AKLLR Aleliunas, Karp, Lipton, Lovász, Rackoff, STOC '79, pp 218-223.

Setting Let G be an undirected graph, $G=(V,E)$, $V=\{1,\dots,n\}$.

Let $d(i)$ = degree of vertex i. $m = |E| = \#$ of edges

A random walk starts at vertex i_0.

When at vertex i, transitions to a randomly chosen neighbor j of i.

Transition probability

$$p_{ij} = \begin{cases} 1/d(i) & \text{if } (i,j) \in G \\ 0 & \text{otherwise.} \end{cases}$$

Transition matrix $P = (p_{ij})_{ij}$ - $n \times n$ matrix.

P has row sums equal to 1, a stochastic matrix.

Defn: The hitting time (aka mean first passage time) h_{ij} is the expected time for a random walk starting at vertex i to reach vertex j.

C_i := expected time for a random walk starting at i to reach every vertex of G at least once

The cover time $C := \max_i C_i$.

The commute time $C_{ij} := h_{ij} + h_{ji}$ is the expected time for random walk, starting at vertex i to reach vertex j and return to vertex i.

We assume G is connected so these values are well-defined and finite.

Goal: Give good upper bounds on these quantities.

A <u>probability distribution</u> τ is a function $\tau : [n] \to [0,1]$ s.t. $\sum_i \tau(i) = 1$.

A <u>stationary distribution</u> is a distribution π s.t. $\pi = \pi P$.

<u>Intuition</u>: If τ = Prob distribution of being at vertex i at time t,
then τP = " " " " " " $i+1$ at time $t+1$.

<u>Defn</u>: Let $\pi(i) = \left\{ \frac{d(i)}{2m} \right.$.

<u>Thm</u> $\pi = (\pi(1), \dots, \pi(n))$ is the (unique) stationary distribution.

<u>Pf</u> (a) π is stationary:

$$(\pi P)_i = \sum_j P_{ji} \pi_j = \sum_{j : i \in N(j)} P_{ji} \pi_j = \sum_{j : i \in N(j)} \frac{d(j)}{2m} \frac{1}{d(j)}$$

$$= \sum_{j \in N(i)} \frac{1}{2m} = |N(i)| \cdot \frac{1}{2m} = \frac{d(i)}{2m} = \pi_i$$

where $N(i) = \{ j : (i,j) \in E \}$.

(b) $\sum_i \pi_i = \sum_i \frac{d(i)}{2m} = \frac{(\# \text{ of edge ends})}{2m} = \frac{2m}{2m} = 1$.

(c) π is unique:

Let τ be another stationary distribution.

Define $r_i \doteq \frac{\tau(i)}{\pi(i)}$. Let $V' = \{ i : r_i = \max_j \{ r_j \} \}$. Note $r_i > 1$.

Since G is connected and since $\sum \tau_i = 1$, there is an i_0

s.t. $r_i = \max\{ r_j \}$ but not all i's neighbours satisfy this property.

This gives a contradiction #

q.e.d.

Fundamental Theorem: Let π be the stationary distribution.

(a) $h_{ii} = 1/\pi_i$

(b) π_i gives the expected frequency of visits to vertex i, i.e. for any initial distribution τ, let $N_\tau(i,t)$ = expected # of visits to vertex i in t steps, then

$$\lim_{t\to\infty} \frac{N_\tau(i,t)}{t} = \pi_i .$$

Pf (sort-of)

(a) Let $H = (h_{ij})_{ij}$ an $n\times n$ matrix.

Note $h_{ij} = 1\cdot p_{ij} + \sum_{k\neq j} p_{ik}(h_{kj}+1)$

Claim: $H = P(H - H_{diag}) + E$

where $H_{diag} = (h_{ij}\delta_{ij})_{ij}$ - δ_{ij} - delta function, H_{diag} - diagonal

$E = (1)_{ij}$ - all 1's matrix.

Pf: $h_{ij} = \sum_{k\neq j} p_{ik}(h_{kj}+1) + p_{ij} = \sum_{k\neq j} p_{ik} h_{kj} + 1$

$$= \sum_k p_{ik} h_{kj} - p_{ij} h_{jj} + 1$$

So $H = PH - P\cdot H_{diag} + E$ qed Claim

Thus $\pi H = \pi P(H - H_{diag}) + \pi E$

$$= \pi(H - H_{diag}) + \pi E = \pi(H - H_{diag}) + \vec{1}$$

where $\vec{1}$ = all 1's row vector

So $\pi\cdot H_{diag} = \vec{1}$, i.e.

$\pi_i \cdot h_{ii} = 1$, so $h_{i,i} = 1/\pi_i$.

(b): By linearity of expectation.

$$N_\tau(i,t) = \tau + \tau P + \tau P^2 + \tau P^3 + \cdots$$
$$= \tau(I + P + P^2 + P^3 + \cdots)$$

and $\lim_{t\to\infty} \frac{\tau(I + P + P^2 + \cdots P^{t-1})}{t}$ is a stationary distribution

and hence equal to π. (Omit rest of proof)

QED.

Corollary: The expected frequency of transitions from vertex i to vertex j along (directed) edge (i,j) equals

$$\pi_i \cdot P_{ij} = \frac{d(i)}{2m} \cdot \frac{1}{d(i)} = \frac{1}{2m}$$

Thus all directed edges are traversed w/ the same expected frequency.

Lemma ~~If~~ there is an edge from i to j,

$$h_{ij} + h_{ji} \le 2m$$

Pf. By linearity, $h_{ij} + h_{ji} = \sum_{\substack{(i',j') \\ \text{a directed} \\ \text{edge of } G}}$ (expected # of occurrences of transition from i' to j' in a commute from i to j + back to i).

By corollary, and since any infinite random walk is made up of i-j-i commutes (with probability 1), each term of the summation is equal. Then

$$h_{ij} + h_{ji} = 2m\,(\text{expected \# of transitions from } i' \text{ to } j' \text{ in an } i\text{-}j\text{-}i \text{ commute})$$
~~A~~ all $(i',j') \in E$,

Since $(i'=i,\ j'=j)$, edge (i,j) is transitioned at most once

$$h_{ij} + h_{ji} \le 2m$$

q.e.d.

Theorem The cover time C is $\leq 2m(n-1) \leq 2n^3$.

Pf For i a vertex, need to show $C_i \leq 2m(n-1)$.

Let T be a spanning tree of G with root i.

T has $n-1$ edges.

Do a depth-first traversal of T as $i = i_0, i_1, i_2, \ldots, i_{2n-2}$
when i_{2n-2} is the last child of i.

By the lemma, the cover time C_i satisfies

$$C_i \leq T(i_0, i_1) + T(i_1, i_2) + \cdots + T(i_{2n-3}, i_{2n-2})$$

$$\leq \sum_{(i',j') \in T} \left(T(i',j') + T(j',i') \right)$$ (by reordering terms — each edge traversed twice)

$$\leq (n-1)\, 2m$$

q.e.d.

Defn RLP = set of languages L for which there is a probabilistic TM M s.t.
M uses log space and polynomial time and, for all x,

(☆) $x \in L \Rightarrow M(x)$ accepts with probability ≥ 2

$x \notin L \Rightarrow M(x)$ does not accept. (i.e. accepts w/ probability $\emptyset$).

[No longer assume G is connected.]

Defn USTCON is the decision problem

$(G, s, t) \in$ USTCON $\Leftrightarrow$ $\exists$ path from s to t in G

$\Leftrightarrow$ s, t in the same connected component of G.

Theorem: USTCON $\in$ RLP.

Pf: Algorithm: Starting from s, do random walk for $4n^3$ steps
Accept iff vertex t is encountered.

By Markov inequality, $(s,t,G) \in$ USTCON $\Rightarrow$ Prob[M accepts] $\geq 1/2$

Let G be a d-regular graph. A walk on G is specified by its initial vertex and a sequence $\sigma \in \{0, \ldots d-1\}^{\ell}$ where σ_i is the choice of outgoing edge chosen at step i.

Defn σ is n-universal iff $\forall$ ~~connected~~ graphs G, all s, t in same connected component, the walk specified by initial vertex s and by σ contains t.

Intuition: "derandomization"

Theorem For all d, there is an n-universal sequence of length $O(n^3 \log n)$

Pf The cover time for a d-regular graph is $\le 2dn(n-1) \le 2dn^2 - 1$

The total number of d-regular graphs is $\le \binom{n}{d}^n \le (n^d)^n = 2^{dn \cdot \log n}$.

~~Hence there is a collection of $\le dn \cdot \log n + 1$ many traversal sequences of length $4dn^2$ s.t. every d-regular graph~~ ~~and any given choice of initial vertex~~ ~~is covered by at least one traversal sequence in the collection.~~

~~The concatenation of these~~

~~Correction: $dn \log n + \log n + 1$ since have to take the starting vertex into account.~~

~~$n \cdot \binom{n}{d}^n$ many graphs G with designated start vertex s~~

Above argument doesn't quite work.

Instead Consider a random choice of $dn \log n + 1$ many traversal sequences and concatenate them.

For a fixed G, probability that this does not traverse G is $\le \left(\frac{1}{2}\right)^{dn \log n + 1}$

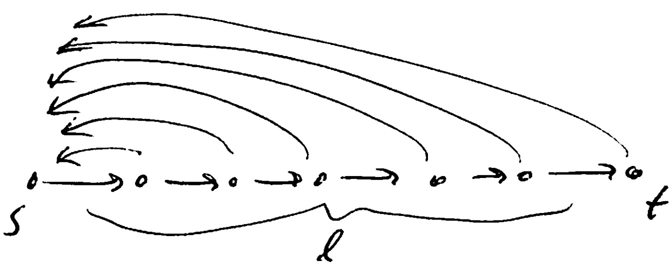
Since there are $\le 2^{dn \log n}$ many G's, the probability it does not traverse all of them is $\le \left(\frac{1}{2}\right)^{dn \log n + 1} \cdot 2^{dn \log} \le 1/2$.

QED

Directed graphs

AKLLR - show: Similar results for strongly connected, directed graphs which have indegree d and outdegree d at each vertex.

For non-fixed degree, here is an example of how things can fail.



Probability of reaching t from s before returning to s is $2^{-\ell}$.
Hence random walk needs time $\approx 2^{-\ell}$ to reach t from s.

Worse, the graph might not be strongly connected (hence not ergodic).

For this case consider following algorithm

Input. G, s, t - n vertices

Loop: Starting at s, do random walk of length n
If encounter t, accept.
Else, continue.

This has (worst-case) exponential runtime if ~~the~~ t is reachable from s.

If t not reachable from s, it never halts.

To make it halt, want to run n^n many ~~steps~~ times thru the loop.
Then reject if t is not encountered.
This makes a log space, exponential time algorithm that satisfies conditions ☆ for $x \in L$.

Problem Counting to n^n takes space $\log n^n = n \log n$, not log space.

Fix: Instead of counting to n^n, instead select (and store) $\log_2(n^n)$ many random bits. If they are all equal to 1, halt (and reject). Else continue

New algorithm

Loop

Starting at s, do random walk of length n

If encounter t, accept

Flip $n \log_2(n)$ many coins,

if all equal heads, reject.

Else continue.

This does satisfy condition (☆):

$x \in L \Rightarrow \text{Prob}[M(x) \text{ accepts}] \geq \frac{1}{2}$

$x \notin L \Rightarrow \text{Prob.}[M(x) \text{ accepts}] = 0.$