Problem # 11066

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Let R be a ring such that for any $x, y \in R$ there exist nonnegative integers $m = m(x, y)$ and $n = n(x, y)$ such that

$$x^{m+1}y^{n+1} = x^m yxy^n$$

Prove that R is commutative.

Proof.

First we prove the following lemmas.

Lemma 1 *Let R be a ring, $x, y \in R$ and m, n nonnegative integers such that*

$$x^m y = (x+1)^n y = 0$$

Then $y = 0$.

Proof of Lemma1.

Let k denote the minimal nonnegative integer such that $x^k y = 0$. We prove that $k = 0$ which implies $y = 0$.

Suppose $k \geq 1$. We have that

$$0 = (x+1)^n y = \sum_{i=0}^{n} \binom{n}{i} x^i y$$

By multiplying to the left by x^{k-1} we get that

$$0 = x^{k-1}y + \sum_{i=1}^{n} \binom{n}{i} x^{k+(i-1)} y = x^{k-1}y$$

which is a contradiction to the definition of k.

Hence $k = 0$ and $y = 0$. This concludes the proof of Lemma 1.

Lemma 2 *Let R be a ring, $x, y \in R$ and m, n nonnegative integers such that*

$$xy^m = x(y+1)^n = 0$$

Then $x = 0$.

The proof of Lemma 2 is the same as the proof of Lemma 1.

We can now prove our initial problem.

Let $x, y \in R$.

Then there exist nonnegative integers n and m such that

$$x^m(xy - yx)y^n = 0 \tag{1}$$

By replacing x by $x+1$, it follows that there exist nonnegative integers p and q such that

$$(x+1)^p(xy - yx)y^q = (x+1)^p((x+1)y - y(x+1))y^q = 0 \tag{2}$$

Multiplying equation (1) to the right by y^q and equation (2) to the right by y^n we get that

$$x^m[(xy - yx)y^{n+q}] = (x+1)^p[(xy - yx)y^{n+q}] = 0$$

By applying Lemma 1, we obtain that $(xy - yx)y^{n+q} = 0$.

Thus, we obtained that for every x and y in R, there exists a nonnegative integer r such that

$$(xy - yx)y^r = 0$$

By replacing y by $y+1$, we get that there exists a nonnegative integer s such that

$$(xy - yx)(y+1)^s = (x(y+1) - (y+1)x)(y+1)^s = 0$$

Thus

$$(xy - yx)y^r = (xy - yx)(y+1)^s = 0$$

Applying Lemma 2, we obtain that $xy - yx = 0$. Thus, $xy = yx$ for all x and y in R. Hence R is commutative.