

Note

Extreme eigenvalues of nonregular graphs

Sebastian M. Cioabă¹

*Department of Mathematics, University of California, San Diego, La Jolla, CA
92093-0112*

David A. Gregory²

*Department of Mathematics, Queen's University at Kingston, Ontario, K7L 3N6,
Canada*

Vladimir Nikiforov

*Department of Mathematical Sciences, University of Memphis, Memphis TN
38152, USA*

Abstract

Let λ_1 be the greatest eigenvalue and λ_n the least eigenvalue of the adjacency matrix of a connected graph G with n vertices, m edges and diameter D . We prove that if G is nonregular, then

$$\Delta - \lambda_1 > \frac{n\Delta - 2m}{n(D(n\Delta - 2m) + 1)} \geq \frac{1}{n(D + 1)}$$

where Δ is the maximum degree of G .

The inequality improves previous bounds of Stevanović and of Zhang. It also implies that a lower bound on λ_n obtained by Alon and Sudakov for (possibly regular) connected nonbipartite graphs also holds for connected nonregular graphs.

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Email addresses: sebi@mast.queensu.ca (Sebastian M. Cioabă),
gregoryd@mast.queensu.ca (David A. Gregory), vnikifrv@memphis.edu
(Vladimir Nikiforov).

1 Background

Let G be a simple graph with vertex set $[n] = \{1, 2, \dots, n\}$, maximum degree Δ and minimum degree δ . Let λ_1 denote the largest eigenvalue of the adjacency matrix A of G . The Perron-Frobenius Theorem [2, p. 178] implies that λ_1 has an eigenvector x with nonnegative entries which must be positive if G is connected. Then $Ax = \lambda_1 x$ and, so $\lambda_1 \sum_i x_i = \sum_j d_j x_j$ where d_1, d_2, \dots, d_n is the degree sequence of G . This implies the well-known result that

$$\delta \leq \lambda_1 \leq \Delta$$

and that, when G is connected, equality holds in either case if and only if G is regular.

Stevanović [3] proved that if a connected graph G is nonregular, then

$$\Delta - \lambda_1 > \frac{1}{2n(n\Delta - 1)\Delta^2}$$

In the final remark of [3], Stevanović asked whether or not the power of Δ could be improved. Recently, Zhang [4] obtained the finer bound

$$\Delta - \lambda_1 > \frac{(\sqrt{\Delta} - \sqrt{\delta})^2}{nD\Delta}$$

where D is the diameter of G . In this note, we prove the following stronger inequality for a connected nonregular graph G :

$$\Delta - \lambda_1 > \frac{n\Delta - 2m}{n(D(n\Delta - 2m) + 1)} \quad (1)$$

The Perron-Frobenius Theorem also implies that if λ_n is the least eigenvalue of a graph, then $\lambda_1 \geq -\lambda_n$ with equality in the connected case if and only if the graph is bipartite [2, p. 178]. For graphs that are connected and nonbipartite (and possibly regular), Alon and Sudakov [1] proved that

$$\Delta + \lambda_n > \frac{1}{n(D + 1)} \quad (2)$$

Because the lower bound in (1) is monotone increasing in $n\Delta - 2m$, it follows that, for a connected nonregular graph G ,

$$\Delta + \lambda_n \geq \Delta - \lambda_1 > \frac{n\Delta - 2m}{n(D(n\Delta - 2m) + 1)} \geq \frac{1}{n(D + 1)} \quad (3)$$

Thus, the bound (2) of Alon and Sudakov also holds for connected nonregular graphs and, in that case, is refined by the inequalities (3).

2 The proof

In this section, we present a proof of inequality (1).

Theorem 1 *If G is a connected nonregular graph with diameter D , then*

$$\Delta - \lambda_1 > \frac{n\Delta - 2m}{n(D(n\Delta - 2m) + 1)}$$

PROOF. Let x be the unique unit positive eigenvector of A with eigenvalue λ_1 . Let E denote the edge set of G and, for each $i \in [n]$, let d_i denote the degree of vertex i . Then

$$\begin{aligned} \lambda_1 &= x^t A x = 2 \sum_{ij \in E} x_i x_j = \sum_{ij \in E} (x_i^2 + x_j^2) - \sum_{ij \in E} (x_i - x_j)^2 \\ &= \sum_{i=1}^n d_i x_i^2 - \sum_{ij \in E} (x_i - x_j)^2 \end{aligned}$$

This implies

$$\Delta - \lambda_1 = \sum_{ij \in E} (x_i - x_j)^2 + \sum_{i=1}^n (\Delta - d_i) x_i^2 \quad (4)$$

Choose vertices $s, t \in [n]$ so that $x_s = \max_i x_i$ and $x_t = \min_i x_i$. Then $t \neq s$ because G is nonregular. If $s = i_0, i_1, \dots, i_{k-1}, i_k = t$ are consecutive vertices of a shortest path from s to t in G , it follows from the Cauchy-Schwarz inequality that

$$\sum_{j=0}^{k-1} (x_{i_j} - x_{i_{j+1}})^2 \geq \frac{1}{k} \left(\sum_{j=0}^k (x_{i_j} - x_{i_{j+1}}) \right)^2 = \frac{1}{k} (x_s - x_t)^2 \geq \frac{1}{D} (x_s - x_t)^2.$$

Thus, from (4) we obtain

$$\Delta - \lambda_1 \geq \frac{(x_s - x_t)^2}{D} + (n\Delta - 2m)x_t^2$$

where the right-hand side is a quadratic function of x_t that attains its minimum when $x_t = \frac{x_s}{D(n\Delta - 2m) + 1}$. It follows that

$$\Delta - \lambda_1 \geq \frac{(n\Delta - 2m)x_s^2}{D(n\Delta - 2m) + 1}$$

where $x_s^2 > \frac{1}{n}$ since $x^T x = 1$. The statement in Theorem 1 now follows. \square

A further refinement of Theorem 1 may be possible. For example, a computer search shows that $\Delta - \lambda_1 > (\Delta - \delta)^{\frac{1}{2}}/nD$ for all connected nonregular graphs of order at most 8.

3 Examples

The lower bound on $\Delta - \lambda_1$ in Theorem 1 is of order $O(\frac{1}{nD})$. The bipartite graphs in the following proposition suggest that for connected nonregular graphs with $\Delta - \delta = 1$, this order is best possible.

Let G_1, \dots, G_k be k disjoint copies of the complete bipartite graph $K_{\Delta, \Delta}$. Remove an edge, say $v_{2i-1}v_{2i}$, from each G_i , and join v_{2i} to v_{2i+1} for each $i = 1, \dots, k-1$. Let $G_{\Delta, k}$ denote the resulting chain of bipartite graphs. Clearly $G_{\Delta, k}$ has $n = 2k\Delta$ vertices, maximum degree Δ , minimum degree $\delta = \Delta - 1$ and diameter $D = 4k - 1$.

Proposition 2 For the graphs $G_{\Delta, k}$, $\Delta - \lambda_1 < \frac{4\pi^2}{nD}$.

PROOF. Let A be the adjacency matrix of $G_{\Delta, k}$ and let $U = \{v_1, v_{2k}\}$, the two vertices of degree $\delta = \Delta - 1$. For each unit vector z , because $\lambda_1 \geq z^T A z$, the argument used to prove (4) gives

$$\Delta - \lambda_1 \leq \sum_{ij \in E} (z_i - z_j)^2 + \sum_{i \in V} (\Delta - d_i) z_i^2 = \sum_{ij \in E} (z_i - z_j)^2 + \sum_{i \in U} z_i^2 \quad (5)$$

Let y be the unit eigenvector corresponding to the greatest eigenvalue $2 \cos \frac{\pi}{k+1}$ of the path P_k on k vertices. By (4),

$$\sum_{i=1}^{k-1} (y_i - y_{i+1})^2 + y_1^2 + y_k^2 = 2 - \lambda_1(P_k) < \frac{\pi^2}{(k+1)^2} \quad (6)$$

For $v \in V(G_{\Delta, k})$, let $z_v = y_i/\sqrt{2\Delta}$ when $v \in G_i$. Then $z^T z = 1$. Substituting z into (5), we obtain

$$\Delta - \lambda_1 \leq \sum_{i=1}^{k-1} \left(\frac{y_i - y_{i+1}}{\sqrt{2\Delta}} \right)^2 + \frac{y_1^2}{2\Delta} + \frac{y_k^2}{2\Delta} < \frac{\pi^2}{2\Delta(k+1)^2} < \frac{4\pi^2}{nD}$$

□

We noted above that Theorem 1 implies that there is a constant c such that $\Delta - \lambda_1 > \frac{c}{nD}$ for all connected nonregular graphs. It is easy to check that the theorem implies that $c \geq 2/3$. Taking a graph formed from the binary

tree on 7 vertices, together with a 4-cycle on the vertices of degree 1, we get $c \leq nD(\Delta - \lambda_1) \approx 1.355$. We conjecture that $c \geq 1$ or, equivalently, that $\Delta - \lambda_1 > 1/nD$ for all connected nonregular graphs.

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References

- [1] N. Alon and B. Sudakov, Bipartite Subgraphs and the Smallest Eigenvalue, *Combin. Probab. Comput.*, **9** (2000), no. 1, 1-12.
- [2] C. Godsil and G. Royle, Algebraic Graph Theory, Springer-Verlag, (2001).
- [3] D. Stevanović, The largest eigenvalue of nonregular graphs, *J. of Combin. Theory Ser. B*, **91** (2004), no. 1, 143-146.
- [4] X.-D. Zhang, Eigenvectors and eigenvalues of non-regular graphs, *Linear Algebra and its Applications*, **409** (2005), 79-86.