (uniform NC^1) $\subseteq L \subseteq NL \subseteq AC^1 \subseteq NC^2 \subseteq AC^2 \subseteq NC^3 \subseteq \cdots$

$AC = \bigcup AC^i$

so $NC = AC$.

$\boxed{ACC^i[m]}$ – similar to AC^i but also w/ unbounded fanin mod m gates

SC hierarchy "SC" = "Steve's class"
- named by Nick Pippenger

Remark: that TM's have "direct access" to their input tape

$SC^i = \{$ languages L recognizable by a Turing machine that uses polynomial time and $O(\log^i n)$ space$\}$.

$SC = \bigcup_i SC^i$ - simultaneous poly time + poly log space

$= TISP(n^{O(1)}, (\log n)^{O(1)})$.

<u>Open question</u>: Is directed st-connectivity, (STCON), in SC?

<u>Note</u> STCON $\in$ P (by Depth-first search) (or Breadth-first search, etc)

and STCON $\in$ SPACE$((\log n)^{O(1)})$ by Savitch's Theorem.

<u>Defin</u> L = LOGSPACE = SPACE($\cancel{O(}\log n$)

NL = Nondeterministic LogSpace) = NSPACE($\log n$).

<u>Remark</u> An NL machine has $2^{O(\log n)} = n^{O(1)}$ time, and thus this many ~~no~~ nondeterministic choices.
This is too much to remember!

<u>Thm</u> [Savitch] $NL \subseteq SPACE((\cancel{O}\log n)^2)$.

Thm: STCON is many-one complete for NL.

Pf STCON $\in$ NL by ~~an~~ nondeterministically picking the path
STCON is many-one complete for NL by:
(a) Add a clock to an NL machine, so graph is a dag ~~direct~~
(b) Nodes of graph are configurations,
edges correspond to a single step (directed edges)
(c) Wlog the NL-machine has a unique accepting configuration

q.e.d.

Open Question (rephrased) Is NL $\subseteq$ SC ?

Optional: Mention [BBRS] here

Definition USTCON, undirected st connectivity
Instance An undirected graph G, and nodes s, t.
Output : Is there a path in G from s to t?

Definition SL = { L : L $\leq_m$ USTCON by a logspace many-one reduction }.
SL: Symmetric Logspace = Sym-L [Lewis+Papadimitriou '82]

Note L would correspond to "out-degree 1 STCON".

Thm [Reingold '05] L=SL.

Corollary [Nisan+Ta-Shma '94] SL=coSL

RL aka RLP = {languages L accepted by probabilistic Turing machines that use log space and polynomial time and are accepted w/ 1-sided error}

i.e. $x \in L \Rightarrow \text{Prob}[M \text{ accepts } x] \geq 1/2$ (or $1-2^{-n}$)
$x \notin L \Rightarrow \text{Prob}[M \text{ accepts } x] = 0$.

Without the restriction of polynomial time, just get NL.

Thm [easy] $RL \subseteq NL$

(*) ⇒ Thm [Nisan '92] $RL \subseteq SC$. (In fact $RL \subseteq TISP(n^{O(1)}, (\log n)^2)$)

Natural Conjecture (?): $L = RL$.

Note again that an RL machine can make more random choices than it can remember.

(*) ⇒ Thm [Nisan '92] $BPL \subseteq SC$
where

Defn: BPL is defined like RL, but with two sided error

i.e. $x \in L \Rightarrow \text{Prob}[M \text{ accepts } x] > 2/3$
$x \notin L \Rightarrow \text{Prob}[M \text{ accepts } x] < 1/3$
} as usual, can amplify.

~~Nisan proposed a derandomization of~~

Remark $RL \subseteq BPL$

~~Thm~~

Nisan also obtains

Thm [Nisan '92] $USTCON \in TISP(n^{O(1)}, (\log n)^2)$

a result which was superceded by Reingold '05 as mentioned above. This used a derandomization of

Thm [Aleliunas - Karp - Lipton - Lovasz, Rackoff '79]
$USTCON \subseteq RL$.

$$L = SL \subseteq RL \subseteq NL \begin{array}{l} \subseteq SC \subseteq P \quad \cdots \quad \\ \subseteq AC^1 \subseteq NC^2 \cdots \quad \subseteq P \end{array}$$

~~By the way: Thm SC ≠ P~~
~~Pf P has a language e.g. (CVP) which is complete under logspace many-one logspace reductions.~~

By the way Thm PolylogSpace $\neq P$
Pf P has a language many-one complete under logspace reductions.
By, the space hierarchy, Polylog Space does not.
q.e.d.!

(*) ⟹ Thm [Barnes, J. Buss, Ruzzo, Schieber '95] $STCON \in TISP(n^{O(1)}, \frac{n}{2^{\Theta(\sqrt{\log n})}})$

Recall STCON is many-one complete for NL.
Also recall $NL \subseteq SC$? is open.

(*) ⟹ Thm [Saks - Zhou '99], $RL \subseteq SPACE(n^{3/2})$, $BPL \subseteq SPACE(n^{3/2})$
- Best ~~space~~ space bound for deterministic simulation of RL.

To repeat

Thm [Saks-Zhou '99] $RL \subseteq SPACE(n^{3/2})$; $BPL \subseteq SPACE(n^{3/2})$

This is the best purely space bound for deterministic simulation of RL / BPL.

$$L \subseteq SL \subseteq RL \subseteq BPL \subseteq SC$$
$$\subseteq NL \subseteq AC^1 \subseteq NC^2 \subseteq \subseteq P$$

Derandomization of RL / BPL: [Nisan '92]

A RL/BPL algorithm can be simulated by a logspace algorithm that uses only $O((\log n)^2)$ many random bits.

Thm [Immerman '88 - Szelepcsenyi '87] $NL = coNL$

Combinatorial Tangle

INW

Reversible Computation

Def'n A TM is reversible iff each configuration has a unique predecessor configuration.

st-connectivity where each node has a unique incoming edge + unique outgoing edge – is the corresponding graph property.

Thm [Bennett '89] $TISP(T, S) \subseteq Rev\text{-}TISP(T^{1+\varepsilon}, S\log T)$

Corollary LogSpace $\subseteq$ Rev-SPACE($\log^2 n$).

Thm [Lange-McKenzie-Tapp '98] $SPACE(S) \subseteq Rev\text{-}SPACE(S)$

Corollary LogSpace $\subseteq$ Rev-LogSpace

Corollary [Crescenzi-Papadimitriou '95] $NSPACE(S) \subseteq Rev\text{-}SPACE(S^2)$