# COMBINATORICS OF RATIONAL FUNCTIONS AND POINCARÉ-BIRCHOFF-WITT EXPANSIONS OF THE CANONICAL $U(\mathfrak{n}_-)$-VALUED DIFFERENTIAL FORM

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Abstract. We study the canonical $U(\mathfrak{n}_-)$-valued differential form, whose projections to different Kac-Moody algebras are key ingredients of the hypergeometric integral solutions of KZ-type differential equations and Bethe ansatz constructions. We explicitly determine the coefficients of the projections in the simple Lie algebras $A_r, B_r, C_r, D_r$ in a conveniently chosen Poincaré-Birchoff-Witt basis.

## 1. Introduction

For a Kac-Moody algebra $\mathfrak{g}$, let $V$ be the tensor product $V_{\Lambda_1} \otimes \ldots \otimes V_{\Lambda_n}$ of highest weight $\mathfrak{g}$-modules. The $V$-valued hypergeometric solutions of Knizhnik-Zamolodchikov-type differential equations have the form [SV1], [SV2]:

$$I(z) \;=\; \int_{\gamma(z)} \Phi(t,z)\; \Omega^V(t,z)\;. \tag{1}$$

Here $t=(t_1,\ldots,t_k)$, $z=(z_1,\ldots,z_n)$, $\Phi$ is a scalar multi-valued (master) function, $\gamma(z)$ is a suitable cycle in $t$-space depending on $z$, and $\Omega^V$ is a $V$-valued rational differential $k$-form.

The same $\Phi$ and $\Omega^V$ have applications to the Bethe Ansatz method. It is known [RV] that the values of $\Omega^V$ at the critical points of $\Phi$ (with respect to $t$) give eigenvectors of the Hamiltonians of the Gaudin model associated with $V$.

For every $V = V_{\Lambda_1} \otimes \ldots \otimes V_{\Lambda_n}$, the $V$-valued differential form $\Omega^V$ is constructed out of a single $U(\mathfrak{n}_-)$-valued differential form $\Omega^{\mathfrak{g}}$, where $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ is the Cartan decomposition of $\mathfrak{g}$, and $U(\mathfrak{n}_-)$ denotes the universal enveloping algebra of the Lie algebra $\mathfrak{n}_-$, see Appendix. In applications, it is important to have convenient formulas for $\Omega^{\mathfrak{g}}$, and this is the goal of the present paper.

In [Mat], Matsuo suggested a formula $\int \Phi(t,z)\widetilde{\Omega}^V(t,z)$ for solutions of the KZ equations for $\mathfrak{g} = sl_{r+1}$. His differential form $\widetilde{\Omega}^V$ also can be constructed from a $U(\mathfrak{n}_-)$-valued form $\widetilde{\Omega}^{sl_{r+1}}$ in

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the same way as $\Omega^V$ from $\Omega^{\mathfrak{g}}$. It is known that for $sl_2$, the two forms

$$\Omega^{sl_2} = \sum_{k=0}^{\infty} \Big( \sum_{\pi \in \Sigma_k} \operatorname{sgn}(\pi) \cdot \frac{dt_{\pi(1)}}{t_{\pi(1)}} \wedge \frac{d(t_{\pi(2)} - t_{\pi(1)})}{t_{\pi(2)} - t_{\pi(1)}} \wedge \ldots \wedge \frac{d(t_{\pi(k)} - t_{\pi(k-1)})}{t_{\pi(k)} - t_{\pi(k-1)}} \Big) \otimes f^k,$$

$$\widetilde{\Omega}^{sl_2} = \sum_{k=0}^{\infty} \Big( \bigwedge_{i=1}^{k} \frac{dt_i}{t_i} \Big) \otimes f^k$$

coincide. For $r > 1$, the form $\Omega^{sl_{r+1}}$ is a polynomial in $f_1, \ldots, f_r$'s with scalar differential forms as coefficients, while the Matsuo form $\widetilde{\Omega}^{sl_{r+1}}$ is a sum over a Poincaré-Birchoff-Witt basis of $U(\mathfrak{n}_-)$ with coefficients of the same type. Both forms have some advantages. The form $\Omega^{\mathfrak{g}}$ is given by the same formula for any $\mathfrak{g}$. The formula for $\widetilde{\Omega}^{sl_{r+1}}$ has less terms and less apparent poles (see above for the apparent poles at $t_i - t_j = 0$). The advantage of having an expression in terms of a PBW basis is most spectacular for representations with 1-dimensional weight-subspaces.

In this paper, we prove that $\widetilde{\Omega}^{sl_{r+1}} = \Omega^{sl_{r+1}}$ and give similar Poincaré-Birchoff-Witt expansions for the differential form $\Omega^{\mathfrak{g}}$ for the simple Lie algebras $\mathfrak{g}$ of types $B_r$, $C_r$, $D_r$.

As a byproduct, we obtain results on the combinatorics of rational functions. Namely, some non-trivial identities are established among certain rational functions with partial symmetries. The results are far reaching generalizations of the prototype of these formulas, the "Jacobi-identity"

$$\frac{1}{(x-y)(x-z)} + \frac{1}{(y-x)(y-z)} + \frac{1}{(z-x)(z-y)} = 0.$$

In all $A_r, B_r, C_r, D_r$ cases, the coefficients of $\Omega^{\mathfrak{g}}$ can be encoded by diagrams relevant to sub-diagrams of the Dynkin diagram of $\mathfrak{g}$. One may expect that the same phenomenon occurs in a more general Kac-Moody setting, too.

According to the formulas for $\Phi$ and $\Omega^V$ in [SV1], [SV2], the poles of $\Omega^V$ contain the singularities of $\Phi$. From our PBW expansion formulas it follows that the poles of $\Omega^V$ *coincide* with the singularities of $\Phi$, hence it makes sense to consider the values of $\Omega^V$ at (e.g.) the critical points of $\Phi$, as is needed in the Bethe ansatz applications.

It was shown in [MaV] that the Matsuo type hypergeometric solutions of the $sl_{r+1}$ KZ-equations satisfy the complementary dynamical difference equations. According to our result $\Omega^{sl_{r+1}} = \widetilde{\Omega}^{sl_{r+1}}$ , the hypergeometric solutions (1) also satisfy the dynamical difference equation for $\mathfrak{g} = sl_{r+1}$.

In [FV], hypergeometric solutions $I(z, \lambda) = \int \Phi^{ell}(t, z)\Omega^{V,ell}(t, z, \lambda)$ of the KZB equations were constructed. Here $\Phi^{ell}$ is the elliptic scalar master function of the same $t, z$ and $\Omega(t, z, \lambda)$ is the elliptic analogue of $\Omega^V$, it is a $V$-valued differential form depending also on $\lambda \in \mathfrak{h}$. It would be useful to find PBW type expansions of $\Omega^{V,ell}$ similar to our PBW expansions of $\Omega^V$.

Hypergeometric solutions of qKZ, the quantum version of KZ equations, for $sl_{r+1}$-modules $V$ were described in [TV1], [TV2] as $I(z) = \int \Phi_q(t, z)\Omega_q^V(t, z)$. There the $V$-valued differential form $\Omega_q^V$ was given in a PBW expansion. Our PBW formulas for the B, C, D series may suggest integral formulas for solutions of B, C, D type qKZ equations.

The structure of the cycles $\gamma$ in (1) for arbitrary $\mathfrak{g}$ was analyzed in [V1]. The cycles were presented as linear combinations of multiple loops, and that presentation established a connection between multi-loops and monomials $f_{i_k} \dots f_{i_1}$ in $U_q(\mathfrak{n}_-)$, where $U_q(\mathfrak{n}_-)$ is the $\mathfrak{n}_-$-part of the quantum group $U_q(\mathfrak{g})$. That connection in particular gives an identification of the monodromy of the KZ equations with the $R$-matrix representations associated with $U_q(\mathfrak{g})$. Our PBW expansions of $\Omega^{\mathfrak{g}}$ suggest that there might be an interesting PBW type geometric theory of cycles for each $\mathfrak{g}$, in which the cycles are presented by linear combinations of cells corresponding to elements of the PBW basis in the corresponding $U_q(\mathfrak{n}_-)$.

It would also be interesting to compare our PBW formulas with Cherednik's formulas for solutions of the trigonometric KZ equations [Ch].

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## 2. Symmetrizers, signs and other conventions

2.1. **Symmetrizers.** For a nonnegative integer $r$ and $k = (k_1, \dots, k_r) \in \mathbb{N}^r = \{0, 1, 2, \dots\}^r$, we will often consider various 'objects' $x(t_j^{(i)})$ (functions, differential forms, flags), depending on the $r$ sets of variables

$$t_1^{(1)}, t_2^{(1)}, \dots, t_{k_1}^{(1)}, \qquad t_1^{(2)}, t_2^{(2)}, \dots, t_{k_2}^{(2)}, \qquad \dots \qquad t_1^{(r)}, t_2^{(r)}, \dots, t_{k_r}^{(r)}. \tag{2}$$

Let $G_k$ be the product $\prod_i \Sigma_{k_i}$ of symmetric groups. We define the action of $\pi \in G_k$ on $x$ by permuting the $t_j^{(i)}$'s with the same upper indices. Then we define the symmetrizer and antisymmetrizer operators

$$\mathrm{Sym}_k\, x(t_j^{(i)}) = \sum_{\pi \in G_k} \pi \cdot x, \qquad \mathrm{ASym}_k\, x(t_j^{(i)}) = \sum_{\pi \in G_k} \mathrm{sgn}(\pi)\ \pi \cdot x.$$

Let $|k| = \sum_i k_i$. For a function ('multi-index') $J = \{1, \dots, |k|\} \to \{1, \dots, r\}$ with $\#J^{-1}(i) = k_i$, let $c : \{1, \dots, |k|\} \to \mathbb{N}$ be the unique map whose restriction to $J^{-1}(i)$ is the increasing function onto $\{1, \dots, k_i\}$. Then $J$ defines an identification of $(t_1, \dots, t_{|k|})$ with the variables in (2) by identifying

$$t_u \qquad \text{with} \qquad t_{c(u)}^{(J(u))}. \tag{3}$$

Thus, if $x$ depends on $t_1, \dots, t_{|k|}$ and $J$ is given, we can consider $x$ depending on the variables in (2). For example, we can (anti)symmetrize $x$:

$$\mathrm{Sym}_k^J\, x(t_u) = \mathrm{Sym}_k\, x(t_j^{(i)}), \qquad \mathrm{ASym}_k^J\, x(t_u) = \mathrm{ASym}_k\, x(t_j^{(i)}).$$

2.2. **The sign of a multi-index; volume forms.** Let $J_0$ be the unique *increasing* function $\{1, \dots, |k|\} \to \{1, \dots, r\}$ with $\#J_0^{-1}(i) = k_i$, and let $J$ be any function $\{1, \dots, |k|\} \to \{1, \dots, r\}$ with $\#J^{-1}(i) = k_i$. Then the identifications defined in (3) for $J$ and $J_0$ together define a permutation of $1, \dots, |k|$. The sign of this permutation will be denoted by $\mathrm{sgn}(J)$. E.g. $\mathrm{sgn}(1 \mapsto 1, 2 \mapsto 1, 3 \mapsto 2) = 1$, $\mathrm{sgn}(1 \mapsto 1, 2 \mapsto 2, 3 \mapsto 1) = -1$.

Define the 'standard volume form' $dV_k$ by $dt_1^{(1)} \wedge \dots \wedge dt_{k_1}^{(1)} \wedge dt_1^{(2)} \wedge \dots \wedge dt_{k_2}^{(2)} \wedge \dots \wedge dt_1^{(r)} \wedge \dots \wedge dt_{k_r}^{(r)}$. Observe that if we use the identification (3), then $dt_1 \wedge dt_2 \wedge \dots \wedge dt_{|k|}$ is equal to $\mathrm{sgn}(J) \cdot dV_k$.

2.3. **The star multiplication.** For $k \in \mathbb{N}^r$, let $\mathcal{F}_k$ be the vector space of rational functions in the variables in (2) which are symmetric under the action of $G_k$. We define a multiplication (c.f. [V2, 6.4.2]) $* : \mathcal{F}_k \otimes \mathcal{F}_l \to \mathcal{F}_{k+l}$ by

$$(f * g)(t_1^{(1)}, \ldots, t_{k_1+l_1}^{(1)}, \ldots, t_1^{(r)}, \ldots, t_{k_r+l_r}^{(r)}) =$$

$$\frac{1}{\prod_i k_i! l_i!} \operatorname{Sym}_{k+l} \Big( f(t_1^{(1)}, \ldots, t_{k_1}^{(1)}, \ldots, t_1^{(r)}, \ldots, t_{k_r}^{(r)}) \cdot g(t_{k_1+1}^{(1)}, \ldots, t_{k_1+l_1}^{(1)}, \ldots, t_{k_r+1}^{(r)}, \ldots, t_{k_r+l_r}^{(r)}) \Big).$$

For example, if we write $t$ for $t^{(1)}$ and $s$ for $t^{(2)}$, then

$$\frac{1}{t_1 t_2} * \frac{1}{t_1(s-t_1)} = \frac{1}{t_1 t_2} \cdot \frac{1}{t_3(s-t_3)} + \frac{1}{t_1 t_3} \cdot \frac{1}{t_2(s-t_2)} + \frac{1}{t_2 t_3} \cdot \frac{1}{t_1(s-t_1)}.$$

This multiplication makes $\oplus_k \mathcal{F}_k$ an associative and commutative algebra.

## 3. Arrangements. The Orlik-Solomon algebra and its dual. Discriminantal arrangements and their symmetries

Let $\mathcal{C}$ be a hyperplane arrangement in $\mathbb{C}^n$. In this section we recall two algebraic descriptions of the cohomology of the complement $U = \mathbb{C}^n - \cup_{H \in \mathcal{C}} H$, as well as properties of the discriminantal arrangement which will be needed later. The general reference is [SV2].

3.1. **The Orlik-Solomon algebra.** For $H \in \mathcal{C}$, let $\omega_H$ be the logarithmic differential form $df_H/f_H$, where $f_H = 0$ is a defining equation of $H$. Let $\mathcal{A} = \mathcal{A}(\mathcal{C})$ be the graded $\mathbb{C}$-algebra with unit element generated by all $\omega_H$'s, $H \in \mathcal{C}$. The elements of $\mathcal{A}$ are closed forms on $U$, hence they determine cohomology classes. According to Arnold and Brieskorn, the induced map $\mathcal{A} \to H^*(U; \mathbb{C})$ is an isomorphism. The degree $p$ part of $\mathcal{A}$ will be denoted by $\mathcal{A}^p$.

3.2. **Flags.** Non-empty intersections of hyperplanes in $\mathcal{C}$ are called edges. A $p$-flag of $\mathcal{C}$ is a chain of edges

$$F = [\mathbb{C}^n = L^0 \supset L^1 \supset L^2 \supset \ldots \supset L^{p-1} \supset L^p],$$

where codim $L^i = i$. Consider the complex vector space generated by all $p$-flags of $\mathcal{C}$ modulo the relations

$$\sum_L [L^0 \supset \ldots \supset L^{i-1} \supset L \supset L^{i+1} \supset \ldots \supset L^p] = 0, \qquad (0 < i < p),$$

where the summation runs over all codim $i$ edges $L$ that contain $L^{i+1}$ and are contained in $L^{i-1}$. This vector space is denoted by $Fl^p = Fl^p(\mathcal{C})$, and let $Fl$ be the direct sum $\oplus_p Fl^p$.

3.3. **Iterated residues.** According to [SV2, Th. 2.4], $\mathcal{A}$ and $Fl$ are dual graded vector spaces. The value of a differential form on a flag is given by an iterated residue operation

$$\operatorname{Res} : Fl \otimes \mathcal{A} \to \mathbb{C},$$

defined as follows. Let $F = [L^i]$ be a $p$-flag and $\omega \in \mathcal{A}$ a $p$-form on $U$. Then

$$\operatorname{Res}_F \omega = \operatorname{Res}_{L^p} \Big( \operatorname{Res}_{L^{p-1}} \big( \ldots \operatorname{Res}_{L^2}(\operatorname{Res}_{L^1}(\omega)) \ldots \big) \Big) \quad \in \mathbb{C}.$$

3.4. **The discriminantal arrangement and its symmetries.** The discriminantal arrangement $\mathcal{C}^n$ in $\mathbb{C}^n$ is defined as the collection of hyperplanes

$$t_i = 0 \quad (i = 1, \dots, n) \qquad\qquad \text{and} \qquad\qquad t_i - t_j = 0 \quad (1 \le i < j \le n).$$

Let us fix $r$ non-negative integers (weights) $k = (k_1, \dots, k_r)$ with $\sum k_i = |k|$ and consider $\mathbb{C}^{|k|}$ with coordinates

$$(t_1^{(1)}, \dots, t_{k_1}^{(1)}, t_1^{(2)}, \dots, t_{k_2}^{(2)}, \dots\dots\dots, t_1^{(r)}, \dots, t_{k_r}^{(r)}).$$

The group $G_k = \prod \Sigma_{k_i}$ then acts on $\mathbb{C}^{|k|}$ (by permuting the coordinates with the same upper indices) which then induces an action of $G_k$ on $\mathcal{A}(\mathcal{C}^{|k|})$ and $Fl(\mathcal{C}^{|k|})$.

The skew-invariant subspaces (i.e. the collection of $x$'s for which $\pi \cdot x = \mathrm{sgn}(\pi) x \ \forall \pi \in G_k$) of $\mathcal{A}^{|k|}(\mathcal{C}^{|k|})$ and $Fl^{|k|}(\mathcal{C}^{|k|})$ will be denoted by $\mathcal{A}^{G_k}$ and $Fl^{G_k}$, respectively. The duality stated in 3.3 is consistent with the group-action in the sense that $\mathcal{A}^{G_k}$ and $Fl^{G_k}$ are dual vector spaces.

3.5. **Flags of the discriminantal arrangement.** Let $U_r$ be the free associative algebra generated by $r$ symbols $\tilde{f}_1, \tilde{f}_2, \dots, \tilde{f}_r$. It is multigraded by $\mathbb{N}^r$; the $(k_1, \dots, k_r)$-degree part will be denoted by $U_r[k] = U_r[k_1, \dots, k_r]$. For any non-zero homogeneous element in $U_r[k]$, we define its *content* to be $k$. It is proved in [SV2, Th. 5.9] that $U_r[k]$ is isomorphic to $Fl^{G_k}$ under the following map. For $J : \{1, \dots, |k|\} \to \{1, \dots, r\}$ with $\#J^{-1}(i) = k_i$, the monomial $\tilde{f}_J = \tilde{f}_{J(|k|)} \tilde{f}_{J(|k|-1)} \dots \tilde{f}_{J(2)} \tilde{f}_{J(1)} \in U_r[k_1, \dots, k_r]$ corresponds to $\frac{\mathrm{sgn}(J)}{\prod_i k_i!} \mathrm{ASym}_k^J(F) \in Fl^{G_k}$, where $F$ is the $|k|$-flag

$$[\mathbb{C}^{|k|} \supset (t_1 = 0) \supset (t_1 = t_2 = 0) \supset \dots \supset (t_1 = \dots = t_{|k|-1} = 0) \supset (t_1 = \dots = t_{|k|} = 0)]$$

with its variables $t_u$ identified with $t_j^{(i)}$'s as defined by (3).

**Example 3.1.** For $r = 2$, $k = (1, 1)$, we have the correspondence

$$\tilde{f}_2 \tilde{f}_1 \leftrightarrow [\mathbb{C}^2 \supset (t_1^{(1)} = 0) \supset (t_1^{(1)} = t_1^{(2)} = 0)], \qquad \tilde{f}_1 \tilde{f}_2 \leftrightarrow -[\mathbb{C}^2 \supset (t_1^{(2)} = 0) \supset (t_1^{(2)} = t_1^{(1)} = 0)].$$

For $r = 2$, $k = (2, 1)$, we have the correspondence

$$\tilde{f}_1^2 \tilde{f}_2 \leftrightarrow \frac{1}{2}\Big( [\mathbb{C}^3 \supset (t_1^{(2)} = 0) \supset (t_1^{(2)} = t_1^{(1)} = 0) \supset (t_1^{(2)} = t_1^{(1)} = t_2^{(1)} = 0)] - [\mathbb{C}^3 \supset (t_1^{(2)} = 0) \supset (t_1^{(2)} = t_2^{(1)} = 0) \supset (t_1^{(2)} = t_2^{(1)} = t_1^{(1)} = 0)] \Big).$$

## 4. The canonical differential form

Using the identifications of Section 3, the tensor product

$$\mathcal{A}^{G_k} \otimes U_r[k_1, \dots, k_r]$$

is the tensor product of a vector space with its dual space. Therefore the canonical element, $\sum_i b_i^* \otimes b_i$ for any basis $\{b_i\}$ of $U_r[k]$ and the dual basis $\{b_i^*\}$ of $\mathcal{A}^{G_k}$, is well defined—it does not depend on the choice of the basis of $U_r[k]$. We will call this element the *canonical differential form of weight* $k$ and denote it by $\Omega_k$. Tracing back the identifications of Section 3, we get the explicit form.

**Theorem 4.1.** [SV2] *Let $t_0 = 0$. The canonical differential form is*

$$\Omega_k = \sum_J \mathrm{sgn}(J)\cdot \mathrm{ASym}_k^J\Big(\bigwedge_{u=1}^{|k|} \mathrm{dlog}(t_u - t_{u-1})\Big)\otimes \tilde f_J$$
$$= \sum_J \mathrm{Sym}_k^J\Big(\prod_{u=1}^{|k|}\frac{1}{t_u - t_{u-1}}\Big) dV_k \otimes \tilde f_J \qquad \in \mathcal{A}^{G_k}\otimes U_r[k],$$

*where the summation runs over all $J : \{1,\dots,|k|\}\to\{1,\dots,r\}$ with $\#J^{-1}(i) = k_i$. (Recall that the variables $t_u$ are identified with $t_i^{(j)}$'s using (3).)*

**Example 4.2.** *Let $r = 2$, $k = (2,1)$, and write $t$ for $t^{(1)}$ and $s$ for $t^{(2)}$. Then*

$$\Omega_{(2,1)} = \Big(\frac{1}{t_1(t_2-t_1)(s-t_2)} + \frac{1}{t_2(t_1-t_2)(s-t_1)}\Big) dt_1\wedge dt_2\wedge ds \ \otimes \tilde f_2\tilde f_1^2 +$$
$$\Big(\frac{1}{t_1(s-t_1)(t_2-s)} + \frac{1}{t_2(s-t_2)(t_1-s)}\Big) dt_1\wedge dt_2\wedge ds \ \otimes \tilde f_1\tilde f_2\tilde f_1 +$$
$$\Big(\frac{1}{s(t_1-s)(t_2-t_1)} + \frac{1}{s(t_2-s)(t_1-t_2)}\Big) dt_1\wedge dt_2\wedge ds \ \otimes \tilde f_1^2\tilde f_2. \tag{4}$$

Similar rational functions will often appear in this paper. It will be convenient to encode them with diagrams as follows: $\Omega_{(2,1)} =$

$$\mathrm{Sym}\Big(\ast\!-\!\overset{t_1}{\bullet}\!-\!\overset{t_2}{\bullet}\!-\!\overset{s}{\bullet}\Big) dV_k\otimes \tilde f_2\tilde f_1^2 + \mathrm{Sym}\Big(\ast\!-\!\overset{t_1}{\bullet}\!-\!\overset{s}{\bullet}\!-\!\overset{t_2}{\bullet}\Big) dV_k\otimes \tilde f_1\tilde f_2\tilde f_1 + \mathrm{Sym}\Big(\ast\!-\!\overset{s}{\bullet}\!-\!\overset{t_1}{\bullet}\!-\!\overset{t_2}{\bullet}\Big) dV_k\otimes \tilde f_1^2\tilde f_2.$$

## 5. Properties of the differential forms

In this section we present the two key properties needed in Section 6.

5.1. **The residue of the canonical differential form.** For $k = (k_1,\dots,k_r)$, we denote $(k_1,\dots,k_{i-1},k_i-1,k_{i+1},\dots,k_r)$ by $k-1_i$.

**Lemma 5.1.** *Let $k\in\mathbb{N}^r$ and $i\in[1,\dots,r]$. Then the maps*

$$R : \mathcal{A}^{G_k} \to \mathcal{A}^{G_{k-1_i}}, \qquad \omega \mapsto \mathrm{Res}_{t^{(i)}_{k_i}=0}\,\omega,$$

*and*

$$\psi : U_r[k-1_i] \to U_r[k], \qquad x \mapsto (-1)^{k_1+\ldots+k_i-1} x\tilde f_i,$$

*are dual.*

*Proof.* Let $\omega\in\mathcal{A}^{G_k}$ and $\tilde f_J \in U_r[k-1_i]$. We need to check that the residue with respect to the flag corresponding to $\tilde f_J$ of $\mathrm{Res}_{t^{(i)}_{k_i}=0}\,\omega$ is equal to $(-1)^{k_1+\ldots+k_i-1}$ times the residue with respect to the flag corresponding to $\tilde f_J\tilde f_i$ of $\omega$. This follows from the definitions (and the sign conventions). □

**Theorem 5.2.**

$$\mathrm{Res}_{t^{(i)}_{k_i}=0}\,\Omega_k = (-1)^{k_1+k_2+\ldots+k_i-1}\cdot \Omega_{k-1_i}\cdot(1\otimes\tilde f_i).$$

*Proof.* Let $\{b_u\}$ be a basis of $U_r[k-1_i]$, hence $\Omega_{k-1_i}=\sum b_u^*\otimes b_u$. Since the map $\psi$ in Lemma 5.1 is an embedding, the images $\psi(b_u)$ can be extended to a basis $\{\psi(b_u),c_v\}$ of $U_r[k]$. Then $\Omega_k=\sum\psi(b_u)^*\otimes\psi(b_u)+\sum c_v^*\otimes c_v$. We have $(R\otimes 1)\Omega_k=\sum R(\psi(b_u)^*)\otimes\psi(b_u)+\sum R(c_v^*)\otimes c_v$, which, according to Lemma 5.1, is $\sum b_u^*\otimes\psi(b_u)=(1\otimes\psi)\Omega_{k-1_i}$, as required. □

**Example 5.3.** For $r=2$, we write $t$ for $t^{(1)}$ and $s$ for $t^{(2)}$. Then $\mathrm{Res}_{s=0}\,\Omega_{(1,1)}=$

$$\mathrm{Res}_{s=0}\left(\frac{dt}{t}\wedge\frac{d(s-t)}{s-t}\otimes\tilde f_2\tilde f_1-\frac{ds}{s}\wedge\frac{d(t-s)}{t-s}\otimes\tilde f_1\tilde f_2\right)=0\otimes\tilde f_2\tilde f_1-\frac{dt}{t}\otimes\tilde f_1\tilde f_2=-\Omega_{(1,0)}\tilde f_2.$$

5.2. **The multiplication of differential forms.** Recall that $U_r$ is equipped with a standard Hopf algebra structure. The co-multiplication $\Delta:U_r\to U_r\otimes U_r$ is defined for degree one elements $x$ as $\Delta(x)=1\otimes x+x\otimes 1$; e.g. $\Delta(\tilde f_1)=1\otimes\tilde f_1+\tilde f_1\otimes 1$. Then $\Delta(\tilde f_1\tilde f_2)=1\otimes\tilde f_1\tilde f_2+\tilde f_1\otimes\tilde f_2+\tilde f_2\otimes\tilde f_1+\tilde f_1\tilde f_2\otimes 1$.

The dual $\Delta^*$ of $\Delta$ is therefore a multiplication on the dual space $U_r^*=\sum_k\mathcal{A}^{G_k}$. Our goal is to express explicitly this multiplication of differential forms.

**Theorem 5.4.** *For $k,l\in\mathbb{N}^r$, let $\omega dV_k\in\mathcal{A}^{G_k}$ and $\eta dV_l\in\mathcal{A}^{G_l}$ be differential forms. Then*

$$\Delta^*(\omega dV_k\otimes\eta dV_l)=(\omega*\eta)\ dV_{k+l},\tag{5}$$

*(see Section 2.3).*

*Proof.* We will need the following concept. Call a triple $(S_1,S_2,J)$ a *shuffle* of $J_1:\{1,\dots,|k|\}\to\{1,\dots,r\}$ and $J_2:\{1,\dots,|l|\}\to\{1,\dots,r\}$ if

- $S_1,S_2$ are subsets of $\{1,\dots,|k+l|\}$, $\#S_1=|k|$, $\#S_2=|l|$,
- $\{1,\dots,|k+l|\}$ is the disjoint union of $S_1$ and $S_2$,
- $J$ is a map from $\{1,\dots,|k+l|\}$ to $\{1,\dots,r\}$,
- for the increasing bijections $s_1:S_1\to\{1,\dots,|k|\}$ and $s_2:S_2\to\{1,\dots,|l|\}$ we have

$$J(i)=\begin{cases}J_1\circ s_1(i) & i\in S_1\\ J_2\circ s_2(i) & i\in S_2.\end{cases}$$

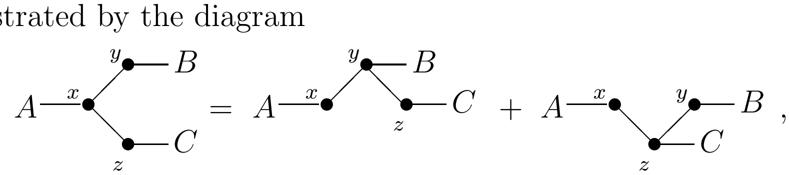
The collection of $\tilde f_J$'s form a basis of $U_r$. Let the dual basis be $\{\tilde f_J^*\}$. We only need to check (5) for this dual basis. Hence, let $\omega dV_k=\tilde f_{J_1}^*$, $\eta dV_l=\tilde f_{J_2}^*$ with $\tilde f_{J_1}\in U_r[k]$, $\tilde f_{J_2}\in U_r[l]$.

The definition of $\Delta$ implies that $\Delta^*(\tilde f_{J_1}^*\otimes\tilde f_{J_2}^*)$ is $\sum\tilde f_J^*$, where the summation runs over all shuffles of $J_1$ and $J_2$; e.g. $\Delta^*(\tilde f_1^*\otimes\tilde f_2^*)=(\tilde f_1\tilde f_2)^*+(\tilde f_2\tilde f_1)^*$, $\Delta^*(\tilde f_1^*\otimes\tilde f_1^*)=2(\tilde f_1^2)^*$.

On the other hand, the right-hand-side in (5) is also $\sum\tilde f_J^*$, with the summation running over the shuffles of $J_1$ and $J_2$. This can be seen by an iterated application of the identity

$$\frac{1}{A(y-x)B(z-x)C}=\frac{1}{A(y-x)B(z-y)C}+\frac{1}{A(z-x)C(y-z)B},$$

which can be illustrated by the diagram




c.f. [MuV, Lemma 4.4]. □

## 6. The canonical differential form for the simple Lie algebras $A, B, C, D$

The following projections of the canonical differential form are used in integral solutions of the KZ equations and in the Bethe ansatz construction. Let $\mathfrak{g}$ be a simple Lie algebra of rank $r$, with Cartan decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$. The universal enveloping algebra $U(\mathfrak{n}_-)$ of $\mathfrak{n}_-$ is generated by $r$ elements $f_1, \dots, f_r$ (the standard Chevalley generators) subject to the Serre relations; i.e. there is the quotient epimorphism $q : U_r \to U(\mathfrak{n}_-)$. We say that a non-zero element $x \in U(\mathfrak{n}_-)$ has content $k \in \mathbb{N}^r$ if $x \in q(U_r[k])$.

**Definition 6.1.** *The canonical differential form $\Omega_k^{\mathfrak{g}}$ of a simple Lie algebra $\mathfrak{g}$ is defined as the image of $\Omega_k$ under the map*

$$\mathrm{id} \otimes\, q : \mathcal{A}^{G_k} \otimes U_r[k] \to \mathcal{A}^{G_k} \otimes U(\mathfrak{n}_-)[k].$$

The Lie algebra $\mathfrak{n}_-$ is a direct sum of 1-dimensional weight spaces $\mathfrak{n}_\beta$ labelled by the positive roots $\beta$ : $\mathfrak{n}_- = \oplus_\beta \mathfrak{n}_\beta$. Let $F_\beta \in \mathfrak{n}_\beta$ be a generator. If we choose these generators and an ordering of the positive roots: $\beta_1 \dots, \beta_m$, then according to the Poincaré-Birchoff-Witt theorem, a $\mathbb{C}$-basis of the algebra $U(\mathfrak{n}_-)$ is given by the collection of elements $F_{\beta_1}^{p_1} F_{\beta_2}^{p_2} \dots F_{\beta_m}^{p_m}$, where $m = \dim \mathfrak{n}_-$ and $p = (p_1, \dots, p_m) \in \mathbb{N}^m$.

**Example 6.2.** For $sl_3$, the Lie algebra of all $3 \times 3$ traceless matrices, the positive roots $\alpha_1$, $\alpha_2$, $\alpha_1 + \alpha_2$ correspond to the matrix entries at the positions $(2, 1)$, $(3, 2)$, and $(3, 1)$, and in turn, to the basis $F_{\alpha_1} = f_1, F_{\alpha_2} = f_2, F_{\alpha_1+\alpha_2} = [f_2, f_1]$ of $\mathfrak{n}_-$. A PBW basis of $U(\mathfrak{n}_-)$ is $f_1^{p_1}[f_2, f_1]^{p_2} f_2^{p_3}$ with $p = (p_1, p_2, p_3) \in \mathbb{N}^3$.

After fixing an order $\beta_1 < \beta_2 < \dots < \beta_m$ of the positive roots of $\mathfrak{g}$, the canonical differential form of $\mathfrak{g}$ can be written in the form of

$$\Omega_k^{\mathfrak{g}} \;=\; \sum_p \;\omega_p\; dV_k \otimes F_{\beta_1}^{p_1} F_{\beta_2}^{p_2} \dots F_{\beta_m}^{p_m}\;. \tag{6}$$

Here the summation is over $p$ such that the content of $F_{\beta_1}^{p_1} F_{\beta_2}^{p_2} \dots F_{\beta_m}^{p_m}$ is $k$; $\omega_p dV_k$ is a differential form in $\mathcal{A}^{G_k}$, and $\omega_p$ is a rational function.

**Theorem 6.3.** (Product formula.) *For $l = 1, \dots, m$, let the content of $F_{\beta_l}$ be $k^{(l)}$. Then there exist rational functions $\eta_{\beta_l}$ in the variables $(t_j^{(i)})_{j=1,\dots,k_i^{(l)}}$, symmetric under $G_{k^{(l)}}$, such that*

$$\omega_p = \frac{1}{\prod_l p_l!} \cdot \overbrace{\eta_{\beta_1} * \dots * \eta_{\beta_1}}^{p_1} * \overbrace{\eta_{\beta_2} * \dots * \eta_{\beta_2}}^{p_2} * \dots * \overbrace{\eta_{\beta_r} * \dots * \eta_{\beta_r}}^{p_r}$$

*Proof.* Denote $F^p = F_{\beta_1}^{p_1} \dots F_{\beta_m}^{p_m}$. The coproduct $\Delta$ can be expressed in the PBW basis as:

$$\Delta(F^p) = \sum_{p'+p''=p} \;\prod_{i=1}^m \frac{p_i!}{p_i'! p_i''!} \cdot F^{p'} \otimes F^{p''}\;.$$

Then for the dual multiplication we have

$$\Delta^*(\omega_{p'} dV_k \otimes \omega_{p''} dV_l) = \Delta^*(F^{p'*} \otimes F^{p''*}) = \prod_i \frac{(p_i' + p_i'')!}{p_i'! p_i''!} \cdot F^{p'+p''*}.$$

Using Theorem 5.4, we obtain

$$\omega_{p'} * \omega_{p''} = \prod_i \frac{(p'_i + p''_i)!}{p'_i! p''_i!} \cdot \omega_{p'+p''},$$

from which the result follows (put $\eta_{\beta_l} = \omega_{1_l}$). □

**Example 6.4.** For $\mathfrak{g} = sl_3$ and the ordering $\alpha_1 < \alpha_1 + \alpha_2 < \alpha_2$, we have $\omega_{\alpha_1} = 1/t_1$, $\omega_{\alpha_1+\alpha_2} = 1/(t_1(s_1 - t_1))$ and $\omega_{\alpha_2} = 1/s_1$. (Again, we write $t$ for $t^{(1)}$ and $s$ for $t^{(2)}$.) Then the differential form corresponding to $f_1^2[f_2, f_1]$ is

$$\omega_{(2,1,0)} dt_1 \wedge dt_2 \wedge dt_3 \wedge ds = \frac{1}{2!1!0!} \cdot \frac{1}{t_1} * \frac{1}{t_1} * \frac{1}{t_1(s_1 - t_1)} =$$

$$= \frac{1}{2} \operatorname{Sym}_{(3,1)} \left( \frac{1}{t_1 t_2 t_3 (s - t_3)} \right) dV_{(3,1)} = \frac{3s^2 - 2s(t_1 + t_2 + t_3) + (t_1 t_2 + t_1 t_3 + t_2 t_3)}{t_1 t_2 t_3 (s - t_1)(s - t_2)(s - t_3)} dV_{(3,1)}.$$

This means that $\Omega_k^{\mathfrak{g}}$ is determined once we know its 'atoms', i.e. the $\eta_\beta$'s for the positive roots $\beta$. In the remainder of this section, we will compute them for the infinite series $A$, $B$, $C$, $D$ of simple Lie algebras. For each of these we will recall (1) the positive roots, (2) the simple roots, and (3) the expression of the positive roots in terms of the simple roots. Then we choose (4) an ordering of the positive roots (in all cases it will be a *normal* ordering) and fix (5) the elements $F_\beta$'s (choice of a constant). Then we describe the elements $\eta_\beta$'s with the choices (4), (5).

Let $\epsilon_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{C}^r$ (1 occurs at the $i$th position from the left). For a multi-index $J = (J(1), J(2), \ldots, J(n))$, let $[f_J] = [f_{J(1)}[f_{J(2)}[\ldots, [f_{J(n-1)}, f_{J(n)}] \ldots]]]$.

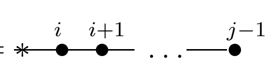
### 6.1. The simple Lie algebra $A_{r-1}$.

(1) The positive roots are $\epsilon_i - \epsilon_j$ for $1 \le i < j \le r$.
(2) The simple roots are $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $1 \le i < r$.
(3) $\epsilon_i - \epsilon_j = \sum_{u=i}^{j-1} \alpha_u$.
(4) Let $\epsilon_i - \epsilon_j < \epsilon_{i'} - \epsilon_{j'}$ if either $i + j < i' + j'$ or $i + j = i' + j'$ and $i < i'$;
(5) For a positive root $\beta = \epsilon_i - \epsilon_j$, let $F_\beta = [f_{(j-1, j-2, \ldots, i)}]$.

**Theorem 6.5.** *For $\beta = \epsilon_i - \epsilon_j = \alpha_i + \alpha_{i+1} + \ldots + \alpha_{j-1}$, we have*

$$\eta_\beta = \frac{1}{t_1^{(i)} (t_1^{(i+1)} - t_1^{(i)}) \cdot \ldots \cdot (t_1^{(j-1)} - t_1^{(j-2)})}.$$

The result can be visualized by the following string-diagram (the labels of the vertices in the diagram indicate the superscripts of the corresponding $t$'s).

$\eta_{\alpha_i + \ldots + \alpha_j} =$ 

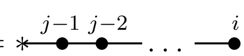
6.2. **The simple Lie algebra $B_r$.**

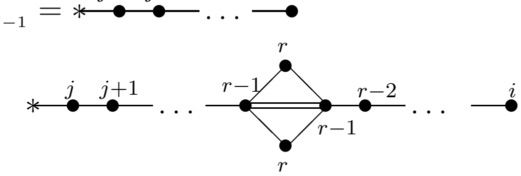
(1) The positive roots are $\epsilon_i$ $(1 \le i \le r)$ and $\epsilon_i - \epsilon_j$, $\epsilon_i + \epsilon_j$ $(1 \le i < j \le r)$.
(2) The simple roots are $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $i = 1, \dots, r-1$ and $\alpha_r = \epsilon_r$ (the 'short' root).
(3)
$$\epsilon_i = \sum_{u=i}^{r} \alpha_u \ (1 \le i \le r), \quad \epsilon_i - \epsilon_j = \sum_{u=i}^{j-1} \alpha_u, \quad \epsilon_i + \epsilon_j = \sum_{u=i}^{j-1} \alpha_u + 2\sum_{u=j}^{r} \alpha_u \ (1 \le i < j \le r).$$
(4) Let $\beta$ be one of $\epsilon_i$, $\epsilon_i - \epsilon_j$, or $\epsilon_i + \epsilon_j$, and let $\beta'$ be one of $\epsilon_{i'}$, $\epsilon_{i'} - \epsilon_{j'}$, or $\epsilon_{i'} + \epsilon_{j'}$ Then we put $\beta < \beta'$ if $i > i'$, and if $i = i'$ then $\epsilon_i + \epsilon_j < \epsilon_i + \epsilon_{j'} < \epsilon_i < \epsilon_i - \epsilon_{j'} < \epsilon_i - \epsilon_j$ for $i < j < j'$.
(5) $F_{\alpha_i + \dots + \alpha_{j-1}} = [f_{(j-1,j-2,\dots,i)}]$ and $F_{\epsilon_i+\epsilon_j} = [[f_{(i,i+1,\dots,r)}],[f_{(r,r-1,\dots,j)}]]$.

The vector $(\overbrace{0,\dots,0}^{u_0},\overbrace{1,\dots,1}^{u_1}\dots)$ will be abbreviated by $(0^{u_0}1^{u_1}\dots)$.

**Theorem 6.6.** *We have*
$$\eta_{\alpha_i+\dots+\alpha_{j-1}} = \frac{1}{t_1^{(j-1)}(t_1^{(j-2)} - t_1^{(j-1)}) \cdot \ldots \cdot (t_1^{(i)} - t_1^{(i+1)})} \qquad (\textit{for both roots } \epsilon_i \textit{ and } \epsilon_i - \epsilon_j),$$
$$\eta_{\epsilon_i+\epsilon_j} = \frac{1}{2}\mathrm{Sym}_{(0^{i-1}1^{j-i}2^{r-j+1})}\Big(\frac{t_1^{(r-1)} - t_2^{(r-1)}}{t_1^{(j)}(t_1^{(r)} - t_1^{(r-1)})(t_2^{(r)} - t_1^{(r-1)})(t_2^{(r-1)} - t_1^{(r)})(t_2^{(r-1)} - t_2^{(r)})} \cdot \frac{1}{\prod_{k=j+1}^{r-1}(t_1^{(k)} - t_1^{(k-1)})\prod_{k=i}^{r-2}(t_2^{(k)} - t_2^{(k+1)})}\Big).$$

The structure of these functions is better understood via the following pictures.

$\eta_{\alpha_i+\dots+\alpha_{j-1}} =$ 


$\eta_{\sum_{i\le k<j}\alpha_k + 2\sum_{j\le k\le r}\alpha_k} = \frac{1}{2}\,\mathrm{Sym}$ 


In the second picture, the double edge means that the corresponding difference is in the numerator.

6.3. **The simple Lie algebra $C_r$.**

(1) The positive roots are $\epsilon_i - \epsilon_j$, $\epsilon_i + \epsilon_j$ for $1 \le i < j \le r$ and $2\epsilon_i$ for $1 \le i \le r$.
(2) The simple roots are $\alpha_i = \epsilon_i - \epsilon_{i+1}$ $(1 \le i < r)$ and $\alpha_r = 2\epsilon_r$ (the 'long' root).
(3)
$$\epsilon_i - \epsilon_j = \sum_{u=i}^{j-1}\alpha_u, \qquad \epsilon_i + \epsilon_j = \sum_{u=i}^{j-1}\alpha_u + 2\sum_{u=j}^{r-1}\alpha_u + \alpha_r, \qquad 2\epsilon_i = 2\sum_{u=i}^{r-1}\alpha_u + \alpha_r.$$
(4) Let $\beta$ be one of $\epsilon_i - \epsilon_j$, $\epsilon_i + \epsilon_j$, or $2\epsilon_i$, and let $\beta'$ be one of $\epsilon_{i'} - \epsilon_{j'}$, $\epsilon_{i'} + \epsilon_{j'}$, or $2\epsilon_{i'}$. Then we put $\beta < \beta'$ if $i > i'$; or if $i = i'$ then $\epsilon_i + \epsilon_j < \epsilon_i + \epsilon_{j'} < 2\epsilon_i < \epsilon_i - \epsilon_{j'} < \epsilon_i - \epsilon_j$ for $i < j < j'$.

(5) $F_{\epsilon_i-\epsilon_j} = [f_{(i,i+1,\dots,j-1)}]$, $F_{\epsilon_i+\epsilon_j} = [f_{i,i+1,\dots,r-1,r,r-1,\dots,j}]$, $F_{2\epsilon_i} = [[f_{(i,i+1,\dots,r-1)}],[f_{(i,i+1,\dots,r)}]]$.

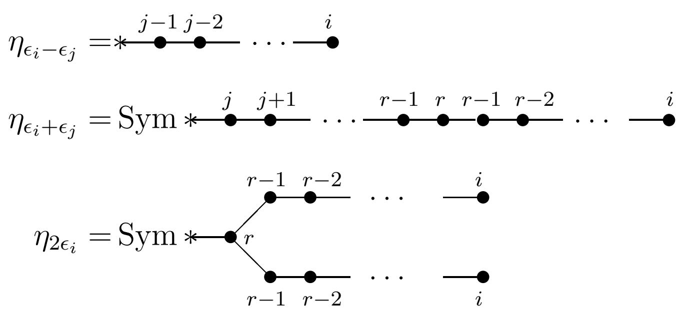
**Theorem 6.7.** *We have*

$$\eta_{\epsilon_i-\epsilon_j} = \frac{1}{t_1^{(j-1)}\prod_{u=i}^{j-2}(t_1^{(u)}-t_1^{(u+1)})},$$

$$\eta_{\epsilon_i+\epsilon_j} = \mathrm{Sym}_{(0^{i-1}1^{j-i}2^{r-j}1)}\Big(\frac{1}{t_1^{(j-1)}\prod_{u=j+1}^{r}(t_1^{(u)}-t_1^{(u-1)})\prod_{u=i}^{r-1}(t_2^{(u)}-t_2^{(u+1)})}\Big),$$

$$\eta_{2\epsilon_i} = \mathrm{Sym}_{(0^{i-1}2^{r-i}1)}\Big(\frac{1}{t_1^{(r)}(t_1^{(r-1)}-t_1^{(r)})(t_2^{(r-1)}-t_1^{(r)})\prod_{u=i}^{r-2}(t_1^{(u)}-t_1^{(u+1)})\prod_{u=i}^{r-2}(t_2^{(u)}-t_2^{(u+1)})}\Big).$$

The result can be visualized by the following diagrams (labels mean upper indices).




6.4. **The simple Lie algebra $D_r$.**

(1) The positive roots are $\epsilon_j-\epsilon_i$ and $\epsilon_j+\epsilon_i$ for $1 \le i < j \le r$.
(2) The simple roots are $\alpha_1 = \epsilon_1+\epsilon_2$ and $\alpha_i = \epsilon_i - \epsilon_{i-1}$ $(1 < i \le r)$.
(3)

$$\epsilon_j-\epsilon_i = \sum_{u=i+1}^{j}\alpha_u, \quad \epsilon_i+\epsilon_j = \alpha_1+\alpha_2+2\sum_{u=3}^{i}\alpha_u+\sum_{u=i+1}^{j}\alpha_u \ (1<i<j\le r), \quad \epsilon_1+\epsilon_j = \alpha_1+\sum_{u=3}^{j}\alpha_u.$$

(4) Let $\beta$ be one of $\epsilon_j-\epsilon_i$ or $\epsilon_j+\epsilon_i$, and let $\beta'$ be one of $\epsilon_{j'}-\epsilon_{i'}$ or $\epsilon_{j'}+\epsilon_{i'}$. Then we put $\beta<\beta'$ if $j<j'$; or if $j=j'$ then $\epsilon_j+\epsilon_i<\epsilon_j+\epsilon_{i'}<\epsilon_j-\epsilon_{i'}<\epsilon_j-\epsilon_i$ for $i'<i<j$.
(5) $F_{\epsilon_j-\epsilon_i} = [f_{(j,j-1,\dots,i+1)}]$, $F_{\epsilon_1+\epsilon_j} = [f_{(j,j-1,\dots,3,1)}]$, $F_{\epsilon_i+\epsilon_j} = [f_{(j,j-1,\dots,2,1,3,4,\dots,i-1,i)}]$.

**Theorem 6.8.** *We have*

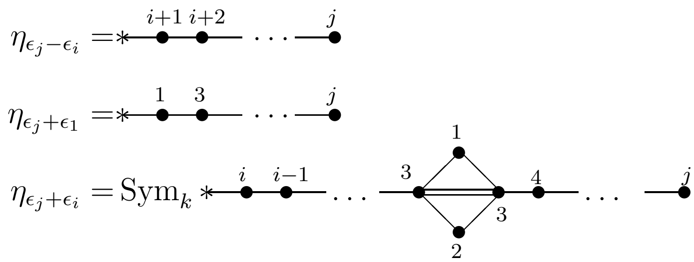
$$\eta_{\epsilon_j-\epsilon_i} = \frac{1}{t_1^{(i+1)}\prod_{u=i+2}^{j}(t_1^{(u)}-t_1^{(u-1)})},$$

$$\eta_{\epsilon_j+\epsilon_1} = \frac{1}{t_1^{(1)}\prod_{u=3}^{j}(t_1^{(u)}-t_1^{(u-1)})}.$$

*For* $1<i<j$, $\eta_{\epsilon_j+\epsilon_i} = \mathrm{Sym}_{(1^2 2^{i-2}1^{j-i}0^{r-j})}$

$$\frac{t_1^{(3)}-t_2^{(3)}}{t_2^{(i)}(t_1^{(1)}-t_2^{(3)})(t_1^{(2)}-t_2^{(3)})(t_1^{(3)}-t_1^{(1)})(t_1^{(3)}-t_1^{(2)})\prod_{u=3}^{i+1}(t_2^{(u)}-t_2^{(u+1)})\prod_{u=4}^{j}(t_1^{(u)}-t_1^{(u-1)})}.$$

The result can be visualized by the following diagrams (labels mean upper indices).




6.5. **The proofs of the theorems 6.5-6.8.** Let $\mathfrak{g}$ be one of the simple Lie algebras $A_r$, $B_r$, $C_r$, $D_r$, and let $\beta$ be one of its positive roots. In one of the Theorems 6.5–6.8 (the one referring to $\mathfrak{g}$), we state a formula for $\eta_\beta$; let us denote the function on the right-hand-side of that formula by $\overline{\eta}_\beta$. In this section we will prove that $\eta_\beta = \overline{\eta}_\beta$, by proving Theorems 6.10 and 6.12 below.

**Lemma 6.9.** *Under the correspondence between $U_r[k]$ and $Fl^{G_k}$ of Section 3.5, we have*

$$[\tilde{f}_i, \tilde{f}_j] \leftrightarrow \pm[\mathbb{C}^2 \supset (t_1^{(i)} = t_1^{(j)}) \supset 0]$$

$$[\tilde{f}_i, [\tilde{f}_i, \tilde{f}_j]] \leftrightarrow \pm \operatorname{ASym}_{(2,1)} \Big( [\mathbb{C}^3 \supset (t_1^{(i)} = t_1^{(j)}) \supset (t_1^{(i)} = t_2^{(i)} = t_1^{(j)}) \supset 0] \Big)$$

$$[\tilde{f}_i, [\tilde{f}_i, [\tilde{f}_i, \tilde{f}_j]]] \leftrightarrow$$

$$\pm \operatorname{ASym}_{(3,1)} \Big( [\mathbb{C}^4 \supset (t_1^{(i)} = t_1^{(j)}) \supset (t_1^{(i)} = t_1^{(j)} = t_2^{(i)}) \supset (t_1^{(i)} = t_1^{(j)} = t_2^{(i)} = t_3^{(i)}) \supset 0] \Big).$$

*Proof.* For $i < j$ we have

$$\tilde{f}_i\tilde{f}_j - \tilde{f}_j\tilde{f}_i \leftrightarrow -[\mathbb{C}^2 \supset (t_1^{(j)} = 0) \supset 0] - [\mathbb{C}^2 \supset (t_1^{(i)} = 0) \supset 0] = [\mathbb{C}^2 \supset (t_1^{(j)} = t_1^{(i)}) \supset 0],$$

which proves the first statement. The others follow from similar calculations. □

Let the content of $F_\beta$ be $k$.

**Theorem 6.10.** *Let $F \in Fl^{G_k}$ be a linear combination of flags corresponding to an element $\sum c_J \tilde{f}_J$ in $U_r[k]$ of content $k$. If $\sum_j c_J \tilde{f}_j$ belongs to the ideal generated by the Serre relations, then* $\operatorname{Res}_F \overline{\eta}_\beta dV_k = 0$.

*Proof.* We show the proof for $\mathfrak{g}$ of type $A$. In this case there are two kinds of Serre relations: $[f_i, f_j] = 0$ if $(\alpha_i, \alpha_j) = 0$, and $[f_i, [f_i, f_j]] = 0$ if $(\alpha_i, \alpha_j) = -1$. We will consider the linear combination of flags corresponding to multiples of $[\tilde{f}_i, \tilde{f}_j]$ and $[\tilde{f}_i, [\tilde{f}_i, \tilde{f}_j]]$. By Lemma 6.9, any multiple of $[\tilde{f}_i, \tilde{f}_j]$ corresponds to a linear combination $F$ of flags, with each term of the form $\pm[\mathbb{C}^{|k|} \supset \ldots \supset L_{u+2} \supset L_{u+1} \supset L_u \supset \ldots \supset 0]$ with

$$[L_{u+2}/L_u \supset L_{u+1}/L_u \supset L_u/L_u] \;\simeq\; [\mathbb{C}^2 \supset (t_{v_1}^{(i)} = t_{v_2}^{(j)}) \supset 0].$$

The rational function $\overline{\eta}_\beta$ does not have a factor of type $t^{(i)} - t^{(j)}$ in the denominator for $(\alpha_i, \alpha_j) = 0$. Thus $\operatorname{Res}_F \overline{\eta}_\beta dV_k = 0$.

By Lemma 6.9, any multiple of $[\tilde{f}_i[\tilde{f}_i, \tilde{f}_j]]$ corresponds to a linear combination $F$ of flags, with each term of the form $\pm[\mathbb{C}^{|k|} \supset \ldots \supset L_{u+3} \supset L_{u+2} \supset L_{u+1} \supset L_u \supset \ldots \supset 0]$ with

$$[L_{u+3}/L_u, L_{u+2}/L_u \supset L_{u+1}/L_u \supset L_u/L_u] \;\simeq\; [\mathbb{C}^3 \supset (t_{v_1}^{(i)} - t_{v_2}^{(j)}) \supset (t_{v_1}^{(i)} = t_{v_3}^{(i)} = t_{v_2}^{(j)}) \supset 0].$$

Since all $k_i$ are 0 or 1 for the positive root $\beta$, this type of flags cannot occur in $Fl^{G_k}$.

The proof for the types $B, C, D$ are analogous. □

**Remark 6.11.** By Theorem 6.10, the differential forms $\Omega^{\mathfrak{g}}$ (and $\Omega^V$, see the Introduction and Section 7 below) do not have poles at the $t^{(i)} = t^{(j)}$ type hyperplanes if the corresponding simple roots are orthogonal. Hence the poles of $\Omega^{\mathfrak{g}}$ coincide with the singularities of the master function, see the Introduction.

Now let $F^p = F_{\beta_1}^{p_1} \dots F_{\beta_m}^{p_m}$ be another element of content $k$ in the Poincare-Birchoff-Witt basis, different from $F_\beta$. Let us choose preimages of $F_\beta$ and $F^p$ under the projection $U_r[k] \to U(\mathfrak{n}_-)$, and let $Flag_\beta$ and $Flag^p \in Fl^{G_k}$ be the corresponding linear combinations of flags.

**Theorem 6.12.**

- *The residue of the differential form $\overline{\eta}_\beta dV_k$ with respect to $Flag^p$ is 0.*
- *The residue of the differential form $\overline{\eta}_\beta dV_k$ with respect to $Flag_\beta$ is 1.*

**Lemma 6.13.** *For $i = 1, \dots, r$, $j = 1, \dots, k_i$, we have*

$$\mathrm{Res}_{t_j^{(i)}=0}\, \eta_\beta\, dV_k \;=\; \mathrm{Res}_{t_j^{(i)}=0}\, \overline{\eta}_\beta\, dV_k\ .$$

*Proof.* It is enough to consider $j = k_i$. Then the left hand side can be calculated from Theorem 5.2, and the right hand side is given explicitly. For example, for type $A$ we obtain

$$\mathrm{Res}_{t_1^{(i)}=0}\, \eta_\beta dV_k = \mathrm{Res}_{t_1^{(i)}=0}\, \overline{\eta}_\beta dV_k \;=\; \begin{cases} \eta_{\beta-\alpha_i} dV_{k-1_i} & \text{if } \beta - \alpha_i > \beta, \text{ and it is a positive root} \\ 0 & \text{otherwise.} \end{cases}$$

For the other types similar expressions are valid. □

*Proof of Theorem 6.12.* The second statement follows from the explicit forms for $\overline{\eta}_\beta$.

Let $\overline{\Omega}_k^{\mathfrak{g}}$ be the form obtained from $\Omega_k^{\mathfrak{g}}$ by replacing the term $\eta_\beta \otimes F_\beta$ by $\overline{\eta}_\beta \otimes F_\beta$. Then $\mathrm{Res}_{Flag^p}\, \overline{\Omega}_k^{\mathfrak{g}} = \mathrm{Res}_{Flag^p}\, \Omega_k^{\mathfrak{g}}$ because of Lemma 6.13. This latter is $1 \otimes F^p$ because $(U_r[k])^* = \mathcal{A}^{G_k}$. Therefore we have $\mathrm{Res}_{Flag^p}\, \overline{\eta}_\beta = 0$, which is what we wanted to prove. □

## 7. Appendix: Representation-valued canonical differential form

For the convenience of the reader, we give formulas from [SV1], [SV2] for the $V$-valued differential form $\Omega^V$ which appears in the hypergeometric solutions to the KZ equations and the Bethe ansatz method.

Consider the formula for the canonical differential form from Theorem 4.1, without putting $t_0 = 0$. Instead, put $t_0 = z$ and denote this form by $\Omega_k(z)$, e.g. $\Omega_{(1,1)}(z) = \frac{dt}{t-z} \wedge \frac{ds}{s-t}$. The projection to $\mathfrak{g}$ of this form will be denoted by $\Omega_k^{\mathfrak{g}}(z)$.

The proofs in Section 6 can be modified to get PBW expansions of $\Omega_k^{\mathfrak{g}}(z)$. The only change in the PBW-coefficient results is that the * of the diagrams has to be decorated by $z$ (instead of 0). E.g. for $k = (2)$, instead of $\mathrm{ASym}_{(2)}(dt_1/t_1 \wedge d(t_2 - t_1)/(t_2 - t_1)) = 1/(t_1 t_2) dt_1 \wedge dt_2$ we have $\mathrm{ASym}_{(2)}(dt_1/(t_1 - z) \wedge d(t_2 - t_1)/(t_2 - t_1)) = 1/((t_1 - z)(t_2 - z)) dt_1 \wedge dt_2$. For a simple Lie algebra $\mathfrak{g}$, let $V_\Lambda$ be a highest weight $\mathfrak{g}$-module with highest weight $\Lambda \in \mathfrak{h}^*$ and generating vector $v_\Lambda$. Recall that the map $U(\mathfrak{n}_-) \to V_\Lambda$, $x \to x \cdot v_\Lambda$ is surjective.

**Definition 7.1.** *Let* $k^{(1)}, k^{(2)}, \ldots, k^{(n)} \in \mathbb{N}^r$, $k = \sum k^{(i)}$. *We extend the star multiplication from Section 2.3 as follows:*

$$* : (\mathcal{A}^{G_{k^{(1)}}} \otimes V_{\Lambda_1}) \otimes (\mathcal{A}^{G_{k^{(2)}}} \otimes V_{\Lambda_2}) \otimes \ldots \otimes (\mathcal{A}^{G_{k^{(n)}}} \otimes V_{\Lambda_n}) \to \mathcal{A}^{G_k} \otimes (V_{\Lambda_1} \otimes V_{\Lambda_2} \otimes \ldots \otimes V_{\Lambda_n})$$

*by*

$$(\Omega_1 \otimes v_1) * (\Omega_2 \otimes v_2) * \ldots * (\Omega_n \otimes v_n) = (\omega_1 * \ldots * \omega_n) dV_k \otimes (v_1 \otimes v_2 \otimes \ldots \otimes v_n),$$

*where* $\Omega_i = \omega_i dV_{k^{(i)}}$.

Let $V = V_{\Lambda_1} \otimes \cdots \otimes V_{\Lambda_n}$. We define the $V$-valued differential form of degree $k$ (c.f. [MuV, (4)]) by

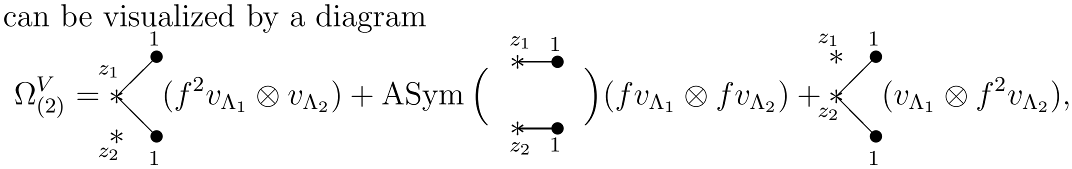
$$\Omega_k^V = \bigoplus_{k^{(1)}+\ldots+k^{(n)}=k} \Omega^{\mathfrak{g}}_{k^{(1)}}(z_1) v_{\Lambda_1} * \ldots * \Omega^{\mathfrak{g}}_{k^{(n)}}(z_n) v_{\Lambda_n}.$$

**Example 7.2.** For $n = 2$, $r = 1$ (i.e. $\mathfrak{g} = sl_2$), we have

$$\Omega^V_{(2)} = \Omega^{\mathfrak{g}}_{(2)}(z_1) v_{\Lambda_1} * \Omega^{\mathfrak{g}}_{(0)}(z_2) v_{\Lambda_2} + \Omega^{\mathfrak{g}}_{(1)}(z_1) v_{\Lambda_1} * \Omega^{\mathfrak{g}}_{(1)}(z_2) v_{\Lambda_2} + \Omega^{\mathfrak{g}}_{(0)}(z_1) v_{\Lambda_1} * \Omega^{\mathfrak{g}}_{(2)}(z_2) v_{\Lambda_2} =$$

$$\begin{aligned} &\mathrm{ASym}_{(2)} \left( \frac{dt_1}{t_1 - z_1} \wedge \frac{d(t_2 - t_1)}{t_2 - t_1} \right) \otimes f^2 v_{\Lambda_1} \otimes v_{\Lambda_1} + \\ &\mathrm{ASym}_{(2)} \left( \frac{dt_1}{t_1 - z_1} \wedge \frac{dt_2}{t_2 - z_2} \right) \otimes f v_{\Lambda_1} \otimes f v_{\Lambda_2} + \\ &\mathrm{ASym}_{(2)} \left( \frac{dt_1}{t_1 - z_2} \wedge \frac{d(t_2 - t_1)}{t_2 - t_1} \right) \otimes v_{\Lambda_1} \otimes f^2 v_{\Lambda_2}. \end{aligned}$$

This can be visualized by a diagram




where we also used the PBW expansions from Section 6.1.

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