Expanders + Eigenvalues - lecture by S. Buss 2/19/2013

<u>Sources:</u> [Arora - Barak] Chapter 21.

S. Hoory, N. Linial, A. Wigderson, "Expander Graphs and their Applications", ~~Bulletin~~ Bulletin AMS 43,4 (2006) 439-561.

I. <u>Expander Graphs</u>

We'll work with graphs that are <u>undirected</u>, d-regular, and may contain self-loops (and multiple edges).

When we work w/ families of graphs $\{G_i\}_i$, d is constant.

<u>Def'n</u>: Let S, T be (disjoint) subsets of the vertices V of $G=(V,E)$.

$E(S,T) = \{(u,v)\in E : u\in S, v\in T\}$

= set of edges from vertices in S to vertices in T.

$\partial S = E(S,\bar{S})$, the <u>edge boundary</u> of S in G.

<u>Def'n</u> An n-vertex, ~~d-edge~~ d-regular $G=(V,E)$ is an (n,d,ρ)-<u>edge expander</u> iff $\forall S \subseteq V$, $|S| \le n/2$,

$$|\partial S| = |E(S,\bar{S})| \ge \rho d |S|$$

A family of graphs $\{G_i\}_i$ are (n,d,ρ)-edge expanders if each member G_i is an (n,d,ρ)-edge expander.

The value ρd is called the <u>Cheeger constant</u>.

A probabilistic construction (of randomly chosen d-regular graphs) gives:

<u>Thm</u> For any $\rho = \frac{1}{2}-\varepsilon$, $\varepsilon>0$, $\exists d=d(\varepsilon)$, $\exists n_0$ $\forall n>n_0$ there is a (n,d,ρ)-edge expander.

<u>Pf</u> omitted.

Explicit example of expanders

[Margulis '73] ⓐ For $m \geq 1$. $G_m = \mathbb{Z}_m \times \mathbb{Z}_m$, 8-regular with edges connecting vertex (i,j) to vertices

$$\left.\begin{array}{l}(i+j, j), (i-j, j), (i+j+1, j), (i-j+1, j) \\ (i, j+i), (i, j-i), (i, j+i+1), (i, j-i+1)\end{array}\right\} \bmod m$$

ⓑ For p prime, $G_p = \mathbb{Z}_p$ with ~~some~~ edges connecting $x \in \mathbb{Z}_p$ to $x+1$, $x-1$, x^{-1} mod p $(0^{-1} = 0)$.

II. Adjacency Matrix / Random Walk Matrix

Let $G = (V, E)$ be d-regular & have n vertices (as above)

Def'n ~~the~~ The adjacency matrix $Adj(G)$ is the $n \times n$ matrix whose entries adj_{ij} = # of edges between i & j.

The random walk matrix $A(G) = \frac{1}{d} Adj(G)$.

(This is the same as the transition matrix defined for the AKLLR lecture)

Lemma: $A(G)$ is symmetric + doubly stochastic (its row sums and column sums all equal one).

Pf: Immediate from the definitions.

Lemma: The all 1's vector $\vec{1} = (1, 1, \ldots 1)^T$ is an eigenvector for $A = A(G)$. That $A\vec{1} = \vec{1}$.

Pf: Since the row sums of A equal one.

Some linear algebra facts + notation

(1) As a symmetric real matrix, A has a system of n orthonormal eigenvectors $\vec{v}^1, \ldots \vec{v}^n$, with eigenvalues $\lambda_1, \ldots, \lambda_n$.

So $A\vec{v}^i = \lambda_i \vec{v}^i$; $(\vec{v}^i)^T A = \lambda_i \vec{v}^{i\,T}$ (by symmetry)

(2) Define $\langle \vec{u}, \vec{v} \rangle = \vec{u}^T \vec{v} = \vec{v}^T \vec{u} = \sum u_i v_i$

$\|u\|^2 = \langle u, u \rangle = \sum_i u_i^2$ - the 2-norm of u.

$\|u\|_1 = \sum_i |u_i|$

$\|u\| = \|u\|_2 = \sqrt{\|u\|^2}$.

(3) $\frac{\|v\|_1}{\sqrt{n}} \leq \|v\| \leq \|v\|_1$

Lemma $|\lambda_i| \leq 1$ for every eigenvalue λ_i of A.

Pf: Let j be the max component of $\vec{v}_i$ in absolute value.

I.e. $|\vec{v}^i_j| \geq |\vec{v}^i_{j'}|$ for all j'.

Consider $(A\vec{v}^i)_j$ = jth component of $A\vec{v}^i$

This value is a linear (~~affine~~) combination of values (with absolute value $\leq |\vec{v}^i_j|$), namely a weighted average.

Thus $|A\vec{v}^i_j| \leq |\vec{v}^i_j|$, proving $|\lambda_i| \leq 1$.

qed.

Examples Disconnected graph can have multiple eigenvalues equal to 1.

Bipartite graph can have -1 as an eigenvalue.

Order the eigenvectors/eigenvalues so that $\vec{v}^1 = \vec{1}$, $\lambda_1 = 1$, and

$$|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|.$$

<u>Def'n</u> $\lambda(G) = |\lambda_2|$ is the second largest eigenvalue of the matrix $A = A(G)$. We often write $\lambda = \lambda(G)$. The <u>spectral gap</u> of G is $1 - \lambda(G)$.

<u>Remark</u> Many authors use $Adj(G)$ instead of $A(G)$, and then λ and the spectral gap are multiplied by a factor of d.

<u>Def'n</u> G is a (n, d, λ) expander graph if G has n vertices, is ~~d~~-regular and $\lambda(G) \leq \lambda$.

<u>Remark</u> λ as small as $(1 - o(1)) \frac{2\sqrt{d-1}}{d}$ is possible (the Alon-Boppana bound) and can be achieved by Ramanujan graphs.

<u>Theme</u> Good (n, d, ρ) edge-expander graphs, with ρ large correspond to

Good (n, d, λ) expander graphs, with λ small and hence the spectral gap big.

Theorem Let G be an (n, d, λ)-expander. Then G is an $(n, d, (1-\lambda)/2)$ edge expander.

Lemma: Let $S, T \subseteq V$, $G = (V, E)$ as above. ~~Then~~ Suppose $S \cap T = \emptyset$. ~~Then~~ and $S \cup T = V$. $\boxed{\text{I.e. } \overline{S} = T,}$

$$|E(S,T)| \geq (1-\lambda)\, d \, \frac{|S| \cdot |T|}{|S| + |T|}$$

Pf Let $\vec{u}$ be the vector:

$$u_i = \begin{cases} |T| & \text{if } i \in S \\ -|S| & \text{if } i \in T. \end{cases}$$

So $\vec{u} \perp \vec{1}$ and

$$\|u\|^2 = |T|^2 \cdot |S| + |S|^2 \cdot |T| = |S| |T| \cdot (|S| + |T|).$$

~~D~~Define

$$Z = \sum_{ij} A_{ij} (u_i - u_j)^2$$

So

$$Z = \sum_{i,j : (i,j) \in G} \frac{1}{d} (u_i - u_j)^2$$

$$= \sum_{i,j : (i,j) \in G} \frac{1}{d} (|S| + |T|)^2$$

$$= \frac{2}{d} |E(S,T)| \cdot (|S| + |T|)^2.$$

Sublemma

Also,

$$Z = \sum_{ij} A_{ij} u_i^2 - 2 \sum_{ij} A_{ij} u_i u_j + \sum_{ij} A_{ij} u_j^2$$

$$= 2 \sum_i \Big(\sum_j A_{ij}\Big) u_i^2 - 2 \langle \vec{u}, A\vec{u} \rangle$$

$$= 2 \sum_i u_i^2 - 2 \langle \vec{u}, A\vec{u} \rangle$$

$$= \|u\|^2 - 2 \langle \vec{u}, A\vec{u} \rangle \geq 2\|u\|^2 - 2\lambda \|u\|^2$$

$$\geq (2 - 2\lambda) |S| \cdot |T| (|S| + |T|) = 2(1-\lambda) |S| \cdot |T| (|S| + |T|)$$

since $\langle \vec{u}, A\vec{u} \rangle \leq \lambda \cdot \|u\|^2$

States sublemma $Z = \|\vec{u}\|^2 - 2\langle \vec{u}, A\vec{u} \rangle$.

And if $Z' = \sum_{ij} A_{ij} (u_i + u_j)^2$

Then $Z' = \|u\|^2 + 2 \langle \vec{u}, A\vec{u} \rangle$.

The lemma follows immediately.
qed Lemma.

Proof of Theorem: By the lemma + since $|S| \leq |\bar{S}| = |T|$,

$$\frac{|E(S,T)|}{|S|} \geq (1-\lambda) d \frac{|T|}{|S|+|T|} \geq \frac{1-\lambda}{2} \cdot d$$

giving $\rho \geq \frac{1-\lambda}{2}$ as desired.
Q.E.D.

Theorem: Suppose G is a (n,d,ρ) edge expander.
(a) Then $\lambda_2 \leq 1 - \rho^2/2$.
(b) If G has all (n many) self loops, then
G is a $(n, d, 1-\varepsilon)$-expander for $\varepsilon = \min\left\{\frac{2}{d}, \frac{\rho^2}{2}\right\}$.

Proof of (b); assuming (a) holds.
Let G' be G minus all its self loops. Let A' be G''s random walk matrix.
Then $A = \frac{d-1}{d} A' + \frac{1}{d}(Id)$

A''s smallest eigenvalue is ≥ -1.
Thus A's smallest eigenvalue is $\geq -\frac{d-1}{d} + \frac{1}{d} = -1 + 2/d$.
Then by part (a),
$$-1 + 2/d \leq \lambda_2 \leq 1 - \frac{\rho^2}{2}$$
so the G's spectral gap is $\geq \min\left\{\frac{\rho^2}{2}, \frac{2}{d}\right\}$.

Proof of (a)

Let $\lambda = \lambda_2$ (redefinition)

$u = v_2$

Let

$v = u^+$ i.e. $v_i = \max\{0, u_i\}$

$w = u^-$ i.e. $w_i = \min\{0, u_i\}$

so $u = v + w$.

Wlog. $v_i \neq 0$ for $\le n/2$ many coordinates (o/w replace u by $-v_2$).

Defn $Z = \sum_{i,j} A_{ij} |v_i^2 - v_j^2| = 2 \sum_{i<j} A_{ij} |v_i^2 - v_j^2|$.

Lemma 1: $Z \geq 2\rho \|v\|^2$

Lemma 2: $Z \leq \sqrt{8(1-\lambda)} \|v\|^2$.

Proof of (a) from the lemmas:

We have

$$2\rho \leq \sqrt{8(1-\lambda)}$$
$$4\rho^2 \leq 8(1-\lambda)$$
$$1-\lambda \geq \rho^2/2$$
$$\lambda \leq 1 - \rho^2/2.$$

Proof of Lemma 1: Wlog $v_1 \geq v_2 \geq \dots v_{n/2} \geq v_{n/2+1} = 0 \dots$

So

$$Z = 2\sum_{i<j} A_{ij} (v_i^2 - v_j^2)$$
$$= 2\sum_{i<j} A_{ij} \sum_{k=i}^{j-1} (v_k^2 - v_{k+1}^2) \quad \text{(telescoping sum)}$$
$$= 2\sum_{k=1}^{\lfloor n/2 \rfloor} (v_k^2 - v_{k+1}^2) \sum_{i\leq k} \sum_{j>k} A_{ij} \quad \text{(reordering summations)}$$

$$= 2 \sum_{k=1}^{\lfloor n/2 \rfloor} (v_k^2 - v_{k+1}^2) \cdot \frac{1}{d} |E([k], [n] \setminus [k])|$$

where $[k] = \{1, 2, \ldots k\}$

using definition of A_{ij}

$$\geq 2 \sum_{k=1}^{\lfloor n/2 \rfloor} (v_k^2 - v_{k+1}^2) \frac{1}{d} \rho dk \qquad \text{by the edge expansion of } G$$

$$= 2 \sum_{k=1}^{\lfloor n/2 \rfloor} \rho \cdot v_k^2 (k - (k-1)) \qquad \text{by regrouping terms}$$

$$= 2\rho \sum_{k=1}^{\lfloor n/2 \rfloor} v_k^2 = 2\rho \|v\|^2$$

qed Lemma 1.

<u>Proof of Lemma 2</u>:

We have

$$\langle Av, v \rangle + \langle Aw, v \rangle = \langle A(v+w), v \rangle \qquad \text{by linearity}$$

$$= \langle Au, v \rangle \qquad \text{by def'n of } v, w$$

$$= \langle \lambda u, v \rangle \qquad \text{by eigenvector}$$

$$= \langle \lambda (v+w), v \rangle$$

$$= \lambda \|v\|^2 \qquad \text{by } \langle v, w \rangle = 0.$$

Since $w \leq 0$, $Aw \leq 0$, so $\langle Aw, v \rangle \leq 0$

Therefore $\langle Av, v \rangle \geq \lambda \|v\|^2$

Thus

$$(\star) \qquad 2 \|v\|^2 (1-\lambda) \geq 2(\|v\|^2 - \langle Av, v \rangle) = \sum_{ij} A_{ij} (v_i - v_j)^2$$

by the "Sublemma" of page 5.

We multiply both sides of (☆) by

$$2\|v\|^2 + 2\langle Av, v\rangle = \sum_{ij} A_{ij}(v_i+v_j)^2$$

The LHS(☆) becomes:

$$2\left(2\|v\|^2 + 2\langle Av,v\rangle\right)\|v\|^2(1-\lambda)$$
$$\leq 2\cdot\left(4\|v\|^2\right)\|v\|^2(1-\lambda) \quad \text{(by stochasticity of } A\text{)}$$
$$= 8\|v\|^4(1-\lambda).$$

the RHS of ☆ becomes

$$\left(\sum_{ij} A_{ij}(v_i-v_j)^2\right)\left(\sum_{ij} A_{ij}(v_i+v_j)^2\right)$$
$$\geq \left(\sum_{ij} A_{ij}(v_i-v_j)(v_i+v_j)\right)^2$$

by Cauchy-Schwartz applied to $\sqrt{A_{ij}}(v_i-v_j)$ and $\sqrt{A_{ij}}(v_i+v_j)$

$\langle x,y\rangle \leq \|x\|\cdot\|y\|.$

$$= Z^2.$$

Thus $\quad Z^2 \leq 8\|v\|^4(1-\lambda)$

so $\quad Z \leq \sqrt{8(1-\lambda)}\cdot\|v\|^2$

qed Lemma 2

QED Theorem.

Theorem If G is a d-regular, connected graph with all n-many self-loops then

$$\lambda(G) \le 1 - \frac{1}{12n^2}.$$

Thus, G is a $(n, d, \frac{1}{12n^2})$-expander graph.

Pf: Let $\vec{u} \perp \vec{1}$, i.e. $\langle \vec{u}, \vec{1} \rangle = 0$, and that $\|u\| = 1$.

Let $\vec{v} = A\vec{u}$. WWTS $\|\vec{v}\| \le 1 - \frac{1}{12n^2}$.

It suffices to show $\|v\|^2 \le 1 - \frac{1}{6n^2}$, i.e. $\frac{1}{6n^2} \le 1 - \|v\|^2 = \|u\|^2 - \|v\|^2$

Claim: $\|u\|^2 - \|v\|^2 = \sum_{ij} A_{ij} (u_i - v_j)^2$

Pf
$$\sum_{ij} A_{ij} (u_i - v_j)^2 = \sum_{ij} A_{ij} u_i^2 - 2 \sum_{ij} A_{ij} u_i v_j + \sum_{ij} A_{ij} v_j^2$$
$$= \|u\|^2 - 2 \langle Au, v \rangle + \|v\|^2 = \|u\|^2 - 2\|v\|^2 + \|v\|^2$$
$$= \|u\|^2 - \|v\|^2.$$

q.e.d. Claim.

So, WWTS: $\sum_{ij} A_{ij} (u_i - v_j)^2 \ge \frac{1}{6n^2}$.

Choose i s.t. $u_i > 0$ and j s.t. $u_j < 0$.

Also, since $\|u\| = 1$, some $|u_i| > 1/\sqrt{n}$ for at least one i, so

wlog $u_i - u_j > 1/\sqrt{n}$.

Let D = diameter of graph. Note $D \le n-1$, in fact $D \le \frac{3n}{d+1}$.

So $\exists$ path of length D from i to j in G. $\odot$ . . $\odot$ . . .

So Let it be $u = u_1 - u_2 - \cdots - u_{D+1} = u_j$

$$\frac{1}{\sqrt{n}} \le u_1 - u_{D+1} = (u_1 - v_1) + (v_1 - u_2) + (u_2 - v_2) + \cdots + (v_D - u_{D+1})$$
$$\le \sqrt{(u_1 - v_1)^2 + \cdots (v_D - u_{D+1})^2} \sqrt{2D+1}$$

by $\|\vec{x}\|_1 \le \frac{\|\vec{x}\|_2}{\sqrt{n}}$.

Now
$$\sum_{ij} A_{ij} (u_i - v_j)^2 \ge \frac{1}{d} \frac{(u_1 - v_1)^2 + \cdots + (v_D - u_{D+1})^2}{\cancel{2D+1}} \ge \frac{1}{d} \frac{1}{n(2D+1)}$$
$$\ge \frac{1}{d} \frac{1}{n} \frac{1}{2 \frac{3n}{d+1}} \ge \frac{1}{6n^2}.$$

$\boxed{\text{qed}}$

Def'n Matrix Norm: $\|A\| = \max\{ \|Av\| : \|v\|=1 \}$.

Fact $\|A\cdot B\| \le \|A\|\cdot\|B\|$. $\|A+B\| \le \|A\| + \|B\|$

Pf Trivial.

Fact If G is a d-regular graph and $A = A(G)$ then $\|A\| = 1$.

Pf: Immediate from earlier results.

Lemma: Let G be an (n,d,λ)-expander and $A = A(G)$.

Let J be the random walk matrix for the "very complete" graph G which is a n-clique plus all self-loops. $(J_{ij})_{ij} = (1/n)_{ij}$

Then

$$A = (1-\lambda)J + \lambda C$$

where $\|C\| \le 1$.

Intuition: J - gives the completely random next vertex in a random walk.

Note: C will have negative entries so is not a random walk (transition) matrix.

Proof Let $C = \frac{1}{\lambda}(A - (1-\lambda)J)$.

WWTS, $\forall v$, $\|Cv\| \le \|v\|$.

Write $v = \alpha\vec{1} + \vec{u}$. where $\vec{u} \perp \vec{1}$ i.e. $\sum_i u_i = 0$

$C\vec{1} = \frac{1}{\lambda}(\vec{1} - (1-\lambda)\vec{1}) = \vec{1}$.

and $C\vec{u} = \frac{1}{\lambda}(A\vec{u} - (1-\lambda)J\vec{u}) = \frac{1}{\lambda}A\vec{u} = \frac{1}{\lambda}\vec{u}'$ where $\|u'\| \le \lambda\|u\|$.

Thus

$$\|Cu\| = \frac{1}{\lambda}\|\vec{u}'\| \le \frac{1}{\lambda}\lambda\|u\| = \|u\|.$$

Note $J\vec{u} = \vec{0}$ since $\vec{u} \perp \vec{1}$

qed.

"W" = Witness

<u>Intuition</u> for using expander graph to reduce randomness:

Suppose we have one-sided error RP problem:

<u>Input</u> x of length n, gives search space of size 2^m, $m = m(n)$.

$x \in L \Leftrightarrow \exists y \in W \cap 2^m$, where $W(x) \equiv W(x,y)$

$x \in L \Rightarrow |W \cap 2^m| \geq \alpha 2^m$

<u>Algorithm</u> Using randomness, sample k members of 2^m uniformly at random. For each member y, check if $y \in G(x)$

If find $y \in G(x)$, accept. ~~Else~~ reject.

Prob of answering correctly is $1-(1-\alpha)^k = 1-\beta^k$

Number of random bits used: $m \cdot k$.

Prob of answering incorrectly is $\leq \beta^k = \beta^{(m \cdot k)/m} = (\beta^{1/m})^{\#\text{ of bits}}$

<u>Expander graph alternative</u>:

Let G be an expander graph on 2^m, so

G is a $(2^m, d, \lambda)$-expander.

<u>Algorithm</u>: Select $y_1 \in 2^m$ at random

For $i = 2, \ldots k$, choose y_i a random neighbor of y_{i-1} in G.

Accept if some $y_i \in G(x)$.

Reject otherwise

Prob of answering incorrectly $= ? \leq (\beta)^k$ ~~when~~

Number of random bits used $= m + k \cdot (\log_2 d)$

Prob. of answering ~~answering~~ incorrectly $\leq \beta^k = \beta^{(R-m)/\log_2 d}$

if $R = c \cdot m$, $\leq \left[(\beta')^{\frac{c-1}{\log_2 d}}\right]^R = \left(\tfrac{1}{2}\right)^{\Theta(R)}$

Let $B = 2^m \setminus W$, "B" = "Bad".

<u>Lemma</u>: Given G, W, B as above

$$\Pr_{\substack{y_1 \ldots y_k \\ \text{chosen as above}}} \left[\forall i \in \{1, \ldots k\}\ y_i \in B \right] \leq (\beta')^{k-1} \text{ where } \beta' = \left((1-\lambda)\sqrt{\beta} + \lambda\right)$$

<u>Remark</u> $\beta' < 1$ is a constant if $\bar{\lambda}$ and β are.

<u>Proof:</u> Let B_i be the <u>event</u> that $y_i \in B$.
We wish to bound $Pr[B_1 \wedge \cdots \wedge B_k] \le (\beta')^{k-1}$.
Now,

$$Pr[B_1 \wedge \cdots \wedge B_k] = Pr[B_1] \cdot Pr[B_2 | B_1] \cdot Pr[B_3 | B_1 B_2] \cdots \cdot Pr[B_k | B_1 \cdots B_{k-1}].$$

Let χ_B be the linear transformation s.t.

$$(\chi_B \vec{u})_i = \begin{cases} u_i & \text{if } i \in B \\ 0 & \text{if } i \notin B \end{cases}$$

So

$$\chi_B = \begin{array}{cc} & \begin{array}{cc} \end{array} \\ \begin{array}{c} B \\ W \end{array} & \left(\begin{array}{c|c} Id & 0 \\ \hline 0 & 0 \end{array}\right) \\ & \begin{array}{cc} B & W \end{array} \end{array} \qquad (n \times n)$$

Add an extra vertex H ("H" = Halt + Accept Witness)
Let

$$A^H = \left(\begin{array}{c|c} A & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \\ \hline 0\,0\,0 & 1 \end{array}\right) \quad (n+1) \times (n+1)$$

$$\chi_B = \begin{pmatrix} Id & 0 & \begin{matrix} 0 \\ 0 \end{matrix} \\ 0 & 0 & \begin{matrix} 0 \\ 0 \end{matrix} \\ 0\,0\,0 & 1\,1\,1\,1 & 1 \end{pmatrix}$$

Both stochastic matrices with row sum 1.

The algorithm then is equivalent to computing

n-vector $\longrightarrow$ $$\vec{p} = \underbrace{\chi_B A \chi_B A \cdots \chi_B A \chi_B A}_{k-1 \text{ random steps}} \chi_B (\vec{1}) = (\chi_B A)^{k-1} \chi_B \vec{1}$$

initially chose random vertex

A - ~~transition~~ step in expander graph

χ_B - halt and accept if $\notin B$.

(n+1) vector $$\begin{pmatrix} \vec{p} \\ p' \end{pmatrix} = (\chi_B^H A^H)^{k-1} \chi_B \vec{1}.$$

and

$$\|\vec{p}\|_1 = Pr[B_1 \wedge \cdots \wedge B_k].$$

$\|(1 \cdots 1)\|_1 = n$

$\|(1 \cdots 1)\|_2 = \sqrt{n}$

Since $\|p\|_2 \geq \frac{\|p\|_1}{\sqrt{n}}$, suffices to prove,

$\|p\|_2 \leq \frac{(\beta')^{k-1}}{\sqrt{n}}$ to show $\|p\|_1 \leq (\beta')^{k-1}$.

I.e.

$$\|(\chi_B A)^{k-1} B \vec{1}\|_2 \leq \frac{(\beta')^{k-1}}{\sqrt{n}}$$

Now $\chi_B A = \chi_B((1-\lambda)J + \lambda C)$ when $\|C\| \leq 1$.

and $\|\chi_B A\| \leq (1-\lambda)\|\chi_B J\| + \lambda \|\chi_B C\|$.

Since $\|\chi_B \vec{1}\|_2 = \|\underbrace{(1 \cdots 1}_{\beta n} 000)\|_2 = \sqrt{\beta n} = \sqrt{\beta}\|1\|$ and $J\vec{u} = \alpha \vec{1}$, $\forall \vec{u}$,

$\|\chi_B J\vec{u}\|_2 \leq$ ~~$\sqrt{\beta n} \cdot \|J\|$ $\sqrt{\beta n} \cdot \|J\vec{u}\|$~~

$\|\chi_B \cdot J\vec{u}\|_2 \leq \sqrt{\beta}\,\|J\vec{u}\| \leq \sqrt{\beta}\,\|\vec{u}\|$ since $\|J\| = 1$.

so $\|\chi_B \cdot J\| \leq \sqrt{\beta}$.

And

$$\|\chi_B \cdot C\| \leq \|\chi_B\| \cdot \|C\| \leq \|C\| \leq 1.$$

Then

$$\|\chi_B A\| \leq (1-\lambda)\sqrt{\beta} + \lambda$$

So

$$\|(\chi_B A)^{k-1} B \vec{1}\|_2 \leq \left[(1-\lambda)\sqrt{\beta} + \lambda\right]^{k-1}.$$

~~$\leq \left[(1-\lambda)\sqrt{\beta} + \lambda\right]^{k-1} \leq \left[(1-\lambda)\sqrt{\beta} + \lambda\right]^{k-1}$~~

Application AKLLR can be proved from the fact that any regular connected graph w/ self loops has $\lambda(G) \le 1 - 1/12n^2$, and $\beta = (1 - 1/n)$ taking $k = n^4$ (say)

Pf
$$(1-\lambda)\sqrt{\beta} + \lambda \le \left(\frac{1}{12n^2}\right)\sqrt{1 - 1/n} + 1 - \frac{1}{12n^2}$$
$$\approx \left(\frac{1}{12n^2}\right)\left(1 - \frac{2}{n}\right) + 1 - \frac{1}{12n^2}$$
$$= 1 - \frac{1}{6n^3}$$

and $\left(1 - \frac{1}{6n^3}\right)^{n^3} \approx e^{-1/6}$

so $\left(1 - \frac{1}{6n^3}\right)^{n^4} \le c^{-n}$ for constant $c < 1$.

Graph Products

Matrix squaring:

Let G be an (n, d, λ)-expander graph.

Define G^2 to be the (n, d^2, λ^2)-expander graph by

$$\text{Edges}(G^2) = \{ (x,z) : \exists y \in [n] \ (x,y) \in \text{Edges}(G) \ \& \ (y,z) \in \text{Edges}(G) \}.$$

Note G^2 has all self loops. Can also have multiedges.

And, G^2 has n vertices and is ~~d~~ d^2-regular.

If the edges incident to x, in G are labelled $1, \ldots, d$ ("edge ends")

then the edges incident to x in G^2 can be labelled

$(1,1), (1,2), \ldots \quad (d, d-1)(d,d)$ and

the (i,j)-edge adjacent to x is (x,z) s.t.

for y s.t. y is the i-th edge of x and

z s.t. z is the j-th edge of y.

Claim G^2 is an (n, d^2, λ^2)-expander

Pf Let $A = A(G)$. If $u \perp 1$, then $A(u) \perp \vec{1}$

Thus, for $u \perp 1$,

$$\|A^2 \vec{u}\| \leq |\lambda| \cdot |\lambda| \cdot \|u\| = \lambda^2 \|u\|,$$

qed.

Tensor product

Let G be an (n, d, λ)-expander graph, and
G' be an (n', d', λ')-expander graph.

The tensor product $G \otimes G'$ is the $(n \cdot n', d \cdot d', \max\{\lambda, \lambda'\})$ expander graph defined by:

~~Edg~~ Vertices of $G \otimes G' = \{\langle x, y \rangle : x \in [n], y \in [n']\}$
Edges of $G \otimes G'$ incident to $\langle x, y \rangle$
are $(\langle x, y \rangle, \langle x', y' \rangle) : (x, x') \in G, (y, y') \in G'\}$

If (x, x') is the i-th edge ~~edg~~ incident to x
and (y, y') " " j-th edge " " y,
then $(\langle x, y \rangle, \langle x', y' \rangle)$ is the $\langle i, j \rangle$ edge incident to $\langle x, y \rangle$.

Let $A = A(G)$, ~~A~~ $A' = A(G')$.
Then

$$A(G'G) = A \otimes A' = \begin{pmatrix} a_{11}A' & a_{12}A' & a_{1n}A' \\ a_{21}A' & \ddots & \vdots \\ \vdots & & \\ a_{n1}A' & \ldots & a_{nn}A' \end{pmatrix}$$

Pf: By inspection. ~~E,~~ If at ~~x ∈ V_i of~~ $\langle x, y \rangle = \langle v_i, v_j' \rangle \in G \otimes G$,
then ~~go~~ transition to $\langle x', y' \rangle$ with probability $a_{ii'} a'_{jj'}$
$\langle v_{i'}, v'_{j'} \rangle \in G \otimes G$.

Eigenvectors of $A \otimes A'$ are $u \otimes u'$ for u, u' eigenvectors of A, A', resp.
with corresponding eigenvalues $\lambda \lambda'$

$$(u \otimes u') = \begin{pmatrix} u_1 \vec{u}' \\ \vdots \\ u_n \vec{u}' \end{pmatrix}$$

Pf: By inspection.

Rotation map of G:

the "edge ends" of G are $\langle x, i\rangle$ denoting the ith edge incident on x, Edge ends = $[n]\times[d]$

Defin The rotation map on $[n]\times[d]$ of G is the bijection

$$\hat{G} : \langle x, i\rangle \mapsto \langle y, j\rangle$$

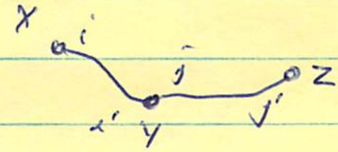
provided (x,y) is the ith edge incident on x and the j-th edge incident on y.

Rotation Map for Matrix Product

If

$$\hat{G} : \langle x, i\rangle \mapsto \langle y, i'\rangle$$

and $\hat{G} : \langle y, j\rangle \mapsto \langle z, j'\rangle$



Then

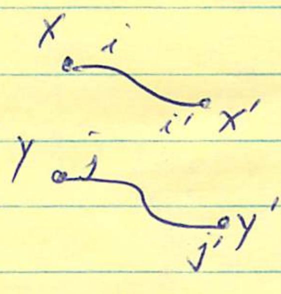
$$\widehat{G^2} : \langle x, \langle i, j\rangle\rangle \mapsto \langle z, \langle j', i'\rangle\rangle$$

Rotation Map for Tensor Product

If

$$\hat{G} : \langle x, i\rangle \mapsto \langle x', i'\rangle$$

and $\hat{G'} : \langle y, j\rangle \mapsto \langle y', j'\rangle$



Then $\widehat{G\times G'} : \langle\langle x, y\rangle, \langle i, j\rangle\rangle$

$$\to \langle\langle x', y'\rangle, \langle i', j'\rangle\rangle$$

Definition Replacement Product (aka "Bounded Replacement Product")

Let G be an $(n, D, 1-\varepsilon)$-expander

Let G' be ~~an~~ $(D, d, 1-\delta)$-expander

$G ® G'$ is an $(nD, 2d, 1-\varepsilon\delta^2/24)$-expander, defined by:

Vertices of $G ® G'$ are $\langle u, i \rangle$, $u \in G$, $i \in G'$.

Edges of $G ® G'$ are:

(a) $(\langle u,i\rangle, \langle v,j\rangle)$ where $\hat{G}: \langle u,i\rangle \to \langle v,j\rangle$.

i.e. i-th edge from u is j-th edge of v.

→ Make d copies of this edge (!)

(b) $(\langle u,i\rangle, \langle u,j\rangle)$ s.t. $(i,j) \in E(G')$.

Intuition Each vertex of degree D in G is replaced by a copy of G'.

The edges of G attach to different vertices of the copy of G'. So i-th edge of G adjacent u now attaches to the i-th vertex of ~~the~~ u's copy of G'.

Plus these edges are "parallelized" d times.

Clearly $G®G'$ has nD vertices and ~~too~~ is $2d$-regular (+ undirected)

Rotation Map for $G®G'$:

Label edges adjacent to $\langle u,i\rangle \in (G®G')$ with an ordered pair $\langle \ell, b\rangle \in [d] \times \{0,1\}$.

Then

$$\widehat{G®G'} : \langle\langle u,i\rangle, \langle \ell, b\rangle\rangle \longrightarrow \begin{cases} \langle\langle u, i'\rangle, \langle \ell', b\rangle\rangle \text{ if } b=0 \\ \quad \text{when } \hat{G'}(i, \ell) = \langle i', \ell'\rangle \\ \langle\langle u', i'\rangle, \langle \ell, b\rangle\rangle \text{ if } b=1 \\ \quad \text{where } \hat{G}(\langle u,i\rangle) = \langle u', i'\rangle. \end{cases}$$

Point Use w/ large D to reduce degree to $2d \ll D$.

Transition Matrix (Random Walk Matrix) for $G Ⓡ G'$ is

$$A(G Ⓡ G') = A Ⓡ A' = \frac{1}{2}\hat{A} + \frac{1}{2} I_n \otimes A'$$

where $A = A(G)$, $A' = A(G')$, A is $n \times n$; A' is $D \times D$.

$\hat{A}$ = the permutation matrix corresponding the rotation matrix of G.

I_n = $n \times n$ identity matrix.

Theorem $\lambda(G Ⓡ H) \leq 1 - \frac{\varepsilon \delta^2}{24}$.

Pf: It suffices to prove that $(\lambda(G Ⓡ G'))^3 \leq 1 - \frac{\varepsilon \delta^2}{8}$.

(since $\lambda(G^k) = (\lambda(G))^k$, for general graph G, (e.g., $G Ⓡ G'$.)

Let $C = A((G Ⓡ G')^3) = (A(G Ⓡ G'))^3$

So

$$C = \left(\frac{1}{2}\hat{A} + \frac{1}{2} I_n \otimes A'\right)^3$$

By Lemma on p 11. $\|B\| \leq 1$

$$A' = (1-\delta)B + \delta J_D \quad \text{when } (J_D)_{i,j} = 1/D \ \forall i,j.$$

So $$C = \left(\frac{1}{2}\hat{A} + \frac{1}{2} I_n \otimes [(1-\delta)B + \delta J_D]\right)^3$$

$$= \left(\frac{1}{2}\hat{A} + \frac{1-\delta}{2} I_n \otimes B + \frac{\delta}{2} I_n \otimes J_D\right)^3$$

$= 27$ terms

Note $\|\hat{A}\| = 1$

$\|I_n \otimes B\| \leq 1$ since $\|B\| \leq 1$

and $\|I_n \times J_n\| = 1$.

Thus $C = \frac{\delta^2}{8}(I_n \otimes J_D)\hat{A}(I_n \otimes J_D)$

$+ \{$ 26 terms each is a coefficient $a_1 \ldots a_{26} \in \mathbb{R}$ times a product of 3 matrices of norm ≤ 1, and $\sum_{i=1}^{26} a_i = 1 - \frac{\delta^2}{8}$ $]$

i.e. $C = \left(1 - \frac{\delta^2}{8}\right)C' + \frac{\delta^2}{8}(I_n \otimes J_n)\hat{A}(I_n \otimes J_D)$

<u>Claim</u> $(I_n \otimes J_D)\hat{A}(I_n \otimes J_D) = A \otimes J_D$

Thus, by Tensor product (p 17) and since $\lambda(J_D) = 0$,

$$\lambda(C) \le 1 - \frac{\delta^2}{8}(1 - \lambda(A \otimes J_D))$$

$$\le 1 - \frac{\delta^2}{8}(1 - \max\{\lambda(A), \lambda(J_D)\})$$

$$\le 1 - \frac{\delta^2}{8}(1 - (1 - \varepsilon))$$

$$= 1 - \frac{\varepsilon\delta^2}{8}.$$

<u>Pf</u> of Claim. Note these are $nD \times nD$ matrices

<u>RHS</u> This the transition matrix for a random walk step in ~~A~~ $G \times K_D$ from $(u, i) \longmapsto (v, j)$

v a neighbor of u

$j \in D$ chosen at random

LHS: $I_n \otimes J_D$ - a random walk step from $\langle u,i\rangle$ to $\langle u,j\rangle$ when $j \in D$ is chosen at random.

$\hat{A}$ - a random step $\langle u,i\rangle$ to $\langle v,j\rangle$ as given by the rotation maps of G.

By inspection,

$(I_n \otimes I_D)\,\hat{A}\,(I_n \otimes J_D)$ has the same effect as $A \otimes J_D$.

This proves the Claim and the Theorem. QED ☑