

# On the extreme eigenvalues of regular graphs

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*To the memory of Dom de Caen*

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## Abstract

In this paper, we present an elementary proof of a theorem of Serre concerning the greatest eigenvalues of  $k$ -regular graphs. We also prove an analogue of Serre's theorem regarding the least eigenvalues of  $k$ -regular graphs: given  $\epsilon > 0$ , there exist a positive constant  $c = c(\epsilon, k)$  and a nonnegative integer  $g = g(\epsilon, k)$  such that for any  $k$ -regular graph  $X$  with no odd cycles of length less than  $g$ , the number of eigenvalues  $\mu$  of  $X$  such that  $\mu \leq -(2 - \epsilon)\sqrt{k - 1}$  is at least  $c|X|$ . This implies a result of Winnie Li.

*Key words:* Alon-Boppana theorem, eigenvalues of graphs

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## 1 Preliminaries

Let  $X$  be a graph and let  $v_0$  be a vertex of  $X$ . A *closed walk* in  $X$  of length  $r \geq 0$  starting at  $v_0$  is a sequence  $v_0, v_1, \dots, v_r$  of vertices of  $X$  such that  $v_r = v_0$  and  $v_{i-1}$  is adjacent to  $v_i$  for  $1 \leq i \leq r$ . For  $r \geq 0$ , let  $\Phi_r(X)$  denote

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<sup>1</sup> Research partially supported by an Ontario Graduate Scholarship

the number of closed walks of length  $r$  in  $X$ . A *cycle* of length  $r$  in  $X$  is a subgraph of  $X$  whose vertices can be labeled  $v_0, \dots, v_r$  such that  $v_0, \dots, v_r$  is a closed walk in  $X$  and  $v_i \neq v_j$  for all  $i, j$  with  $0 \leq i < j \leq r$ . The *girth*, denoted  $\text{girth}(X)$ , of  $X$  is the length of a smallest cycle in  $X$  if such a cycle exists and  $\infty$  otherwise; the *oddgirth*, denoted  $\text{oddg}(X)$ , of  $X$  is the length of a smallest odd cycle in  $X$  if such a cycle exists and  $\infty$  otherwise. The adjacency matrix of  $X$  is the matrix  $A = A(X)$  of order  $|X|$ , where the  $(u, v)$  entry is 1 if the vertices  $u$  and  $v$  are adjacent and 0 otherwise. It is a well known fact that  $\Phi_r(X) = \text{Tr}(A^r)$ , for any  $r \geq 0$ . The eigenvalues of  $X$  are the eigenvalues of  $A$ . If  $X$  is  $k$ -regular, then it is easy to see that  $k$  is an eigenvalue of  $X$  with multiplicity equal to the number of components of  $X$  and that any eigenvalue  $\lambda$  of  $X$  satisfies  $|\lambda| \leq k$ . For  $l \geq 1$ , we denote by  $\lambda_l(X)$  the  $l$ th greatest eigenvalue of  $X$  and by  $\mu_l(X)$  the  $l$ th least eigenvalue of  $X$ .

## 2 An elementary proof of Serre's theorem

Serre has proved the following theorem (see [4,5,7,15]) using Chebyshev polynomials. See also [2] for related results. In this section, we present an elementary proof of Serre's result.

**Theorem 1** *For each  $\epsilon > 0$ , there exists a positive constant  $c = c(\epsilon, k)$  such that for any  $k$ -regular graph  $X$ , the number of eigenvalues  $\lambda$  of  $X$  with  $\lambda \geq (2 - \epsilon)\sqrt{k - 1}$  is at least  $c|X|$ .*

For the proof of this theorem we require the next lemma which can be deduced from McKay's work [11], Lemma 2.1. For the sake of completeness, we include a short proof here.

**Lemma 2** *Let  $v_0$  be a vertex of a  $k$ -regular graph  $X$ . Then the number of closed walks of length  $2s$  in  $X$  starting at  $v_0$  is greater than or equal to  $\frac{1}{s+1} \binom{2s}{s} k(k-1)^{s-1}$ .*

**Proof of Lemma 2** The number of closed walks of length  $2s$  in  $X$  starting at  $v_0$  is at least the number of closed walks of length  $2s$  starting at a vertex  $u_0$  in the infinite  $k$ -regular tree. To each closed walk in the infinite  $k$ -regular tree, there corresponds a sequence of nonnegative integers  $\delta_1, \dots, \delta_{2s}$ , where  $\delta_i$  is the distance from  $u_0$  after  $i$  steps. The number of such sequences is the  $s$ -th Catalan number  $\frac{1}{s+1} \binom{2s}{s}$ . For each sequence of distances, there are at least  $k(k-1)^{s-1}$  closed walks of length  $2s$  since for each step away from  $u_0$  there are  $k-1$  choices ( $k$  if the walk is at  $u_0$ ).

□

By Stirling's bound on  $s!$  or by a simple induction argument it is easy to see that  $\binom{2s}{s} \geq \frac{4^s}{s+1}$ , for any  $s \geq 1$ . Hence, for any  $k$ -regular graph  $X$  and for any  $s \geq 1$ , we have by Lemma 2

$$\mathrm{Tr}(A^{2s}) \geq |X| \frac{1}{s+1} \binom{2s}{s} k(k-1)^{s-1} > |X| \frac{1}{(s+1)^2} (2\sqrt{k-1})^{2s} \quad (1)$$

**Proof of Theorem 1** Let  $X$  be  $k$ -regular graph of order  $n$  with eigenvalues  $k = \lambda_1 \geq \dots \geq \lambda_n \geq -k$ . Given  $\epsilon > 0$ , let  $m$  be the number of eigenvalues  $\lambda$  of  $X$  with  $\lambda \geq (2-\epsilon)\sqrt{k-1}$ . Then  $n-m$  of the eigenvalues of  $X$  are less

than  $(2 - \epsilon)\sqrt{k - 1}$ . Thus

$$\begin{aligned}\mathrm{Tr}(kI + A)^{2s} &= \sum_{i=1}^n (k + \lambda_i)^{2s} \\ &< (n - m)(k + (2 - \epsilon)\sqrt{k - 1})^{2s} + m(2k)^{2s} \\ &= m((2k)^{2s} - (k + (2 - \epsilon)\sqrt{k - 1})^{2s}) + n(k + (2 - \epsilon)\sqrt{k - 1})^{2s}\end{aligned}$$

On the other hand, the binomial expansion and relation (1) give

$$\begin{aligned}\mathrm{Tr}(kI + A)^{2s} &= \sum_{i=0}^{2s} \binom{2s}{i} k^i \mathrm{Tr}(A^{2s-i}) \\ &\geq \sum_{j=0}^s \binom{2s}{2j} k^{2j} \mathrm{Tr}(A^{2s-2j}) \\ &> \frac{n}{(s+1)^2} \sum_{j=0}^s \binom{2s}{2j} k^{2j} (2\sqrt{k-1})^{2s-2j} \\ &= \frac{n}{2(s+1)^2} ((k + 2\sqrt{k-1})^{2s} + (k - 2\sqrt{k-1})^{2s}) \\ &> \frac{n}{2(s+1)^2} (k + 2\sqrt{k-1})^{2s}\end{aligned}$$

Thus,

$$\frac{m}{n} > \frac{\frac{1}{2(s+1)^2} (k + 2\sqrt{k-1})^{2s} - (k + (2 - \epsilon)\sqrt{k-1})^{2s}}{(2k)^{2s} - (k + (2 - \epsilon)\sqrt{k-1})^{2s}}$$

for any  $s \geq 1$ . Since

$$\begin{aligned}\lim_{s \rightarrow \infty} \left( \frac{(k + 2\sqrt{k-1})^{2s}}{2(s+1)^2} \right)^{\frac{1}{2s}} &= k + 2\sqrt{k-1} \\ &> k + (2 - \epsilon)\sqrt{k-1} = \lim_{s \rightarrow \infty} \left( 2(k + (2 - \epsilon)\sqrt{k-1})^{2s} \right)^{\frac{1}{2s}}\end{aligned}$$

it follows that there exists  $s_0 = s_0(\epsilon, k)$  such that for all  $s \geq s_0$

$$\frac{(k + 2\sqrt{k-1})^{2s}}{2(s+1)^2} - (k + (2 - \epsilon)\sqrt{k-1})^{2s} > (k + (2 - \epsilon)\sqrt{k-1})^{2s}$$

Hence, if

$$c(\epsilon, k) = \frac{(k + (2 - \epsilon)\sqrt{k-1})^{2s_0}}{(2k)^{2s_0} - (k + (2 - \epsilon)\sqrt{k-1})^{2s_0}}$$

then  $c(\epsilon, k) > 0$  and  $m > c(\epsilon, k)n$ .

□

The proofs of Serre's theorem given in [4,5,7] don't allow an easy estimation of the constant  $c(\epsilon, k)$  in terms of  $\epsilon$  and  $k$ . We relegate the detailed analysis of the constant obtained by those arguments to a future work [3]. We should mention that Serre's theorem can be also deduced from the work of Friedman [6] or Nilli [13]. Friedman's results imply an estimate of  $\left(\frac{1}{2}\right)^{O\left(\frac{\log k}{\sqrt{\epsilon}}\right)}$  for the proportion of the eigenvalues that are at least  $(2 - \epsilon)\sqrt{k - 1}$ . Nilli's work provides a bound of  $\left(\frac{1}{2}\right)^{O\left(\frac{\log k}{\arccos(1-\epsilon)}\right)}$ . Their methods provide better bounds on  $c(\epsilon, k)$  than ours. From our proof of Serre's theorem, we obtain that a proportion of  $\left(\frac{1}{2}\right)^{O\left(\frac{\sqrt{k}}{\epsilon} \log\left(\frac{\sqrt{k}}{\epsilon}\right)\right)}$  of the eigenvalues are at least  $(2 - \epsilon)\sqrt{k - 1}$ . This is because in Theorem 1 we pick  $s_0$  such that  $\frac{s_0}{\log s_0} = \Theta\left(\frac{\sqrt{k}}{\epsilon}\right)$ .

Theorem 1 has the following consequence regarding the asymptotics of the greatest eigenvalues of  $k$ -regular graphs.

**Corollary 3** *Let  $(X_i)_{i \geq 0}$  be a sequence of  $k$ -regular graphs such that*

*$\lim_{i \rightarrow \infty} |X_i| = \infty$ . Then for each  $l \geq 1$ ,*

$$\liminf_{i \rightarrow \infty} \lambda_l(X_i) \geq 2\sqrt{k - 1}$$

This corollary has also been proved directly by Serre in an appendix to [8] using the eigenvalue distribution theorem in [16]. When  $l = 2$ , we obtain the asymptotic version of the Alon-Boppana theorem (see [1,10,12,14] for more details).

### 3 Analogous theorems for the least eigenvalues of regular graphs

The analogous result to Theorem 1 for the least eigenvalues of a  $k$ -regular graph is not true. For example, the eigenvalues of line graphs are all at least  $-2$ . However, by adding an extra condition to the hypothesis of Theorem 1, we can prove an analogue of Serre's theorem for the least eigenvalues of a  $k$ -regular graph.

**Theorem 4** *For any  $\epsilon > 0$ , there exist a positive constant  $c = c(\epsilon, k)$  and a non-negative integer  $g = g(\epsilon, k)$  such that for any  $k$ -regular graph  $X$  with  $\text{oddg}(X) > g$ , the number of eigenvalues  $\mu$  of  $X$  with  $\mu \leq -(2 - \epsilon)\sqrt{k-1}$  is at least  $c|X|$ .*

**Proof** Let  $X$  be a  $k$ -regular graph of order  $n$  with eigenvalues  $-k \leq \mu_1 \leq \mu_2 \leq \dots \leq \mu_n = k$ . Given  $\epsilon > 0$ , let  $m$  be the number of eigenvalues  $\mu$  of  $X$  with  $\mu \leq -(2 - \epsilon)\sqrt{k-1}$ . Then  $n - m$  of the eigenvalues of  $X$  are greater than  $-(2 - \epsilon)\sqrt{k-1}$ . Thus

$$\begin{aligned} \text{Tr}(kI - A)^{2s} &= \sum_{i=1}^n (k - \mu_i)^{2s} < (n - m)(k + (2 - \epsilon)\sqrt{k-1})^{2s} + m(2k)^{2s} \\ &= m((2k)^{2s} - (k + (2 - \epsilon)\sqrt{k-1})^{2s}) + n(k + (2 - \epsilon)\sqrt{k-1})^{2s} \end{aligned}$$

In the previous section, we proved that there exists  $s_0 = s_0(\epsilon, k)$  such that for all  $s \geq s_0$

$$\frac{(k + 2\sqrt{k-1})^{2s_0}}{2(s_0 + 1)^2} - (k + (2 - \epsilon)\sqrt{k-1})^{2s_0} > (k + (2 - \epsilon)\sqrt{k-1})^{2s_0}$$

Let  $g(\epsilon, k) = 2s_0$ . If  $\text{oddg}(X) > 2s_0$ , then for  $0 \leq j \leq s_0 - 1$ , the number of closed walks of length  $2s_0 - 2j - 1$  in  $X$  is 0. Hence,  $\text{Tr}(A^{2s_0 - 2j - 1}) = 0$ , for

$0 \leq j \leq s_0 - 1$ . Using also (1), we obtain

$$\begin{aligned} \mathrm{Tr}(kI - A)^{2s_0} &= \sum_{j=0}^{s_0} \binom{2s_0}{2j} k^{2j} \mathrm{Tr}(A^{2s_0-2j}) - \sum_{j=0}^{s_0-1} \binom{2s_0}{2j+1} k^{2j+1} \mathrm{Tr}(A^{2s_0-2j-1}) \\ &= \sum_{j=0}^{s_0} \binom{2s_0}{2j} k^{2j} \mathrm{Tr}(A^{2s_0-2j}) > \frac{n}{(s_0+1)^2} \sum_{j=0}^{2s_0} \binom{2s_0}{2j} k^{2j} (2\sqrt{k-1})^{2s_0-2j} \\ &> \frac{n}{2(s_0+1)^2} (k + 2\sqrt{k-1})^{2s_0}. \end{aligned}$$

From the previous inequalities, it follows that if

$$c(\epsilon, k) = \frac{(k + (2 - \epsilon)\sqrt{k-1})^{2s_0}}{(2k)^{2s_0} - (k + (2 - \epsilon)\sqrt{k-1})^{2s_0}}$$

then  $c(\epsilon, k) > 0$  and  $m > c(\epsilon, k)n$ .

□

The next result is an immediate consequence of Theorem 4.

**Corollary 5** *Let  $(X_i)_{i \geq 0}$  be a sequence of  $k$ -regular graphs such that*

*$\lim_{i \rightarrow \infty} \mathrm{oddg}(X_i) = \infty$ . Then for each  $l \geq 1$*

$$\limsup_{i \rightarrow \infty} \mu_l(X_i) \leq -2\sqrt{k-1}$$

When  $l = 1$ , we get the main result from [8]. Also, Corollary 5 holds when  $l = 1$  and  $\lim_{i \rightarrow \infty} \mathrm{girth}(X_i) = \infty$ . This special case of Corollary 5 was proved directly in [9] using orthogonal polynomials and is also a consequence of the eigenvalue distribution theorem from [11].

A theorem stronger than Corollary 5 has been proved by Serre in [8] using the eigenvalue distribution results from [16]. We now present an elementary proof of this theorem. For  $r \geq 0$ , let  $c_r(X)$  be the number of cycles of length  $r$  in a graph  $X$ .

**Theorem 6** Let  $(X_i)_{i \geq 0}$  be a sequence of  $k$ -regular graphs such that  $\lim_{i \rightarrow \infty} |X_i| = \infty$ . If  $\lim_{i \rightarrow \infty} \frac{c_{2r+1}(X_i)}{|X_i|} = 0$  for each  $r \geq 1$ , then for each  $l \geq 1$

$$\limsup_{i \rightarrow \infty} \mu_l(X_i) \leq -2\sqrt{k-1}$$

**Proof** Let  $l \geq 1$ . For a graph  $X$  and  $r \geq 1$ , let  $n_{2r+1}(X)$  denote the number of vertices  $v_0$  in the graph  $X$  such that the subgraph of  $X$  induced by the vertices at distance at most  $r$  from  $v_0$  is bipartite. Thus,  $|X| - n_{2r+1}(X)$  is the number of vertices  $u_0$  of  $X$  such that the subgraph of  $X$  induced by the vertices at distance at most  $r$  from  $u_0$  contains at least one odd cycle. Since each such vertex is no further than  $r$  from each of the vertices of an odd cycle of length at most  $2r+1$ , it follows that

$$|X| - n_{2r+1}(X) \leq \sum_{l=1}^{r-1} \alpha_{l,r} c_{2l+1}(X)$$

where  $0 \leq \alpha_{l,r} \leq 3(2l+1)(k-1)^r$ . Thus, we have the following inequalities

$$1 - \sum_{l=1}^{r-1} \alpha_{l,r} \frac{c_{2l+1}(X_i)}{|X_i|} \leq \frac{n_{2r+1}(X_i)}{|X_i|} \leq 1$$

for all  $r \geq 1, i \geq 0$ . Hence, for each  $r \geq 1$

$$\lim_{i \rightarrow \infty} \frac{n_{2r+1}(X_i)}{|X_i|} = 1 \tag{2}$$

For  $i \geq 0$ , let  $A_i = A(X_i)$ . Then, for  $i \geq 0$  and  $r \geq 1$ , we have

$$\text{Tr}(A_i^{2r+1}) = n_{2r+1}(X_i) \cdot 0 + (|X_i| - n_{2r+1}(X_i))\theta_{2r+1}(X_i) \tag{3}$$

where  $0 \leq \theta_{2r+1}(X_i) \leq k^{2r+1}$ . From (2) and (3), we obtain that for each  $r \geq 1$

$$\lim_{i \rightarrow \infty} \frac{\text{Tr}(A_i^{2r+1})}{|X_i|} = 0 \tag{4}$$

By using relation (1), it follows that for each  $r \geq 1$

$$\liminf_{i \rightarrow \infty} \frac{\text{Tr}(A_i^{2r})}{|X_i|} \geq \frac{(2\sqrt{k-1})^{2r}}{(r+1)^2} \quad (5)$$

Now for each  $i \geq 0$ , we have

$$\text{Tr}(kI - A_i)^{2s} = \sum_{j=1}^{|X_i|} (k - \lambda_j(X_i))^{2s} \leq (|X_i| - l)(k - \mu_l(X_i))^{2s} + l(2k)^{2s}$$

Once again, the binomial expansion gives us

$$\text{Tr}(kI - A_i)^{2s} = \sum_{j=0}^{2s} \binom{2s}{j} k^j (-1)^{2s-j} \text{Tr}(A_i^{2s-j})$$

From the previous two relations, we get that

$$(k - \mu_l(X_i))^{2s} + \frac{4^s l k^{2s}}{|X_i| - l} \geq \sum_{j=0}^{2s} \binom{2s}{j} k^j (-1)^{2s-j} \frac{\text{Tr}(A_i^{2s-j})}{|X_i| - l}$$

Using relations (4) and (5), it follows that

$$\begin{aligned} k - \limsup_{i \rightarrow \infty} \mu_l(X_i) &\geq \left( \sum_{j=0}^s \binom{2s}{2j} k^{2j} \frac{(2\sqrt{k-1})^{2s-2j}}{(s-j+1)^2} \right)^{\frac{1}{2s}} \\ &> \left( \frac{1}{(s+1)^2} \sum_{j=0}^s \binom{2s}{2j} k^{2j} (2\sqrt{k-1})^{2s-2j} \right)^{\frac{1}{2s}} \\ &> \left( \frac{1}{2(s+1)^2} \right)^{\frac{1}{2s}} (k + 2\sqrt{k-1}) \end{aligned}$$

for any  $s \geq 1$ . By taking the limit as  $s \rightarrow \infty$ , we get

$$k - \limsup_{i \rightarrow \infty} \mu_l(X_i) \geq k + 2\sqrt{k-1}$$

which implies the inequality stated in the theorem.

□

## Acknowledgments

This paper is part of my Ph.D. thesis at Queen's University. I am grateful to my thesis advisors: David Gregory, Ram Murty and David Wehlau for their help and support and to Chris Godsil and Shlomo Hoory for their comments. I thank the referees for many useful suggestions.

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