Eigenvalues of graphs and a simple proof of a theorem of Greenberg

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Abstract

In his Ph.D. thesis, Greenberg proved that if $\rho(\tilde{X})$ is the spectral radius of the universal cover $\tilde{X}$ of a finite graph X, then for each $\epsilon > 0$, a positive proportion (depending only on $\tilde{X}$ and ϵ) of the eigenvalues of X have *absolute value* at least $\rho(\tilde{X}) - \epsilon$. In this paper, we show that the same result holds true if we remove *absolute* from the previous result. We also prove an analogue result for the smallest eigenvalues of X.

Key words: spectral radius, universal cover, eigenvalues
AMS Classifications: 05C50, 15A18

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1 Preliminaries

Our graph theoretic notation is standard, see West [18]. The graphs discussed in this paper are simple and connected unless stated otherwise. For a graph X, we denote by $\lambda_1(X) \geq \lambda_2(X) \geq \cdots \geq \lambda_n(X)$ the eigenvalues of the adjacency matrix of X.

Serre has proved the following theorem (see [4,5,11,17]) using Chebyschev polynomials. The simplest self-contained proof of this theorem is given in [4] and it is fairly involved. See [2,3] for a simple proof of Serre's theorem and other related results.

Theorem 1 (Serre) *For each $\epsilon > 0$, there exists a positive constant $c = c(\epsilon, k)$ such that for any k-regular graph X, the number of eigenvalues λ of X with $\lambda \geq (2 - \epsilon)\sqrt{k-1}$ is at least $c|X|$.*

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[1] Research partially supported by an Ontario Graduate Scholarship
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Serre's theorem is a generalization of the asymptotic Alon and Boppana theorem, see [1,6,16] for more details.

Theorem 2 *If* $(X_n)_n$ *is an infinite sequence of* k*-regular graphs, then* $\liminf \lambda_2(X_n) \geq 2\sqrt{k-1}$.

Theorem 1 is also related to a result obtained by Greenberg in [9] whose proof has not appeared to our knowledge in any journal as of yet. Greenberg's result is cited in many places, [14] and [15] for example. For a graph X, we denote by $\rho(X)$ its spectral radius and by $\mathcal{C}(X)$ the family of all finite graphs that are covered by X. In Section 2, we describe these notions in more detail.

Theorem 3 (Greenberg) *Let* X *be a connected, infinite graph with finite maximum degree. Given* $\epsilon > 0$*, there exists* $c = c(X, \epsilon) > 0$*, such that for every* $Y \in \mathcal{C}(X)$*,*

$$|\{\lambda \in \textit{spectrum of } Y : |\lambda| \geq \rho(X) - \epsilon\}| \geq c|V(Y)|. \tag{1}$$

In Section 3, we present a proof of Theorem 3. Note that Theorem 3 implies a weaker form of Serre's theorem. This is because if X is the infinite k-regular tree, then $\rho(X) = 2\sqrt{k-1}$. Thus, if Y is a finite k-regular graph we obtain that for each $\epsilon > 0$, a positive proportion (that depends only on ϵ and k) of the eigenvalues of Y have *absolute value* at least $(2-\epsilon)\sqrt{k-1}$. This is slightly weaker than Theorem 1.

In Section 4, we present a slight improvement of Greenberg's theorem. We prove that given X and $\epsilon > 0$, there is a positive $c = c(X, \epsilon)$ such that $|\{\lambda \in \text{spectrum of } Y : \lambda \geq \rho(X) - \epsilon\}| \geq c|V(Y)|$ for each finite graph Y covered by X. We also prove a similar result regarding the smallest eigenvalues of general (not necessarily regular) graphs.

2 Graphs and coverings

If X is a connected graph (not necessarily finite) such that the maximum degree of X is finite, let $l^2(X)$ denote the space of functions $f : V(X) \to \mathbb{R}$ with $\sum_{x \in V(X)} |f(x)|^2 < +\infty$. Let $\delta : l^2(X) \to l^2(X)$ be the adjacency operator of X, i.e., $(\delta f)(x) = \sum_{yx \in E(X)} f(y)$. If $x \in V(X)$, let $t_s(x)$ denote the number of closed walks of length s that start at x. Denote by $\rho(X)$ the *spectral radius* of X:

$$\rho(X) = \sup\{|\lambda| : \lambda \in \text{spectrum of } \delta\}$$

Lemma 4 *Let* X *be a connected graph. Then* $\limsup_{s \to +\infty} \sqrt[s]{t_s(x)}$ *is independent* *on the vertex $x \in V(X)$.*

PROOF. Since X is connected, it is enough to prove that

$$\limsup_{s\to+\infty} \sqrt[s]{t_s(x)} = \limsup_{s\to+\infty} \sqrt[s]{t_s(y)}$$

for any adjacent vertices x and y. The previous assertion will follow easily from the fact that

$$t_{s+2}(y) \geq t_s(x) \geq t_{s-2}(y)$$

for each $s \geq 2$.

It is well known (cf. Lubotzky [13], Chapter 4) that

$$\rho(X) = \limsup_{s\to+\infty} \sqrt[s]{t_s(x)}, \tag{2}$$

Given two graphs X_1 and X_2, a *homomorphism* from X_1 to X_2 is a function $f : V(X_1) \to V(X_2)$ such that $xy \in E(X_1)$ implies $f(x)f(y) \in E(X_2)$ for each $x, y \in V(X_1)$. If f is bijective and f and f^{-1} are both homomorphisms, then f is called an *isomorphism* from X_1 to X_2. An isomorphism from a graph X to itself is called an *automorphism* of X. The automorphisms of X form a group, called the *automorphism group* of X that we denote by $\mathrm{Aut}(X)$. If x is a vertex of X, then the *automorphism orbit* of x is $\mathrm{Orb}(x) = \{y \in V(X) : \exists f \in \mathrm{Aut}(X) \text{ such that } f(x) = y\}$. If x is a vertex in the graph X, then we denote by $N_X(x)$ the set of neighbours of x in X.

If X_1 and X_2 are two graphs, a homomorphism $\pi : V(X_1) \to V(X_2)$ is called a *cover map* if it is surjective and for each $x \in V(X_1)$, π induces an isomorphism from $N_{X_1}(x)$ to $N_{X_2}(\pi(x))$. It follows from (2) that if $\pi : V(X_1) \to V(X_2)$ is a cover map, then $\rho(X_1) \leq \rho(X_2)$. If π is a finite cover, then $\rho(X_1) = \rho(X_2)$. Denote by $\mathcal{C}(X)$ the family of finite graphs covered by X.

Using a result of Leighton [12], the next theorem is also proved by Greenberg [9] (see also Lubotzky [14]).

Theorem 5 *Let X be a connected graph with finite maximum degree. Then for each X_1 and X_2 in $\mathcal{C}(X), \rho(X_1) = \rho(X_2)$. This common value is denoted by $\chi(X)$.*

For a finite graph Z, its *universal cover* $\tilde{Z}$ is the graph with the property that for any graph Y with a cover map $\pi : V(Y) \to V(Z)$, there exists a cover map $\pi' : V(\tilde{Z}) \to V(Y)$. The universal cover of any finite graph is an infinite tree. For example, the universal cover of a k-regular graph is the infinite k-regular tree. However, not every infinite tree X covers a finite graph. It is easy to see that a necessary condition for covering a finite graph is that $\mathrm{Aut}(X)$ has finitely many orbits. See Lubotzky [14] for more details on the universal covers of finite graphs.

3 A proof of Greenberg's theorem

In his Ph.D. thesis [9], Greenberg proved Theorem 3. This result is also cited in [14] and [15] (Theorem 2.3), but it seems that no proof of it exists in the literature other than in Greenberg's thesis. The proof given below is a simplified version of the original proof.

Proof of Theorem 3 Let $\epsilon > 0$ and $Y \in \mathcal{C}(X)$. Because X has finitely many automorphism orbits, it follows that there exists $r_0 = r(X, \epsilon)$ such that

$$t_{2r}(y) \geq \left(\rho(X) - \frac{\epsilon}{2}\right)^{2r}$$

for each vertex $y \in V(Y)$ and $r \geq r_0$. Let c be the proportion of eigenvalues of Y that have absolute value $\geq \rho(X) - \epsilon$. Using the previous inequality, we obtain

$$\left(\rho(X) - \frac{\epsilon}{2}\right)^{2r} \leq \min_{y \in V(Y)} t_{2r}(y) \leq \frac{\mathrm{tr}(A^{2r}(Y))}{|V(Y)|} \leq c\chi^{2r}(X) + (1-c)\left(\rho(X) - \epsilon\right)^{2r}$$

for each $r \geq r_0$. This implies

$$c \geq \frac{\left(\rho(X) - \frac{\epsilon}{2}\right)^{2r} - \left(\rho(X) - \epsilon\right)^{2r}}{\chi^{2r}(X) - \left(\rho(X) - \frac{\epsilon}{2}\right)^{2r}}$$

for each $r \geq r_0$. Letting $r = r_0$, this proves the theorem.

□

4 An slight improvement of Greenberg's theorem

In this section, we present a slight improvement of Theorem 3. Our proof is similar to the previous one and we obtain the required estimate for the largest eigenvalues by shifting the spectra of Y up by a constant.

Theorem 6 *Let X be a connected, infinite graph with finite maximum degree. Given $\epsilon > 0$, there exists $c = c(X, \epsilon) > 0$, such that for every $Y \in \mathcal{C}(X)$,*

$$|\{\lambda \in \textit{spectrum of } Y : \lambda \geq \rho(X) - \epsilon\}| \geq c|V(Y)|$$

Proof. Let $Y \in \mathcal{C}(X)$ with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and cover map $f : V(X) \to V(Y)$. Given $\epsilon > 0$, let $m = |\{i : \lambda_i \geq \rho(X) - \epsilon\}|$.

From (2) we know that $\rho(X) = \limsup\limits_{s \to +\infty} \sqrt[2s]{t_{2s}(x)}$, for any vertex $x \in V(X)$. Since $\rho(X) > 0$, it follows that there exists an integer $N = N(X, \epsilon) > 1$ such that $\rho(X) > \frac{\epsilon}{N}$. Obviously, if x and y are in the same orbit of $\mathrm{Aut}(X)$, then $t_r(x) = t_r(y)$ for any non-negative integer r. Hence, the fact that X has finitely many automorphism orbits implies that there exists a non-negative integer $s_0 = s(X, \epsilon)$ such that $t_{2s}(x) \geq \left(\rho(X) - \frac{\epsilon}{N}\right)^{2s}$, for each $s \geq s_0$ and any $x \in V(X)$.

It is easy to see that the number of closed walks of length r in X starting at vertex $x \in V(X)$ is less than or equal to the number of closed walks of length r in Y starting at vertex $f(x) \in V(Y)$. Hence, $\Phi_r(Y) \geq \sum\limits_{\substack{y \in V(Y) \\ y = f(x)}} t_r(x)$ for each non-negative integer r. From the previous two relations it follows that $\Phi_{2s}(Y) \geq n\left(\rho(X) - \frac{\epsilon}{N}\right)^{2s}$ for $s \geq s_0$.

Let K be a positive constant that does not depend on Y and is larger than $\lambda_1(Y)$. We can take $K = \Delta(X)$, the maximum degree of X, for example. From the previous inequality we deduce

$$\begin{aligned}
\mathrm{tr}(K \cdot I + A(Y))^{2l} &= \sum_{i=0}^{2l} \binom{2l}{i} K^{2l-i} \Phi_i(Y) \geq \sum_{j=s_0}^{l} \binom{2l}{2j} K^{2l-2j} \Phi_{2j}(Y) \\
&\geq n \sum_{j=s_0}^{l} \binom{2l}{2j} K^{2l-2j} \left(\rho(X) - \frac{\epsilon}{N}\right)^{2j} \\
&\geq n \sum_{j=0}^{l} \binom{2l}{2j} K^{2l-2j} \left(\rho(X) - \frac{\epsilon}{N}\right)^{2j} - n \sum_{j=0}^{s_0-1} \binom{2l}{2j} K^{2l-2j} \left(\rho(X) - \frac{\epsilon}{N}\right)^{2j} \\
&\geq \frac{n}{2}\left(K + \rho(X) - \frac{\epsilon}{N}\right)^{2l} - n s_0 \binom{2l}{2s_0} K^{2l}
\end{aligned}$$

for each $l \geq 2s_0$. Since $K \cdot I$ and $A(Y)$ commute, it follows that

$$\begin{aligned}
\mathrm{tr}(K \cdot I + A)^{2l} &= \sum_{i=1}^{n} (K + \lambda_i(Y))^{2l} \\
&\leq (n - m)(K + \rho(X) - \epsilon)^{2l} + m(2K)^{2l}.
\end{aligned}$$

Hence, we obtain

$$\frac{m}{n} \geq \frac{\frac{\left(K+\rho(X)-\frac{\epsilon}{N}\right)^{2l}}{2} - (K+\rho(X)-\epsilon)^{2l} - s_0\binom{2l}{2s_0}K^{2l}}{(2K)^{2l} - (K+\rho(X)-\epsilon)^{2l}} \tag{3}$$

for each $l \geq 2s_0$.

Now

$$\lim_{l\to\infty} \sqrt[2l]{\frac{\left(K+\rho(X)-\frac{\epsilon}{N}\right)^{2l}}{2}} = K+\rho(X)-\frac{\epsilon}{N}$$

and

$$\lim_{l\to\infty} \sqrt[2l]{2\left(K+\rho(X)-\epsilon\right)^{2l} + s_0\binom{2l}{2s_0}K^{2l}} = \max(K+\rho(X)-\epsilon, K) < K+\rho(X)-\frac{\epsilon}{N}$$

imply that there exists $l_0 = l(X,\epsilon)$ such that

$$\frac{\left(K+\rho(X)-\frac{\epsilon}{N}\right)^{2l}}{2} - \left(K+\rho(X)-\epsilon\right)^{2l} - s_0\binom{2l}{2s_0}K^{2l} > (K+\rho(X)-\epsilon)^{2l},$$

for each $l \geq l_0$. Hence,

$$\frac{m}{n} > \frac{(K+\rho(X)-\epsilon)^{2l_0}}{(2K)^{2l_0} - \left(K+\rho(X)-\epsilon\right)^{2l_0}} = c(X,\epsilon) > 0$$

□

By using a similar argument as before, we can also prove a similar result to Theorem 6 for the smallest eigenvalues. Note that we need an extra hypothesis since there are classes of graphs that have eigenvalues bounded from below by a constant. For examples, line graphs have all their eigenvalues at least -2.

Theorem 7 *Let X be a connected, infinite graph with finite maximum degree. Given $\epsilon > 0$, there exist a non-negative integer $g = g(X,\epsilon)$ and $c = c(X,\epsilon) > 0$, such that for every graph $Y \in \mathcal{C}(X)$ with no odd cycles of length less than g,*

$$|\{\mu \in \textit{spectrum of } Y : \mu \leq -(\rho(X)-\epsilon)\}| \geq c|V(Y)|$$

Proof. Let $Y \in \mathcal{C}(X)$ with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and cover map $f : V(X) \to V(Y)$. Given $\epsilon > 0$, let $m = |\{i : \lambda_i \leq -(\rho(X)-\epsilon)\}|$.

The proof continues now similarly to the proof of Theorem 6. There exist $N = N(X, \epsilon) > 1$, $s_0 = s(X, \epsilon)$ and $l_0 = l(X, \epsilon)$ with $l_0 \geq 2s_0$ such that

$$\frac{\left(K + \rho(X) - \frac{\epsilon}{N}\right)^{2l_0}}{2} - (K + \rho(X) - \epsilon)^{2l_0} - s_0 \binom{2l_0}{2s_0} K^{2l_0} > (K + \rho(X) - \epsilon)^{2l_0}$$

Consider

$$\operatorname{tr}(K \cdot I - A(Y))^{2l} = \sum_{i=1}^{n} (K - \lambda_i)^{2l} \leq (n - m)(K + \rho(X) - \epsilon)^{2l} + m(2K)^{2l}$$

Let $g(X, \epsilon) = 2l_0$. If Y has no odd cycles of length less than $2l_0$, then

$$\begin{aligned}
\operatorname{tr}(K \cdot I - A(Y))^{2l_0} &= \sum_{j=0}^{l_0} \binom{2l_0}{2j} K^{2l_0 - 2j} \Phi_{2j}(Y) \\
&\geq \sum_{j=s_0+1}^{l_0} \binom{2l_0}{2j} K^{2l_0 - 2j} \Phi_{2j}(Y) \\
&\geq n \sum_{j=0}^{2l_0} \binom{2l_0}{2j} K^{2l_0 - 2j} \left(\rho(X) - \frac{\epsilon}{N}\right)^{2j} - s_0 \binom{2l_0}{2s_0} K^{2l_0} \\
&\geq \frac{n}{2} \left(K + \rho(X) - \frac{\epsilon}{N}\right)^{2l_0} - s_0 \binom{2l_0}{2s_0} K^{2l_0}
\end{aligned}$$

From the previous two inequalities, we deduce

$$\begin{aligned}
\frac{m}{n} &\geq \frac{\frac{\left(K + \rho(X) - \frac{\epsilon}{N}\right)^{2l_0}}{2} - (K + \rho(X) - \epsilon)^{2l_0} - s_0 \binom{2l_0}{2s_0} K^{2l_0}}{(2K)^{2l_0} - (K + \rho(X) - \epsilon)^{2l_0}} \\
&> \frac{(K + \rho(X) - \epsilon)^{2l_0}}{(2K)^{2l_0} - (K + \rho(X) - \epsilon)^{2l_0}}
\end{aligned}$$

This proves the theorem.

□

In his Ph.D. thesis [9], Greenberg also introduced the notion of Ramanujan graph for general finite graphs (not necessarily regular). See also Lubotzky [14] for more details. A finite graph Y is called *Ramanujan* if for any eigenvalue $\lambda \neq \pm\chi(\tilde{Y})$ of Y, the inequality $|\lambda| \leq \rho(\tilde{\tilde{Y}})$ holds. If Y is regular, then we obtain the definition given by Lubotzky, Phillips and Sarnak in [16] where an infinite sequence of regular Ramanujan graphs is constructed when the degree equals $p + 1$, $p \equiv 1 \pmod 4$ prime number. In [4], this construction is extended and an infinite sequence of regular Ramanujan graphs is produced for the degree equal to a prime plus one. We are not aware of any similar results for irregular graphs as of yet.

Friedman [8] has recently proved that almost all regular graphs are almost Ramanujan. For similar results regarding irregular graphs, see [7]. Hoory [10] proved that if Y is a finite graph with average degree d, then $\rho(\tilde{Y}) \geq 2\sqrt{d-1}$. Hoory used this result to prove a generalization of the asymptotic Alon-Boppana theorem. Denote by $B_r(v)$ the ball of radius r around v. A graph Y has an r-robust average-degree d if for every vertex v the graph induced on $V(Y) \setminus B_r(v)$ has average degree at least d. Hoory's generalization is the following result.

Theorem 8 *Let Y_i be a sequence of graphs such that Y_i has an r_i-robust average degree $d \geq 2$, where $\lim_{i\to+\infty} r_i = +\infty$. Then*

$$\liminf_{i\to\infty} \lambda(Y_i) \geq 2\sqrt{d-1}$$

Acknowledgements

I am grateful to my Ph.D. advisors: David Gregory, Ram Murty and David Wehlau for their support and encouragement. I thank Shlomo Hoory for translating and adapting Greenberg's argument from Section 3 and an anonymous referee for his(her) comments.

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