HOLISTIC PROBLEMS WITH PEDAGOGICAL COMMENTARY

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1. Introduction

This collection of problems is designed for mathematics teachers of all levels. Problems are everywhere – as teachers, we needn't reinvent the wheel. The key is how we use the problems. Our intention in creating this document was not to provide an exhaustive list of problems, but rather to share problems which can be used to supplement the existing curriculum and moreover to share an approach to using problems with our students. Our hope was to accompany each included problem with enough pedagogical discussion so that the approach to utilizing problems described herein can become second nature to the reader. The reader can then apply the approach to problems from other sources.

The problems themselves have been used in various professional development programs for both in-service and pre-service mathematics teachers, including the Algebraic Thinking Institute, an intensive summer professional development institute for in-service teachers, held at the University of California, San Diego from 1999 until 2004. Some of the problems were designed from scratch, whereas others were adapted from textbooks. The unifying theme behind the problems and the approach to using the problems is a theoretical framework called DNR [H08]. DNR aims at helping educators to make decisions as to what mathematics to teach and how to teach it effectively. In particular, DNR asserts that we should design our instructional objectives not only in terms of particular content, but also in terms of habits of mind that are crucial to practicing mathematics, for example, problem solving approaches, attention to structure, deductive reasoning, reasoning from definitions, and so on. These habits of mind, referred to in DNR as "ways of thinking", are similar in spirit to the Common Core "Standards for Mathematical Practice" [CCS10]. Another feature of the problems is that they are *holistic*; a holistic problem refers to a problem where one must figure out from the problem statement the elements needed for its solution – it does not include hints or cues as to what is needed to solve it. In contrast, a non-holistic problem is one which is broken down into small parts, each of which attends to one or two isolated elements [H].

The problems in this collection are organized into seven sections: *ratio and proportion, reasoning quantitavely, absolute value and inequalities, problems about patterns, thinking in terms of functions, problems about quadratic functions,* and *geometry*. Within a given

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section, each problem or cluster of grouped problems is immediately followed by a four part pedagogical analysis addressing:

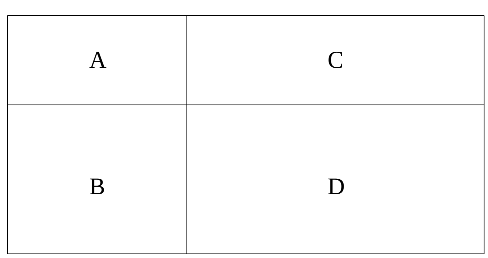
(1) What students need to know in order to solve the problem(s)
(2) What students gain from solving the problem(s)
(3) Expected difficulties that students will face when solving the problem(s) and suggestions for dealing with these difficulties
(4) Relevant Common Core Standards for Mathematical Content

In using this booklet with your students, you can choose whichever problems you like in whichever order you prefer. Each pedagogical analysis is self-contained, and therefore if you read most of the analyses, you will find certain blocks of texts repeated because they are relevant to different problems.

We have left some space for you to solve the problems (although you may need more room!), and we recommend that you solve them prior to reading the corresponding pedagogical analysis so that your creativity and problem solving approaches will not be influenced by the analysis. The analyses do not contain full solutions to the problems, but in some cases we outline various possible approaches. The problems can usually be solved in multiple ways, and we don't want to dictate one particular approach. Similarly, when we use these problems with our students, we always give them the opportunity to approach the problems on their own. Some of the most powerful learning experiences in our classrooms come from analyzing erroneous solutions, so allowing students to make errors is an important part of the learning process. Even more generally, the underlying principle guiding our methods is that we impart the responsibility of learning to our students.

We close the introduction with a sample problem and pedagogical analysis. For a more extensive discussion of this problem, see [H10].

Rectangular Land Problem: A man owns a rectangular piece of land. The land is divided into four rectangular pieces, known as Region A, Region B, Region C, and Region D (see the figure below). One day his daughter, Nancy, asked him, what is the area of our land? The father replied: I will only tell you that the area of Region B is 200 ft^2 larger than the area of Region A; the area of Region C is 400 ft^2 larger than the area of Region B; and the area of Region D is 800 ft^2 larger than area of Region C. What answer to her question will Nancy derive from her father's statement?



Your Solution to Rectangular Land Problem:

Analysis of Rectangular Land Problem:

What students need to know in order to solve the problem:

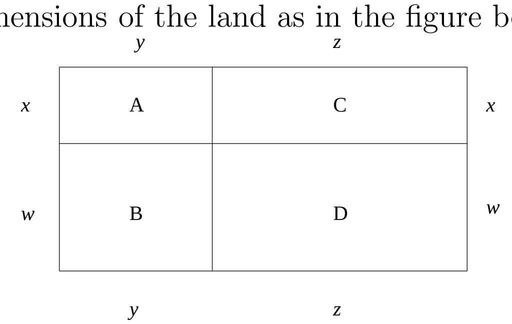
Students should have experience with systems of equations in more than one variable.

Students need to know how to compute the area of a rectangle.

What students gain from solving the problem:

In solving this problem, students will advance their abilities to "tell a given problem to algebra", i.e. to set up a system of equations which represent the problem and all of the constraints of the problem. In our experience, students often neglect to include in their algebraic representation the constraint that neighboring regions share a common side (see the expected difficulties), so the problem can be particularly powerful in demonstrating that all problem constraints must be represented.

There are multiple approaches to this problem, but upon suspecting that the area of the land is uniquely determined (see the expected difficulties), a relatively common approach is to set up a system of four equations in five unknowns. For example, if we label the dimensions of the land as in the figure below,



and let A denote the area of region A, then we have the system:

$$xy = A,\ wy = A + 200,\ xz = A + 600,\ wz = A + 1400.$$

Students usually approach this system haphazardly, adding or subtracting pairs of equations. Many students erroneously think the resulting equations correspond to new constraints, but realizing that this is not so reinforces the idea of what an equation actually is. In this case, dividing the first and the third equations by the second and fourth, respectively, one obtains two expressions in A equalling x/w, and the problem is reduced to solving a linear equation in A. Thus, this problem is an excellent opportunity for students to see the value in pausing before attacking and manipulating systems of equations purposefully.

This problem is a multi-step *holistic* problem; in particular, it does not include hints or cues as to what must be done in order to solve it.

The question – "What answer will Nancy derive...." is intentionally worded so as not to give any suggestions or hints as how to how to approach the problem. Considering the possibilities of the nature of a solution to a problem and determining how to start are extremely important mathematical practices.

Expected difficulties that students will face when solving the problem and suggestions for dealing with these difficulties:

In our experience, students usually conclude that the given information does not determine the area of the rectangular land. In particular, letting A, B, C, and D denote the areas of regions A, B, C, and D, respectively, students translate the farmer's statement into the following system of equations:

$$B = A + 200, \qquad C = B + 400, \qquad D = C + 800.$$

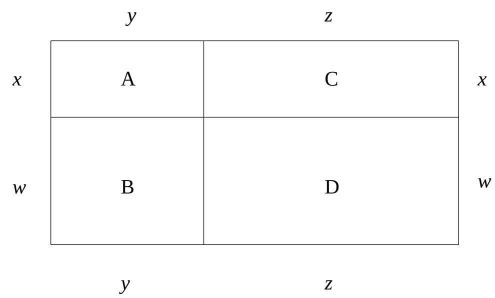
Upon arriving at this point, some students attempt to construct a fourth equation such as

$$\mathrm{B+C+D} = (\mathrm{A}+200)+(\mathrm{B}+400)+(\mathrm{C}+800)$$

and then manipulate the four equations in hopes of of determining a unique value of each unknown. When they cannot find a unique solution to their system, the students conclude that the total area is $4A + 2200$, and hence, there are infinitely many values for the area of the land, each value being dependent on the choice of the value A.

At this point, we suggest that the teacher present the students with a new task, namely: Pick two different values for A. For each value, construct a figure that illustrates your solution. Students may need help understanding that this task entails not only substituting two random values for A into the expression $4A + 2200$, but that one must also show that the areas of the four regions entailed by the choice of A must be such that the geometric configuration of the four regions is preserved.

For example, the teacher may suggest taking $A = 100$. The students should conclude that $B = 300$, $C = 700$, and $D = 1500$. Next they need to come up with dimensions for each portion of the land. Here we encourage the teacher to have the students do this individually and then discuss their findings together as a class.



For example, with the figure labelled as above, if $x = 5$, then $xy = A = 100$, so $y = 20$. In turn, $wy = B = 300$, so $w = 15$. Since $xz = C = 700$, $z = 140$. Then $D = wz = 1500$. But $wz = 15 \times 140 = 2100$, so something must be wrong. Different students should suggest various values for x, and each time they will lead to a contradiction (the teacher may even suggest trying an irrational value!).

The students will eventually suspect that $A = 100$ is not possible, and one can justify this conjecture by using a parameter such as t to represent the width of A so that the length is A/t, and the length of B is $(A + 200)/t$, and so on.

Once the students see that $A = 100$ is not possible, they may also suspect that other values of A are not possible either. At this point, the teacher can ask the class what went wrong in their first approach to the problem. Eventually the students will realize that their initial system of equations left out an essential constraint, namely, the fact that neighboring regions share a common side.

Relevant standards:

4.MD3: Apply the area and perimeter formulas for rectangles in real world and mathematical problems.

A-CED3: Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or nonviable options in a modeling context.

A-CED4: Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations.

2. Ratio and Proportion

Proportional reasoning is a unifying form of mathematical reasoning that can be applied to a wide range of problems such as problems about fractions and percentages. Many students are taught to solve problems about percentages by using rote strategies to choose an appropriate formula from one of a collection; unfortunately this matching strategy does little for the students' reasoning skills and is unlikely to contribute to their success in future mathematics courses. In contrast, if students can reason proportionally, they can solve problems without relying on formulas; thus, the ability to reason proportionally empowers students.

Jill and Jack's Investment Problems: Jill and Jack invested money in a mutual fund for one year. Jill invested $\$23000$, and Jack invested $\$22950$. Their broker deducted a 5.5% commission before turning the rest of the money over to the mutual fund. During the year, the value of each share of the mutual fund increased by 11.85%.

(1) What percentage return on their investments did Jill and Jack realize?
(2) John, Jill and Jack's friend, invested $\$17342$ with the same broker. What percentage return on his investment did John realize?

Your Solutions to Jill and Jack's Investment Problems:

Analysis of Jill and Jack's Investment Problems

What students need to know in order to solve the problems:	Students must have a solid grasp of the concepts of percentage and percentage return, as well as a *computational fluency* in calculating percentages. Here it is important to recognize that students have different ways of understanding the concept of percentage. For example, suppose that Juan successfully completes 75% of his free throw shots. One student might understand this to mean that for every 100 free throws that Juan shoots, he makes 75. Another student might understand this to mean that to find the number of successful free throws that Juan shot, we take the total number of free throws and divide by 100 and then multiply the result by 75. Moreover, by computational fluency, we mean that students can calculate percentages using proportional reasoning as opposed to blindly applying formulas. For example, when asked what is 20% of 80, a student can apply the formula $.2 \times 80$ without understanding why the resulting quantity is actually 20% of 80. This is not computational fluency. Computational fluency entails that the student understand, for example, the following line of reasoning: 20% of 100 is 20. But we only have 80. So we should find which portion of 100 is 80, and then 20% of 80 will be that same portion of 20. Since 80 is 8/10 of 100, 20% of 80 is 8/10 of 20. It is critical that students be able to articulate how the computation of a percentage is entailed from a definition of percentage (refer to the discussion of computational fluency in the previous point.)
What students gain from solving the problems:	In solving these problems, students will solidify their understanding of the concept of percentage in a natural way by repeatedly reasoning through a particular definition of percentage. To this end, it is crucial that the teacher bring the students to perform the reasoning behind the formulas rather than blindly employing formulas. For example, for Problem 1, if a student claims that the broker will deduct $.055 \times 23000$ dollars from Jills money, the teacher should ask the student why this is so. In particular, WHY does $.055 \times 23000$ give us 5.5% of \$23000? (Possible answer: 5.5% of 100 is 5.5, and 23000 is 230×100, so 5.5% of 23000 is 230×5.5.)

	This problem also has the potential for eliciting the concept of variable in a natural way as follows. Experience suggests that most students will solve Question 1 making separate calculations for Jack and Jill. The students may be slightly surprised by the answers, but many will likely attribute the equal percentage returns on the two investments to the small difference in the initial investments. Making an independent calculation for Question 2 will once again lead them to puzzlement and an implicit Question 3: Why is the percentage return independent of the initial investment? Allowing the students to arrive at Question 3 independently is paramount to the exercise. If the teacher poses the question before the students discover it, it will weaken the element of surprise, and the question, along with its resolution, will be less legitimate. To resolve Question 3 (refer to the previous comment), students may assign a variable to represent the principal.
Expected difficulties that students will face when solving the problems and suggestions for dealing with these difficulties:	Students may have difficulty sorting out the meaning of the phrase "the value of each share increased by 11.85%", and in general with the notion of percentage increases/decreases. We suggest that before starting the investment problems, the class do some warm-up problems such as: (1) A pair of jeans costs $130. The store selling the jeans decides to have a 20% off sale. What is the sale price of the jeans? (percentage decrease by 20%) (2) The price of the jeans is further reduced by 15%. What is the new price of the jeans? (percentage decrease by 15% after decrease by 20%) (3) You decide to buy the jeans. The sales tax on the jeans is 8%. The cashier computes the sales tax before taking the discounts. Should you complain? (percentage increase by 8% followed by two percentage decreases) In working through these examples, the class should observe that percentage increases and decreases are multiplicative, i.e. increasing a quantity P by $a\%$ and then $b\%$ gives the same result as increasing first by $b\%$ and then $a\%$.
Relevant Standards:	7.RP3: Use proportional relationships to solve multistep ratio and percent problems.

More Investment Problems:

(1) Jack wants to invest $\$50,000$ in a five year CD. He found out that the three banks in his neighborhood sell five year CD's with the same annual interest rate of 7%. However, the three banks have different investment programs. At Bank A the compounding period is annual, at Bank B the compounding period is quarterly, and at Bank C the compounding period is monthly. In addition, Bank A is running a promotional program in which they will add a bonus of 0.5% of the principal for each year of investment (the bonus is paid in a lump sum at the end of the five year term). What percentage bonuses do Bank B and Bank C need to offer in order to remain competitive with Bank A?

(2) A recent report on the cost of college education predicts that in 18 years, the cost will reach $\$63,000$ per year and will continue to increase about 7.5% per year thereafter. You decide to buy a certificate of deposit for your newly born child that will cover at least 75% and at most 87% of the total expenses for four years of college. Your bank offers a certificate of deposit whose interest rate of 10% compounded quarterly is guaranteed for a period of 18 years. If you buy this certificate, what should your investment be?

(3) Suppose that the monthly statement from your mutual fund reports a beginning balance of $\$17,396.17$ and a closing balance of $\$21,034.25$ for 29 days. Using this rate compounded daily, how long do you need to wait to purchase your dream car, a Ferrari, which costs $\$342,000$? If you do not want to wait more than one year, what should have been your minimum beginning balance?

Your Solutions to Investment Problems:

Analysis of Investment Problems

What students need to know in order to solve the problems:	Students must have a solid grasp of the concepts of percentage, as well as a *computational fluency* in calculating percentages (e.g. see analysis of Jill and Jack's Investment problems). It is critical that students be able to articulate how the computation of a percentage is entailed from a definition of percentage. Students must be able to manipulate expressions and solve equations containing exponents. In solving Question 3, students will need to use logarithms. Students must understand the meaning of compound interest: for example interest compounded annually, quarterly, etc. It is critical that students be able to articulate how the frequency of compounding entails the standard formulas for calculating the return on a given investment.
What students gain from solving the problems:	In solving these problems, students will solidify their understanding of the concept of percentage in a natural way by repeatedly reasoning through a definition of percentage. To this end, it is necessary that the teacher bring the student to perform the reasoning behind the formulas rather than blindly employing formulas. For example, for Question 2, if a student claims that the cost of one year of college 19 years in the future will be $\$63,000 \times 1.075$, the teacher should ask the student why this is so. In particular, WHY does $63,000 \times 1.075$ give us 107.5% of 63,000? (Possible answer: 7.5% of 100 is 7.5, and 63,000 is $630 * 100$, so 7.5% of 63,000 is 630×7.5.)

	In solving these problems, students will realize that that the way interest is calculated makes a difference as well as develop fluency calculating compound interest. As usual, computational fluency entails that students understand and are able to articulate WHY different formulas are used to calculate compound interest for compounding intervals of different lengths. To this end, we urge the teacher to bring the students to perform the reasoning behind the formulas rather than blindly employing formulas. For example, in Question 2, students must be able to explain why the balance in bank A after one year equals $50,000 \times (1 + .07)$ whereas the balance is bank B after one year equals $50,000 \times (1 + .07/4)^4$. In particular, what does the quantity $50,000 \times (1 + .07)$ represent? (Possible answers include that $50,000 \times 1$ is the principal, and $50,000 \times .07$ is the interest accrued). Similarly for the balances after two years, etc.
Expected difficulties that students will face when solving the problems and suggestions for dealing with these difficulties:	These problems require careful reading, but this is a *good difficulty* as this difficulty is consistent with the nature of mathematics. The problems are designed to be lengthy in their presentation so that in solving them, students will advance their ability to coordinate information and be empowered to solve future problems. We anticipate that students will have difficulty with the terms "at most" and "at least" in Question 2. We recommend that before starting Question 2, the class do some straightforward warm-up problems using these terms. For example, "There are 20 students in my class. If at least 45% of the students are girls, how many girl students could there be?"
Relevant standards:	7.RP3: Use proportional relationships to solve multistep ratio and percent problems. A-CED1: Create equations and inequalities in one variable and use them to solve problems. Include equations arising from linear and quadratic functions, and simple rational and exponential functions.

Flooded Basement Problem: My basement flooded and there are 2.5 inches of water in it. Last time when it flooded there was $3/8$ inch of water, and it took my pump 45 minutes to pump it out.

(1) How long will it take this time?

(2) When I started my pump, I realized that the pump has enough gasoline for one hour. According to the pump manual, the capacity of the gasoline tank is 0.5 gallon, which is sufficient for 5 hours work. What is the least amount of gasoline I need to add to my pump to ensure that it can pump all of the water out of my basement?

(3) Unfortunately, I have only $1/5$ gallon of gasoline to put in my pump. My neighbor has a portable pump which holds the same amount of gasoline as my pump and pumps water out at the same rate. It has $1/4$ gallon of gasoline. Using all of the gasoline available, would the two pumps be able to pump 100% of the water out of my basement?

(4) If there were enough gasoline and if the two pumps were working together, how long would it take them to pump 100% of the water out of my basement?

(5) My neighbor isn't home, so I have to borrow a pump from my brother. My brother's pump pumps out water at a rate $4/5$ as fast as my pump. If the two pumps were working together, how many inches of water would be pumped out of the basement at any given moment from the start of their working together?

(6) If my and my brother's pumps were working together, how long would it take them to pump all of the water out of the basement?

Your Solution to Flooded Basement Problem:

Analysis of Flooded Basement Problem

What students need to know in order to solve the problem:	Students should have a familiarity with different representations of numbers, e.g. as fractions and decimals, as well as the ability to manipulate numbers in these various forms and convert from one form to another. Students should know how to multiply and divide numbers expressed as fractions and decimals.
What students gain from solving the problem:	Students will strengthen their skills in converting between different representations of numbers. In the first four questions, numbers appear as both decimals and fractions, thus presenting natural situations in which the students must convert from one representation to another. For example, in solving Question 3, one needs to compare the quantities 1/4 gallon and 0.2 gallon. This problem is designed to strengthen students' abilities to reason proportionally. To this end, it is crucial that the teacher encourage proportional reasoning rather than the use of formulas or algorithms such as setting up a ratio and cross-multiplying. A conceptual tool for the proportional reasoning is that of a unit rate, in particular, using the following line of reasoning to find out how long it takes for the pump to pump out 1 inch of water: The pump pumps out 3/8 of an inch in 45 minutes, that is, in 3/4 of an hour. So it pumps out 1/8 of an inch in $3/4 \div 3$ hours. So it pumps put $8/8 = 1$ inch in $3/4 \div 3 \times 8 = 3/4 \times 1/3 \times 8 = 2$ hours. Thus it will take $2 \times 2.5 = 5$ hours to pump out 2.5 inches. As a second example of encouraging proportional reasoning over the blind use of formulas, consider the following solution to Question 4: if one pump takes 5 hours to pump out all of the water, then two pumps working together will take one half of the time, i.e. $5 \times 1/2$ hours. Contrast this with the approach: the amount of time required to empty the water from the basement is the solution of the equation $2t/5 = 1$. Certainly equations are useful tools and ultimately our students must be able to use them appropriately, but in this case, equations are not necessary to solve the problem and in fact, students can use equations to circumvent the actual reasoning required to solve the problem.

We also note that calculators should be avoided here as these calculations will help strengthen students' abilities with algebraic manipulations. Finally, teachers should ensure that the students are indicating the units of inches and hours in their reasoning as the reasoning is quantitative.

This problem also has the potential for eliciting the concept of function in a natural way (Question 5); notice that the wording "at any given moment" encourages the notion of a function as a process which transforms a collection of input objects into output objects, rather than simply as a formula.

This problem is a multi-step *holistic* problem; in particular, it does not include hints or cues as to what must be done in order to solve it.

Expected difficulties that students will face when solving the problem and suggestions for dealing with these difficulties:

To solve Question 6, one approach is to reason that in one hour, the first pump empties 20% of the water and the second pump empties 16% of the water. Thus, finding the amount of time to empty the entire basements amounts to a quotitive division problem, namely how many times does 20%+16% go into 100%?

Another common approach to solving Question 6 is to set up and solve the equation $t/5 + 4t/25 = 1$. This approach is useful provided that the student is not blindly using formulas but rather thinking in terms of functions; in particular the teacher can help the student come to realize that Question 6 is the "reverse" question to Question 5: in Question 5, we want to know how much water has been emptied at any given moment, whereas in Question 6, given an amount of water that has been emptied, how long did it take? Experience suggests that for students who are not thinking in terms of functions, the most difficult aspect of this second approach is the 1 on the right hand side of the equation. The difficulty with interpreting the right hand side comes when one does not view the process of pumping out the water as dynamic. To deal with this difficulty, we suggest that the teacher vary the question before asking how long it will take to pump out all of the water; i.e. first ask: How long before 1/4 of the water has been pumped out? How long before 1/3 of the water has been pumped out? How long before 1/2 of the water has been pumped out?

Relevant standards:	6.RP2: Understand the concept of a unit rate a/b associated with a ratio $a : b$ with $b \neq 0$, and use rate language in the context of a ratio relationship. 7.EE3: Solve multi-step real-life and mathematical problems posed with positive and negative rational numbers in any form (whole numbers, fractions, and decimals), using tools strategically. Apply properties of operations to calculate with numbers in any form; convert between forms as appropriate; and assess the reasonableness of answers using mental computation and estimation strategies.

Carmen's Store Problem: Carmen, a store owner, sells her product according to the following strategy. The percentage she gains on any product she sells is twice what she pays for the product.

(1) The most expensive item in her store sells for $\$600$. How much did she pay for this item?

(2) She paid $\$25$ for one of the items in her store. What is the price tag of this item?

Your Solution to Carmen's Store Problem:

Analysis of of Carmen's Store Problem

What students need to know in order to solve the problem:	Students must have a solid grasp of the concept of percentage, as well as a *computational fluency* in calculating percentages. Here it is important to recognize that students have different ways of understanding the concept of percentage. For example, suppose that Juan successfully completes 75% of his free throw shots. One student might understand this to mean that for every 100 free throws that Juan shoots, he makes 75. Another student might understand this to mean that to find the number of successful free throws that Juan shot, we take the total number of free throws and divide by 100 and then multiply the result by 75. Moreover, by computational fluency, we mean that students can calculate percentages using proportional reasoning as opposed to blindly applying formulas. For example, when asked what is 20% of 80, a student can apply the formula $.2 \times 80$ without understanding why the resulting quantity is actually 20% of 80. This is not computational fluency. Computational fluency entails that the student understand, for example, the following line of reasoning: 20% of 100 is 20. But we only have 80. So we should find which portion of 100 is 80, and then 20% of 80 will be that same portion of 20. Since 80 is 8/10 of 100, 20% of 80 is 8/10 of 20. It is critical that students be able to articulate how the computation of a percentage is entailed from a definition of percentage (refer to the discussion of computational fluency in the previous comment). Students will need to solve quadratic equations.
What students gain from solving the problem:	In solving this problem, students will solidify their understanding of the concept of percentage in a natural way by repeatedly reasoning through a particular definition of percentage. This problem also has the potential for eliciting the concept of variable in a natural way. Some students may set up an equation in two unknowns: both the price Carmen paid for an item and the price she charges in her store. Students who don't write down a general equation will still need to create an equation in one variable for each part of the problem.

Expected difficulties that students will face when solving the problem and suggestions for dealing with these difficulties:	Experience suggests that students will have difficulty conceptualizing percentages using variables. We suggest that the teacher address this difficulty by ensuring that the process of creating expressions and equations is consistent with the students' aforementioned notions of percentage (i.e. as a quantity per 100) in order that the students have an anchor in this more abstract setting. For example, if P denotes the price Carmen pays for an item and Q denotes the price she charges in her store, then P and Q satisfy the equation $100(Q - P)/P = 2P$. It is extremely important that the teacher attend to the meaning of the left hand side of the equation. In particular, the left hand side should represent what percent $Q - P$ is of P. In order to find the percent that $Q - P$ is of P, we can find how many pieces of size $P/100$ go into $Q - P$, in other words, we want to know $(Q - P) \div (P/100)$, which equals the left hand side. We discourage using tricks to create expressions or equations that de-emphasize the meaning.
Relevant standards:	6.RP3c: Find a percent of a quantity as a rate per 100 (e.g., 30% of a quantity means 30/100 times the quantity); solve problems involving finding the whole, given a part and the percent. A-CED1: Create equations and inequalities in one variable and use them to solve problems. Include equations arising from linear and quadratic functions, and simple rational and exponential functions. A-CED2: Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales. A-REI4: Solve quadratic equations in one variable.

TV Rating Problem: During the Evening Prime Time (between 7:00 PM and 9:00 PM) 70% of all TVs in Greenville are turned on, whereas during the Morning Prime Time (between 6:00 AM and 8:00 AM) only 60% of all TVs in Greenville are turned on. Greenville has only four local TV stations: Star Trek Enterprise (STE), Fair and Balance News (FBN), Public Broadcasting Service (PBS), and Quality Family Broadcasting (QFB). John, a local news reporter, obtained from an advertisement firm the "Rating" and "Share" of each station for each of the Prime Times. *The "Rating" shows the percentage of all television sets in the market that are tuned to a given show. The "Share" tells us the percentage of just the television sets in use at a given time that are tuned to a given show.* John intended to publish his data in an article about the four local TV stations. He chose a table like the following to report his data:

	STE		FBN		PBS		QFB	
	Morning Prime Time	Evening Prime Time	Morning Prime Time	Evening Prime Time	Morning Prime Time	Evening Prime Time	Morning Prime Time	Evening Prime Time
Rating								
Share								

Unfortunately for John, he lost some of the data he obtained from the advertisement firm. Here is the data he has left:

	STE		FBN		PBS		QFB	
	Morning Prime Time	Evening Prime Time	Morning Prime Time	Evening Prime Time	Morning Prime Time	Evening Prime Time	Morning Prime Time	Evening Prime Time
Rating	25%			14%		16.75%		
Share		30%			27.5%		11.2%	

(1) The advertisement firm closed down before John completed his article. Can John derive the missing information from what he has left? Help John fill in as many of the other entries in the table as possible.

(2) John published his article on the basis of whatever data he was able to derive. However, in his report he intended, but forgot, to mention the percentage of all TVs in Greenville that is turned on during the Prime Times. Kim, a local junior high school student, read the article and was wondering if she could determine the percentage of all TVs in Greenville that are turned on during the Prime Times. Help Kim by either determining those percentages or explaining to her why you cannot.

Your Solution to TV Rating Problem:

Analysis of TV Rating Problem

What students need to know in order to solve the problem:

Students must have a solid grasp of the concept of percentage, as well as a *computational fluency* in calculating percentages. Here it is important to recognize that students have different ways of understanding the concept of percentage. For example, suppose that Juan successfully completes 75% of his free throw shots. One student might understand this to mean that for every 100 free throws that Juan shoots, he makes 75. Another student might understand this to mean that to find the number of successful free throws that Juan shot, we take the total number of free throws and divide by 100 and then multiply the result by 75. Moreover, by computational fluency, we mean that students can calculate percentages using proportional reasoning as opposed to blindly applying formulas. For example, when asked what is 20% of 80, a student can apply the formula $.2 \times 80$ without understanding why the resulting quantity is actually 20% of 80. This is not computational fluency. Computational fluency entails that the student understand, for example, the following line of reasoning: 20% of 100 is 20. But we only have 80. So we should find which portion of 100 is 80, and then 20% of 80 will be that same portion of 20. Since 80 is 8/10 of 100, 20% of 80 is 8/10 of 20.

It is critical that students be able to articulate how the computation of a percentage is entailed from a definition of percentage (refer to the discussion of computational fluency in the previous comment).

What students gain from solving the problem:

In solving this problem, students will solidify their understanding of the concept of percentage in a natural way by repeatedly reasoning through a definition of percentage.

This problem also has the potential for eliciting the concept of variable in a natural way. Since the total number of televisions is not specified, students who rely on counting to find percentages will be stuck unless they assign a variable to represent the total number of televisions in Greenville.

In solving this problem, students will also have to reason using the previously unfamiliar definitions of *rating* and *share*. Reasoning from definitions is a crucial habit of mind when practicing mathematics.

	This problem is a multi-step *holistic* problem; in particular, it does not include hints or cues as to what must be done in order to solve it.
Expected difficulties that students will face when solving the problem and suggestions for dealing with these difficulties:	A conceptual difficulty for the students is that the problem does not provide them with any formulas. This is a *good difficulty* because the lack of formulas requires the students to reason proportionally, and hence working through the problems empowers them to solve future problems. Students will likely have trouble sorting out the definitions of *rating* and *share*. It will take some discussion before they are familiar with these concepts. We recommend that the teacher spend some time helping to familiarize the students with rating and share by giving a few problems where the students compute rating and share using actual numbers before presenting the students with the TV Rating Problem.
Relevant standards:	7.RP3: Use proportional relationships to solve multistep ratio and percent problems. A-CED1: Create equations and inequalities in one variable and use them to solve problems. Include equations arising from linear and quadratic functions, and simple rational and exponential functions.

3. Reasoning Quantitatively, Without and WIth the Aid of Equations

Reasoning Without the Aid of Equations: Before considering the problems with analysis, we begin with the following five problems which we recommend to be used with the class before the other problems in this section. These problems are intended for use with students of any level, and students who have already taken algebra should be directed to solve them without using variables and equations. We have included these problems to emphasize that students should be educated to reason quantitatively *both prior to and throughout algebra courses*. Often students don't build coherent images of problems because they resort to the use of symbolism too quickly. Moreover, it becomes difficult for students to appreciate the power of symbolism and equations when they are told to use variables and equations in problems in which it is not necessary to use them.

(1) Kim walks 80 minutes per day at a constant pace. Suppose she wants to increase the distance she walks by 5% and that she wants to cover this new distance in 70 minutes instead of 80. Help Kim determine how much faster her new pace should be.
(2) Team A of workers can complete a task in 9 days. Team B is slower and can complete the same task in 12 days. Team A worked on the task for 3 days and then Team B joined them. How many days did it take to complete the task?
(3) Towns A and B are 280 miles apart. At noon, a car leaves A and drives toward B, and a truck leaves B and drives toward A. The car drives at 80 mph and the truck at 60 mph. When will they meet?
(4) Towns A and B are 219.51 miles apart. At noon, a car leaves A and drives toward B, and a truck leaves B and drives toward A. The car drives at 80.23 mph and the truck at 60.59 mph. When will they meet?
(5) A train passes through a tunnel 7 miles long in 2 minutes. It takes 15 seconds from the time the front of the train is out of the tunnel until the whole train is out of the tunnel. What are the length and speed of the train?

Next we present the problems with analysis.

Father/Son Ages Problem: A father is 35 years old, and his son is 2. In how many years will the father be four times as old as his son?

Your Solution to Father/Son Ages Problem:

Analysis of Father/Son Ages Problem

What students need to know in order to solve the problem:	Students need to be able to *build an image* of this problem. To assess whether or not our students have an image of a problem, we can ask them to explain the problem in their own words or see if they can act out the problem. Based on their images, some students may solve the problem using tables of values. Students who want to use algebra to solve the problem will need to "tell the problem to algebra", i.e. create an equation which represents the problem. One of the goals of this problem is in fact advancing students' abilities to tell a given problem to algebra (see below).
What students gain from solving the problem:	The main goal of this problem is advancing students' abilities to tell a given problem to algebra, i.e. create an equation which represent the problem. As noted above, students may solve the problem by reasoning without introducing variables and equations, and these students' solutions should be shared with the class and applauded. Students who prefer to use tables to solve the problem can ultimately be brought to use algebraic solutions as follows. If the teacher continues to pose several variations of the problem, e.g. by changing the ages of the father and the son, the students will eventually tire of making the tables as they notice that their peers who are setting up equations finish the problems much more quickly, i.e. they will naturally realize that the algebraic approach is more efficient.
Expected difficulties that students will face when solving the problem and suggestions for dealing with these difficulties:	In our experience, many students have difficulty realizing that the son's and father's ages are changing at the same rate, for example, some students write that in t years, the son's age will be $t + 2$ and the father's age will still be 35 or will be $s + 35$ for some s. The teacher can address this by asking the student "How old are you now? And what about your father? How old will you each by 5 years from now?" This sequence of questions will shift the student's attention back to the fact that the father and son are aging at the same rate.
Relevant standards:	ACED1: Create equations and inequalities in one variable and use them to solve problems. Include equations arising from linear and quadratic functions, and simple rational and exponential functions.

Father and Son Problem: A father and his son are workers, and they walk from home to the plant. The father covers the distance in 40 minutes, and the son in 30 minutes. In how many minutes will the son overtake the father if the father leaves home 5 minutes earlier than the son?

Your Solution to Father and Son Problem:

Analysis of Father and Son Problem

What students need to know in order to solve the problem:

Students need to be able to *build an image* of this problem. To assess whether or not our students have an image of a problem, we can ask them to explain the problem in their own words or see if they can act out the problem.

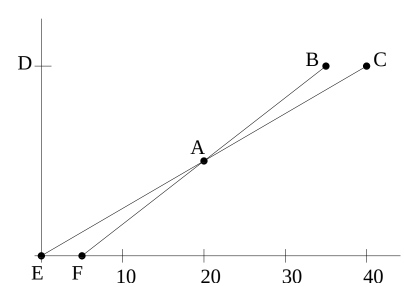
Based on their images, some students may solve the problem with tables of values, geometrically, or algebraically. Students who want to use algebra to solve the problem will need to "tell the problem to algebra", i.e. create an equation which represents the problem.

What students gain from solving the problem:

This problem can serve to advance students' abilities to tell a given problem to algebra.

Students who solve this problem algebraically can learn that we don't always need two equations in order to solve a problem with two unknowns (see the expected difficulties).

There is a beautiful geometric solution to the problem, so this problem can also serve as an opportunity for the class to discuss equations of lines and congruent triangles. In particular, if one graphs the father's and son's positions as a function of time from the moment of the father's departure, then the graphs are line segments which form a pair of congruent triangles. For example, in the figure below, D denotes the distance from home to the factory. The triangles $\triangle ABC$ and $\triangle AFE$ are congruent. One can demonstrate the congruence by either finding equations for the lines or by using theorems from Euclidean geometry.



Even if none of the students use geometry to solve this problem, the teacher may choose to present this solution to the class.

Expected difficulties that students will face when solving the problem and suggestions for dealing with these difficulties:	Some students may solve the problem through quantitative reasoning without the use of variables. For example, in each minute, the father travels 1/40 of the total distance to the plant, and the son travels 1/30 of the total distance. Thus, in each minute, the son "gains" 1/120 on the father. Since at the five minute mark (when the son starts), the father will be 1/8 of the way to the plant, the son will catch up with the father after $1/8 \div 1/120$. Students often have difficulty with the last part of this argument, namely why we should divide 1/8 by 1/120 in order to find the time when the father and son meet. We recommend that the teacher ask the students the following question: "If the father and son were walking at the same speed, what would be the distance between them at any time?" The students should see that in this case, the distance would always be 1/8 of the total distance to the plant. Then the teacher can ask the students to reconsider the case of the different speeds and ask the students what is the distance between the father and the son at minute six, at minute seven, and so on. In our experience, students who do create equations to solve the problem typically meet confusion when they write down one equation with two variables. For example, if d represents the distance to the plant and t represents time, then we have the equation $(t + 5)d/40 = td/30$. Since the students see two unknowns, they often think that they need to find another equation in order to solve the problem. We recommend that the teacher let the students spend a little time looking for another equation before suggesting that they try to solve for t (a natural question since are interested in the time when the son and father meet). The students will likely be surprised that the d's cancel, so this can serve as an excellent opportunity for them to see WHY the answer is independent of the distance to the plant. To illustrate this point, the teacher can suggest that the students consider specific examples, for instance if the distance is two miles and then if the distance is three miles.
Relevant standards:	ACED1: Create equations and inequalities in one variable and use them to solve problems. Include equations arising from linear and quadratic functions, and simple rational and exponential functions. ACED4: Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations.

Rabbits and Cages Problems:

(1) A farmer has some rabbits and some cages. When he puts 2 rabbits in each cage, there are 2 rabbits left over. When he puts 3 rabbits in each cage, there are 16 cages (but no rabbits) left over. How many rabbits and how many cages are there?

(2) A pet shop owner had some rabbits and some cages. When she put 2 rabbits in each cage, there were 4 rabbits left over. When she put 4 rabbits in each cage, there were 2 cages left over. What happened when she put 3 rabbits in each cage? (Assume the number of rabbits remains constant!) Follow-up Question: Is it possible to solve the problem without finding the actual number of rabbits and cages?

Brick Layer Problem: A mason is building walls around a square garden. There are 18 bricks for each layer in the wall. If he completes 3 of the walls, he has 22 bricks left over. OR. . . he could build all 4 of the walls with 3 fewer layers than the total needed and have 4 bricks left over. How many bricks does he have altogether, and how many layers are needed to build a wall?

Your Solutions to Rabbits and Cages Problems and Brick Layer Problem:

Analysis of Rabbits and Cages Problems and Brick Layer Problem

What students need to know in order to solve the problems:	Students need to be able to *build images* of these problems. To assess whether or not our students have an image of a problem, we can ask them to explain the problem in their own words or see if they can act out the problem. Based on their images, some students may solve the problem using tables of values or diagrams. Students who want to use algebra to solve the problem will need to "tell the problem to algebra", i.e. set up a system of equations which represent the problem. One of the goals of these problems is in fact advancing students' abilities to tell a given problem to algebra, as will be seen below.
What students gain from solving the problems:	The main goal of these problems is advancing students' abilities to tell a given problem to algebra, i.e. to set up a system of equations which represent the problem. To this end, these problems are carefully worded so that rote strategies for translating sentences into equations (e.g. "and" corresponds to addition, etc.) do not work well. As noted above, students may solve the problems by reasoning without introducing variables and equations, and these students' solutions should be shared with the class and applauded. These students can ultimately be brought to use algebraic solutions if the teacher then varies the problem by increasing the number of rabbits and cages or bricks and layters to the point that the non-algebraic approaches become too tedious. In solving these problems, many students may write down a simple system of two linear equations in two unknowns which can either be solved via back substitution or the variable elimination method. Thus, these problems can and should serve as an opportunity for students to understand WHY these methods work. For example, if a student uses the substitution, the teacher should ask the student to explain WHY the resulting point is a solution of both equations.

Expected difficulties that students will face when solving the problems and suggestions for dealing with these difficulties:	As noted above, the wording of these problems is such that students who rely on rote strategies for translating sentences into equations will have difficulty telling the problems to algebra. Specifically, students cannot pick out key words to translate into plus or minus, and so on. This is a *good difficulty* because the students will be forced to build a coherent image of the problem rather than resorting too quickly to symbolism. Once the teacher notices that some students have set up equations, we suggest that these students write their equations on the board so that the class as a whole can consider and compare the equations and decide if they accurately represent the situation.
Relevant standards:	8.EE8: Analyze and solve pairs of simultaneous linear equations. ACED2: Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.

Exam Problem:

(1) Your class' next test is worth 100 points and contains 29 problems. There are two point problems and five point problems. How many of each are there?

(2) Other than 29, how many problems could be on a 100 point test made up of two and five point problems?

Two Digit Number Problem:

(1) The digits of a two digit number add up to 12. Interchanging the digits gives a new number which is less than the original one by 36. Find the number.

(2) If you were a teacher designing a similar problem for your class, what numbers could you use instead of 12 and 36 and produce a solvable problem?

Three Digit Number Problem: When the digits of a three digit number A are reversed, a new number B is obtained. The difference, $B - A$ is 99. What are the numbers A and B?

Your Solutions to Exam Problem, Two Digit Number Problem, and Three Digit Number Problem:

Analysis of Exam Problem, Two Digit Number Problem, and Three Digit Number Problem

What students need to know in order to solve the problems:	The first parts of the Exam Problem and the Two Digit Number Problem can be solved using tables of values, but to solve all of the problems completely, students need to be able to "tell the problems to algebra", in particular to set up equations and inequalites which represent the problems and all of the constraints of the problems. Students must be able to solve a system of two linear equations in two unknowns. Here it is critical that they understand what it means to solve a system, specifically that the resulting point is a solution of both equations. Students need to understand place value in order to solve the Two Digit and Three Digit Number Problems.
What students gain from solving the problems:	In solving these problems, students will advance their abilities to tell a given problem to algebra, i.e. to set up a system of equations and inequalities which represent the problems and all of the constraints of the problems. In solving these problems, students will need to solve systems of two linear equations in two unknowns, which can either be solved via back substitution or the variable elimination method. Thus, this problem can and should serve as an opportunity for students to understand WHY these methods work. For example, if a student uses substitution, the teacher should ask the student to explain WHY the resulting point is a solution of both equations. In solving these problems, students will use variables to represent and solve for unknown quantities, but the problems also provide natural opportunities for students to use variables for reasoning. For example, in Question 2 of the Exam Problem, one can introduce a variable Q to represent the total number of problems on the exam. For Question 2 of the Two Digit Number Problem, one can introduce variables S and D to represent the sum of the digits of the number and the difference between the original two digit number and the new number obtained from interchanging its digits.

	In solving Question 2 of the Two Digit Number Problem, students will naturally be led to a case by case analysis: if we let S and D be as above, one considers the cases of S even and S odd. Case by case analysis pervades all levels of mathematics; it is the logical basis of the technique of proof by exhaustion.
Expected difficulties that students will face when solving the problems and suggestions for dealing with these difficulties:	Solving the second parts of both of the Exam Problem and Two Digit Number Problem requires careful organization of all of the information. We suggest that the teacher help the students to collect and organize the information as follows. First the teacher can ask the students to "state clearly what we know". The teacher can then ask the students to "state clearly what we are trying to find/show". Then, the teacher can help the students with spatial organization of this information and the consequences/implications of this information. For example, the teacher can make a list on the blackboard or student's paper entitled "What we know". Each piece of information can be enumerated and arrows can be drawn to identify the implications of these pieces of information. The teacher can then make a similar list entitled "What we are trying to find/show". Although students will initially feel overwhelmed by the second parts of these problems, they will benefit greatly from learning how to clearly organize their thoughts. Organizing information, both mentally and spatially, is key to learning mathematics.
Relevant standards:	8.EE8: Analyze and solve pairs of simultaneous linear equations. A-CED2: Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales. A-CED3: Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or nonviable options in a modeling context. A-CED4: Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations.

Clock Hands Problems:

(1) The time now is 12:00 PM. When is the next time the clock hands will be on top of each other?
(2) Find all possible times when the clock hands are on top of each other.
(3) Let the distance between the clock hands be the angle between them, measured clockwise from the small hand to the big hand. Find the distance between the clock hands at any moment from 12:00 PM until the next time they are on top of each other. When is the distance between the hands 180^o?
(4) At what time after 4:00 will the minute hand of a clock overtake the hour hand?
(5) At what time after 7:30 will the hands of a clock be perpendicular?
(6) Between 3:00 and 4:00 Noreen looked at her watch and noticed that the minute hand was between 5 and 6. Later, Noreen looked again and noticed that the hour hand and the minute hand had exchanged places. What time was it in the second case?
(7) Ernie works in a factory where all of the clocks are synchronized at midnight. The clock in Ernie's office is defective, though, and the hands overlap exactly every 65 minutes.
    (a) If, according to Ernie's clock, he begins working at midnight and finishes at 8:00 AM, how long does he work, according to an accurate clock?
    (b) If, according to Ernie's clock, he begins working at 9:00 AM and finishes at 5:00 AM, how long does he work, according to an accurate clock?

Your Solutions to Clock Hands Problems:

Analysis of Clock Hands Problems

What students need to know in order to solve the problems:	Students must be familiar with a good old-fashioned analog clock.
What students gain from solving the problems:	These problems are an excellent opportunity for students to build images of a problem and create corresponding equations. These problems are amenable to multiple to solution strategies. Each student may have a slightly different image of a clock, so the teacher should be prepared for students to approach the problems in different ways. For example, various approaches to Question 1 are: • Some students may use purely proportional reasoning rather than introducing variables: At 1:00, the hour hand is 1/12 of the way around the circle and the minute hand is 0 of the way around the circle. Each minute that passes, the minute hand moves 1/60 of the way around the circle and the hour hand moves 1/720 of the way around the circle. So each minute that passes, the distance between the minute and the hour hand is decreased by 11/720. Thus it will take (1/12)/(11/720) minutes until the distance between the clock hands is 0. • Another approach is to note that there are 11 times when the clock hands will meet, and the meeting points are equally spaced around the circle. Thus, the first meeting point after 12:00 is at 360/11 degrees (clockwise) from 12. • Students may introduce variables and set up simple equations: Let x be the distance from 12 to the meeting point, and let t be the rate of the hour hand, and so on. • Some students may even use modular arithmetic. Thus, these problems can help establish the classroom culture that each person thinks differently and has different ideas. These problems also invite students to think in terms of different units. For example, students may think in terms of degrees, minutes, hours, and even of the clock as one whole unit.

Expected difficulties that students will face when solving the problems and suggestions for dealing with these difficulties:	We recommend that the class spend a good deal of time on Question 1 as it is an introduction to the rest. In solving it, students will become accustomed to thinking about the hour hand and the minute hand, but we expect that many of them will initially fail to take the relative rates of travel of the hour and minute hands into consideration. The teacher can aim to help the students to develop the anchor that the minute hand moves twelve times as fast as the hour hand by asking them specific questions such as "If we start at 12:00 and end at 12:30, then how far do the hour and minute hands move?" Students may say that the answer to Question 1 is 1:05. This is a *good difficulty* because it is an example where we can see that the answer is not possible. We suggest that the teacher simply ask the students whether it is possible for the hands to meet at 1:05. The students will realize that it is not, since the hour hand is at the 1 at 1:00 and must move as time passes to 1:05. We expect students to have difficulty solving Question 7. The key is that according to Ernie's clock, 65 and 5/11 minutes have passed, when in reality, only 65 minutes have passed. Thus, Ernie "steals" 5/11 minutes every 65 minutes, and we need to know how many total minutes that he steals during his shift. To help the students arrive at this key observation, the teacher should assist the students in acting out the problem. The teacher should also clarify that the 65 minutes in the problem statement are 65 minutes according to an accurate clock, not Ernie's clock. Once the students understand part (a), in solving part (b), they should see that the starting times and ending times are actually irrelevant – the important factor is the length of Ernie's shift.
Relevant standards:	6.RP3b: Solve unit rate problems including those involving unit pricing and constant speed. 6.RP3d: Use ratio reasoning to convert measurement units; manipulate and transform units appropriately when multiplying or dividing quantities. ACED1: Create equations and inequalities in one variable and use them to solve problems. Include equations arising from linear and quadratic functions, and simple rational and exponential functions.

Track Problem: Jill and Jack start running at the same time from a point A on a circular track in opposite directions. They each run at a constant speed and finish running when they first meet at A. Jill runs at 9 feet per second and Jack at 5 feet per second. How many times do they meet from the time they start running to the time they meet at A?

Your Solution to Track Problem:

Analysis of Track Problem

What students need to know in order to solve the problem:	Students need to be able to build an image of this problem. Based on their image, some students may solve the problem using a table of values, whereas others may solve the problem using variables and equations.
What students gain from solving the problem:	This problem is amenable to multiple solution strategies. For example, some students may solve the problem without introducing variables by dividing the track into fourteen equal pieces and making a table of Jill's and Jack's positions each time they meet. Other students may introduce variables and set up simple equations. Some students may even use modular arithmetic. Thus, these problems can help establish the classroom culture that each person thinks differently and has different ideas. This problem also invites students to think in terms of different units. For example, students may think in terms of degrees, units of time, or of the track as one whole unit.
Expected difficulties that students will face when solving the problem and suggestions for dealing with these difficulties:	This is a non-standard problem in terms of time, distance, and speed, since the problem asks how many times Jack and Jill meet (as opposed to how long it takes them to meet, for example). To aid students in conceptualizing the problem, the teacher should lead the students to act it out (mentally, not in actuality) as though they were the runners themselves. As mentioned, some students may draw very careful pictures and use them to count the number of meeting times. The teacher should embrace these types of empirical solutions and as a next step, pose a new problem to the students which will require them to think more generally in order to solve the problem, i.e. in terms of the underlying structure rather than the specific numbers. For example, the teacher can ask the students what will happen in the track problem if Jill's speed is 593 feet per minute and Jack's speed is 450 feet per minute. Another interesting extension is to ask whether it is possible that Jill and Jack run at rates so that they will never meet at point A again.
Relevant standards:	ACED1: Create equations and inequalities in one variable and use them to solve problems. Include equations arising from linear and quadratic functions, and simple rational and exponential functions.

4. Absolute Value and Inequalities

Absolute value is an excellent context in which to strengthen the concept of function. For instance, in considering a simple inequality such as $|x| > 4$, it suffices to understand absolute value in terms of distance. In contrast, considering an inequality such as $|x-1|+|2x+3| > 5$ compels one to think about the algebraic meaning of absolute value, and in turn the concept of function. Inequalities provide a rich opportunity for logical reasoning in a natural context; for example, $ab > 0$ if a is positive AND b is positive OR if a is negative AND b is negative. Moreover, in solving problems involving inequalities, one can understand that certain actions are not reversible; in particular, in solving a system of inequalities, substitution can lead to an inequivalent system. Both absolute value and inequalities are natural contexts for case by case analysis.

Budget Discrepancy Problem: "Budget Discrepancy is defined as $|a-b|$, where a is the amount of money allocated and b is the actual amount of money spent. The total budget of a certain institution is divided into four categories: A, B, C, and D. $1/7$ of the budget was allocated to Category A, $2/9$ to Category B, $5/14$ to Category C, and the rest to Category D. The actual amount of money spent in each category is as follows: 250K for Category A, 340.5K for category B, 400K for category C, and 17.75K for Category D. What is the exact budget if you know that the sum of the four discrepancies is 600K?

Your Solution to Budget Discrepancy Problem:

Analysis of Budget Discrepancy Problem

What students need to know in order to solve the problem:	Students must have a solid understanding of the concept of absolute value both algebraically (e.g. $\|x\| = x$ if $x \geq 0$ and $\|x\| = -x$ if $x < 0$) and geometrically, i.e. as a distance.
What students gain from solving the problem:	In solving this problem, students will solidify their understanding of the concept of absolute value in a natural way by repeatedly reasoning through the algebraic definition. In particular, the budget discrepancy $\|a-b\| = a-b$ if $a \geq b$ and $\|a-b\| = b-a$ if $a < b$. This problem also provides an opportunity for students to work with fractions in a natural context. If teachers avoid incorporating fractions into problems and examples because they think fractions are difficult for the students, then fractions will become difficult beyond their inherent difficulty; the more opportunities that students have to work with fractions, the more natural it will become for them. In solving this problem, students will meet an equation involving the absolute value of four unknown quantities and will therefore need to perform a case by case analysis in a natural context. Case by case analysis pervades all levels of mathematics – it is the logical basis of the technique of proof by exhaustion. In this problem, there are, a priori, sixteen cases. but careful analysis reveals that certain cases are subsumed by others; there are actually only five cases.

Expected difficulties that students will face when solving the problem and suggestions for dealing with these difficulties:	The algebraic definition of absolute value, specifically the fact that $\|x\| = -x$ if $x < 0$, is inherently difficult for students since the negative sign on the right hand side seemingly contradicts their understanding that absolute value is positive (or at least non-negative). A useful exercise to help students confront and ultimately overcome this difficulty is to play around with questions such as: What is $\|-4\|$? Students will respond that the answer is 4, but then the teacher should prompt them to explain the process that leads to the result of 4. For example, the teacher can ask the students, "I have a number in my hand that you can't see. You want to know the absolute value of the number. What would you like to know about the number?" Eventually, the students should ask if the number is positive or negative, which leads to examining the process that defines the absolute value function. Understanding functions as processes is an important step in thinking in terms of functions.
Relevant standards:	5.NF2: Solve word problems involving addition and subtraction of fractions referring to the same whole, including cases of unlike denominators, e.g. by using visual fraction models or equations to represent the problem. Use benchmark fractions and number sense of fractions to estimate mentally and assess the reasonableness of answers. 6.NS7c: Understand the absolute value of a rational number as its distance from 0 on the number line; interpret absolute value as magnitude for a positive or negative quantity in a real-world situation. 7.NS1c: Understand subtraction of rational numbers as adding the additive inverse, $pq = p + (q)$. Show that the distance between two rational numbers on the number line is the absolute value of their difference, and apply this principle in real-world contexts.

Power Line Problem: The state wants to build a power station along an east-west highway to supply power to four small towns, A, B, C, and D, located along the highway. Town C is 100 miles west of D and 85 miles east of B. Town A is 30 miles west of B. The cost of connecting the power station to any town is $\$1,000$ per mile. The total budget cannot exceed $\$230,000$. Where can you locate the power station? Note that there must be a separate power line from the power station to each town.

Inequalities and Absolute Value Problem 1: In the previous problem we encountered inequality that involves absolute values. For what values of x do the following inequalities hold?

(1) $|x+2| - \left|2x - \frac{2}{3}\right| \leq 1$

(2) $\left|-x + \frac{3}{2}\right| - \left|\frac{2}{5} - 2x\right| + x \geq 0$

(3) $-\frac{1}{7}\left|\frac{2}{3} - x\right| - 2x < 3$

Inequalities and Absolute Value Problem 2: Solve the inequality

$$2x + \left|-\frac{4}{3}x + 5\right| - \frac{1}{6}|6x| - \frac{1}{8}|3x - 5| < 2.$$

Illustrate your solution graphically.

Your Solutions to Power Line Problem and Inequalities and Absolute Value Problems:

Analysis of Power Line Problem and Inequalities and Absolute Value Problems

What students need to know in order to solve the problems:

Students must have a solid understanding of the concept of absolute value both algebraically (e.g. $|x| = x$ if $x \geq 0$ and $|x| = -x$ if $x < 0$) and geometrically, i.e. as a distance.

Students must be able to solve linear inequalities in one variable. Here it is critical that the teacher check that students understand what it means to solve an inequality. In particular, do the students understand the problem of solving an inequality as a puzzle – searching for something that is to be found (in this case a solution set)? Or do the students only view the problem of solving an inequality as a process of symbol pushing where the goal is to isolate the variable on one side of the inequality sign? In order to ascertain the students' levels of understandings, the teacher can ask them to carefully articulate

- why a given number is or is not in the solution set of a given inequality
- why an inequality can have more than one solution.

What students gain from solving the problems:

Students will solidify their understanding of the concept of absolute value in a natural way by repeatedly reasoning through the algebraic definition. For example, in part (a) of the second problem, students must reason that $|x + 2| = x + 2$ if $x \geq -2$ and $|x + 2| = -x - 2$ if $x < -2$, and similarly for $\left|2x - \frac{2}{3}\right|$.

In solving these problems, students will naturally be led to a case by case analysis, which pervades all levels of mathematics. We note that students may not initially realize that certain cases are subsumed by others; for example in Problem 1 Question 1, students may initially think there are four distinct cases: two possibilities for the sign of $x + 2$ times two possibilities for the sign of $2x - 2/3$. We suggest that the teacher allow students to proceed in this manner for one or two problems and then point out that they can shorten the process by carefully analyzing the situation if they observe that there are actually only three cases. We note that it is important to allow the students to do a few examples the longer way, or they will not have any reason to appreciate the value of how conceptualization can save calculation.

These problems also provide the opportunity for students to think in terms of functions as processes (see the second comment under expected difficulties).

Expected difficulties that students will face when solving the problems and suggestions for dealing with these difficulties:

Students frequently treat inequalities as equations; for example, to solve the inequality $x^2 > 1$, many students replace it with the equation $x^2 = 1$, find the solutions $x = \pm 1$, and then conclude that $x > 1$ or $x < -1$. This obstacle to understanding inequalities is actually more of a *didactical* obstacle than a conceptual one, i.e. it stems from the way students are taught to deal with inequalities rather than a difficulty inherent to the concept of inequality. Students are less apt to fall prey to this difficulty if as teachers, we emphasize the meaning of inequalities as opposed to teaching only procedures and algorithms to solve them.

The algebraic definition of absolute value, specifically the fact that $|x| = -x$ if $x < 0$, is inherently difficult for students since the negative sign on the right hand side seemingly contradicts their understanding that absolute value is positive (or at least non-negative). A useful exercise to help students confront and ultimately overcome this difficulty is to play around with questions such as: What is $|-4|$? Students will respond that the answer is 4, but then the teacher should prompt them to explain the process that leads to the result of 4. For example, the teacher can ask the students, "I have a number in my hand that you can't see. You want to know the absolute value of the number. What would you like to know about the number?" Eventually, the students should ask if the number is positive or negative, which leads to examining the process that defines the absolute value function. Understanding functions as processes is an important step in thinking in terms of functions.

Most of these problems involve inequalities that contain absolute values of at least two distinct expressions, and we anticipate that students will have difficulty devising a strategy to solve such complex inequalities. Thus, we recommend that the class start with easier problems solving inequalities of the form $|ax+b| \leq c$. It is important for us as teachers to be sensitive to the difficulties that students have with the expression $|x+2|$ as compared to $|x|$, for example.

Relevant standards:

6.NS7c: Understand the absolute value of a rational number as its distance from 0 on the number line; interpret absolute value as magnitude for a positive or negative quantity in a real-world situation.

6.EE5: Understand solving an equation or inequality as a process of answering a question: which values from a specified set, if any, make the equation or inequality true? Use substitution to determine whether a given number in a specified set makes an equation or inequality true.

7.NS1c: Understand subtraction of rational numbers as adding the additive inverse, $pq = p + (q)$. Show that the distance between two rational numbers on the number line is the absolute value of their difference, and apply this principle in real-world contexts.

7.EE4b: Solve word problems leading to inequalities of the form $px + q > r$ or $px + q < r$, where p, q, and r are specific rational numbers. Graph the solution set of the inequality and interpret it in the context of the problem.

A-REI3: Solve linear equations and inequalities in one variable, including equations with coefficients represented by letters.

Coins Problem 1: Ann, Ben, Cari, and Dale collect old coins. Ann has less than twice as many coins as Ben, Ben has less than 3 times as many coins as Cari, and Cari has less than 4 times a many coins as Dale. Dale has less than 100 coins. What is the maximum number of coins Ann can have?

Coins Problem 2: Ann, Ben, and Cari collect old coins. The number of Anns coins is less than 9 times the number of Bens coins. Furthermore, 9 times the combined number of coins owned by Ann and Ben is less than 4 times the number of Caris coins. Cari has fewer than 50 coins. What is the maximum number of coins that Ann can have?

Your Solutions to Coins Problems:

Analysis of Coins Problems

What students need to know in order to solve the problems:	Students must be able to solve linear inequalities. Here it is critical that the teacher check that students understand what it means to solve an inequality. In particular, do the students understand the problem of solving an inequality as a puzzle – searching for something that is to be found (in this case a solution set)? Or do the students only view the problem of solving an inequality as a process of symbol pushing where the goal is to isolate the variable on one side of the inequality sign? In order to ascertain the students' levels of understandings, the teacher can ask them to carefully articulate • why a given number is or is not in the solution set of a given inequality • why an inequality can have more than one solution. Students must be able to solve a system of linear inequalities. As described in the previous comment, it is critical that students understand what it means to solve a system. For example, students should be able to articulate WHY the solution set of a system of linear inequalities in two variables (as in Problem 2) is the intersection of the half-planes which are the solution sets of each of the inequalities, respectively.
What students gain from solving the problems:	In solving these problems, students will advance their ability to create (and subsequently solve) systems of inequalities which represent relationships between quantities. In solving systems of equations, we use techniques such as substitution in order to arrive at new equivalent systems of equations, i.e. systems with the same solution set. In solving systems of inequalities though, substitution may lead us to inequivalent systems; Coins Problem 2 can serve as a powerful illustration of this. For example, consider the following solution: Let A, B, and C be the number of coins of Ann, Ben, and Cari, respectively. Note that A, B, and C are all non-negative integers. We have: (1) $A < 9B$, (2) $9A + 9B < 4C$ and (3) $C \leq 49$. So (2) and (3) tell us: (4) $A + B \leq 196/9$.

	Notice already that we have lost information since (4) does not tell us (2) and (3). From (1) and (4) we get: (5)$A + A/9 < A + B \leq 196/9$. (We have lost information again.) Thus, $10A/9 \leq 196/9$, or $A \leq 19.6$. So we see that a necessary condition on a solution is that $A \leq 19$, but this condition is not sufficient! Plugging $A = 19$ into (1) tells us that $B \geq 3$, and plugging $A = 19$ into (4) tells us that $B \leq 2$. Thus, we cannot have $A \leq 19$, and in fact we must have $A = 18$.
Expected difficulties that students will face when solving the problems and suggestions for dealing with these difficulties:	Students frequently treat inequalities as equations; for example, to solve the inequality $x^2 > 1$, many students replace it with the equation $x^2 = 1$, find the solutions $x = \pm 1$, and then conclude that $x > 1$ or $x < -1$. This obstacle to understanding inequalities is actually more of a *didactical* obstacle than a conceptual one, i.e. it stems from the way students are taught to deal with inequalities rather than a difficulty inherent to the concept of inequality. Students are less apt to fall prey to this difficulty if as teachers, we emphasize the meaning of inequalities as opposed to teaching only procedures and algorithms to solve them. In both of the problems, solutions should be integers, and we anticipate that some students will ignore this requirement. We suggest that the class compare several students' solutions to the problems to arrive at a common understanding of the correct solutions.
Relevant standards:	6.EE5: Understand solving an equation or inequality as a process of answering a question: which values from a specified set, if any, make the equation or inequality true? Use substitution to determine whether a given number in a specified set makes an equation or inequality true. A-CED3: Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or nonviable options in a modeling context. A-REI12: Graph the solutions to a linear inequality in two variables as a halfplane (excluding the boundary in the case of a strict inequality), and graph the solution set to a system of linear inequalities in two variables as the intersection of the corresponding half-planes.

Fraction Problem: Take a fraction whose numerator and denominator are both nonzero integers. Add some nonzero number to the numerator, and divide the denominator by the same number. Under what conditions will the new fraction be greater than the original fraction?

Pleasant Fractions Problem: We call fractions with the following three properties pleasant:

- The fraction is greater than zero.
- The denominator of the fraction is 4 more than the numerator.
- Subtracting 1 from both the numerator and denominator of the fraction forms a new fraction that is at least as large as the reciprocal of the numerator of the original fraction.

(1) Among all pleasant fractions, is there one with a smaller numerator than any other? If there is one, what is its value?

(2) Among all pleasant fractions, is there one with a larger numerator than any other? If there is one, what is its value?

(3) Among all pleasant fractions for which the numerator is positive, is there one with a smaller numerator than any other? If there is one, what is its value?

Your Solutions to Fraction Problem and Pleasant Fractions Problem:

Analysis of Fraction Problem and Pleasant Fraction Problem

What students need to know in order to solve the problems:	Students must be able to solve polynomial and rational inequalities. Here it is critical that the teacher check that students understand what it means to solve an inequality. In particular, do the students understand the problem of solving an inequality as a puzzle – searching for something that is to be found (in this case a solution set)? Or do the students only view the problem of solving an inequality as a process of symbol pushing where the goal is to isolate the variable on one side of the inequality sign? In order to ascertain the students' levels of understandings, the teacher can ask them to carefully articulate • why a given number is or is not in the solution set of a given inequality • why an inequality can have more than one solution.
What students gain from solving the problems:	In solving these problems, students will gain fluency in solving polynomial and rational inequalities. As usual, fluency entails that students understand and be able to articulate WHY one performs the particular steps to solve such inequalities. To this end, it is crucial that the teacher bring the students to perform the reasoning behind the algorithms rather than blindly employing algorithms. For example, in solving the pleasant fraction problem, one needs to solve the inequality $\dfrac{x^2-2x-3}{x^2+3x} \geq 0$. Students must understand that the quotient is zero if and only if the numerator is zero. Students must also be able to explain how the quotient can be positive: both the numerator AND the denominator are positive OR both the numerator AND the denominator are negative (see the next bullet point). Students must also be able to explain WHY the first step of the algorithm used to solve the inequality is to find the roots of the numerator and denominator. In solving these problems, students will naturally be led to a case by case analysis. Case by case analysis pervades all levels of mathematics – it is the logical basis of the technique of proof by exhaustion.

	This problem provides a natural opportunity for students to understand the difference between a fraction and a rational number as follows. Students may notice that 3/7 is a pleasant fraction (for example), but that 6/14 is not – although these are equal rational numbers, they are indeed distinct fractions. If students do not make this observation, we suggest that the teacher draw it to their attention.
Expected difficulties that students will face when solving the problems and suggestions for dealing with these difficulties:	Students frequently treat inequalities as equations; for example, to solve the inequality $x^2 > 1$, many students replace it with the equation $x^2 = 1$, find the solutions $x = \pm 1$, and then conclude that $x > 1$ or $x < -1$. This obstacle to understanding inequalities is actually more of a *didactical* obstacle than a conceptual one, i.e. it stems from the way students are taught to deal with inequalities rather than a difficulty inherent to the concept of inequality. Students are less apt to fall prey to this difficulty if as teachers, we emphasize the meaning of inequalities as opposed to teaching only procedures and algorithms to solve them. Rational inequalities afford more opportunities for logical complexities. We recommend that before starting these two problems, the class discuss the following questions: (1) When is a fraction $a/b = 0$? When is $a/b > 0$? When is $a/b < 0$? (2) Suppose we have a fraction a/b. If we decrease (respectively, increase) a and leave b the same, how does our new fraction compare to the original fraction? What if we decrease a and we decrease b? What if we decrease a and we increase b? Now suppose we decrease (respectively, increase) b and leave a the same; how does our new fraction compare to the original fraction? What if we increase a and we decrease b? What if we increase a and we increase b?
Relevant standards:	Quadratic and rational inequalities are not mentioned in the common core standards.

Auditorium Problem: The seating capacity of an auditorium is 600. For a certain performance, with the auditorium not filled to capacity, the returns were $\$330.00$. Admission prices were 75 cents for adults and 25 cents for children. What is the minimum number of adults that could have attended this performance?

Your Solution to Auditorium Problem:

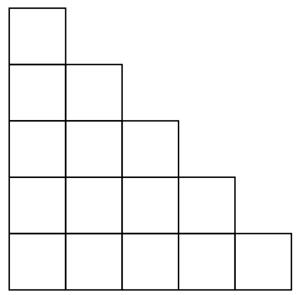
Analysis of Auditorium Problem

What students need to know in order to solve the problem:	Students must be able to solve linear equations and inequalities.
What students gain from solving the problem:	In solving this problem, students will advance their ability to create (and subsequently solve) systems of equations and inequalities which represent relationships between quantities. Textbooks include few (if any) opportunities for students to solve mixed systems such as this.
Expected difficulties that students will face when solving the problem and suggestions for dealing with these difficulties:	Students frequently treat inequalities as equations; for example, to solve the inequality $x^2 > 1$, many students replace it with the equation $x^2 = 1$, find the solutions $x = \pm 1$, and then conclude that $x > 1$ or $x < -1$. This obstacle to understanding inequalities is actually more of a *didactical* obstacle than a conceptual one, i.e. it stems from the way students are taught to deal with inequalities rather than a difficulty inherent to the concept of inequality. Students are less apt to fall prey to this difficulty if as teachers, we emphasize the meaning of inequalities as opposed to teaching only procedures and algorithms to solve them. Here an equality and inequality are both present, thus giving students a concrete opportunity to realize the difference. Since students are not used to mixed systems, we expect them to have some difficulty solving the system; they may struggle with how to incorporate the inequality into the equality or vice versa.
Relevant standards:	A-CED3: Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or nonviable options in a modeling context.

5. Problems about Patterns

Identifying and explaining patterns is critical to algebraic thinking. In particular, it is essential that students are educated not only to observe patterns, but to understand the underlying structure which is causing a pattern. Recognizing patterns is an important skill, but the follow-up question of what makes the pattern to be so is indispensable. The problems in this section are designed to compel students to explain the process responsible for a given pattern.

Stair-like Structure Problems: A figure such as the one below is called a stair-like structure:



(1) You have 1176 identical square pieces. Can you use all the pieces to construct a stair-like structure?
(2) You want to build a stair-like structure out of toothpicks. Is it possible to use a total of 2628 identical toothpicks to create a stair-like structure?
(3) You have constructed a stair-like structure with a base of 98 pieces (squares). How many squares does your structure have?
(4) The number of rows (or columns) in a stair-like structure is called the number of steps; for example, the figure above has five steps. You want to use between 2345 and 8789 toothpicks to build a stair-like structure. What is the minimum number of steps your structure can have? What is the maximum number of steps your structure can have?

Your Solutions to Stair-like Structure Problems:

Analysis of Stair-like Structure Problems

What students need to know in order to solve the problems:	Students must know the quadratic formula.
What students gain from solving the problems:	Identifying and explaining patterns is critical to algebraic thinking. These problems are designed to compel students to justify the existence of patterns based on the underlying structure. To this end, it is crucial that students be brought to carefully explain how to count the number of squares and toothpicks in a stair-like structure with n steps. Here the role of the teacher is to ask questions to shift the students' focus from the result of the pattern to the process underlying the pattern, which is what ultimately justifies the pattern itself. For example, if a student states that there are three squares in a stair-like structure with two steps, six squares in a stair-like structure with three steps, ten squares in a stair-like structure with four steps, the teacher can ask the student where these numbers are coming from. The student should eventually explain that there is one square in the top row, two squares in the second row from the top, etc, and that he or she is summing the number of squares in each row (note that the emphasis of the explanation has now shifted to the process). Upon realizing the explicit process, the student can now count the number of squares in a stair-like structure with n steps – since the k^{th} row contributes k squares, there is a total of $1 + 2 + \cdots + n$ squares. Similarly, there are various methods to count the number of toothpicks, and it is the role of the teacher to direct the class dialogue to flesh out each of these methods until the students can explain the patterns in terms of the processes. For example, a student may count as follows: 4 toothpicks for a stair-like structure with one step, $4 + (4 + 2)$ toothpicks for a stair-like structure with two steps, $(4) + (4+2) + (4+2+2)$ toothpicks for a stair-like structure with one step, etc. To explain this pattern in terms of the process, a student must be able to explain that after counting the toothpicks in the rows above the k^{th} row, $4 + 2(k - 1)$ toothpicks must be added to complete the k^{th} row in order to avoid double counting.

These problems lend themselves to various counting strategies because each person has his or her own image of a stair-like structure. For example, some students use their image of a stair-like structure as a portion of a larger square in order to count the number of small squares in the structure; specifically, they complete the structure to a square, find half of the area of the square, and then add in half of the number of squares on a diagonal. Thus, in solving this problem, students can gain confidence that utilizing their own images (as opposed to that of the teacher or a classmate) to devise a counting strategy can lead to success.

In counting the number of squares or toothpicks in a stair-like structure, the sum $1 + 2 + \cdots + n$ arises in a natural way. Thus these problems provide an opportunity to find and justify a formula for this sum in a natural context (for example, as described in the previous comment).

In solving these problems, students will use variables to represent and solve for unknown quantities, but the problems also provide natural opportunities for students to use variables for reasoning, in particular when conjecturing and justifying the formulas for the number of squares and toothpicks in a stair-like structure with n steps.

Expected difficulties that students will face when solving the problems and suggestions for dealing with these difficulties:

Students may develop erroneous formulas for the number of toothpicks because of double counting. To deal with this difficulty, the teacher can ask a student to carefully explain his or her process of counting; upon explaining the process, the student will likely recognize the error.

	These problems are designed to compel students to justify the existence of a pattern based on the underlying structure – it is usually initially quite difficult for students to realize that this is different than simply recognizing a pattern (from a table of numbers, for example). The role of the teacher is to help them realize the difference by asking them to explain identified patterns – what makes the pattern to be so? The teacher can make the point that looking at a pattern and extracting why it works is not the same as just seeing how the pattern continues. Reasoning through the structure behind a pattern will be a gradual process for most students; given many problems of this nature, students will ultimately realize how we gain insight from the process underlying a pattern.
Relevant standards:	6.EE6: Use variables to represent numbers and write expressions when solving a real-world or mathematical problem; understand that a variable can represent an unknown number, or, depending on the purpose at hand, any number in a specified set. A-SSE2: Use the structure of an expression to identify ways to rewrite it. A-CED1: Create equations and inequalities in one variable and use them to solve problems. Include equations arising from linear and quadratic functions, and simple rational and exponential functions. A-REI4: Solve quadratic equations in one variable. F-BF2: Write arithmetic and geometric sequences both recursively and with an explicit formula, use them to model situations, and translate between the two forms.

Pattern Problems:

(1) Take the arithmetic sequence of positive integers: $1,\ 2,\ 3, \ldots$ Divide it into the sets: $\{1\}$, $\{2, 3\}$, $\{4, 5, 6\}$, $\{7, 8, 9, 10\}, \ldots$ You get a sequence of sets in which the first set consists of one item, the second of two items, the third of three items, and so on.
   (a) How many items are there in the 55^{th} set? What is their sum?
   (b) How many items are there in the n^{th} set? What is their sum?
   (c) Which set contains the number 1000?

(2) Take the arithmetic progression of the odd integers: $1,\ 3,\ 5, \ldots$ Divide it into the sets, $\{1\}$, $\{3, 5, 7\}$, $\{9, 11, 13, 15, 17\}, \ldots$ You get a sequence of sets in which the first set consists of one term, the second of three terms, the third of five terms, and so on.
   (a) How many terms are there in the 178th set? What is their sum?
   (b) What is the sum of the terms in the nth set?
   (c) What is the sum of all terms in the first n sets?
   (d) Do sets with the following property exist?
      (i) The sum of the terms in the set is 10000?
      (ii) The sum of the terms in the set is 30071?
      (iii) The sum of the terms in the set is even?
      (iv) The sum of the terms in the set is a perfect square?
   (e) The sum of the terms in a set exceeds 5234. What is the position of this set in the sequence?

(3) For any positive integer n, we define n sequences as follows: Each sequence has n terms. The first terms of the sequences are $1,\ 3,\ 5, \ldots,$ respectively. The differences of the sequences are $1,\ 2,\ 3, \ldots,$ respectively. Compute the sum of all the terms in these sequences.

Your Solutions to Pattern Problems:

Analysis of Pattern Problems

What students need to know in order to solve the problems:	In solving these problems, students will need to evaluate sums of the form $1+2+\cdots+n$. If students have not seen or forgotten the formula for such a sum, this problem can serve as an opportunity to develop and justify the formula in a natural context. Students must be able to solve quadratic inequalities.
What students gain from solving the problems:	Identifying and explaining patterns is critical to algebraic thinking. These problems are designed to compel students to reason through the structure underlying a pattern – in contrast to simply recognizing a pattern. For example, even students who find the terms in the 55^{th} set in Problem 1 part (a) via the brute force pattern recognition approach will eventually need to attend to the underlying structure in order to find the terms in the n^{th} set. Similarly, in Problem 3, students must be able to describe the n^{th} sequence, and in particular its first and last terms. In solving these problems, students will use variables to represent and solve for unknown quantities, but the problems also provide natural opportunities for students to use variables for reasoning. For example, in both the second and third problems, one must represent an odd number as a number of the form $2k-1$.
Expected difficulties that students will face when solving the problems and suggestions for dealing with these difficulties:	Problems 1 and 2 are difficult for students because the students must coordinate between the sets, the content of the sets, and the type of sequence in the background. Problem 3 presents similar conceptual difficulties – students must coordinate between the first term of the sequence and the difference that defines the sequence. Students are likely to feel overwhelmed, so the teacher can help the students get started on Problems 1 and 2 by asking questions that will lead the students to "act out" or begin to develop an image of the problem. For example, for Problems 1 and 2, the teacher may ask about some simple examples with questions such as "What are the elements in the fourth set?" and "How would we find the elements in the fourth set if the sequence were different?" For Problem 3, the teacher may ask the student to describe the third sequence, the fourth sequence, etc.

Relevant standards:	6.EE6: Use variables to represent numbers and write expressions when solving a real-world or mathematical problem; understand that a variable can represent an unknown number, or, depending on the purpose at hand, any number in a specified set. F-IF8a: Use the process of factoring and completing the square in a quadratic function to show zeros, extreme values, and symmetry of the graph, and interpret these in terms of a context. F-BF2: Write arithmetic and geometric sequences both recursively and with an explicit formula, use them to model situations, and translate between the two forms.

Quilt Problem: A company makes square quilts. Each quilt is made out of small congruent squares, where the squares on the main diagonals of the quilt are black and the rest of the squares are white. The cost of a quilt is calculated as follows:

Materials: $\$1.00$ for each black square and $\$0.50$ for each white square.
Labor: $\$0.25$ for each square

To order a quilt, one must specify the number of black squares, or the number of white squares, or the total number of squares on the order form below:

Number of black squares	Number of white squares	Total number of squares

April, Bonnie, and Chad wanted to order identical quilts. Each of the three filled out a different order form. April entered the number of black squares correctly on her order form. The other two entered the same number as on April's form, but Bonnie accidentally wrote it in the "white squares" column, and Chad wrote it in the "total squares" column. April was charged $\$139.25$. How much money were Bonnie and Chad charged?

Your Solution to Quilt Problem:

Analysis of Quilt Problem

What students need to know in order to solve the problem:	Students must be able to solve a quadratic equation.
What students gain from solving the problem:	In solving this problem, students will assign variables to represent the dimensions of the quilt and set up expressions and equations related to the cost of the quilt, thus reinforcing the approach of representing a problem algebraically. Moreover, this problem provides natural opportunities for students to use variables for reasoning as well as as unknowns. For example, students may use a purely algebraic argument (as opposed to appealing to the symmetry of the quilt) to show that the number of white squares in a quilt is always divisible by 4, by showing that it equals $4k$ for some integer k. Since the number of white and black squares in a given quilt depends on whether or not the number of squares along each side is even or odd, students will need to perform a case by case analysis in a natural context. Case by case analysis pervades all levels of mathematics – it is the logical basis of the technique of proof by exhaustion. This problem is amenable to multiple solutions; students can use various strategies to count the number of black and white squares in a given quilt. For example: • some students make tables and look for patterns • other students count the number of white squares by counting the number of black squares and subtracting it from the number of total squares • students may also count the number of white squares directly using summation formulas, area arguments, or rearranging the squares to more easily count the white ones (in the even case one can slide the black squares around to form a plus shape, and in the odd case one can slide the black squares to form two columns). As different students present different counting arguments, we strongly encourage the teacher to take advantage of this natural opportunity for students to compare empirical and deductive reasoning. Solutions using tables should be applauded since recognizing patterns is an important and useful skill, but the

	teacher should encourage the students to go one step further – *What is causing this pattern that we see in the table? What makes the pattern to be so?* The solutions which use deductive reasoning provide answers to these questions. This problem is a multi-step *holistic* problem; in particular, it does not include hints or cues as to what must be done in order to solve it.
Expected difficulties that students will face when solving the problem and suggestions for dealing with these difficulties:	Sometimes students miss the plural "diagonals" in the problem statement and began the problem with an incorrect image of the quilt, for example with only one black diagonal. This error manifests itself quite quickly though since most of the students start by drawing pictures, so we recommend that the teacher look over the students' work early on and suggest the necessary corrections. It is not uncommon for students to overlook the issue of the two separate cases of even and odd dimension and to assume that there will be a universal formula for the number of black and white squares in terms of the side length. If students make this error, we suggest that the teacher ask them to check their formulas in some simple examples. Many students are surprised to discover that no quilt exists with Bonnie's specified number of white squares, and sometimes they think they have made an error or that there is some kind of typo in the problem statement. Here the teacher can simply ask them how they think the company would respond upon receiving an order for an "inconstructible" quilt, and of course there are various possible answers to this question.
Relevant standards:	6.EE6: Use variables to represent numbers and write expressions when solving a real-world or mathematical problem; understand that a variable can represent an unknown number, or, depending on the purpose at hand, any number in a specified set. A-CED1: Create equations and inequalities in one variable and use them to solve problems. Include equations arising from linear and quadratic functions, and simple rational and exponential functions. A-REI4: Solve quadratic equations in one variable.

Sequence Construction Problems:

(1) Build a sequence that is both arithmetic and geometric. How many such sequences can you build?
(2) Can you build an arithmetic sequence with the following property – the sum of any number of consecutive terms, starting with the first term, is three times the square of the number of consecutive terms?
(3) Can you build an arithmetic sequence with the following property – there are distinct positive integers m and n so that the sum of the first m terms equals the sum of the first n?
(4) Can you build a geometric sequence with the following property – there are three consecutive terms in the sequence which correspond to the side lengths of a right triangle?

Your Solutions to Sequence Construction Problems:

Analysis of Sequence Construction Problems

What students need to know in order to solve the problems:	Students must be familiar with both arithmetic and geometric sequences. Solutions to Question 4 use the Pythagorean Theorem.
What students gain from solving the problems:	These are all construction problems. Construction problems have an embedded task of proving that one has constructed the right thing. Thus, in solving these problems, students will advance their proof skills in a natural way. In solving these problems, students will use variables to represent and solve for unknown quantities, but the problems also provide natural opportunities for students to use variables for reasoning. For example, given an arithmetic sequence with initial term a, we can represent the k^{th} term in the sequence as $a + (k-1)d$ for some number d.
Expected difficulties that students will face when solving the problems and suggestions for dealing with these difficulties:	The majority of textbooks contain very few, if any, construction problems. Thus, an expected difficulty is that the students will not understand what the problem is asking them to do. We recommend that the class do some simple warm-up problems before starting these problems, for example "Build a geometric sequence" or "Build an arithmetic sequence".
Relevant standards:	6.EE6: Use variables to represent numbers and write expressions when solving a real-world or mathematical problem; understand that a variable can represent an unknown number, or, depending on the purpose at hand, any number in a specified set. 8.G7: Apply the Pythagorean Theorem to determine unknown side lengths in right triangles in real-world and mathematical problems in two and three dimensions. F-BF2: Write arithmetic and geometric sequences both recursively and with an explicit formula, use them to model situations, and translate between the two forms.

Three Digit Number Problems:

(1) What is the sum of all positive integers between 100 and 500 that are divisible by 6?
(2) What is the sum of all positive integers between 100 and 500 that are divisible by 6 and whose last digit is 6?
(3) What is the sum of all positive integers between 100 and 500 that are divisible by 6 and whose last digit is not 6?
(4) What is the sum of all positive integers between 100 and 500 that are not divisible by 6 and whose last digit is 6?

Your Solutions to Three Digit Number Problems:

Analysis of Three Digit Number Problems

What students need to know in order to solve the problems:	Students must be familiar with arithmetic sequences (specifically, the formula for the n^{th} term in an arithmetic sequence.) Students must be familiar with the notion of divisibility. In solving these problems, students will need to evaluate sums of the form $1+2+\cdots+n$. If students have not seen or forgotten the formula for such a sum, this problem can serve as an opportunity to develop and justify the formula in a natural context.
What students gain from solving the problems:	These problems provide natural opportunities for students to use variables for reasoning (rather than as unknowns for which they are trying to solve). For example, given an arithmetic sequence with initial term a, we can represent the k^{th} term in the sequence as $a + (k - 1)d$ for some number d. In solving these problems, studying the complement of a set arises in a natural way in Problems 3 and 4. Reasoning logically using the complement of a set is an extremely important way of approaching many problems in probability and combinatorics. For example, suppose one wants to know how many ways it is possible to arrange twelve people in a line with the constraint that a particular pair of them are not standing next to one another. To solve this problem, it is much easier to count the numbers of ways to arrange the people so that the pair are standing next to one another and then subtract this number from the total number of ways to arrange twelve people in a line.
Expected difficulties that students will face when solving the problems and suggestions for dealing with these difficulties:	The key to solving these problems in a generalizable way is to realize the numbers which are divisible by 6 as an arithmetic sequence. We expect that some students will instead try to solve them in a brute force fashion by listing all of the numbers within the given constraints that are divisible by 6. To compel the students to approach the problems in a more conceptual way, the teacher may choose to change the upper bound to a much larger number, for example 5000.
Relevant standards:	6.EE6: Use variables to represent numbers and write expressions when solving a real-world or mathematical problem; understand that a variable can represent an unknown number, or, depending on the purpose at hand, any number in a specified set.

A-SSE2: Use the structure of an expression to identify ways to rewrite it.

F-BF2: Write arithmetic and geometric sequences both recursively and with an explicit formula, use them to model situations, and translate between the two forms.

S-CP1: Describe events as subsets of a sample space (the set of outcomes) using characteristics (or categories) of the outcomes, or as unions, intersections, or complements of other events ("or","and","not").

6. Thinking in Terms of Functions

The concept of function is one of the single most important concepts in mathematics; students should not begin a calculus course without a solid understanding of function. The current curriculum contains no shortage of problems which compel students to understand functions as formulas, such as problems asking to state the domain or range of a given function. In contrast, the goal of the following problems is to advance our students' understanding of a function as a dynamic input/output process.

Pharmacist Problems:

(1) A pharmacist is to prepare 15 milliliters of special eye drops for a glaucoma patient. The eye drop solution must have a 2% active ingredient, but the pharmacist only has 10% solution and 1% solution in stock. Can the pharmacist use the solutions she has in stock to fill the prescription?

(2) The same pharmacist receives a large number of special eye drops for glaucoma patients. The prescriptions vary in volume but each requires a 2% active ingredient. Help the pharmacist find a convenient way to determine the exact amounts of 10% solution and 1% solution needed for a given volume of eye drops.

(3) The pharmacist calculates the cost of the special eye drops as follows: T milliliters of the 1% solution costs $T/3 + 1.93$ dollars, and T milliliters of the 10% solution costs $2.87T + 3.07$ dollars.
   (a) How much does 18.93 milliliters of the special solution cost?
   (b) You paid \$$29.19$ for a bottle of the special solution. How many milliliters of the special solution did you purchase?
   (c) If the solution is only sold in *whole* milliliter quantities, what is the maximum amount of solution that you can purchase for \$$15.11$?

(4) A different pharmacist calculates the cost of the special eye drops as follows: S milliliters of the 1% solution costs $\dfrac{|S-17|}{3} + 2$ dollars, and S milliliters of the 10% solution costs $4\left|S - 15\frac{2}{7}\right| + 1$ dollars.
   (a) How much does 18.93 milliliters of the special solution cost?
   (b) You paid \$$29.19$ for a bottle of the special solution. How many milliliters of the special solution did you purchase?
   (c) If the solution is only sold in *whole* milliliter quantities, what is the maximum amount of solution that you can purchase for \$$15.11$?

Your Solutions to Pharmacist Problems:

Analysis of Pharmacist Problems

What students need to know in order to solve the problems:	Students should have some experience solving systems of two linear equations in two unknowns. Students should have experience solving equations and inequalities involving absolute value.
What students gain from solving the problems:	These problems have the potential to advance and strengthen students' abilities to think in terms of functions. Functions arise naturally when one has a large collection of data and attempts to globally relate the inputs and outputs (instead of one by one). Functions have three ingredients: inputs, ouputs, and the rule that relates them, but in order to think in terms of functions, one must have the ability to abstract from particular input/output pairs. Standard questions about the domain and ranges of functions are artificial and usually feel purposeless to students and as a consequence, do not prepare them for the process of abstraction. In these questions, inputs and outputs appear in a concrete context that is meaningful; for example the abstraction from Problem 1 to Problem 2 is natural and meaningful. Notice the formulation of the question in Problem 1: "*Can* the pharmacist..." The question does not tell the students which direction to pursue; i.e. the students must determine whether or not to the pharmacist can fill the prescription. Thus, this problem has the added value that the students must determine the direction to take in solving the problem. In solving Problems 3 and 4, students will solidify their understanding of absolute value and increase their fluency in solving equations and inequalities involving absolute value. The choices of T and S as variables in Problems 3 and 4 are intentional. The ultimate goal is that students realize that when one describes a function, it does not matter which variable is chosen to represent the input. This problem is a multi-step *holistic* problem; in particular, it does not include hints or cues as to what must be done in order to solve it.

Expected difficulties that students will face when solving the problems and suggestions for dealing with these difficulties:	Students may have difficulty understanding what percent means in this context. In order to prepare them to correctly interpret percent, before starting this problem, the class can do some simple problems that emphasize the meaning of percent, for example from the first section of this booklet. We expect several students to reason additively: combining a 10% solution and a 1% solution yields an 11% solution. Students can quickly realize the error here, especially if the teacher asks them about some other situations which they can act out more easily. For example, "If it takes Amy 6 hours to paint one room, and it takes Alex 12 hours to paint the same room, how long would it take for them to paint the room together?" Reasoning additively, it will take 18 hours, but students will be perplexed: how can it take *more* time if each of them have *less* work?
Relevant standards:	8.EE8c: Solve real-world and mathematical problems leading to two linear equations in two variables. A-REI3: Solve linear equations and inequalities in one variable, including equations with coefficients represented by letters. F-BF1: Write a function that describes a relationship between two quantities.

Train Problems:

(1) Assume that the trains between Los Angeles and San Diego leave each city every hour on the hour with no stops. A one-way trip in each direction takes four hours. How many trains going from San Diego to Los Angeles will a train pass on its run from Los Angeles to San Diego?

(2) Amtrak has hundreds of schedules similar to those shown in the table below. Due to safety reasons, Amtrak's engineers must know the number of passing points for a given train. What suggestion would you give Amtrak for how to compute the number of passing points for a given train?

	Route	Frequency on each end	One-way travel time	Number of passing points
1	San Diego – Los Angeles	Every 1 hour	4 hours	
2	Washington – New York	Every 1 hour	3 hours	
3	Chicago – Cincinnati	Every 1 hour	7 hours	
...	...	...	...	...

(3) Another type of schedule is shown in the following table. Again, what suggestion would you five Amtrak for how to compute the number of passing points for each train?

	Route	Frequency on each end	One-way travel time	Number of passing points
1	Chicago – Indianapolis	Every 30 minutes	5 hours	
2	Los Vegas – Los Angeles	Every 3 hours	9 hours	
3	Seattle – Salt Lake City	Every 140 minutes	14 hours	
...	...	...	...	...

Your Solutions to Train Problems:

Analysis of Train Problems

What students need to know in order to solve the problems:	Students need to be able to act out a problem, i.e. to build a coherent image of the problem situation.
What students gain from solving the problems:	These problems have the potential to advance and strengthen students' abilities to think in terms of variables and functions in a natural way. As the students move through the sequence of problems, certain quantities change from fixed to variable. In Problem 1, the frequencies and one-way travel times are each one hour. In Problem 2, the frequencies are all one hour, but the one way travel times vary from train to train. In Problem 3, both quantities vary.
Expected difficulties that students will face when solving the problems and suggestions for dealing with these difficulties:	In solving Problem 1, often it does not occur to students to focus on one departing train. The teacher should let the students struggle with this issue for a bit, and then he or she may suggest to the students to pick one departing train (for example the train that leaves Los Angeles at noon) in order to understand what is happening. Upon focusing on one departing train, often it does not occur to students to consider the trains traveling in the opposite direction that left before it. Again, the teacher should wait for students to develop their own ideas, and if they make this mistake, the teacher can ask them about the trains that departed before. For example, if the student fixes a train which leaves Los Angeles at noon, the teacher can ask, "Will it pass the train that left from San Diego at 11:00 am?" Problem 1 evokes the rate/distance/time schema, so some students are likely to try and solve it by setting up equations. It is certainly more "painful" to solve the problem in this way, so in this sense it is an expected difficulty. However, solving the problem in this way gives much more information, for example where and what time the trains meet. We recommend that the class compare these solutions with the simpler solutions. The students should agree that using equations requires more work, but the payoff is that we get more information. Even if the the question did not ask for the the information obtained, one can ask for this information, whereby the student becomes the creator of a new problem.
Relevant standards:	F-BF1a: Determine an explicit expression, a recursive process, or steps for calculation from a context.

Hill Problems: Jack and Jill run 10 kilometers. They start at the same point, run 5 kilometers up a hill, and return to the starting point by the same route. Jack has a 10 minute head start and runs at the rate of 15 km/hr uphill and 20 km/hr downhill. Jill runs 16 km/hr uphill and 22 km/hr downhill.

(1) How far from the top of the hill are they when they pass going in opposite direction?
(2) What is the distance between Jack and Jill at any given moment from the time Jill leaves until Jack arrives?
(3) At what time(s) from Jills departure is the distance between Jill and Jack no more than $3/7$ km and no less than $2/7$ km?

Your Solutions to Hill Problems:

Analysis of Hill Problems

What students need to know in order to solve the problems:	Students should have experience representing problems algebraically.
What students gain from solving the problems:	Notice the formulation of the question in Problem 2: "What is the distance between Jack and Jill *at any given moment...*", and contrast this with the usual wording of similar questions in textbooks, e.g. "Find a formula for the distance..." The wording here is designed so that students conceive of a function as a process which transforms a collection of input objects into output objects, rather than simply a formula, whereby encouraging attention to meaning, especially of algebraic symbols. Moreover, here we have a context in which a piecewise function emerges in a natural way. In solving these problems, students will have the opportunity to solidify their understanding of the concept of absolute value, as well as to gain experience solving equations on intervals and solving inequalities involving absolute values. Here the teacher should be sure to take the opportunities to ask the students to articulate what it means to solve an equation on an interval, and so on.
Expected difficulties that students will face when solving the problems and suggestions for dealing with these difficulties:	The nonstandard wording of Problem 2 may leave students unsure as to how to proceed. Here the teacher may suggest to the students to find the distance between Jack and Jill at particular moments, i.e. the teacher can ask the students for the distance between them after one minute, and then after two minutes, and so on. Sometimes students approach this problem geometrically, but instead of graphing the distances of Jack and Jill from the starting point as functions of time, they graph their respective distances traveled as functions of time. Based on our experience, when students make this error, the teacher usually does not need to intervene, since the students will correct themselves when they see that the two graphs do not intersect.

Relevant standards:	6.EE5: Understand solving an equation or inequality as a process of answering a question: which values from a specified set, if any, make the equation or inequality true? Use substitution to determine whether a given number in a specified set makes an equation or inequality true. B-BF1: Write a function that describes a relationship between two quantities. F-IF7b: Graph square root, cube root, and piecewise-defined functions, including step functions and absolute value functions.

7. Problems about Quadratic Functions

Comparing the treatment of linear functions and quadratic functions in the current curriculum reveals a certain asymmetry. A line has both an algebraic definition: the points (x, y) in a plane satisfying an equation of the form $y = mx + b$, and a geometric characterization: for example, the set of points in a plane that are equidistant from two given points in the same plane, or the set of all points (x, y) in a plane whose relation to a given point (x_0, y_0) in the same plane is $y - y_0 = K(x - x_0)$. In contrast, a parabola is defined algebraically as the set of points (x, y) in a plane satisfying an equation of the form $y = ax^2 + bx + c$. The following problems first develop the geometric definition of a problem and then address the question of why the geometric and algebraic objects are the same. Understanding that geometry and algebra are representations of one another is a crucial habit of mind when practicing mathematics.

Parabola Problems:

(1) Object A is moving in an x-y plane. Following its movement, it was observed that the object's distance from the point $(3/2, -2/3)$ is always equal to the object's distance from the line $y = 5/6$. Sketch a graph of the object's positions in the plane.

(2) Object B is moving in an x-y plane. The equation of the graph of the object's positions is $y = 2x^2 - 5x - 12$. Is the nature of the movement of Object B the same as the nature of the movement of Object A?

(3) A parabola is the set of all points in the plane whose distance from a given point is equal to their distance from a given line. A quadratic function is a function of the form $y = ax^2 + bx + c$.
   (a) Prove that the graph of a quadratic function is a parabola.
   (b) Prove that the equation of a parabola is a quadratic function.

Your Solutions to Parabola Problems:

Analysis of Parabola Problems:

What students need to know in order to solve the problems:	Students should know how to find the distance between two points in the plane. Students should know how to find the distance between a point and a line in the plane. Students should be able to solve simple systems of equations. Students should be familiar with the idea of substitution, i.e. if two expressions are equal for each value of the variables, then they are equal for any particular choice of the variables. Students also need a good sense of functions, and in particular that two functions with the same domain are equal if their outputs are equal at every input.
What students gain from solving the problems:	Students at this level are likely to have seen the formula for the distance between a point and a line, and this problem serves as an opportunity to make sense of and solidify their understanding of the formula. The teacher can ask the students "What makes sense as the distance between a point and a line?" and can act out the shortest path by walking towards the wall or a row of desks. The students will find the distance between a point and a line in each of the questions, and it is important that the teacher ask them to repeatedly reason to obtain the formula, rather than simply memorizing and plugging in; in this way the students will understand and internalize the formula so that they can recreate it each time they need it. The substitution in Question 1 and the equating coefficients in Questions 2 and 3 are opportunities to promote a deeper understanding of the concept of function and specifically the notion of a function as a process which transforms a collection of input objects into output objects, rather than simply as a formula. Question 3 affords an excellent opportunity to for students to engage in a concrete situation in which one needs to "unpack" a complex logical statement: What does it mean that the graph of every quadratic function is a parabola? It means that we can find a fixed point P and a fixed line l so that all of the points satisfying the associated quadratic equation are the same

distance from P as they are from l. Students who master Problem 3 would have an easier time understanding the definition of a limit, for example.

As a whole, this sequence of problems reinforces the extremely important notion that mathematical topics have both geometric and algebraic formulations. Important features visible in either of these aspects should be visible in the other as well, and one should search for and exploit these correspondences.

Expected difficulties that students will face when solving the problems and suggestions for dealing with these difficulties:

We expect students to be unsure as to how to approach Question 1, and it is likely that the class will need to revisit the definition of the distance between a point and a line (see above). After that, it is likely that students will take an "eyeballing" approach. The idea is that the teacher let the students take this approach and then ultimately provoke a more rigorous idea. In particular, the teacher can ask the students "how to tell the eyeballing approach to algebra?", i.e. given a point (x, y), how do we tell algebra that (x, y) is the same distance from $(3/2, -2/3)$ as it is from the line $y = 5/6$? Once the students set up the corresponding equation, the teacher can suggest that they set up a table of x and y values that satisfy the equation and finally graph the corresponding points in the plane.

We expect students to be confused by the question "Is the nature of the movement of Object B the same as the nature of the movement of Object A?" in Question 2. The class as a whole should resolve this confusion. We suggest that the teacher ask the students about the nature of the movement of Object A, and after some discussion, the class should agree that Problem 2 is asking if we can find a point and a line so that Object B is always the same distance from the point as it is from the line. Once the class agrees that this is what Problem 2 is asking, the teacher can formalize the question by writing it on the board, for example, as follows: Is there a point (N, R) and a line $y = K$ so that the distance from Object B to the point (N, R) is always equal to the distance from Object B to the line $y = K$?

Upon understanding what Question 2 is asking, students still may be unsure as to proceed, and we recommend that the teacher ask them "how to tell the question to algebra"? Here again, the students will need to utilize the definition of the distance between a point and a line. Upon translating to algebra, the question becomes whether or not there are real numbers N,

R, and K which satisfy the equation

$$\sqrt{(x-N)^2+(2x^2-5x-12-R)^2}=\sqrt{(2x^2-5x-12-K)^2},$$

and we expect that the teacher will also be very involved in the process of helping the students to answer this question. It is likely that the students will suggest squaring both sides of the equation, and after that, we recommend that the teacher ask the students what is the meaning of the polynomial equation $ax^2+bx+c=dx^2+ex+f$? With some discussion, the class should agree that it means that for every number x, if we substitute x into the left hand side of the equation, we will get the same number as if we substitute x into the right hand side of the equation.

Then, the teacher can ask the students, why, if the equality holds, must we have $a=d$, $b=e$, and $c=f$? Some students may suggest that we can see this from substituting particular values of x into both sides, but if no students suggest this, the teacher may suggest substituting $x=0$ into both sides, for example. Once the students have reached this point, the problem amounts to solving a system of three equations in three unknowns.

As Question 3 part (a) is an abstraction of Question 2, we recommend that the teacher give the students two or three more problems of the nature of Question 2 (concrete examples) before presenting them with this problem. As the students solve these additional problems, the teacher can scale down his or her role, so that the students gradually become more independent.

Relevant standards:

8.G8: Apply the Pythagorean Theorem to find the distance between two points in a coordinate system.

F-IF7a: Graph linear and quadratic functions and show intercepts, maxima, and minima.

F-IF9: Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions).

G-GPE2: Derive the equation of a parabola given a focus and directrix.

More Parabola Problems:

(1) What does a parabola look like? In particular:
   (a) When does a parabola intersect the x-axis, and when doesn't a parabola intersect the x-axis? When it does intersect the x-axis, what are the intersection points? Please justify your answer.
   (b) It seems that a parabola has a line of symmetry. Is it so? If yes, what is this line of symmetry? Please justify your answer.
   (c) It seems that a parabola has either a highest point or a lowest point. Is it so? If yes, what is this highest or lowest point? Please justify your answer.

(2) When the admission price to a ball game is 50 cents, $10,000$ people attend. For every increase of 5 cents in the admission price, 200 fewer (than the $10,000$) attend. There is an expense of one dollar for every 100 people in attendance. What is the admission price that yields the largest income?

Your Solutions to More Parabola Problems:

Analysis of More Parabola Problems

What students need to know in order to solve the problems:	Students should know the quadratic formula. Students should know how to complete the square. Students should have a geometric understanding of symmetry.
What students gain from solving the problems:	Students are likely to remember the algorithm for completing the square, but the teacher can use Problem 1 as an opportunity for a class discussion of why the algorithm works. This problem can also serve as an opportunity to derive the quadratic formula. In solving Problem 1 part (b), students will solidify and deepen their understanding of symmetry; in particular, how do we tell algebra about the symmetry of a graph? Note that the resolution of this question is another opportunity for students to internalize the definition of the distance from a point to a line which was used several times in the previous group of problems. If students are already familiar with the notions of graph transformations that preserve symmetry, then they can argue the symmetry of the graph of $y = x^2$, complete the square in the expression $y = ax^2+bx+c$ to rewrite it in the form $y = a(x+h)^2+k$, and recognize its graph as the image of the graph of $y = x^2$ after a sequence of symmetry preserving transformations. If students have not yet studied transformations of graphs, this problem can serve as an opportunity to enter the topic in a natural way. Problem 2 gives the students a chance to solidify the concepts from the preceding problems in a concrete context.
Expected difficulties that students will face when solving the problems and suggestions for dealing with these difficulties:	Students who argue Problem 1 part (b) using graph transformations should have a straightforward argument as to the existence of the highest or lowest point on the transformed graph. But it is likely that the students will not know how to approach this problem algebraically, so we suggest that the teacher give them the hint to complete the square in the expression $ax^2 + bx + c$. Once the students sort out which part of the expression is constant, they can argue why an expression of the form $mX^2 + n$, with m and n constant, has either a maximum or a minimum value.

Relevant standards:	A-REI4a: Use the method of completing the square to transform any quadratic equation in x into an equation of the form $(xp)^2 = q$ that has the same solutions. Derive the quadratic formula from this form. F-IF8a: Use the process of factoring and completing the square in a quadratic function to show zeros, extreme values, and symmetry of the graph, and interpret these in terms of a context. F-BF3: Identify the effect on the graph of replacing $f(x)$ by $f(x)+k$, $kf(x)$, $f(kx)$, and $f(x+k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include recognizing even and odd functions from their graphs and algebraic expressions for them.

8. Geometry Problems

These problems provide opportunities for students to apply theorems from geometry within algebraic contexts, so in solving these problems, students have access to both algebraic and geometric tools. Most mathematics curricula separate algebra from geometry, when in fact, realizing and exploiting the connections between the geometric and algebraic aspects of a problem or mathematical object is a crucial habit of mind for practicing mathematics.

Similar Rectangles Problem: I would like to construct a pair of similar rectangles in the following way. Starting with a rectangle $ABCD$, choose points E and F on the longer sides so that $AEFB$ is a square. Then rectangle $EDCF$ should be similar to the original rectangle $ABCD$. (Note that two rectangles are similar if one can be obtained from the other by enlarging all of the sides by the same factor.) Can this method work? Are any particular dimensions for the starting rectangle required?

Your Solution to Similar Rectangles Problem:

Analysis of Similar Rectangles Problem

What students need to know in order to solve the problem:	Since the term "similar rectangles" is defined in the problem statement, this problem assumes no geometric prerequisite knowledge apart from the definitions of rectangle and square.
What students gain from solving the problem:	This is a relatively simple problem that gives students the opportunity to investigate the meaning of similar rectangles, starting from an intuitive point of view. In solving this problem, students will need to use variables for reasoning rather than as unknowns. If we let k and m denote the side lengths of the rectangle, then our goal is to find conditions on k and m so that our construction works, but we don't need to find numeric values for k and m.
Expected difficulties that students will face when solving the problem and suggestions for dealing with these difficulties:	Here we expect that some students will start by setting up an incorrect ratio for the sides of the similar rectangles. For example, if $ABCD$ has side lengths m and k, with $m > k$, some students are likely to write down the ratio $\frac{k}{m} = \frac{k}{m-k}$. This is a perfect opportunity for the students to really think about the similarity; they will realize that since all sides must be enlarged by the *same* factor, then the sides of length $m - k$ and length k in the smaller rectangle must be enlarged to lengths k and m, respectively, to obtain the larger rectangle, and thus, for a pair of rectangles to be similar, the ratio of the long side to the short side must be the same in both.
Relevant standards:	6.EE6: Use variables to represent numbers and write expressions when solving a real-world or mathematical problem; understand that a variable can represent an unknown number, or, depending on the purpose at hand, any number in a specified set.

Sliding Circle Problem: The sides of triangle ABC have lengths 6, 8 and 10. A circle with center P and radius 1 rolls around the inside of triangle ABC, always remains tangent to at least one side of the triangle. When P first returns to its original position, through what distance has P traveled?

Your Solution to Sliding Circle Problem:

Analysis of Sliding Circle Problem

What students need to know in order to solve the problem:	Students need to know congruencies of angles when parallel lines are cut by a transversal. Students need to know the Pythagorean theorem. Students need to know what it means for two triangles to be similar and conditions for determining similarity. Students need to know what it means for a line to be tangent to a circle.
What students gain from solving the problem:	This problem is an opportunity for students to practice geometric knowledge in an algebraic context. Making connections between algebra and geometry is important at every level of mathematics. This problem offers numerous opportunities for students to revisit and enrich their understanding of geometric objects, and in particular, to realize equivalent definitions of geometric objects. For example, students may start the problem with the understanding that a line l is tangent to a circle O at a point P if the line and circle only intersect at P. In order to solve the problem, they will need to realize that this condition is equivalent to the radius $\overline{PO}$ being perpendicular to the line l. Similarly, students will likely understand the angle bisector as the ray through the vertex which divides the angle into two congruent angles, but in solving the problem, they may end up proving that the angle bisector is also the locus of points which are equidistant from the sides of the angle, thus affording another opportunity to revisit the definition of the distance from a point to a line. Also, students often forget that the converse of the Pythagorean theorem is also true, but they will use it to justify why ABC is a right triangle.

	This problem is also particularly amenable to multiple to solution strategies: • some students may set up a coordinate system and find equations of tangent lines • students may use various arguments about similarity • students may also use trigonometry (an excellent opportunity to recall trig identities as well as the definition of the "inverse" trig functions!). Thus, this problem can help establish the classroom culture that each person thinks differently and has different ideas.
Expected difficulties that students will face when solving the problem and suggestions for dealing with these difficulties:	Students often tend to draw incorrect diagrams when solving this problem. Specifically, when the circle is pushed as far as possible into one of the acute angles of ABC, some students draw the chord connecting the points of tangency as a diameter. This is a *good difficulty*, though, as it can serve as a reminder not to rely too heavily on perception. Students commonly assume that the triangle ABC is a 30-60-90 triangle. Here the teacher can ask them "How do you know?" If the students do not realize their mistake, the teacher can ask them about the relationship of ABC with a triangle of side lengths 1, 2, and $\sqrt{3}$.
Relevant standards:	8.G5: Use informal arguments to establish facts about the angle sum and exterior angle of triangles, about the angles created when parallel lines are cut by a transversal, and the angle-angle criterion for similarity of triangles. 8.G7: Apply the Pythagorean Theorem to determine unknown side lengths in right triangles in real-world and mathematical problems in two and three dimensions. G-SRT5: Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometric figures. G-C2: Identify and describe relationships among inscribed angles, radii, and chords. Include the relationship between central, inscribed, and circumscribed angles; inscribed angles on a diameter are right angles; the radius of a circle is perpendicular to the tangent where the radius intersects the circle.

Polygon Problem: A polygon has the following properties: (a) It has exactly two acute angles, (b) the measures of its angles are all positive integers, (c) the measures of its angles form an arithmetic progression whose difference equals its first term. How many segments does this polygon have?

Your Solution to Polygon Problem:

Analysis of Polygon Problem

What students need to know in order to solve the problem:	Students need to know that the number of segments in a polygon equals the number of angles. Students need to know how to find the sum of the angle measures of a polygon.
What students gain from solving the problem:	This problem is an opportunity for students to practice geometric knowledge in an algebraic context. Making connections between algebra and geometry is important at every level of mathematics. This problem affords an opportunity for the students to solve a system of inequalities in a context. If the polygon has n segments, then n must be an integer satisfying the system $$23 \leq \frac{360(n-2)}{n(n+1)} \leq 20.$$ Some students may find n using guess and check, as opposed to solving the system; if the teacher would like to discourage the guess and check approach, he or she can elevate the level of difficulty of the problem by changing part (a) to two acute angles, for example. We note that this problem also has the potential to lead students to think about divisibility in a natural context. In particular, in order for $360(n-2)/n(n+1)$ to be an integer, one must have that $n(n+1)$ divides $2^3 \cdot 3^2 \cdot 5(n-2)$, etc.
Expected difficulties that students will face when solving the problem and suggestions for dealing with these difficulties:	We expect that some students will have difficulty incorporating all of the givens into their considerations, so the instructor may choose to first present the students with a smaller, more manageable version of the problem, for example: A triangle has exactly two acute angles, and the measures of its angles form an arithmetic progression whose difference equals its first term; what are the angles? An example like this will help students realize that exactly means no more and no less, and that the implication of having exactly d acute angles in this context is they must be the *smallest* of the angles.
Relevant standards:	A formula for the sum of the angle measures of a polygon is not mentioned in the common core standards.

Tangent Circles Problem: A circle is tangent to the two sides of an acute angle; a second circle is tangent to the first circle and to the two sides of the angle; a third circle is tangent to the second circle and to the two sides of the angle; and so on. Select any three consecutive circles, and call the smallest one C_1, the middle one C_2, and the largest one C_3. Let the radii of C_1 and C_2 be r and R, respectively.

(1) What is the sum of the areas of C_1, C_2, and C_3?

(2) Show that the ratio of the areas of C_1 and C_2 is $1 : 9$ if and only if the measure of the angle is $60°$.

Your Solution to Tangent Circles Problem:

Analysis of Tangent Circles Problem

What students need to know in order to solve the problem:	Students need to know what it means for two triangles to be similar and conditions for determining similarity. Students need to know the definition of an angle bisector. Students need to know the Pythagorean theorem. Students need to know what it means for a line to be tangent to a circle. Students need to know what it means for two circles to be tangent to one another.
What students gain from solving the problem:	This problem is an opportunity for students to practice geometric knowledge in an algebraic context. Making connections between algebra and geometry is important at every level of mathematics. This problem offers numerous opportunities to revisit and enrich their understanding of geometric objects, and in particular, to realize equivalent definitions of geometric objects. For example, students may start the problem with the understanding that a line l is tangent to a circle O at a point P if the line and circle only intersect at P. In order to solve the problem, they will need to realize that this condition is equivalent to the radius $\overline{PO}$ being perpendicular to the line l. Similarly, students will likely understand the angle bisector as the ray through the vertex which divides the angle into two congruent angles, but in order to solve the problem, they will need to realize that that the angle bisector is also the locus of points which are equidistant from the sides of the angle, thus affording another opportunity to revisit the definition of the distance from a point to a line. The teacher can also choose to modify the problem to bring in a natural opportunity for the students to use the formula for a geometric series as follows. The teacher may ask, "Suppose the radius of the j^{th} circle equals one, what is the sum of the area of the circles C_1, $C_2, \ldots, C_3$?"

Expected difficulties that students will face when solving the problem and suggestions for dealing with these difficulties:	Many students may not know where to begin. We recommend that the teacher suggest that all of the students draw pictures, if they haven't already. Then one hint that the teacher may give is to ask the students why the problem only gives the names to the first two radii, and specifically, for which kinds of sequences does knowing two terms mean that you know all of the rest? Another expected difficulty is that students will rely on their perception and assume too much. For example, they may assume that the points of tangency of the pairs of circles are on the angle bisector, but in fact this needs to be argued. To address this difficulty, the teacher should keep asking the students "Why?" and "How do you know?"
Relevant standards:	8.G7: Apply the Pythagorean Theorem to determine unknown side lengths in right triangles in real-world and mathematical problems in two and three dimensions. G-CO9: Prove theorems about lines and angles. Theorems include: vertical angles are congruent; when a transversal crosses parallel lines, alternate interior angles are congruent and corresponding angles are congruent; points on a perpendicular bisector of a line segment are exactly those equidistant from the segments endpoints. G-SRT5: Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometric figures. G-C2: Identify and describe relationships among inscribed angles, radii, and chords. Include the relationship between central, inscribed, and circumscribed angles; inscribed angles on a diameter are right angles; the radius of a circle is perpendicular to the tangent where the radius intersects the circle.

Congruent Triangles Problem: Your geometry textbook contains the following challenging homework problem: "Can two triangles have five equal parts (sides and angles) and yet not be congruent?" The answer in the back is "YES", with no explanation. Feeling skeptical, and knowing your textbook isn't always reliable, you sit down to settle the issue yourself. What do you conclude?

Your Solution to Congruent Triangles Problem:

Analysis of Congruent Triangles Problem

What students need to know in order to solve the problem:	Students need to know what it means for two triangles to be congruent and conditions for determining congruency. Students need to know what it means for two triangles to be similar and conditions for determining similarity. Students should know the triangle inequality, and in particular, its implications for constructibility.
What students gain from solving the problem:	This problem offers numerous opportunity for students to apply theorems from geometry in a natural context. One of the goals of this problem is that students develop perseverance. Students are accustomed to short problem statements in their textbooks and short solutions, but a large number mathematical problems require perseverance to understand, let alone solve. Empirical reasoning is important in mathematics – we gain insight from examples, but students tend to rely too heavily on empirical reasoning. In solving this problem, students can gain a healthy skepticism of empirical reasoning since many of them will initially suspect that the triangles must be congruent. The lesson is to utilize intuition, but also to be suspicious of it. In solving this problem, students will naturally be led to a case by case analysis. Case by case analysis pervades all levels of mathematics – it is the logical basis of the technique of proof by exhaustion. This problem is free of numbers. In order to consider the possible cases, students will be compelled to assign variables to represent the relevant quantities. The process of assigning variables as a tool to organize and keep information present is a crucial habit of mind when practicing mathematics.

Expected difficulties that students will face when solving the problem and suggestions for dealing with these difficulties:

Based on our experience with this problem, we expect that most students will be quick to conclude that the two triangles must be congruent. Here we recommend the teacher to lead the class through a case by case analysis:

- if three of the five parts are side lengths, then the triangles must be congruent by SSS
- if three of the five parts are angle measures, then the triangles are necessarily similar, but one needs to systematically consider the possibilities for the side lengths.

The teacher can help the students to be systematic by drawing two triangles on the board with the equal angles labelled α, β, and γ. He or she can then label the side lengths of one of the two triangles, for example as a, b, and c, with a opposite α, etc. Supposing the side lengths of the second triangle are a, b, and $\tilde{c}$, the class can systematically consider the possible positions for the side of length a, etc., eliminating those possibilities that make the two triangles congruent. In particular, a cannot be opposite α in the second triangle, or we would have congruence. The teacher should then ask the students what happens if a is opposite β, for example. Using similarity, we see that in this case, $a = \lambda b$, $b = \lambda c$, and $\tilde{c} = \lambda a$, which means that the triangles are not congruent if $\lambda \neq 1$.

The next issue that the students are likely to ignore is whether or not such triangles are constructible. Here we recommend that the teacher ask the students, "So how could we actually make two triangles with five equal parts which are not congruent? What are the steps?" The goal is to lead the students to describe the process, i.e. fix a length a and a positive number λ and let $b = a/\lambda$, etc. The teacher can help the students reach this point by starting with specific questions, such as, "Can we take one of the side lengths to be 2? If so, what should we do next?". Once the students reach this point, the teacher can ask them "Is it possible that it turns out that a, b and $\tilde{c}$ are lengths of sides that can't make a triangle? How do we check if three side lengths can make a triangle?" If the students have forgotten the triangle inequality, the teacher can remind them.

Relevant standards:

G-SRT5: Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometric figures.

The triangle inequality is not mentioned in the common core standards.

Cat and Mouse Problem: A cat and a mouse were standing on two adjacent vertices of a rhombus, $ABCD$ – the cat on A and the mouse on B. The mouse's safe house is located at O, the intersection point of the rhombus diagonals. A fence surrounds the rhombus along its sides with only three holes through which the mouse can get to his safe house: one hole at B, one hole at E, which is the midpoint of the route BC, and one hole at C. The moment the cat and the mouse notice each other, the cat starts running toward B along route AB, and the mouse toward C along route BC. Had the mouse chosen route BO, he would have reached his safe house when the cat is halfway between A and B. Out of excitement, the mouse misses the hole at E, and continues instead toward C. Had he taken route EO, he would have reached his safe house at the same time the cat reaches B. The mouse turns at C and continues running along route CO. Since the cat cannot see the mouse, she enters the hole at E, and runs along the shortest way toward route CO. The moment she reaches route CO, the mouse is six yards away from his safe house. Upon reaching route CO, the cat realizes that with her current speed, she is not going to reach the mouse, and so she increases her speed by 1 yard per minute. The mouse, however, reaches his safe house 15 seconds before the cat does.

(1) How far did the cat and the mouse run?

(2) What was the speed of the cat? What was the speed of the mouse?

Your Solution to Cat and Mouse Problem:

Analysis of Cat and Mouse Problem

What students need to know in order to solve the problem:	Students need to know what it means for two triangles to be similar and conditions for determining similarity. Students need to know the Pythagorean theorem. Students need to know the definition of a rhombus, as well as the property that the diagonals bisect one another and are perpendicular to one another.
What students gain from solving the problem:	This problem offers numerous opportunity for students to apply theorems from geometry in a natural way. In addition to the theorems listed as prerequisite knowledge, students will also need to develop the midsegment theorem to solve this problem. Students who have not seen the midsegment theorem before can deduce it using similar triangles, and students who have seen the theorem will have an opportunity to apply and internalize it. One of the goals of this problem is that students develop perseverance. Simply reading the problem requires perseverance, and then the student must persevere to develop a coherent image of the problem and follow all of the details of the solution process. Students are accustomed to short problem statements in their textbooks, but a large number mathematical problems require perseverance to understand, let alone solve. This problem is virtually free of numbers. Hence, in order for students to maintain and utilize the given information, they are compelled to assign variables to represent the relevant quantities. The process of assigning variables as a tool to organize and keep information present is a crucial habit of mind when practicing mathematics.
Expected difficulties that students will face when solving the problem and suggestions for dealing with these difficulties:	This problem requires careful reading, but this is a *good difficulty* as this difficulty is consistent with the nature of mathematics. The problem is designed to be lengthy in its presentation so that in solving it, students will advance their ability to coordinate information and be empowered to solve future problems.

	Students may not remember or know some of the theorems used in solving this problem. We recommend that the class as a whole recall the definition of a rhombus and use it to deduce the relevant properties of the diagonals.
Relevant standards:	8.G7: Apply the Pythagorean Theorem to determine unknown side lengths in right triangles in real-world and mathematical problems in two and three dimensions. G-SRT5: Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometric figures.

References

[CCS10] *Common Core Standards for Mathematics*, Retrieved in 2010 from http://www.corestandards.org/assets/CCSSI_Math%20Standards.pdf.

[H] G. Harel, *A Review of Four High-School Mathematics Programs*, http://www.strategicteaching.com/washington_state_standards_.html.

[H08] G. Harel, *What is Mathematics? A Pedagogical Answer to a Philosophical Question*, Current Issues in the Philosophy of Mathematics from the Perspective of Mathematicians, Mathematical Association of America, 2008.

[H10] G. Harel, *DNR-Based Instruction in Mathematics as a Conceptual Framework*, in "Theories of Mathematics Education" pp 343–368, Springer, 2010.