The Makeenko–Migdal equation for Yang–Mills theory on compact surfaces

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Abstract

We prove the Makeenko–Migdal equation for two-dimensional Euclidean Yang–Mills theory on an arbitrary compact surface, possibly with boundary. In particular, we show that two of the proofs given by the first,

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third, and fourth authors for the plane case extend essentially without change to compact surfaces.

1 Introduction

The Euclidean Yang–Mills field theory on a surface Σ describes a random connection on a principal bundle over Σ for a compact Lie group K, known as the structure group. Work of A. Sengupta [Sen1, Sen2, Sen3, Sen4] gave a formula for the expectation value of any gauge-invariant function defined in terms of parallel transport along the edges of a graph $\mathbb{G}$ in Σ. (Related work was done by D. Fine [Fine1, Fine2] and E. Witten [Witt1, Witt2].) This theory was then further developed [Lévy1] and generalized [Lévy2] in the work of T. Lévy. Sengupta's formula (generalizing Driver's formula [Dr, Theorem 6.4] in the plane case) is given in terms of the *heat kernel* on the group K. (See Section 2.) One noteworthy feature of the formula is its invariance under area-preserving diffeomorphisms of Σ.

The typical objects of study in the theory are the Wilson loop functionals, given by

$$\mathbb{E}\{\mathrm{trace}(\mathrm{hol}(L))\}, \tag{1}$$

where $\mathbb{E}$ denotes the expectation value with respect to the Yang–Mills measure, $\mathrm{hol}(L)$ denotes the holonomy of the connection around a loop L traced out in a graph $\mathbb{G}$, and the trace is taken in some fixed representation of K. The diffeomorphism-invariance of the theory is reflected in Sengupta's formula: the expectation (1) is given as a function (determined by the topology of the graph and of Σ) of all the areas of the faces of $\mathbb{G}$.

A key identity for calculating Wilson loops is the *Makeenko–Migdal equation* [MM, Equation 3] for Yang–Mills theory. For the plane case, the Makeenko–Migdal equation takes the form (3) below, as worked out by V. A. Kazakov and I. K. Kostov in [KK, Equation 24] (see also [K, Equation 9] and [GG, Equation 6.4]. We take $K = U(N)$ and we use the bi-invariant metric on $U(N)$ whose value on the Lie algebra $\mathfrak{u}(N) = T_e(U(N))$ is a scaled version of the Hilbert–Schmidt inner product:

$$\langle X, Y\rangle = N\mathrm{trace}(X^*Y). \tag{2}$$

We then express the Wilson loop functionals using the *normalized* trace,

$$\mathrm{tr}(A) := \frac{1}{N}\mathrm{trace}(A).$$

We now consider a loop L in the plane with simple crossings, and we let v be one such crossing. We let t_1, t_2, t_3, and t_4 denote the areas of the faces adjacent to v, as in Figure 1. We also let L_1 denote the portion of the loop from the beginning to the first return to v and let L_2 denote the loop from the first return to the end, as in Figure 2. The planar Makeenko–Migdal equation then

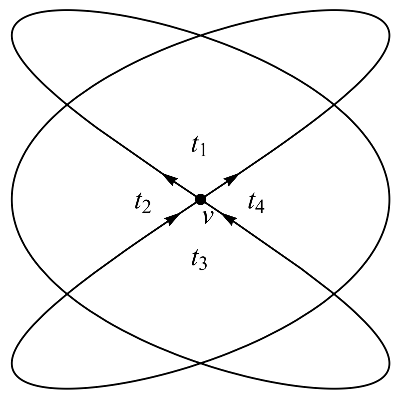

Figure 1: A typical loop L for the Makeenko–Migdal equation

gives a formula for the alternating sum of the derivatives of the Wilson loop functional with respect to these areas:

$$\left(\frac{\partial}{\partial t_1} - \frac{\partial}{\partial t_2} + \frac{\partial}{\partial t_3} - \frac{\partial}{\partial t_4}\right) \mathbb{E}\{\mathrm{tr}(\mathrm{hol}(L))\} = \mathbb{E}\{\mathrm{tr}(\mathrm{hol}(L_1))\mathrm{tr}(\mathrm{hol}(L_2))\}. \quad (3)$$

We follow the convention that if any of the adjacent faces is the unbounded face, the corresponding derivative on the left-hand side of (3) is omitted. Note also that the faces F_1, F_2, F_3, and F_4 are not necessarily distinct, so that the same derivative may occur more than once on the left-hand side of (3).

The first rigorous proof of (3) was given by Lévy in [Lévy3, Proposition 6.24]. A second proof was given by A. Dahlqvist in [Dahl, Proposition 7.2]. Both of these proofs proceed by computing the *individual* time derivatives on the left-hand side of (3). These formulas involve calculations along a sequence of faces proceeding from a face adjacent to v to the unbounded face. After taking the alternating sum of derivatives, both Lévy and Dahlqvist obtain a cancellation that allows the result to simplify to the right-hand side of (3). In [DHK2], three of the authors of the present paper gave three new proofs of (3). All of these proofs were "local" in nature, meaning that the calculations involve only faces and edges adjacent to the crossing v.

The goal of the present paper is to demonstrate that two of the proofs of (3) in [DHK2] can be applied almost without change to the case of an arbitrary compact surface Σ, possibly with boundary. In particular, the local nature of the proofs in [DHK2] mean we do not require the presence of an unbounded face.

Let us say that a graph $\mathbb{G}$ in Σ is **admissible** if $\mathbb{G}$ contains the entire boundary of Σ and each component of the complement of $\mathbb{G}$ is homeomorphic to a disk. (Actually, according to Proposition 1.3.10 of [Lévy2], if each component of the complement is a disk, the graph necessarily contains the entire boundary of Σ.)

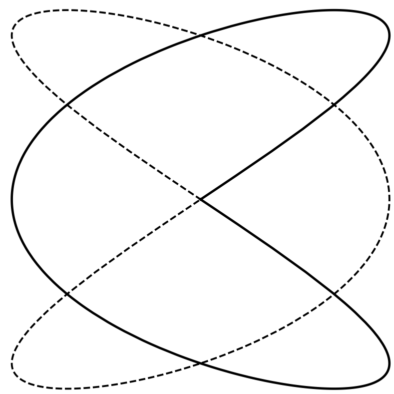

Figure 2: The loops L_1 (black) and L_2 (dashed)

Theorem 1 (Makeenko–Migdal Equation for Surfaces) *Let Σ be a compact surface, possibly with boundary. Let $K = U(N)$ and let $\mathbb{E}$ denote expectation value with respect to the normalized Yang–Mills measure over Σ, possibly with constraints on the holonomies around the boundary components. Suppose that L is a closed curve that can be traced out on an admissible graph $\mathbb{G}$ in Σ. Suppose v is a simple crossing of L in the interior of Σ and let L_1 and L_2 denote the two pieces of the curve cut at v. Then we have*

$$\left(\frac{\partial}{\partial t_1} - \frac{\partial}{\partial t_2} + \frac{\partial}{\partial t_3} - \frac{\partial}{\partial t_4}\right)\mathbb{E}\{\mathrm{tr}(\mathrm{hol}(L))\} = \mathbb{E}\{\mathrm{tr}(\mathrm{hol}(L_1))\mathrm{tr}(\mathrm{hol}(L_2))\}.$$

We will actually prove an abstract Makeenko–Migdal equation (generalizing Proposition 6.22 in [Lévy3]) that applies to an arbitrary structure group K and that implies Theorem 1 as a special case. As in the plane case, the abstract Makeenko–Migdal equation allows one to compute alternating sums of derivatives of other sorts of functions; see Section 2.5 of [DHK2] for examples.

For any fixed N, the Makeenko–Migdal equation in (3) or in Theorem 1 is not especially helpful in computing Wilson loop functionals. After all, even though the loops L_1 and L_2 are simpler than L, the right-hand side of (3) involves the *expectation of a product* of traces rather than a product of expectations. Thus, the right-hand side cannot be considered as a recursively known quantity. In the plane case, however, it is known that the Yang–Mills theory for $U(N)$ has a large-N limit, and that in this limit, all traces become deterministic. (This deterministic limit is known as the master field and was investigated by various authors, including I. M. Singer [Sing], R. Gopakumar and D. Gross [GG], and M. Anshelevich and Sengupta [AS]. A detailed proof of the existence and deterministic nature of the limit was provided by Lévy in [Lévy3, Section 5].) Thus, in the large-N limit in the plane case, there is no difference between the expectation of a product and a product of expectations and (3) becomes

$$\left(\frac{\partial}{\partial t_1} - \frac{\partial}{\partial t_2} + \frac{\partial}{\partial t_3} - \frac{\partial}{\partial t_4}\right) \tau(\mathrm{hol}(L)) = \tau(\mathrm{hol}(L_1))\tau(\mathrm{hol}(L_2)), \tag{4}$$

where $\tau(\cdot)$ is the limiting value of $\mathbb{E}\{\mathrm{tr}(\cdot)\}$.

In the plane case, Lévy also establishes the following "unbounded face condition." If t denotes the area of any face F that adjoins the unbounded face, we have

$$\frac{\partial}{\partial t}\tau(\mathrm{hol}(L)) = -\frac{1}{2}\tau(\mathrm{hol}(L)), \quad (F \text{ adjoins the unbounded face}). \tag{5}$$

(See Axiom Φ_4 in Section 0.5 of [Lévy3] and compare Theorem 2 in [DHK2].) Lévy then shows that the large-N limit of $U(N)$ Yang–Mills theory on the plane is completely determined by the large-N Makeenko–Migdal equation (4) and the unbounded face condition (5), together with some continuity and invariance properties [Lévy3, Section 0.5].

It is currently not rigorously known whether Yang–Mills theory on a compact surface Σ admits a large-N limit. (However, see for example [DK], where the large-N limit of Yang-Mills theory on the 2-sphere is explored non-rigorously.) If the limit does exist and is deterministic (as in the plane case), it is reasonable to expect that the limiting theory would satisfy (4) (this is assumed in [DK]). One would have to justify interchanging the derivatives with the large-N limit in Theorem 1. On the other hand, since Σ does not have an unbounded face, the unbounded face condition in (5) does not even make sense. Thus, even if (4) holds for the large-N limit of Yang–Mills theory on Σ, this relation may not allow for a complete characterization of the limit. Nevertheless, if the large-N limit on Σ exists and satisfies (4), this relation should contain a lot of information about the limiting theory.

The authors thank Ambar Sengupta for many useful discussions of Yang–Mills theory on surfaces.

2 Yang–Mills theory on surfaces

The Yang–Mills measure for a graph $\mathbb{G}$ in a surface Σ has been described by Sengupta, first for closed surfaces in [Sen1] (see also [Sen2]) and then for surfaces with boundary [Sen3], possibly incorporating constraints on the holonomy around the boundary. Related work was done by Fine [Fine1, Fine2] and Witten [Witt1, Witt2]. Sengupta's results were further developed and generalized by Lévy in [Lévy1] and [Lévy2].

We consider a compact, connected surface Σ, possibly with boundary. We do not require that Σ be orientable. We then consider a connected compact group K, called the structure group, equipped with a fixed bi-invariant Riemannian metric. (If K is not simply connected, the Yang–Mills measure as described below may incorporate contributions from inequivalent principal K-bundles over

Σ.) We also consider the heat kernel ρ_t on K at the identity, that is, the unique function such that

$$\frac{\partial \rho_t}{\partial t} = \frac{1}{2}\Delta \rho_t$$

and such that for any continuous function f on K,

$$\lim_{t\to 0} \int_K f(x)\rho_t(x)\ dx = f(\mathrm{id}),$$

where id is the identity element of K and dx is the normalized Haar measure.

2.1 The unconstrained Yang–Mills measure on a graph

We begin by precisely defining the appropriate notion of a graph in Σ. By an edge we will mean a continuous map $\gamma : [0,1] \to \Sigma$, assumed to be injective except possibly that $\gamma(0) = \gamma(1)$. We identify two edges if they differ by an orientation-preserving reparametrization. Two edges that differ by an orientation-reversing reparametrization are said to be inverses of each other. A graph is then a finite collection of edges (and their inverses) that meet only at their endpoints. Given a graph $\mathbb{G}$, we choose arbitrarily one element out of each pair consisting of an edge and its inverse. We then refer to the chosen edges as the positively oriented edges.

We call a graph $\mathbb{G}$ in Σ **admissible** if $\mathbb{G}$ contains the entire boundary of Σ and each face F of $\mathbb{G}$ (i.e., each component of the complement of $\mathbb{G}$ in Σ) is homeomorphic to an open disk. Thus, the boundary of F can be represented by a single loop in $\mathbb{G}$. To each positively oriented edge e in $\mathbb{G}$ we associate an *edge variable* $x \in K$, and then correspondingly associate x^{-1} to the inverse of e. We then form a measure on K^n, where n is the number of edges, as follows. For each face F of $\mathbb{G}$, we consider the "holonomy" h, which is just the product of edge variables (and their inverses) along the boundary of F. We then consider first an un-normalized measure $\tilde{\mu}$ on K^n, given by

$$d\tilde{\mu}(\mathbf{x}) = \left(\prod_i \rho_{|F_i|}(h_i)\right) d\mathbf{x},$$

where $d\mathbf{x}$ is the product of the *normalized* Haar measures in the edge variables. Note: since the Haar measure on K is symmetric (i.e. invariant under $x \mapsto x^{-1}$), the measure $\tilde{\mu}$ is independent of the choice of which edged in $\mathbb{G}$ are positively oriented.

We consider also the normalized measure

$$d\mu(\mathbf{x}) = \frac{1}{Z} d\tilde{\mu}(\mathbf{x}),$$

where

$$Z = \int_{K^n} \left(\prod_i \rho_{|F_i|}(h_i)\right) d\mathbf{x}$$

is the **partition function** of the graph. This formula for μ is *Sengupta's formula* [Sen1, Theorem 5.3], which he derives from a rigorous version of the usual path-integral formula. (As with $\tilde{\mu}$, μ is independent of which edges are chosen to be positively oriented.) We use the notation $\mathbb{E}$ for the expectation value with respect to the normalized Yang–Mills measure:

$$\mathbb{E}\{f\} := \int_{K^n} f(\mathbf{x})\ d\mu(\mathbf{x}).$$

It is known that the partition function Z depends only on the area and diffeomorphism class of Σ and not on the choice of graph; see Proposition 5.2 in [Sen1]. (For the independence of Z from the graph, it is essential that we use normalized Haar measures in the definition of the un-normalized measure $\tilde{\mu}$.) If, for example, $\Sigma = S^2$, then Z is given by

$$Z_{S^2} = \rho_A(\mathrm{id}),$$

where A is the area of the sphere and id is the identity element of K. In particular, for a fixed diffeomorphism class of surface and fixed topological type of the embedded graph, Z depends only on the *sum* of the areas t_i of the faces of $\mathbb{G}$.

Although the formula for the Yang–Mills measure on a surface is similar to the formula [Dr, Theorem 6.4] in the plane case, the two measures behave differently. In the plane case, the holonomies h_i around the bounded faces of a graph are independent heat-kernel distributed random variables [Lévy3, Proposition 4.4]. For a general compact surface Σ, the h_i's are neither independent nor heat kernel distributed. For the case of a simple closed curve in S^2, for example, we may represent the curve by a graph with a single edge, with edge variable x. The holonomies associated to the two faces of the graph are then $h_1 = x$ and $h_2 = x^{-1}$, so that the Yang–Mills measure for this graph is

$$d\mu(x) = \frac{1}{\rho_{s+t}(\mathrm{id})}\rho_s(x)\rho_t(x^{-1})\ dx,$$

where s and t are the areas of the two faces. (This formula may be interpreted as saying that the holonomy around the loop is distributed as a Brownian bridge at time s, where the bridge returns to the identity at time $s + t$.) Even for this simple example, there is no easy way to compute the expected trace of the holonomy around the loop.

Although the Yang–Mills measure on a surface is more difficult to compute with than the measure on the plane, we will show that two of the proofs of the Makeenko–Migdal equation given in [DHK2] go through essentially without change. To illustrate this point, consider the graph in Figure 3, which we regard as being embedded in S^2. If x_i is the edge variable associated to the edge e_i, and t_j is the area of F_j, the un-normalized Yang–Mills measure takes the form

$$d\tilde{\mu}(\mathbf{x}) = \rho_{t_1}(x_2^{-1}x_1)\rho_{t_2}(x_3^{-1}x_6x_2)\rho_{t_3}(x_4^{-1}x_3)\rho_{t_4}(x_1^{-1}x_5^{-1}x_4)\rho_{t_5}(x_6^{-1}x_5)\ d\mathbf{x}. \quad (6)$$

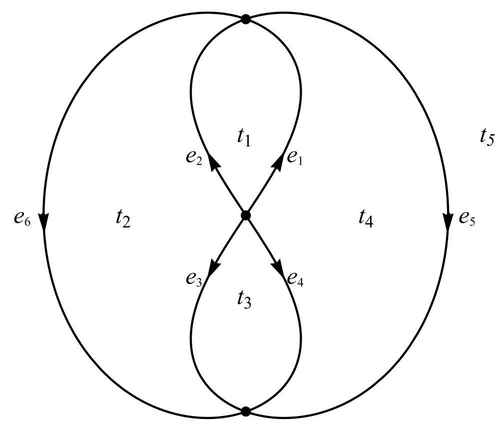

Figure 3: A graph embedded in S^2 with five faces

(Note that the boundary of, say, F_1 is $e_1e_2^{-1}$, but since parallel transport is order-reversing, the holonomy around F_1 is represented as $x_2^{-1}x_1$.)

If the graph were embedded in the plane instead of the sphere, we would simply omit the factor of $\rho_{t_5}(x_6^{-1}x_5)$, since in that case, F_5 would be the unbounded face, which does not contribute to Driver's formula. We see, then, that replacing the plane by some other surface does not change the "local" structure of the un-normalized Yang–Mills measure. If, for example, we wish to establish the Makeenko–Migdal equation for the central vertex in Figure 3, the first two proofs in [DHK2] apply without change, since the "local Makeenko–Migdal equation" in Theorem 6 there can be applied to the integration over the variables $x_1, \ldots, x_4$. (In particular, since our proofs in the plane case make no reference to the unbounded face, the absence of an unbounded face on Σ does not cause a difficulty.) Once the Makeenko–Migdal equation for the un-normalized measure is established, it is then a simple matter to establish it for the normalized measure as well.

2.2 The constrained Yang–Mills measure on a graph

It is possible to modify the construction in the preceding subsection by constraining the holonomy around one or more of the boundary components to lie in a fixed conjugacy class. If the boundary component in question consists of a sequence $e_1, \ldots, e_k$ of edges with edge variables $x_1, \ldots, x_k$, the holonomy around the component will be $x_k x_{k-1} \cdots x_1$, since holonomy is order reversing. (Note that this boundary component will usually not be the boundary of one of the faces of $\mathbb{G}$.) To constrain $x_k x_{k-1} \cdots x_1$ to lie in C, we insert a δ-function $\delta(x_k x_{k-1} \cdots x_1 c^{-1})$ and then integrate over $c \in C$. Thus, integration with respect to the un-normalized constrained measure $\tilde{\mu}$ takes the form

$$\int f(\mathbf{x})\ d\tilde{\mu}(\mathbf{x}) = \int_{K^n}\int_{C_1}\cdots\int_{C_N} f(\mathbf{x})\left(\prod_i \rho_{|F_i|}(h_i)\right) \times \prod_j \delta(x^j_{k_j} x^j_{k_j-1}\cdots x^j_1 c_j^{-1})\ d\mathbf{x}\ d\mathrm{vol}(c_1)\ \cdots\ d\mathrm{vol}(c_N), \qquad (7)$$

where $C_1,\ldots,C_N$ are the conjugacy classes to which various boundary holonomies are constrained and where $d$vol is the normalized, Ad-invariant volume measure on the given conjugacy class. (See Theorem 4 in [Sen3] and compare Section 1.5 of [Lévy1].) In (7), we may interpret $\delta(\cdot)$ as the small-ε limit of $\rho_\varepsilon(\cdot)$. Alternatively, we may think of $\delta(x_k x_{k-1}\cdots x_1 c^{-1})$ as a rule telling us that instead of integrating over x_k, we simply evaluate x_k to $(x_{k-1}\cdots x_1 c^{-1})^{-1}$.

We may then construct a normalized measure by dividing by a normalization constant, which we refer to as the constrained partition function. Similarly to the unconstrained case, the constrained partition function depends only on the area, the topological type of the surface, and the constraints, but not on the choice of graph. (See the formula for $N_T(\mathbf{c})$ in Theorem 4 of [Sen3] and compare Proposition 4.3.5 in [Lévy2] in a more general setting.)

In Figure 4, for example, if the holonomy around the boundary of a disk is constrained to lie in C, the expected trace of the holonomy around the inner loop would be computed as

$$\begin{aligned}&\frac{1}{Z}\int_C\int_{K^3}\mathrm{tr}(x^{-1})\rho_s(x^{-1})\rho_t(y^{-1}zyx)\delta(zc^{-1})\ dx\ dy\ dz\ d\mathrm{vol}(c)\\ &\quad=\frac{1}{Z}\int_C\int_{K^2}\mathrm{tr}(x^{-1})\rho_s(x^{-1})\rho_t(y^{-1}cyx)\ dx\ dy\ d\mathrm{vol}(c)\\ &\quad=\frac{1}{Z}\int_C\int_{K}\mathrm{tr}(x^{-1})\rho_s(x^{-1})\rho_t(cx)\ dx\ d\mathrm{vol}(c),\end{aligned}$$

where in the last expression, we have used the Ad-invariance of $d\mathrm{vol}(c)$ to eliminate the y variable. By contrast, if we left the boundary holonomy unconstrained, we would integrate $\mathrm{tr}(x^{-1})\rho_s(x^{-1})\rho_t(y^{-1}zyx)$ over K^3, in which case the result would simplify to $\int_K \mathrm{tr}(x^{-1})\rho_s(x^{-1})\ dx$ (with no normalization factor necessary).

3 The Makeenko–Migdal equation for surfaces

Throughout this section, we assume $\mathbb{G}$ is an admissible graph in Σ, that is, one containing the boundary of Σ and such that each face of $\mathbb{G}$ is a disk.

3.1 An abstract Makeenko–Migdal equation

Following Lévy [Lévy3, Definition 6.21] for the plane case, we now introduce a natural invariance property that will be crucial in proving the Makeenko–Migdal equation.

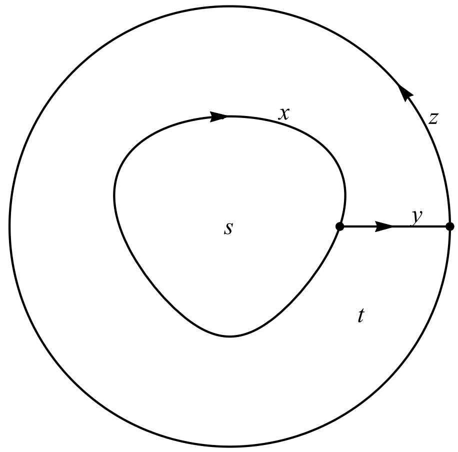

Figure 4: The holonomy z around the boundary of the disk is constrained to lie in C.

Definition 2 *Let $\mathbb{G}$ be an admissible graph in Σ and let v be a vertex of $\mathbb{G}$ in the interior of Σ having exactly four distinct edges, labeled in cyclic order as e_1, e_2, e_3, e_4 and taken to be outgoing edges. Let $f : K^n \to \mathbb{C}$ be a function of the edge variables of $\mathbb{G}$ and let a_1, a_2, a_3, a_4 be the edge variables associated to e_1, e_2, e_3, e_4. Then f has **extended gauge invariance** at v if*

$$f(a_1, a_2, a_3, a_4, \mathbf{b}) = f(a_1 x, a_2, a_3 x, a_4, \mathbf{b}) = f(a_1, a_2 x, a_3, a_4 x, \mathbf{b})$$

for all $x \in K$, where $\mathbf{b}$ is the tuple of all edge variables other than a_1, a_2, a_3, a_4.

With this definition in hand, we may formulate a general version of the Makeenko–Migdal equation for Σ, generalizing Proposition 6.22 in [Lévy3] in the plane case. The result applies to arbitrary structure groups K and to functions that are not necessarily given as the trace of a holonomy. In what follows, we allow the areas of the faces to be arbitrary positive real numbers; if we vary one area with the other areas fixed, we are changing the total area of the surface.

We consider a graph with four distinct edges $e_1, \ldots, e_4$ attached to a vertex v, and we label the four faces surrounding v as $F_1, \ldots, F_4$, as in Figure 5, with the labeling chosen so that e_1 lies between F_4 and F_1.

Theorem 3 (Abstract Makeenko–Migdal Equation for Σ) *Following the notation of Definition 2, assume the four faces and four edges adjacent to v are distinct. Suppose $f : K^n \to \mathbb{C}$ is a smooth function with extended gauge invariance at v. If $t_1, \ldots, t_4$ denote the areas of the faces of $\mathbb{G}$ surrounding v, we have*

$$\left(\frac{\partial}{\partial t_1} - \frac{\partial}{\partial t_2} + \frac{\partial}{\partial t_3} - \frac{\partial}{\partial t_4} \right) \int_{K^n} f \, d\mu = - \int_{K^n} \nabla^{a_1} \cdot \nabla^{a_2} f \, d\mu,$$

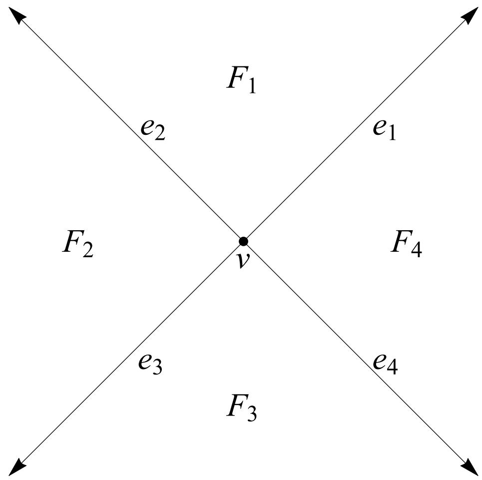

Figure 5: The edges and faces at v

where μ is the normalized Yang–Mills measure, possibly with constraints on the boundary holonomies. The same result holds with μ replaced by the unnormalized measure $\tilde{\mu}$.

Using the arguments in Section 4 of [DHK2], it is possible to prove this result also when the faces are not distinct. It is also possible to formulate and prove a version of the result when the four edges emanating from v are not distinct, although the definition of extended gauge invariance needs some modification in this case. See Section 3.3 for more information.

In the theorem, the gradients are left-invariant gradients with respect to a_1 and a_2 with the other edge variables fixed. Explicitly,

$$(\nabla^{a_1} \cdot \nabla^{a_2} f)(a_1, a_2, a_3, a_4, \mathbf{b}) = \sum_X \frac{\partial^2}{\partial s \partial t} f(a_1 e^{sX}, a_2 e^{tX}, a_3, a_4, \mathbf{b}) \bigg|_{s=t=0},$$

where X ranges over an orthonormal basis for the Lie algebra $\mathfrak{k}$ of K and $\mathbf{b}$ represents the tuple of edge variables other than $a_1, \ldots, a_4$. Using the extended gauge invariance of f, it is easy to show that

$$\nabla^{a_1} \cdot \nabla^{a_2} f = -\nabla^{a_2} \cdot \nabla^{a_3} f = \nabla^{a_3} \cdot \nabla^{a_4} f = -\nabla^{a_4} \cdot \nabla^{a_1} f.$$

Suppose L is a closed curve traced out in $\mathbb{G}$ that has a crossing at v. Specifically, assume L starts at v, leaves v along e_1, returns to v along e_4^{-1}, leaves v again along e_2, and then finally returns to v along e_3^{-1} (with no visits to v besides those just mentioned). Then since holonomy is order-reversing, we will have

$$\mathrm{tr}(\mathrm{hol}(L)) = \mathrm{tr}(a_3^{-1} \alpha a_2 a_4^{-1} \beta a_1),$$

where α and β are words in the **b** variables. Any function of the this form is easily seen to have extended gauge invariance. If $K = U(N)$, we compute that

$$\begin{aligned}\nabla^{a_1} \cdot \nabla^{a_2}[\mathrm{tr}(a_3^{-1}\alpha a_2 a_4^{-1}\beta a_1)] &= \sum_X \mathrm{tr}(a_3^{-1}\alpha a_2 X a_4^{-1}\beta a_1 X) \\ &= -\mathrm{tr}(a_3^{-1}\alpha a_2)\mathrm{tr}(a_4^{-1}\beta a_1) \\ &= -\mathrm{tr}(L_2)\mathrm{tr}(L_1),\end{aligned}$$

where L_1 and L_2 are as in Theorem 1, and where we used the elementary identity $\sum_X XCX = -\mathrm{tr}(C)I$ (e.g., [DHK1, Proposition 3.1]) in the second equality. This calculation shows that the abstract Makeenko–Migdal equation implies the Makeenko–Migdal equation for $U(N)$ (Theorem 1).

3.2 The generic case

Let us assume at first that our loop is traced out in an admissible graph $\mathbb{G}$ and that the vertex v is generic, meaning that the edges $e_1, \ldots, e_4$ and the faces $F_1, \ldots, F_4$ are distinct. We then make use of the following result, which was proven in [DHK2, Theorem 6].

Theorem 4 (Local Abstract Makeenko–Migdal Equation) *Suppose $f : K^4 \to \mathbb{C}$ is a smooth function satisfying the following "extended gauge invariance" property:*

$$f(a_1, a_2, a_3, a_4) = f(a_1 x, a_2, a_3 x, a_4) = f(a_1, a_2 x, a_3, a_4 x)$$

for all $\mathbf{a} = (a_1, a_2, a_3, a_4)$ in K^4 and all x in K. For each fixed $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ in K^4 and $\mathbf{t} = (t_1, t_2, t_3, t_4)$ in $(\mathbb{R}^+)^4$, define a measure $\mu_{\alpha,\mathbf{t}}$ on K^4 by

$$d\mu_{\alpha,\mathbf{t}}(\mathbf{a}) = \rho_{t_1}(a_2^{-1}\alpha_1 a_1)\rho_{t_2}(a_3^{-1}\alpha_2 a_2)\rho_{t_3}(a_4^{-1}\alpha_3 a_3)\rho_{t_4}(a_1^{-1}\alpha_4 a_4)\ d\mathbf{a},$$

where $d\mathbf{a}$ is the normalized Haar measure on K^4. Then for all $\alpha \in K^4$, we have

$$\left(\frac{\partial}{\partial t_1} - \frac{\partial}{\partial t_2} + \frac{\partial}{\partial t_3} - \frac{\partial}{\partial t_4}\right)\int_{K^4} f\ d\mu_{\alpha,\mathbf{t}} = -\int_{K^4} \nabla^{a_1} \cdot \nabla^{a_2} f\ d\mu_{\alpha,\mathbf{t}}.$$

We now come to the proof of the abstract Makeenko–Migdal equation in Theorem 3, in the generic case where the four edges $e_1, \ldots, e_4$ and the four faces $F_1, \ldots, F_4$ are distinct.

Proof of Theorem 3. Let **b** denote the tuple of all edge variables other than $a_1, \ldots, a_4$. The holonomies around the adjoining faces F_i, $i = 1, \ldots, 4$, will have the form

$$h_i = a_{i+1}^{-1}\alpha_i a_i,$$

where α_i is a word in the **b** variables. Let us first consider integration with respect to the un-normalized Yang–Mills measure, $\tilde{\mu}$ with or without constraints on boundary holonomies. Since v lies in the interior of Σ, the edges $e_1, \ldots, e_4$ do

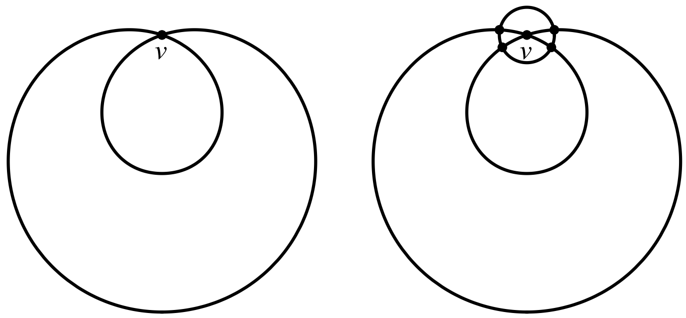

Figure 6: A graph that is non-generic at v (left) and its generic counterpart

not lie on the boundary. Thus, the holonomy around any boundary component will involve only the $\mathbf{b}$ variables. Integration with respect to $\tilde{\mu}$ therefore takes the form of integration over $a_1, \ldots, a_4$ with respect to $\mu_{\alpha,\mathbf{t}}$, where α is a function of the $\mathbf{b}$ variables, followed by integration in the $\mathbf{b}$ variables and possibly another layer of integration with respect to the constraint variables c_j. In the un-normalized measure $\tilde{\mu}$, the only dependence on $t_1, \ldots, t_4$ is in the inner layer of integration. Thus, we may push the time derivatives inside the outer layers of integration and allow them to hit on the integral over K^4. If we then apply the local result in Theorem 4, Theorem 3 for $\tilde{\mu}$ will follow.

For the normalized measure, we must incorporate the partition function Z. Since Z depends only on the total area of the surface (i.e., the sum the areas of all the faces), we see that

$$\left(\frac{\partial}{\partial t_1} - \frac{\partial}{\partial t_2} + \frac{\partial}{\partial t_3} - \frac{\partial}{\partial t_4} \right) Z = 0.$$

Thus, Theorem 3 for the normalized measure easily follows from the corresponding result for the un-normalized measure. ■

3.3 The nongeneric case

Suppose $\mathbb{G}$ is an admissible graph and v is a vertex of $\mathbb{G}$ having four attached edges, where we count an edge twice if both ends of the edge are attached to v. We say that $\mathbb{G}$ is **nongeneric** at v if either the four edges are not distinct or the four faces surrounding v are not distinct. If $\mathbb{G}$ is not generic at v, we can embed $\mathbb{G}$ into another admissible graph $\mathbb{G}'$ that is generic at v, as in Figure 6. If L is a loop traced out on $\mathbb{G}$ with a simple crossing at v, then "the same" loop can also be traced out on $\mathbb{G}'$. In that case, the expectation values of $\mathrm{tr}(\mathrm{hol}(L))$ and of $\mathrm{tr}(\mathrm{hol}(L_1))\mathrm{tr}(\mathrm{hol}(L_2))$—where L_1 and L_2 are as in Theorem 1—are the same whether we work over $\mathbb{G}$ or over $\mathbb{G}'$. This invariance result has two parts. First, there is invariance under subdividing an edge by adding a vertex in the middle of that edge, which is very easy to establish, as shown in Section 4.1 of [DHK2].

(The argument given there applies equally well in the surface case or the plane case.) Second, there is invariance under keeping the vertex set the same and adding a new edge. This invariance result is a consequence of Proposition 4.3.4 in [Lévy2], in the case that the Lévy process there is taken to be Brownian motion on K.

Furthermore, it is not hard to see that area derivatives of expectation values give the same result whether computed over $\mathbb{G}$ or $\mathbb{G}'$. (See Section 4.3 of [DHK2].) Thus, the $U(N)$ version of the Makeenko–Migdal equation for the loop in $\mathbb{G}$ reduces to the corresponding result for the loop in $\mathbb{G}'$, which in turn follows from Theorem 3. In the graph on the left-hand side of Figure 6, for example, F_1 coincides with F_3. Thus, t_3 is just another name for t_1 and the Makeenko–Migdal equation may be written as

$$\left(2\frac{\partial}{\partial t_1} - \frac{\partial}{\partial t_2} - \frac{\partial}{\partial t_4}\right)\mathbb{E}\{\mathrm{tr}(\mathrm{hol}(L))\} = \mathbb{E}\{\mathrm{tr}(\mathrm{hol}(L_1))\mathrm{tr}(\mathrm{hol}(L_2))\}.$$

It is also possible to formulate a version of Theorem 3 itself that holds in the nongeneric situation. If the edges $e_1, \ldots, e_4$ are distinct but the faces $F_1, \ldots, F_4$ are not distinct, Theorem 3 holds with no changes to the statement, and the arguments in Section 4 of [DHK2] show how this result can be reduced to the generic case. If the edges (and possibly also faces) are not distinct, the notion of extended gauge invariance needs some revision [DHK2, Section 4.2], after which one can reduce the result to the generic case. Since this process of reduction requires no changes from the arguments in [DHK2], we do not enter into the details here, but refer the interested reader to Sections 4.2 and 4.3 of [DHK2].

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