ELLIPTIC SELBERG INTEGRALS AND CONFORMAL BLOCKS

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Abstract. We present an elliptic version of Selberg's integral formula.

1. Introduction

The Selberg integral is the integral

$$B_p(\alpha,\beta,\gamma)=\int_{\Delta_p}\prod_{j=1}^{p}t_j^{\;\alpha-1}(1-t_j)^{\beta-1}\prod_{0\le j<k\le 1}(t_j-t_k)^{2\gamma}dt_1\dots dt_p,$$

where $\Delta_p=\{t\in\mathbb{R}^p\,|\,0\le t_p\le\dots\le t_1\le 1\}$. The Selberg integral is a generalization of the beta function. It can be calculated explicitly,

$$B_p(\alpha,\beta,\gamma)=\frac{1}{p!}\prod_{j=0}^{p-1}\frac{\Gamma(1+\gamma+j\gamma)\Gamma(\alpha+j\gamma)\Gamma(\beta+j\gamma)}{\Gamma(1+\gamma)\Gamma(\alpha+\beta+(p+j-1)\gamma)}.$$

The Selberg integral has many applications, see [A1, A2, As, D, DF1, DF2, M, S]. In this paper, we present elliptic versions of the Selberg integral.

2. Conformal Blocks on the Torus

Let $\tau\in\mathbb{C}$ be such that $\operatorname{Im}\tau>0$. Let κ and p be non-negative integers satisfying $\kappa\ge 2p+2$. The KZB-heat equation is the partial differential equation

$$2\pi i\kappa\frac{\partial u}{\partial\tau}(\lambda,\tau)=\frac{\partial^2 u}{\partial\lambda^2}(\lambda,\tau)+p(p+1)\rho'(\lambda,\tau)u(\lambda,\tau). \tag{1}$$

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Here, the prime denotes the derivative with respect to the first argument, and ρ is defined in terms of the first Jacobi theta function,

$$\vartheta_1(\lambda,\tau) = 2q^{\frac{1}{8}}\sin(\pi\lambda)\prod_{j=1}^{\infty}(1-q^j e^{2\pi i\lambda})(1-q^j e^{-2\pi i\lambda})(1-q^j), \quad \rho(\lambda,\tau) = \frac{\vartheta_1'(\lambda,\tau)}{\vartheta_1(\lambda,\tau)},$$

where $q = e^{2\pi i\tau}$. Holomorphic solutions of the KZB-heat equation with the properties,

(i) $u(\lambda+2,\tau) = u(\lambda,\tau)$,
(ii) $u(\lambda+2\tau,\tau) = e^{-2\pi i\kappa(\lambda+\tau)}\,u(\lambda,\tau)$,
(iii) $u(-\lambda,\tau) = (-1)^{p+1}\,u(\lambda,\tau)$,
(iv) $u(\lambda,\tau) = \mathcal{O}((\lambda-m-n\tau)^{p+1})$ as $\lambda\to m+n\tau$ for any $m,n\in\mathbb{Z}$

are called conformal blocks (or elliptic hypergeometric functions) associated with the family of elliptic curves $\mathbb{C}/\mathbb{Z}+\tau\mathbb{Z}$ with the marked point $z=0$ and the irreducible sl_2 representation of dimension $2p+1$. It is known that the space of conformal blocks has dimension $\kappa-2p-1$.

3. Integral Representations of Conformal Blocks

Introduce special functions

$$\sigma_\lambda(t,\tau) = \frac{\vartheta_1(\lambda-t,\tau)\vartheta_1'(0,\tau)}{\vartheta_1(\lambda,\tau)\vartheta_1(t,\tau)}, \quad E(t,\tau) = \frac{\vartheta_1(t,\tau)}{\vartheta_1'(0,\tau)}.$$

Consider the theta functions

$$\theta_{\kappa,n}(\lambda,\tau) = \sum_{j\in\mathbb{Z}} e^{2\pi i\kappa(j+\frac{n}{2\kappa})^2\tau+2\pi i\kappa(j+\frac{n}{2\kappa})\lambda}, \quad n\in\mathbb{Z}/2\kappa\mathbb{Z}.$$

They form a basis of the space of theta functions of level κ. They satisfy the equations

$$\theta_{\kappa,n}(\lambda+1,\tau) = (-1)^n\theta_{\kappa,n}(\lambda,\tau), \quad \theta_{\kappa,n}(\lambda+\tau,\tau) = e^{-\pi i\kappa(\lambda+\frac{\tau}{2})}\theta_{\kappa,n+\kappa}(\lambda,\tau)$$

and have the modular properties

$$\theta_{\kappa,n}(\lambda,\tau+1) = e^{\pi i\frac{n^2}{2\kappa}}\theta_{\kappa,n}(\lambda,\tau), \quad \theta_{\kappa,n}\left(\frac{\lambda}{\tau},-\frac{1}{\tau}\right) = \sqrt{-\frac{i\tau}{2\kappa}}e^{\pi i\kappa\frac{\lambda^2}{2\tau}}\sum_{m=0}^{2\kappa-1}e^{-\pi i\frac{mn}{\kappa}}\,\theta_{\kappa,m}(\lambda,\tau),$$

where $|\arg(-i\tau)| < \pi/2$. Let $\theta^s_{\kappa,n}$ denote the symmetrization of $\theta_{\kappa,n}$ with respect to λ, $\theta^s_{\kappa,n}(\lambda,\tau) = \theta_{\kappa,n}(\lambda,\tau)+\theta_{\kappa,n}(-\lambda,\tau)$.

Define $u_{\kappa,n}$ by

$$u_{\kappa,n}(\lambda,\tau) = u_{p,\kappa,n}(\lambda,\tau) = J_{p,\kappa,n}(\lambda,\tau)+(-1)^{p+1}J_{p,\kappa,n}(-\lambda,\tau),$$

where

$$J_{p,\kappa,n}(\lambda,\tau) = \int_{\Delta_p} \prod_{j=1}^{p} E(t_j,\tau)^{-\frac{2p}{\kappa}} \prod_{1\le j<k\le p} E(t_j-t_k,\tau)^{\frac{2}{\kappa}} \times \prod_{j=1}^{p} \sigma_\lambda(t_j,\tau)\theta_{\kappa,n}\left(\lambda+\frac{2}{\kappa}\sum_{j=1}^{p} t_j,\tau\right) dt_1\dots dt_p \ .$$

The branch of the logarithm is chosen in such a way that arg $(E(t,\tau)) \to 0$ as $t \to 0^+$, and the integral is understood as the analytic continuation from the region where all of the exponents in the integrand have positive real parts.

Theorem 3.1. [FV1] *For all n, the integrals $u_{\kappa,n}(\lambda,\tau)$ are solutions of the KZB-heat equation having the properties (i)-(iv).*

Theorem 3.2. [FSV2] *We have*

(a) $u_{\kappa,n} = u_{\kappa,n+2\kappa}$ *and* $u_{\kappa,n} = -e^{2\pi i p n/\kappa} u_{\kappa,-n}$.

(b) *The set $\{u_{\kappa,n}(\lambda,\tau)\,|\,n = p+1,\dots,\kappa-p-1\}$ is a basis for the space of conformal blocks. The integrals $u_{\kappa,n}$ are identically zero for all other values of n in the interval from 0 to κ.*

4. Transformations acting on the space of conformal blocks

Introduce four transformations A, B, T, and S defined by

$$Au(\lambda,\tau) = u(\lambda+1,\tau), \quad Bu(\lambda,\tau) = e^{\pi i\kappa(\lambda+\frac{\tau}{2})}u(\lambda+\tau,\tau),$$

$$Tu(\lambda,\tau) = u(\lambda,\tau+1), \quad Su(\lambda,\tau) = e^{-\pi i\kappa\frac{\lambda^2}{2\tau}}\tau^{-\frac{1}{2}-\frac{p(p+1)}{\kappa}}\, u\left(\frac{\lambda}{\tau},-\frac{1}{\tau}\right),$$

where we fix $\arg\tau \in (0,\pi)$.

Proposition 4.1. *If $u(\lambda,\tau)$ is a solution of the KZB-heat equation, then $Au(\lambda,\tau)$, $Bu(\lambda,\tau)$, $Tu(\lambda,\tau)$, and $Su(\lambda,\tau)$ are solutions too. Moreover, the transformations A, B, T and S preserve the properties (i)-(iv).*

The proofs that T and S preserve the space of conformal blocks are given in [EK]. The proofs that A and B also preserve this space are straightforward and follow from the equations

$$\vartheta_1(\lambda+1,\tau) = -\vartheta_1(\lambda,\tau), \quad \vartheta_1(\lambda+\tau,\tau) = -e^{-\pi i(2\lambda+\tau)}\vartheta_1(\lambda,\tau).$$

Lemma 4.2. *Restricted to the space of conformal blocks, the transformations A, B, T, and S satisfy the relations*

$$A^2 = I, \quad B^2 = I, \quad S^2 = (-1)^p i e^{-\pi i\frac{p(p+1)}{\kappa}} I, \quad (ST)^3 = (-1)^p i e^{-\pi i\frac{p(p+1)}{\kappa}} I,$$

$$SAS^{-1} = B, \quad AB = (-1)^\kappa BA, \quad TB = i^\kappa BAT,$$

where I denotes the identity transformation. □

Lemma 4.3. *We have*

$$Au_{\kappa,n}(\lambda,\tau) = (-1)^n u_{\kappa,n}(\lambda,\tau), \quad Bu_{\kappa,n}(\lambda,\tau) = -e^{2\pi i \frac{pn}{\kappa}} u_{\kappa,\kappa-n}(\lambda,\tau). \quad \square$$

Let $(t_{m,n})$ and $(s_{m,n})$ be the matrices of the transformations T and S, respectively, with respect to the basis $\{u_{\kappa,n}(\lambda,\tau)\,|\,p+1 \le n \le \kappa-p-1\}$, namely, $Tu_{\kappa,n} = \sum_{m=p+1}^{\kappa-p-1} t_{m,n}u_{\kappa,m}$, $Su_{\kappa,n} = \sum_{m=p+1}^{\kappa-p-1} s_{m,n}u_{\kappa,m}$. In Theorem 4.4, we give formulas for the matrices of T and S in terms of Macdonald polynomials of type A_1.

The Macdonald polynomials [Ma] of type A_1 are x-even polynomials in terms of ϵ^{mx}, where $m \in \mathbb{Z}$. They depend on two parameters k and n, where k and n are non-negative integers. They are defined by the conditions:

(1) $P_n^{(k)}(x) = \epsilon^{nx} + \epsilon^{-nx} +$ lower order terms, except for $P_0^{(k)}(x) = 1$,
(2) $\langle P_m^{(k)}, P_n^{(k)} \rangle = 0$ for $m \ne n$, where

$$\langle f, g \rangle = \frac{1}{2}\text{Const Term}\left(fg \prod_{j=0}^{k-1} \left(1-\epsilon^{2(j+x)}\right)\left(1-\epsilon^{2(j-x)}\right) \right).$$

Theorem 4.4. [FSV2] *Let $\epsilon = e^{\pi i/\kappa}$. For $p+1 \le m, n \le \kappa-p-1$, we have*

$$t_{m,n} = \epsilon^{\frac{n^2}{2}} \delta_{mn},$$

$$s_{m,n} = \frac{e^{-\frac{\pi i}{4}}}{\sqrt{2\kappa}} \epsilon^{p(n-m)-\frac{p(p+1)}{2}} (\epsilon^{-m} - \epsilon^{m}) \left(\prod_{j=1}^{p} (\epsilon^{-n+j} - \epsilon^{n-j}) \right) P_{n-p-1}^{(p+1)}(m),$$

where $\delta_{mn} = 1$ for $m = n$ and 0 otherwise.

5. Integral identities

To formulate our main result, we need functions η, ϕ_1, ϕ_2, and ϕ_3. The Dedekind η-function is the function $\eta(\tau) = q^{1/24}\prod_{j=1}^{\infty}(1-q^j)$. We have $\vartheta_1'(0,\tau) = 2\pi\eta^3(\tau)$. Consider the functions [W]

$$\phi_1(\tau) = \frac{\eta(\tau)^2}{\eta(\frac{\tau}{2})\eta(2\tau)} = q^{-\frac{1}{48}} \prod_{j=1}^{\infty}\left(1+q^{j-\frac{1}{2}}\right), \quad \phi_2(\tau) = \frac{\eta(\frac{\tau}{2})}{\eta(\tau)} = q^{-\frac{1}{48}} \prod_{j=1}^{\infty}\left(1-q^{j-\frac{1}{2}}\right),$$

$$\phi_3(\tau) = \sqrt{2}\,\frac{\eta(2\tau)}{\eta(\tau)} = \sqrt{2}q^{\frac{1}{24}} \prod_{j=1}^{\infty}(1+q^j).$$

We have

$$\phi_1\left(-\frac{1}{\tau}\right) = \phi_1(\tau), \quad \phi_2\left(-\frac{1}{\tau}\right) = \phi_3(\tau), \quad \phi_3\left(-\frac{1}{\tau}\right) = \phi_2(\tau),$$

$$\phi_1(\tau+1) = e^{-\frac{\pi i}{24}}\phi_2(\tau), \quad \phi_2(\tau+1) = e^{-\frac{\pi i}{24}}\phi_1(\tau), \quad \phi_3(\tau+1) = e^{\frac{\pi i}{12}}\phi_3(\tau).$$

Let

$$c_{\kappa,n} = c_{p,\kappa,n} = (2\pi)^{\frac{p(p+1)}{\kappa}} e^{-\pi i \frac{p(3p-1)}{2\kappa}} e^{\pi i \frac{p+1}{2}} B_p\left(\frac{n+1}{\kappa}, -\frac{2p}{\kappa}, \frac{1}{\kappa}\right) \prod_{j=1}^{p}\left(1 - e^{2\pi i \frac{n+j}{\kappa}}\right).$$

Here, $B_p(\alpha,\beta,\gamma)$ is the Selberg integral.

Theorem 5.1. *We have ten series of identities,*

$$u_{2p+2,p+1}(\lambda,\tau) = c_{2p+2,p+1}\vartheta_1(\lambda,\tau)^{p+1}, \tag{2}$$

$$u_{2p+3,p+1}(\lambda,\tau) = c_{2p+3,p+1}\eta(\tau)^{-\frac{3(p+1)}{2p+3}}\vartheta_1^{p+1}(\lambda,\tau)\theta_{1,0}(\lambda,\tau), \tag{3}$$

$$u_{2p+3,p+2}(\lambda,\tau) = c_{2p+3,p+2}\eta(\tau)^{-\frac{3(p+1)}{2p+3}}\vartheta_1^{p+1}(\lambda,\tau)\theta_{1,1}(\lambda,\tau), \tag{4}$$

$$u_{2p+4,p+2}(\lambda,\tau) = 2^{-\frac{2(p+1)}{2p+4}} c_{2p+4,p+2}\left(\phi_3(\tau)\eta(\tau)^{-1}\right)^{\frac{4(p+1)}{2p+4}}\vartheta_1^{p+1}(\lambda,\tau)\theta^s_{2,1}(\lambda,\tau), \tag{5}$$

$$\begin{aligned} u_{2p+4,p+1}(\lambda,\tau) + (-1)^{p+1} e^{2\pi i \frac{p(p+1)}{2p+4}} u_{2p+4,p+3}(\lambda,\tau) = \\ c_{2p+4,p+1}\left(\phi_2(\tau)\eta(\tau)^{-1}\right)^{\frac{4(p+1)}{2p+4}}\vartheta_1^{p+1}(\lambda,\tau)(\theta_{2,0}(\lambda,\tau) - \theta_{2,2}(\lambda,\tau)), \end{aligned} \tag{6}$$

$$\begin{aligned} u_{2p+4,p+1}(\lambda,\tau) + (-1)^{p} e^{2\pi i \frac{p(p+1)}{2p+4}} u_{2p+4,p+3}(\lambda,\tau) = \\ c_{2p+4,p+1}\left(\phi_1(\tau)\eta(\tau)^{-1}\right)^{\frac{4(p+1)}{2p+4}}\vartheta_1^{p+1}(\lambda,\tau)(\theta_{2,0}(\lambda,\tau) + \theta_{2,2}(\lambda,\tau)), \end{aligned} \tag{7}$$

$$\begin{aligned} u_{2p+6,p+1}(\lambda,\tau) + (-1)^{p+1} e^{2\pi i \frac{p(p+1)}{2p+6}} u_{2p+6,p+5}(\lambda,\tau) = \\ 2^{\frac{3(p+1)}{2p+6}} c_{2p+6,p+1}\left(\phi_3(\tau)\eta(\tau)\right)^{-\frac{6(p+1)}{2p+6}}\vartheta_1^{p+1}(\lambda,\tau)(\theta_{4,0}(\lambda,\tau) - \theta_{4,4}(\lambda,\tau)), \end{aligned} \tag{8}$$

$$\begin{aligned} u_{2p+6,p+2}(\lambda,\tau) + (-1)^{p} e^{2\pi i \frac{p(p+2)}{2p+6}} u_{2p+6,p+4}(\lambda,\tau) = \\ c_{2p+6,p+2}\left(\phi_2(\tau)\eta(\tau)\right)^{-\frac{6(p+1)}{2p+6}}\vartheta_1^{p+1}(\lambda,\tau)(\theta^s_{4,1}(\lambda,\tau) + \theta^s_{4,3}(\lambda,\tau)), \end{aligned} \tag{9}$$

$$\begin{aligned} u_{2p+6,p+2}(\lambda,\tau) + (-1)^{p+1} e^{2\pi i \frac{p(p+2)}{2p+6}} u_{2p+6,p+4}(\lambda,\tau) = \\ c_{2p+6,p+2}\left(\phi_1(\tau)\eta(\tau)\right)^{-\frac{6(p+1)}{2p+6}}\vartheta_1^{p+1}(\lambda,\tau)(\theta^s_{4,1}(\lambda,\tau) - \theta^s_{4,3}(\lambda,\tau)), \end{aligned} \tag{10}$$

$$\begin{aligned} u_{2p+8,p+2}(\lambda,\tau) + (-1)^{p+1} e^{2\pi i \frac{p(p+2)}{2p+8}} u_{2p+8,p+6}(\lambda,\tau) = \\ c_{2p+8,p+2}\eta(\tau)^{\frac{-8(p+1)}{2p+8}}\vartheta_1^{p+1}(\lambda,\tau)(\theta^s_{6,1}(\lambda,\tau) - \theta^s_{6,5}(\lambda,\tau)). \end{aligned} \tag{11}$$

The integrals in Theorem 5.1 are appropriately called the elliptic Selberg integrals. The identity (2) appears in [FSV1] and in [FV1] for $p = 1$.

6. Differential equations

In Lemmas 6.1–6.3, let $'$ denote the derivative with respect to λ, let $\dot{}$ denote the derivative with respect to τ, and let $v(\lambda,\tau)=\vartheta_1^{p+1}(\lambda,\tau)\sum_{j=0}^{\kappa-2p-2}c_j(\tau)\theta^s_{\kappa-2p-2,j}(\lambda,\tau)$.

Lemma 6.1. *The function $v(\lambda,\tau)$ is a solution of the KZB-heat equation if and only if the differential equation*

$$\frac{\kappa}{p+1}\sum_{j=0}^{\kappa-2p-2}\left(\frac{d}{d\tau}c_j\right)\theta^s_{\kappa-2p-2,j}=(2p+2-\kappa)\frac{\dot\vartheta_1}{\vartheta_1}\sum_{j=0}^{\kappa-2p-2}c_j\theta^s_{\kappa-2p-2,j} -2\sum_{j=0}^{\kappa-2p-2}c_j\dot\theta^s_{\kappa-2p-2,j}+\frac{1}{\pi i}\frac{\vartheta_1'}{\vartheta_1}\sum_{j=0}^{\kappa-2p-2}c_j(\theta^s_{\kappa-2p-2,j})' \tag{12}$$

holds.

The proof of Lemma 6.1 uses the identities

$$2\pi i(2p+2)(\dot{\vartheta_1^{p+1}})(\lambda,\tau)=(\vartheta_1^{p+1})''(\lambda,\tau)+p(p+1)\rho'(\lambda,\tau)\vartheta_1^{p+1}(\lambda,\tau), \tag{13}$$

$$2\pi i\kappa\dot\theta^s_{\kappa,m}(\lambda,\tau)=(\theta^s_{\kappa,m})''(\lambda,\tau). \tag{14}$$

Proof of Lemma 6.1. Applying the differential operator $2\pi i\kappa\partial/\partial\tau$ to $v(\lambda,\tau)$ gives

$$2\pi i\kappa\Bigg[(p+1)\dot\vartheta_1\vartheta_1^p\sum_{j=0}^{\kappa-2p-2}c_j\theta^s_{\kappa-2p-2,j}+\vartheta_1^{p+1}\sum_{j=0}^{\kappa-2p-2}\left(\frac{d}{d\tau}c_j\right)\theta^s_{\kappa-2p-2,j} +\vartheta_1^{p+1}\sum_{j=0}^{\kappa-2p-2}c_j\dot\theta^s_{\kappa-2p-2,j}\Bigg].$$

Applying the differential operator $\partial^2/\partial\lambda^2+p(p+1)\rho'(\lambda,\tau)$ to $v(\lambda,\tau)$ gives

$$(\vartheta_1^{p+1})''\sum_{j=0}^{\kappa-2p-2}c_j\theta^s_{\kappa-2p-2,j}+2(p+1)\vartheta_1'\vartheta_1^p\sum_{j=0}^{\kappa-2p-2}c_j(\theta^s_{\kappa-2p-2,j})' +\vartheta_1^{p+1}\sum_{j=0}^{\kappa-2p-2}c_j(\theta^s_{\kappa-2p-2,j})''+p(p+1)\rho'\vartheta_1^{p+1}\sum_{j=0}^{\kappa-2p-2}c_j\theta^s_{\kappa-2p-2,j}.$$

Applying (13) and (14), we obtain the result. □

Lemma 6.2. *If $v(\lambda,\tau)$ is a solution of the KZB-heat equation, then the functions $c_j(\tau)$ satisfy the differential equation*

$$\frac{\kappa}{p+1}\sum_{j=0}^{\kappa-2p-2}\left(\frac{d}{d\tau}c_j(\tau)\right)\theta_{\kappa-2p-2,j}(0,\tau) = (2p+2-\kappa)\left(\frac{d}{d\tau}\ln\vartheta_1'(0,\tau)\right)\sum_{j=0}^{\kappa-2p-2}c_j(\tau)\theta_{\kappa-2p-2,j}(0,\tau) + 2(\kappa-2p-3)\sum_{j=0}^{\kappa-2p-2}c_j(\tau)\frac{d}{d\tau}\theta_{\kappa-2p-2,j}(0,\tau).$$

Proof. For any fixed λ, equation (12) gives a differential equation for the functions $c_j(\tau)$. We take the limit of that equation as $\lambda\to 0$. In the ratio $\dot\vartheta_1/\vartheta_1$, both the numerator and denominator tend to zero, so the limit of this term as $\lambda\to 0$ is equal to the limit of the ratio of the derivatives of the numerator and the denominator. The limit of the ratio $(\sum_{j=0}^{\kappa-2p-2}c_j(\theta^s_{\kappa-2p-2,j})')/\vartheta_1$ is calculated in the same way, since each $\theta^s_{\kappa-2p-2,j}$ is a symmetric function and therefore $(\theta^s_{\kappa-2p-2,j})'(0,\tau)=0$. Then the result follows from (14) . □

Lemma 6.3. *If $v(\lambda,\tau)$ is a solution of the KZB-heat equation, then the functions $c_j(\tau)$ satisfy the differential equation*

$$\frac{\kappa}{p+1}\sum_{j=0}^{\kappa-2p-2}\left(\frac{d}{d\tau}c_j(\tau)\right)\theta_{\kappa-2p-2,\kappa-2p-2-j}(0,\tau) = (2p+2-\kappa)\left(\frac{d}{d\tau}\ln\vartheta_1'(0,\tau)\right)\sum_{j=0}^{\kappa-2p-2}c_j(\tau)\theta_{\kappa-2p-2,\kappa-2p-2-j}(0,\tau) + 2(\kappa-2p-3)\sum_{j=0}^{\kappa-2p-2}c_j(\tau)\frac{d}{d\tau}\theta_{\kappa-2p-2,\kappa-2p-2-j}(0,\tau).$$

Proof. We take the limit of (12) as $\lambda\to\tau$. This limit is calculated in terms of the limit $\lambda\to 0$ using the formulas

$$\frac{\partial}{\partial z}\theta_1(z,\tau)|_{z=\lambda+\tau} = e^{-2\pi i\lambda-\pi i\tau}\left(2\pi i\theta_1(\lambda,\tau)-\theta_1'(\lambda,\tau)\right),$$
$$\frac{\partial}{\partial z}\theta_1(\lambda+\tau,z)|_{z=\tau} = e^{-2\pi i\lambda-\pi i\tau}\left(-\pi i\theta_1(\lambda,\tau)+\theta_1'(\lambda,\tau)-\dot\theta_1(\lambda,\tau)\right),$$
$$\frac{\partial}{\partial z}\theta^s_{\kappa,m}(z,\tau)|_{z=\lambda+\tau} = e^{-\pi i\kappa\lambda-\pi i\kappa\frac{\tau}{2}}\left(-\pi i\kappa\theta^s_{\kappa,\kappa-m}(\lambda,\tau)+(\theta^s_{\kappa,\kappa-m})'(\lambda,\tau)\right),$$
$$\frac{\partial}{\partial z}\theta^s_{\kappa,m}(\lambda+\tau,z)|_{z=\tau} = e^{-\pi i\kappa\lambda-\pi i\kappa\frac{\tau}{2}}\left(\pi i\frac{\kappa}{2}\theta^s_{\kappa,\kappa-m}(\lambda,\tau)-(\theta^s_{\kappa,\kappa-m})'(\lambda,\tau)+\dot\theta^s_{\kappa,\kappa-m}(\lambda,\tau)\right).$$

It is straightforward to calculate the limit of the left hand side. Using the above formulas, the limit as $\lambda \to \tau$ of the right hand side is equal to the limit as $\lambda \to 0$ of the expression

$$e^{\pi i(2p+2-\kappa)(\lambda+\frac{\tau}{2})}\Bigg((2p+2-\kappa)\frac{\dot\vartheta_1}{\vartheta_1}\sum_{j=0}^{\kappa-2p-2} c_j\theta^s_{\kappa-2p-2,\kappa-2p-2-j} \\ -2\sum_{j=0}^{\kappa-2p-2} c_j\dot\theta^s_{\kappa-2p-2,\kappa-2p-2-j}+\frac{1}{\pi i}\frac{\vartheta_1'}{\vartheta_1}\sum_{j=0}^{\kappa-2p-2} c_j(\theta^s_{\kappa-2p-2,\kappa-2p-2-j})'\Bigg).$$

This limit is calculated using L'Hôpital's rule. □

7. Identities for theta functions

In the next section, we give the proofs of the integral identities in Theorem 5.1. We will use the following results.

Lemma 7.1. *We have* $\theta^s_{2,1}(\lambda) = \vartheta_1(\lambda + 1/2)$.

Lemma 7.1 is proved by comparing the Fourier series expansions of the functions.

Corollary 7.2. *We have* $2\theta_{2,1}(0) = \eta(\tau)\phi_3(\tau)^2$.

Lemma 7.3. *Let*

$$f_1(\tau) = \frac{\theta_{4,1}(0) - \theta_{4,3}(0)}{\eta(\tau)}, \quad f_2(\tau) = \frac{\theta_{4,1}(0) + \theta_{4,3}(0)}{\eta(\tau)}, \quad f_3(\tau) = \frac{\theta_{4,0}(0) - \theta_{4,4}(0)}{\sqrt{2}\eta(\tau)}.$$

Then $f_1(\tau) = \phi_1(\tau)^{-1}$, $f_2(\tau) = \phi_2(\tau)^{-1}$, $f_3(\tau) = \phi_3(\tau)^{-1}$.

The proof of Lemma 7.3 is based on the following result.

Lemma 7.4. [W] *Suppose* $g_1(\tau)$, $g_2(\tau)$, *and* $g_3(\tau)$ *are holomorphic functions on the upper half plane* $\mathbb{C}_+$ *satisfying the following conditions.*

P1. *The functions* g_1, g_2, *and* g_3 *can be written in the forms*

$$g_1(\tau) = q^{-\frac{a}{48}}\sum_{j=0}^{\infty} a_j q^{\frac{j}{2}}, \quad g_2(\tau) = q^{-\frac{a}{48}}\sum_{j=0}^{\infty}(-1)^j a_j q^{\frac{j}{2}}, \quad g_3(\tau) = q^{\frac{a}{24}}\sum_{j=0}^{\infty} b_j q^j,$$

where a *is an integer,* $a_j, b_j \in \mathbb{C}$, *and* $a_0 = 1$.

P2. *The functions* g_1, g_2, *and* g_3 *have the modular properties*

$$g_1\left(-\frac{1}{\tau}\right) = g_1(\tau), \quad g_2\left(-\frac{1}{\tau}\right) = g_3(\tau), \quad g_3\left(-\frac{1}{\tau}\right) = g_2(\tau).$$

Then $g_i(\tau) = \phi_i(\tau)^a$, *for* $i = 1, 2, 3$.

Proof of Proposition 7.3. We have

$$\theta_{4,1}(0)=\sum_{j\in\mathbb{Z}} q^{4(j+\frac18)^2}=q^{\frac1{16}}\sum_{j\in\mathbb{Z}} q^{\frac12(8j^2+2j)},\quad \theta_{4,3}(0)=\sum_{j\in\mathbb{Z}} q^{4(j+\frac38)^2}=q^{\frac1{16}}\sum_{j\in\mathbb{Z}} q^{\frac12(8j^2+6j+1)},$$
$$\theta_{4,0}(0)=\sum_{j\in\mathbb{Z}} q^{4j^2}=\sum_{j\in\mathbb{Z}} q^{(2j)^2},\quad \theta_{4,4}(0)=\sum_{j\in\mathbb{Z}} q^{4(j+\frac12)^2}=\sum_{j\in\mathbb{Z}} q^{(2j+1)^2}.$$

Hence, the functions

$$\tilde f_1(\tau)=q^{-\frac1{24}}(\theta_{4,1}(0)-\theta_{4,3}(0)),\quad \tilde f_2(\tau)=q^{-\frac1{24}}(\theta_{4,1}(0)+\theta_{4,3}(0)),$$
$$\tilde f_3(\tau)=2^{-\frac12}q^{-\frac1{24}}(\theta_{4,0}(0)-\theta_{4,4}(0))$$

are holomorphic functions on $\mathbb{C}_+$ with the property P1 for $a=-1$ and $g_i=\tilde f_i$. Let $y(q)=\sum_{j=1}^\infty c_jq^j$ be defined by the condition $1+y(q)=q^{-1/24}\eta(\tau)$. Then for $i=1,2,3$,

$$f_i(\tau)=\frac{\tilde f_i(\tau)}{1+y(q)}=\tilde f_i(\tau)(1-y(q)+y(q)^2+\dots)$$

are holomorphic functions on $\mathbb{C}_+$ with the property P1 for $a=-1$ and $g_i=f_i$. One checks that f_1, f_2 and f_3 have the property P2 using the modular properties of $\theta_{4,n}(0)$ and $\eta(\tau)$. □

Lemma 7.5. *We have* $\theta_{6,1}(0)-\theta_{6,5}(0)=\eta(\tau)$.

Lemma 7.5 is proved by comparing the infinite series expansions of the functions.

8. The proof of Theorem 5.1

8.1. **Proof of (2).** For $\kappa=2p+2$, the space of conformal blocks is one-dimensional. The right hand side of (2) is a solution of (1) with the properties (i)-(iv) [FV1]. According to Theorem 3.1, the left hand side also has these properties. Thus the two functions are proportional. The coefficient of proportionality is calculated by comparing the leading terms of ϑ_1^{p+1} and $u_{\kappa,p+1}$ in the limit as $\tau\to i\infty$. The leading term of ϑ_1^{p+1} is $(-i)^{p+1}q^{(p+1)/8}(e^{\pi i\lambda}-e^{-\pi i\lambda})^{p+1}$. Let $dt=dt_1\dots dt_p$. The leading term of $u_{\kappa,p+1}$ is

$$\int_{\Delta_p}\prod_{j=1}^p\left(\frac{e^{\pi it_j}-e^{-\pi it_j}}{2\pi e^{\frac{\pi i}{2}}}\right)^{-\frac{2p}{2p+2}-1}\prod_{1\le j<k\le p}\left(\frac{e^{\pi i(t_j-t_k)}-e^{-\pi i(t_j-t_k)}}{2\pi e^{\frac{\pi i}{2}}}\right)^{\frac{2}{2p+2}}$$
$$\left(\left(\prod_{j=1}^p\frac{e^{\pi i(\lambda-t_j)}-e^{-\pi i(\lambda-t_j)}}{e^{\pi i\lambda}-e^{-\pi i\lambda}}\right)q^{\frac{(p+1)^2}{4(2p+2)}}e^{\pi i(p+1)(\lambda+\frac{2}{2p+2}\sum_{j=1}^p t_j)}\right.$$
$$\left.+(-1)^{p+1}\left(\prod_{j=1}^p\frac{e^{\pi i(\lambda+t_j)}-e^{-\pi i(\lambda+t_j)}}{e^{\pi i\lambda}-e^{-\pi i\lambda}}\right)q^{\frac{(p+1)^2}{4(2p+2)}}e^{\pi i(p+1)(-\lambda+\frac{2}{2p+2}\sum_{j=1}^p t_j)}\right)dt.$$

The above expression is equal to

$$(2\pi e^{\frac{\pi i}{2}})^{\frac{p(p+1)}{2p+2}+p} e^{-\pi i(\frac{2p^2}{2p+2}+p)} q^{\frac{p+1}{8}} (e^{\pi i\lambda} - e^{-\pi i\lambda})^{-p} \sum_{l=0}^{p} I_l \,(e^{\pi i(2l+1)\lambda} - e^{-\pi i(2l+1)\lambda}),$$

where

$$I_l = \int_{\Delta_p} f_l(t_1,\dots,t_p) \prod_{j=1}^{p} e^{2\pi i t_j(\frac{p+2}{2p+2}+\frac{1}{2})}(1-e^{2\pi i t_j})^{-\frac{2p}{2p+2}-1} \prod_{1\le j<k\le p}(e^{2\pi i t_j}-e^{2\pi i t_k})^{\frac{2}{2p+2}}\,dt,$$

for some function f_l, symmetric in the variables $t_1,\dots,t_p$. Comparing the coefficients of $e^{\pi i(p+1)\lambda}$ in the leading terms, we find that $u_{\kappa,p+1} = i^{p+1}(2\pi e^{\frac{\pi i}{2}})^{\frac{p(p+1)}{2p+2}+p} e^{-\pi i(\frac{2p^2}{2p+2}+p)} I_p \vartheta_1^{p+1}$. To complete the proof, it remains to compute I_p. It is not difficult to show that $f_p(t_1,\dots,t_p) = \prod_{j=1}^{p} e^{-\pi i t_j}$. Let $x_j = e^{2\pi i t_j}$. Let $\tilde\Delta_p$ be the image of Δ_p under the map $t_j \mapsto x_j$. We have

$$I_p = (2\pi i)^{-p} \int_{\tilde\Delta_p} \prod_{j=1}^{p} x_j^{\frac{p+2}{2p+2}-1}(1-x_j)^{-\frac{2p}{2p+2}-1} \prod_{1\le j<k\le p}(x_j-x_k)^{\frac{2}{2p+2}}\,dx.$$

Applying the Stokes theorem, we deform the contour $\tilde\Delta_p$ to get

$$I_p = (2\pi i)^{-p} \prod_{j=1}^{p}(e^{2\pi i\frac{j+p+1}{2p+2}}-1) \int_{\Delta_p} \prod_{j=1}^{p} x_j^{\frac{p+2}{2p+2}-1}(1-x_j)^{-\frac{2p}{2p+2}-1} \prod_{1\le j<k\le p}(x_j-x_k)^{\frac{2}{2p+2}}\,dx.$$

Observe that

$$\int_{\Delta_p} \prod_{j=1}^{p} x_j^{\frac{p+2}{2p+2}-1}(1-x_j)^{-\frac{2p}{2p+2}-1} \prod_{1\le j<k\le p}(x_j-x_k)^{\frac{2}{2p+2}}\,dx$$

is the Selberg integral $B_p((p+2)/(2p+2), -2p/(2p+2), 1/(2p+2))$. This completes the proof.

8.2. **Proof of (3) and (4).** Let $\kappa = 2p+3$. Then any solution of (1) with the properties (i)-(iv) has the form $v(\lambda,\tau) = \vartheta_1^{p+1}(\lambda,\tau)(c_0(\tau)\theta_{1,0}(\lambda,\tau) + c_1(\tau)\theta_{1,1}(\lambda,\tau))$. Let A be the transformation introduced in section 4. By Proposition 4.1, $Av(\lambda,\tau) = (-1)^{p+1}\vartheta_1^{p+1}(\lambda,\tau)(c_0(\tau)\theta_{1,0}(\lambda,\tau) - c_1(\tau)\theta_{1,1}(\lambda,\tau))$ is also a solution. Hence, for $j=0$ or 1, the function $v_j(\lambda,\tau) = c_j(\tau)\vartheta_1^{p+1}(\lambda,\tau)\theta_{1,j}(\lambda,\tau)$ gives a solution too. Moreover, $Av_j = (-1)^{p+1+j}v_j$. By Theorem 3.2, the integrals $u_{\kappa,p+1}$ and $u_{\kappa,p+2}$ span the space of conformal blocks. By Lemma 4.3, $Au_{\kappa,p+1} = (-1)^{p+1}u_{\kappa,p+1}$ and $Au_{\kappa,p+2} = (-1)^{p}u_{\kappa,p+2}$. So for $j=0$ or 1, the integral $u_{\kappa,p+1+j}$ is proportional to v_j. By Lemma 6.2, $c_j(\tau)$ must satisfy the differential equation

$$\frac{\kappa}{p+1}\left(\frac{d}{d\tau}c_j(\tau)\right)\theta_{1,j}(0,\tau) = -\left(\frac{d}{d\tau}\ln\vartheta_1'(0,\tau)\right)c_j(\tau)\theta_{1,j}(0,\tau).$$

The function $c_j(\tau) = \left((2\pi)^{-1}\vartheta_1'(0,\tau)\right)^{-(p+1)/\kappa} = \eta(\tau)^{-3(p+1)/\kappa}$ is a solution of this equation for $j=0$ and $j=1$. The coefficients of proportionality are computed in the limit as $\tau \to i\infty$, cf. the proof of (2). This completes the proof.

8.3. **Proof of (5), (6), and (7).** Let $\kappa = 2p+4$. Let $v_j(\lambda,\tau) = \vartheta_1^{p+1}(\lambda,\tau)\theta^s_{2,j}(\lambda,\tau)$, $0 \le j \le 2$. Any solution of (1) with the properties (i)-(iv) has the form $v(\lambda,\tau) = \sum_{j=0}^2 c_j(\tau)v_j(\lambda,\tau)$. Let A be the transformation in section 4. By Proposition 4.1, $Av = \sum_{j=0}^2 (-1)^{p+j+1}c_jv_j$ is also a solution. Hence the function c_1v_1 gives a solution too. Moreover, it is an eigenvector of A with eigenvalue $(-1)^p$. By Theorem 3.2, the space of conformal blocks is three-dimensional with spanning set $\{u_{\kappa,n} \,|\, p+1 \le n \le p+3\}$. According to Lemma 4.3, the eigenspace of A corresponding to the eigenvalue $(-1)^p$ is one-dimensional and is spanned by $u_{\kappa,p+2}$. It follows that $u_{\kappa,p+2}$ is proportional to c_1v_1. By Lemma 6.2, $c_1(\tau)$ must satisfy the differential equation

$$\frac{\kappa}{p+1}\left(\frac{d}{d\tau}c_1(\tau)\right)\theta_{2,1}(0,\tau) = -2\left(\frac{d}{d\tau}\ln\vartheta_1'(0,\tau)\right)c_1(\tau)\theta_{2,1}(0,\tau) + 2c_1(\tau)\frac{d}{d\tau}\theta_{2,1}(0,\tau).$$

The function $c_1(\tau) = \left(4\pi\theta_{2,1}(0,\tau)\vartheta_1'(0,\tau)^{-1}\right)^{2(p+1)/\kappa}$ is a solution of the above equation. By Corollary 7.2, $c_1(\tau) = \left(\phi_3(\tau)\eta(\tau)^{-1}\right)^{4(p+1)/\kappa}$. The coefficient of proportionality is computed in the limit $\tau \to i\infty$, cf. the proof of (2). This proves (5). To prove (6), we apply the transformation S to both sides of (5). To prove (7), we apply the transformation T to both sides of (6). This completes the proof.

8.4. **Proof of (8), (9), and (10).** Let $\kappa = 2p+6$. Let $v_j(\lambda,\tau) = \vartheta_1^{p+1}(\lambda,\tau)\theta^s_{4,j}(\lambda,\tau)$, $0 \le j \le 4$. Any solution of (1) with the properties (i)-(iv) has the form $v(\lambda,\tau) = \sum_{j=0}^4 c_j(\tau)v_j(\lambda,\tau)$. Let A and B be the transformations in section 4. By Proposition 4.1, $Av = \sum_{j=0}^4 (-1)^{p+j+1}c_jv_j$ is also a solution. Hence the function $c_0v_0 + c_2v_2 + c_4v_4$ gives a solution too. Moreover, $B(c_0v_0 + c_2v_2 + c_4v_4) = (-1)^{p+1}(c_4v_0 + c_2v_2 + c_0v_4)$ is also a solution. So there exists a solution of the form $c(\tau)(v_0 - v_4)$. It is an eigenvector of A with eigenvalue $(-1)^{p+1}$ and an eigenvector of B with eigenvalue $(-1)^p$. We show that the subspace of conformal blocks with this property is one-dimensional. By Theorem 3.2, the space of conformal blocks is five-dimensional with spanning set $\{u_{\kappa,n} \,|\, p+1 \le n \le p+5\}$. By Lemma 4.3, the eigenspace of A corresponding to the eigenvalue $(-1)^{p+1}$ is three-dimensional and is spanned by $u_{\kappa,p+1}$, $u_{\kappa,p+3}$, and $u_{\kappa,p+5}$. By Lemma 4.3, the transformation B preserves the subspace $\langle u_{\kappa,p+1}, u_{\kappa,p+3}, u_{\kappa,p+5}\rangle$. The matrix of B restricted to this subspace is

$$\begin{pmatrix} 0 & 0 & -e^{2\pi i\frac{p(p+5)}{\kappa}} \\ 0 & (-1)^{p+1} & 0 \\ -e^{2\pi i\frac{p(p+1)}{\kappa}} & 0 & 0 \end{pmatrix}.$$

Thus the restriction of B to $\langle u_{\kappa,p+1}, u_{\kappa,p+3}, u_{\kappa,p+5}\rangle$ has eigenvalues $(-1)^p$ and $(-1)^{p+1}$ of multiplicities 1 and 2, respectively. The eigenspace corresponding to the eigenvalue

$(-1)^p$ is spanned by the vector $u_{\kappa,p+1} + (-1)^{p+1} e^{2\pi i p(p+1)/\kappa} u_{\kappa,p+5}$. It follows that this vector is proportional to $c(\tau)(v_0 - v_4)$. By Lemma 6.2, $c(\tau)$ must satisfy the differential equation

$$\frac{\kappa}{p+1}\left(\frac{d}{d\tau}c(\tau)\right)(\theta_{4,0}(0,\tau) - \theta_{4,4}(0,\tau)) =$$
$$-4\left(\frac{d}{d\tau}\ln\vartheta_1'(0,\tau)\right)c(\tau)(\theta_{4,0}(0,\tau) - \theta_{4,4}(0,\tau)) + 6c(\tau)\frac{d}{d\tau}(\theta_{4,0}(0,\tau) - \theta_{4,4}(0,\tau)).$$

The function $c(\tau) = \left((2\pi)^2(\theta_{4,0}(0,\tau) - \theta_{4,4}(0,\tau))^3\vartheta_1'(0,\tau)^{-2}\right)^{2(p+1)/\kappa}$ is a solution of the above equation. By Lemma 7.3, we have $c(\tau) = 2^{3(p+1)/\kappa}(\phi_3(\tau)\eta(\tau))^{-6(p+1)/\kappa}$. The coefficient of proportionality is computed as in the proof of (2). This proves (8). To prove (9) and (10), apply the transformations S and TS, respectively, to both sides of (8). This completes the proof.

8.5. **Proof of (11).** Let $\kappa = 2p+8$. Let $v_j(\lambda,\tau) = \vartheta_1^{p+1}(\lambda,\tau)\theta^s_{6,j}(\lambda,\tau)$, $0 \le j \le 6$. Any solution of (1) with the properties (i)-(iv) has the form $v(\lambda,\tau) = \sum_{j=0}^6 c_j(\tau)v_j(\lambda,\tau)$. Let A and B be as in section 4. By Proposition 4.1, $Av = \sum_{j=0}^6(-1)^{p+j+1}c_jv_j$ is also a solution. Hence the function $c_1v_1 + c_3v_3 + c_5v_5$ gives a solution too. The function $B(c_1v_1 + c_3v_3 + c_5v_5) = (-1)^{p+1}(c_5v_1 + c_3v_3 + c_1v_5)$ also gives a solution. So there is a solution of the form $c(\tau)(v_1 - v_5)$ which is an eigenvector of A and B with eigenvalue $(-1)^p$ under both transformations. We show that the subspace of conformal blocks with this property is one-dimensional. By Theorem 3.2, the space of conformal blocks is seven-dimensional with spanning set $\{u_{\kappa,n}\,|\,p+1 \le n \le p+7\}$. By Lemma 4.3, the three-dimensional eigenspace of A corresponding to the eigenvalue $(-1)^p$ is spanned by $u_{\kappa,p+2}$, $u_{\kappa,p+4}$, and $u_{\kappa,p+6}$. The matrix of B restricted to $\langle u_{\kappa,p+2}, u_{\kappa,p+4}, u_{\kappa,p+6}\rangle$ is

$$\begin{pmatrix} 0 & 0 & -e^{2\pi i\frac{p(p+6)}{\kappa}} \\ 0 & (-1)^{p+1} & 0 \\ -e^{2\pi i\frac{p(p+2)}{\kappa}} & 0 & 0 \end{pmatrix}.$$

Thus, B has eigenvalues $(-1)^p$ and $(-1)^{p+1}$ of multiplicities 1 and 2, respectively. The eigenspace corresponding to the eigenvalue $(-1)^p$ is spanned by the vector $u_{\kappa,p+2} + (-1)^{p+1}e^{2\pi i p(p+2)/\kappa}u_{\kappa,p+6}$. So this vector is proportional to $c(\tau)(v_1 - v_5)$. By Lemma 6.2, $c(\tau)$ must be a solution of the differential equation

$$\frac{\kappa}{p+1}\left(\frac{d}{d\tau}c(\tau)\right)(\theta_{6,1}(0,\tau) - \theta_{6,5}(0,\tau)) =$$
$$-6\left(\frac{d}{d\tau}\ln\vartheta_1'(0,\tau)\right)c(\tau)(\theta_{6,1}(0,\tau) - \theta_{6,5}(0,\tau)) + 10c(\tau)\frac{d}{d\tau}(\theta_{6,1}(0,\tau) - \theta_{6,5}(0,\tau)).$$

The function $c(\tau) = ((2\pi)^3(\theta_{6,1}(0,\tau) - \theta_{6,5}(0,\tau))^5\vartheta_1'(0,\tau)^{-3})^{2(p+1)/\kappa}$ is a solution of the preceding equation. By Lemma 7.5, $c(\tau) = (\eta(\tau))^{-8(p+1)/\kappa}$. The coefficient of proportionality is computed as in the proof of (2). Notice that the solution in (11) is invariant with respect to the action of the modular group. □

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