ELLIPTIC SELBERG INTEGRALS

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The Selberg integral is

$$B_p(\alpha,\beta,\gamma)=\int_{\Delta_p}\prod_{j=1}^{p} t_j^{\,\alpha-1}(1-t_j)^{\beta-1}\prod_{0\le j<k\le 1}(t_j-t_k)^{2\gamma},$$

where $\Delta_p=\{t\in\mathbb{R}^p\,|\,0\le t_p\le\cdots\le t_1\le 1\}$. The Selberg integral is a generalization of the beta function. It can be calculated explicitly,

$$B_p(\alpha,\beta,\gamma)=\frac{1}{p!}\prod_{j=0}^{p-1}\frac{\Gamma(1+\gamma+j\gamma)\Gamma(\alpha+j\gamma)\Gamma(\beta+j\gamma)}{\Gamma(1+\gamma)\Gamma(\alpha+\beta+(p+j-1)\gamma)}.$$

The Selberg integral has many applications, see [A1, A2, As, D, DF1, DF2, M, S]. In this paper, we present an elliptic version of the Selberg integral.

Let $\vartheta_1(t,\tau)$ be the first Jacobi theta function [WW],

$$\vartheta_1(t,\tau)=-\sum_{j\in\mathbb{Z}}e^{\pi i(j+\frac12)^2\tau+2\pi i(j+\frac12)(t+\frac12)}.$$

Introduce special functions

$$\sigma_\lambda(t,\tau)=\frac{\vartheta_1(\lambda-t,\tau)\vartheta_1'(0,\tau)}{\vartheta_1(\lambda,\tau)\vartheta_1(t,\tau)},\qquad \rho(t,\tau)=\frac{\vartheta_1'(t,\tau)}{\vartheta_1(t,\tau)},\qquad E(t,\tau)=\frac{\vartheta_1(t,\tau)}{\vartheta_1'(0,\tau)}.$$

Here $'$ denotes the derivative with respect to the first argument.

Let $\kappa\ge 2$ be an integer. The theta functions

$$\theta_{\kappa,m}(\lambda,\tau)=\sum_{j\in\mathbb{Z}}e^{2\pi i\kappa(j+\frac{m}{2\kappa})^2\tau+2\pi i\kappa(j+\frac{m}{2\kappa})\lambda},\qquad m\in\mathbb{Z}/2\kappa\mathbb{Z},$$

form a basis of the theta functions of level κ.

For a positive integer p, the elliptic Selberg integral $I_p(\lambda,\tau)$ is the integral,

$$I_p(\lambda,\tau)=J_p(\lambda,\tau)+(-1)^{p+1}J_p(-\lambda,\tau),$$

where

$$J_p(\lambda,\tau)=\int_{\Delta_p}\prod_{j=1}^{p}E(t_j,\tau)^{-\frac{p}{p+1}}\prod_{1\le j<k\le p}E(t_j-t_k,\tau)^{\frac{1}{p+1}}\qquad\times$$
$$\prod_{j=1}^{p}\sigma_\lambda(t_j,\tau)\theta_{2(p+1),p+1}\left(\lambda+\frac{1}{p+1}\sum_{j=1}^{p}t_j,\tau\right)\quad dt_1\dots dt_p\ .$$

The branch of the logarithm is chosen in such a way that arg $(E(t,\tau))\to 0$ as $t\to 0^+$, and the integral is understood as a natural analytic continuation.[1]

Theorem 1. *We have*

$$I_p(\lambda,\tau)=c_p\,B_p\left(\frac{1}{2}+\frac{1}{2(p+1)},-\frac{p}{p+1},\frac{1}{2(p+1)}\right)\vartheta_1(\lambda,\tau)^{p+1}\tag{1}$$

where

$$c_p=-(2\pi)^{\frac{p}{2}}\,e^{\pi i\frac{p}{p+1}}\,e^{-\pi i\frac{p+2}{4}}\prod_{j=1}^{p}\left(1-e^{-\pi i\frac{j}{p+1}}\right).$$

The theorem is a generalization of theorem 13 in [FV1]. The proof is based on the following remarks. Consider the heat equation

$$4\pi i(p+1)\frac{\partial u}{\partial\tau}(\lambda,\tau)=\frac{\partial^2 u}{\partial\lambda^2}(\lambda,\tau)+p(p+1)\rho'(\lambda,\tau)u(\lambda,\tau).$$

It is known that this equation has a one dimensional space of solutions $u(\lambda,\tau)$ which are holomorphic theta functions of level $2(p+1)$,

$$u(\lambda+2,\tau)=u(\lambda,\tau),\qquad u(\lambda+2\tau,\tau)=e^{-4\pi i(p+1)(\lambda+\tau)}u(\lambda,\tau)$$

and Weyl skew-symmetric, $u(-\lambda,\tau)=(-1)^{p+1}u(\lambda,\tau)$, see [FV1, FV2]. The space is called the space of conformal blocks. Clearly the right hand side of (1) has these properties. According to [FV1], the left hand side of (1) also has these properties. Thus the two functions are proportional. The coefficient of proportionality is easily calculated in the limit $\tau\to i\infty$.

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[1] To define the analytic continuation, we replace the exponential $-\frac{p}{p+1}$ by a, and consider the analytic continuation with respect to a from the region where a is positive.

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