THE COMPLEX-TIME SEGAL–BARGMANN TRANSFORM

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ABSTRACT. We introduce a new form of the Segal–Bargmann transform for a Lie group K of compact type. We show that the heat kernel $(\rho_t(x))_{t>0,x\in K}$ has a space-time analytic continuation to a holomorphic function $(\rho_{\mathbb{C}}(\tau,z))_{\mathrm{Re}\,\tau>0,z\in K_{\mathbb{C}}}$, where $K_{\mathbb{C}}$ is the complexification of K. The new transform is defined by the integral

$$(B_\tau f)(z) = \int_K \rho_{\mathbb{C}}(\tau, zk^{-1}) f(k)\, dk, \quad z \in K_{\mathbb{C}}.$$

If $s > 0$ and $\tau \in \mathbb{D}(s,s)$ (the disk of radius s centered at s), this integral defines a holomorphic function on $K_{\mathbb{C}}$ for each $f \in L^2(K,\rho_s)$. We construct a heat kernel density $\mu_{s,\tau}$ on $K_{\mathbb{C}}$ such that, for all s,τ as above, $B_{s,\tau} := B_\tau|_{L^2(K,\rho_s)}$ is an isometric isomorphism from $L^2(K,\rho_s)$ onto the space of holomorphic functions in $L^2(K_{\mathbb{C}},\mu_{s,\tau})$. When $\tau = t = s$, the transform $B_{t,t}$ coincides with the one introduced by the second author for compact groups and extended by the first author to groups of compact type. When $\tau = t \in (0,2s)$, the transform $B_{s,t}$ coincides with the one introduced by the first two authors.

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1. INTRODUCTION

1.1. **The Classical Segal–Bargmann Transform.** This paper concerns a generalization of the Segal–Bargmann transform over compact-type Lie groups, to allow the time parameter of the transform to be complex. We begin by briefly discussing the history of the transform. For $t > 0$ and $d \in \mathbb{N}$, let ρ_t denote the variance-t Gaussian density on $\mathbb{R}^d$: $\rho_t(x) = (2\pi t)^{-d/2} \exp(-|x|^2/2t)$. This is the *heat kernel* on $\mathbb{R}^d$: the solution u of the heat equation $\partial_t u = \frac{1}{2}\Delta u$ with (sufficiently integrable) initial condition f is given in terms of ρ_t by

$$u(t,x) = \int_{\mathbb{R}^d} \rho_t(y) f(y+x)\, dy. \tag{1.1}$$

Alternatively using $\rho_t(-x) = \rho_t(x)$ and a change of variables, we have

$$u(t,x) = (\rho_t * f)(x) = \int_{\mathbb{R}^d} \rho_t(x-y) f(y)\, dy. \tag{1.2}$$

The function ρ_t admits an explicit entire analytic continuation to $\mathbb{C}^d$, which we call $(\rho_t)_{\mathbb{C}}$: it is simply the function

$$(\rho_t)_{\mathbb{C}}(z) = (2\pi t)^{-d/2} \exp\left(-\frac{z \cdot z}{2t}\right) = (2\pi t)^{-d/2} \exp\left(\frac{1}{2t} \sum_{j=1}^d z_j^2\right).$$

If $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and of sufficiently slow growth, then the integral

$$(B_t f)(z) := \int_{\mathbb{R}^d} (\rho_t)_{\mathbb{C}}(z-y) f(y)\, dy \tag{1.3}$$

converges and defines an entire holomorphic function on $\mathbb{C}^d$.

The map $f \mapsto B_t f$ is equivalent to the **Segal–Bargmann transform**, invented and explored by the eponymous authors of [1, 2, 38, 39, 40]. Note that neither Segal nor Bargmann explicitly connected the transform to the heat kernel, nor did they write the transform precisely as in (1.3). Nevertheless, their transforms can easily be rewritten in the form (1.3) by simple changes of variable; cf. [20].

We consider also the heat kernel on $\mathbb{C}^d \cong \mathbb{R}^{2d}$ (with rescaled variance), which we refer to as μ_t:

$$\mu_t(z) = (\pi t)^{-d} \exp(-|z|^2/t).$$

(Note that the real, positive function μ_t on $\mathbb{C}^d$ is not the same as the holomorphic function $(\rho_t)_{\mathbb{C}}$.) The main theorem is that B_t is an isometric isomorphism from $L^2(\mathbb{R}^d, \rho_t)$ onto $\mathcal{H}L^2(\mathbb{C}^d, \mu_t)$ — the reproducing kernel Hilbert space of holomorphic functions in $L^2(\mathbb{C}^d, \mu_t)$. For more information about the classical Segal–Bargmann transform, see, for example, [20, 24].

1.2. **The Segal–Bargmann Transform for Lie Groups of Compact Type.** In [18], the second author introduced an analog of the Segal–Bargmann transform on an arbitrary compact Lie group. Then, in [6], the first author extended the results of [18] to a Lie group K of compact type (cf. Section 2.2), a class that includes both compact groups and $\mathbb{R}^d$. The idea of [18] and [6] is the same as in the $\mathbb{R}^d$ case: the heat kernel ρ_t on K (cf. (1.5) and Theorem 3.12) has an entire analytic continuation $(\rho_t)_{\mathbb{C}}$ to the *complexification* $K_{\mathbb{C}}$ of K (cf. Section 2.1). The transform B_t is defined by the group convolution formula generalizing (1.3):

$$(B_t f)(z) = \int_K (\rho_t)_{\mathbb{C}}(zk^{-1}) f(k)\, dk. \tag{1.4}$$

The theorem is that B_t is an isometric isomorphism from $L^2(K, \rho_t)$ onto the holomorphic space $\mathcal{H}L^2(K_{\mathbb{C}}, \mu_t)$, where μ_t is the (time-rescaled) heat kernel on $K_{\mathbb{C}}$. If $K = \mathbb{R}^d$, then B_t is precisely the classical Segal–Bargmann transform of Section 1.1.

Later, in [10, 19], the authors made a further generalization related to the time parameter t. One can use a different time $s \neq t$ to measure the functions f in the domain, while still using the analytically continued heat kernel at time t to define the transform, as in (1.4). The resulting map,

$$B_{s,t} \colon L^2(K, \rho_s) \to \mathcal{H}L^2(K_{\mathbb{C}}, \mu_{s,t})$$

is still an isometric isomorphism for an appropriate two-parameter heat kernel density $\mu_{s,t}$, provided $0 < t < 2s$. Note that the formula for the transform $B_{s,t}$ does not depend on s; this parameter only indicates the inner product to be used on the domain and range spaces. In the special case that $K = \mathbb{R}^d$, the two-parameter heat kernel density $\mu_{s,t}$ in the range is a Gaussian measure with different variances in the real and imaginary directions. (Take $u = 0$ in (1.17) below.)

Remark 1.1. For a complex manifold M, let $\mathcal{H}(M)$ denote the space of holomorphic functions on M. If μ is a measure on M having a strictly positive, continuous density with respect to the Lebesgue measure in each holomorphic local coordinate system, it is not hard to show that $\mathcal{H}L^2(M, \mu) := \mathcal{H}(M) \cap L^2(M, \mu)$ is a closed subspace of $L^2(M, \mu)$ and is therefore a Hilbert space. Furthermore, the *pointwise evaluation map* $F \mapsto F(z)$ is continuous for each $z \in M$, and the norm of this functional is locally bounded as a function of z. (See, for example, Theorem 3.2 and Corollary 3.3 in [7] or Theorem 2.2 in [20].)

1.3. **The Complex-Time Segal–Bargmann Transform.** The topic of the present paper is a new generalization that modifies the transform $B_{s,t}$ as well; in particular, we show that the time parameter t can also be extended into the complex plane, and there is still an isomorphism between real and holomorphic L^2 spaces of associated heat kernel measures. This generalization is natural and, in a certain sense, a completion of Segal–Bargmann transform theory, as explained below. (See Theorem 4.2.)

Let K be a compact-type Lie group with Lie algebra $\mathfrak{k}$, and fix an $\mathrm{Ad}(K)$-invariant inner product $\langle\, ,\, \rangle_{\mathfrak{k}}$ on $\mathfrak{k}$ (cf. Section 2.2). This induces a bi-invariant Riemannian metric on K, and an associated Laplace operator Δ_K with $\mathcal{D}(\Delta_K) = C_c^\infty(K)$ (see Definition 3.20), which is bi-invariant, elliptic, and essentially self-adjoint in $L^2(K)$ with respect to any right Haar measure (see Section 3.2 for precise statements and proofs of these properties). There is an associated **heat kernel**, $\rho_t \in C^\infty(K, (0, \infty))$, satisfying

$$\left(e^{\frac{t}{2}\bar{\Delta}_K} f\right)(x) = \int_K \rho_t(k) f(xk)\, dk \qquad \text{for all } f \in L^2(K) \text{ and } t > 0. \tag{1.5}$$

(See Definition 3.22.) The key properties of the heat kernel ρ_t are described in Theorem 3.12 and Corollaries 3.28 and 3.29.

Our first theorem is that the heat kernel can be complexified in both space and time.

Theorem 1.2. *Let K be a connected Lie group of compact type, with a given $\mathrm{Ad}(K)$-invariant inner product on its Lie algebra $\mathfrak{k}$, and let $(\rho_t)_{t>0}$ be the associated heat kernel. Let $\mathbb{C}_+$ denote the right half-plane $\{\tau = t + iu : t > 0, u \in \mathbb{R}\}$. There is a unique holomorphic function*

$$\rho_{\mathbb{C}} : \mathbb{C}_+ \times K_{\mathbb{C}} \to \mathbb{C}$$

such that $\rho_{\mathbb{C}}(t, x) = \rho_t(x)$ for all $t > 0$ and $x \in K \subset K_{\mathbb{C}}$.

Theorem 1.2 is proved in Section 5, as part of Theorem 5.13.

Following the pattern described above, for each $\tau \in \mathbb{C}_+$ and "reasonable" function $f : K \to \mathbb{C}$, we would like to define $(B_\tau f)(z)$ to be the analytic continuation of the function $\mathbb{R}_+ \times K \ni (t, x) \to (e^{\frac{t}{2}\bar{\Delta}_K} f)(x) \in \mathbb{C}$. In order to carry out the analytic continuation, we use the $\mathrm{Ad}(K)$-invariance assumption to rewrite (1.5) as

$$\left(e^{\frac{t}{2}\bar{\Delta}_K} f\right)(x) = \int_K \rho_t(xk^{-1}) f(k)\, dk; \tag{1.6}$$

see Section 3.5 and in particular Corollary 3.29 below. This equation along with Theorem 1.2 motivates the following notation.

Notation 1.3 (Complex-time Segal–Bargmann transform)**.** *For $\tau \in \mathbb{C}_+$ and $z \in K_{\mathbb{C}}$, define*

$$\left(B_\tau f\right)(z) := \int_K \rho_{\mathbb{C}}(\tau, zk^{-1}) f(k)\, dk \qquad \text{for } z \in K_{\mathbb{C}} \tag{1.7}$$

for all measurable functions $f : K \to \mathbb{C}$ satisfying

$$\int_K \left|\rho_{\mathbb{C}}(\tau, zk^{-1}) f(k)\right| dk < \infty. \tag{1.8}$$

Further let $\mathcal{D}(B_\tau)$ denote the vector space of measurable functions $f : K \to \mathbb{C}$ such that (1.8) *holds for all $z \in K_{\mathbb{C}}$ and such that $B_\tau f \in \mathcal{H}(K_{\mathbb{C}})$.*

As defined, $\mathcal{D}(B_\tau)$ is a linear subspace of the measurable $\mathbb{C}$-valued functions on K, and $B_\tau : \mathcal{D}(B_\tau) \to \mathcal{H}(K_{\mathbb{C}})$ is a linear map. The main theorem of this paper (Theorem 1.6) identifies L^2-Hilbert subspaces of $\mathcal{D}(B_\tau)$ and $\mathcal{H}(K_{\mathbb{C}})$ which are unitarily equivalent to one another under the action of B_τ. To describe the relevant subspaces of $\mathcal{H}(K_{\mathbb{C}})$ we need a little more notation.

Definition 1.4. *Let $s > 0$ and $\tau = t + iu \in \mathbb{C}$. The (s, τ)-**Laplacian** $\Delta_{s,\tau}$ on $K_{\mathbb{C}}$ is the left-invariant differential operator*

$$\Delta_{s,\tau} = \sum_{j=1}^{d} \left[\left(s - \frac{t}{2}\right) \widetilde{X}_j^2 + \frac{t}{2} \widetilde{Y}_j^2 - u\, \widetilde{X}_j \widetilde{Y}_j\right] \tag{1.9}$$

where $\{X_j\}_{j=1}^d$ is any orthonormal basis of $\mathfrak{k}$, and $Y_j = JX_j$ where J is operation of multiplication by i on $\mathfrak{k}_{\mathbb{C}} = \mathrm{Lie}(K_{\mathbb{C}})$. Here $\widetilde{X}$ is the left-invariant vector field on $K_{\mathbb{C}}$ whose value at the identity is X.

Remark 1.5. Given $s > 0$ and $\tau = t + iu \in \mathbb{C}_+$, from (1.9), it is not difficult to show that the operator $\Delta_{s,\tau}$ is elliptic if and only if

$$\alpha\,(s, \tau) := \det \begin{bmatrix} s - t/2 & -u/2 \\ -u/2 & t/2 \end{bmatrix} = \frac{1}{4}(2st - t^2 - u^2) > 0. \tag{1.10}$$

This can be written equivalently as

$$2s > t + u^2/t \tag{1.11}$$

or, more succinctly, as $\tau \in \mathbb{D}(s, s)$ (the disk of radius s, centered at s). Further notice that $\mathbb{D}(s, s) \uparrow \mathbb{C}_+$ as $s \uparrow \infty$.

If one (and hence all) of the conditions in Remark 1.5 hold, then (by Theorems 3.7 and 3.12 below) $\Delta_{s,\tau}$ with $\mathcal{D}(\Delta_{s,\tau}) := C_c^\infty(K_\mathbb{C})$ is essentially self-adjoint on $L^2(K_\mathbb{C})$ (with respect to any right Haar measure), and there exists a heat kernel density $\mu_{s,\tau} \in C^\infty(K_\mathbb{C}, (0, \infty))$ such that

$$\left(e^{\frac{1}{2}\bar{\Delta}_{s,\tau}} f\right)(w) = \int_{K_\mathbb{C}} \mu_{s,\tau}(z)\, f(wz)\, dz \quad \text{for all } \ f \in L^2(K_\mathbb{C}).$$

We are now prepared to state the main theorem of this paper.

Theorem 1.6 (Complex-time Segal–Bargmann transform)**.** *Let K be a connected, compact-type Lie group. For $s > 0$ and $\tau \in \mathbb{D}(s, s)$, $L^2(K, \rho_s) \subset \mathcal{D}(B_\tau)$; i.e., $B_\tau f$ is holomorphic on $K_\mathbb{C}$ for each $f \in L^2(K, \rho_s)$. The image of B_τ on this domain is $B_\tau\left(L^2(K, \rho_s)\right) = \mathcal{H}L^2(K_\mathbb{C}, \mu_{s,\tau})$. Moreover,*

$$B_{s,\tau} := B_\tau|_{L^2(K,\rho_s)}$$

is a unitary isomorphism from $L^2(K, \rho_s)$ onto $\mathcal{H}L^2(K_\mathbb{C}, \mu_{s,\tau})$.

Theorem 1.6 is proved in Section 5. The $\tau = t \in \mathbb{R}$ case of Theorem 1.9 was established in [10, Theorem 5.3]. (See also [19, Theorem 2.1].)

Remark 1.7. The condition in [10, 19] for the two-parameter Segal–Bargmann transform $B_{s,t}$ to be a well-defined unitary map was $t > 0$ and $s > t/2$, or equivalently $t \in (0, 2s)$. It is therefore natural that, in complexifying t to τ, the optimal condition is that $\tau \in \mathbb{D}(s, s)$, whose intersection with $\mathbb{R}$ is the interval $(0, 2s)$.

In the case that the group K is compact, there is a limiting $s \to \infty$ variant (Theorem 1.9) of Theorem 1.6. To state this variant, as in [18], we first introduce a one parameter family of "K-averaged heat kernels."

Definition 1.8. *For $t > 0$, define the K**-averaged heat kernel** ν_t on $K_\mathbb{C}$ by*

$$\nu_t(z) = \int_K \mu_{t,t}(zk)\, dk \quad \textit{for all } \ z \in K_\mathbb{C}$$

where dk denotes the Haar probability measure on K.

In fact, one can replace $\mu_{t,t}$ by $\mu_{s,\tau}$ for any $\tau \in \mathbb{D}(s, s)$ in the above integral, and the resulting K-averaged density ν_t is the same: it only depends on $t = \operatorname{Re}\tau$; cf. Lemma 4.11 below.

Theorem 1.9 (Large-s limit)**.** *For all $s > 0$ and $\tau = t + iu \in \mathbb{D}(s, s)$, we have $L^2(K) = L^2(K, \rho_s)$ and $L^2(K_\mathbb{C}, \mu_{s,\tau}) = L^2(K_\mathbb{C}, \nu_t)$ (equalities as sets). Furthermore, for all $f \in L^2(K)$ and all $F \in L^2(K_\mathbb{C}, \nu_t)$, we have*

$$\lim_{s\to\infty} \|f\|_{L^2(K,\rho_s)} = \|f\|_{L^2(K)}$$
$$\lim_{s\to\infty} \|F\|_{L^2(K_\mathbb{C},\mu_{s,\tau})} = \|F\|_{L^2(K_\mathbb{C},\nu_t)}\,.$$

It follows that $B_{\infty,\tau} := B_\tau|_{L^2(K)}$ is a unitary isomorphism from $L^2(K)$ onto $\mathcal{H}L^2(K_\mathbb{C}, \nu_t)$.

This theorem is proved in Section 5.3 below.

Remark 1.10. The unitarity of the map $B_{\infty,\tau}$ was previously established in [14, Prop. 2.3]. Indeed, this unitarity result follows easily from the unitarity of the "C-version" Segal–Bargmann transform in [18] and the unitarity of the Schrödinger operator $e^{iu\Delta/2}$: $L^2(K) \to L^2(K)$. The significance of Theorem 1.9 is that the unitary map $B_{\infty,\tau}$ is, in a strong sense, the $s \to \infty$ limit of the unitary map $B_{s,\tau}$.

1.4. **An Outline of the Proof.** We now give a heuristic proof of the isometricity portion of Theorem 1.6, in the Euclidean case $K = \mathbb{R}^d$, for motivation. The argument is a generalization of the method used in the appendix of [21]. By (1.7), if we restrict to real time $\tau = t > 0$ and look at the transform $(B_{s,t}f)(x)$ at a point $x \in \mathbb{R}^d$, we simply have $(B_{s,t}f)(x) = \int_{\mathbb{R}^d} \rho_t(x-y)f(y)\,dy$; in other words, restricted to real time and K, $B_{s,t}f$ is just the heat operator applied to f, $B_{s,t}f = e^{\frac{t}{2}\Delta}f$ where Δ is the standard Laplacian on $\mathbb{R}^d$ (cf. (3.18)). Therefore, in general the transform can be described as "apply the heat operator, then analytically continue in space and time". But if the function f itself already possesses a holomorphic extension $f_{\mathbb{C}}$ to all of $\mathbb{C}^d$ (e.g., if f is a polynomial), then at least informally we should have

$$B_{s,\tau}f = e^{\frac{\tau}{2}\Delta}f_{\mathbb{C}},$$

where now Δ (the sum of squares of the $\mathbb{R}^d$-derivatives) is acting on functions on $\mathbb{C}^d$.

Let $F = B_{s,\tau}f$; we need to compute $|F|^2 = F\bar{F}$. Since $f_{\mathbb{C}}$ is holomorphic, we have $\frac{\partial}{\partial x_j}f_{\mathbb{C}} = \frac{\partial}{\partial z_j}f_{\mathbb{C}}$, and so $\Delta f_{\mathbb{C}} = \sum_{j=1}^d \frac{\partial^2}{\partial z_j^2}f_{\mathbb{C}} := \partial^2 f_{\mathbb{C}}$; similarly $\Delta \bar{f}_{\mathbb{C}} = \sum_{j=1}^d \frac{\partial^2}{\partial \bar{z}_j^2}\bar{f}_{\mathbb{C}} := \bar{\partial}^2\bar{f}_{\mathbb{C}}$. Again, since $f_{\mathbb{C}}$ is holomorphic and $\bar{f}_{\mathbb{C}}$ is antiholomorphic, $\partial^2\bar{f}_{\mathbb{C}} = 0 = \bar{\partial}^2 f_{\mathbb{C}}$; so we have

$$(F\bar{F}) = (e^{\frac{\tau}{2}\partial^2}f_{\mathbb{C}})(e^{\frac{\bar{\tau}}{2}\bar{\partial}^2}\bar{f}_{\mathbb{C}}) = e^{(\frac{\tau}{2}\partial^2+\frac{\bar{\tau}}{2}\bar{\partial}^2)}f_{\mathbb{C}}\bar{f}_{\mathbb{C}}. \tag{1.12}$$

Now, we measure f in $L^2(\mathbb{R}^d,\rho_s)$; setting $x=0$ in the (additive-form of) (1.5) defining the heat operator, we can compute

$$\|f\|^2_{L^2(\mathbb{R}^d,\rho_s)} = \int_{\mathbb{R}^d}\rho_s(y)|f(y)|^2\,dy = \left(e^{\frac{s}{2}\Delta}|f|^2\right)(0) = \left(e^{\frac{s}{2}\Delta}|f_{\mathbb{C}}|^2\right)(0). \tag{1.13}$$

Similarly, we measure F in $L^2(\mathbb{C}^d,\mu_{s,\tau})$, meaning

$$\|F\|^2_{L^2(\mathbb{C}^d,\mu_{s,\tau})} = \left(e^{\frac{1}{2}\Delta_{s,\tau}}|F|^2\right)(0). \tag{1.14}$$

Combining (1.12) and (1.14), and commuting partial derivatives to combine the exponentials, we therefore have

$$\|B_{s,\tau}f\|^2_{L^2(\mathbb{C}^d,\mu_{s,\tau})} = \left(e^{\frac{1}{2}\Delta_{s,\tau}+\frac{\tau}{2}\partial^2+\frac{\bar{\tau}}{2}\bar{\partial}^2}|f_{\mathbb{C}}|^2\right)(0). \tag{1.15}$$

Comparing (1.13) with (1.15), we see that to prove the isometry in Theorem 1.6, it suffices to have

$$s\Delta = \Delta_{s,\tau} + \tau\partial^2 + \bar{\tau}\bar{\partial}^2.$$

Expressing the operators ∂^2 and $\bar{\partial}^2$ in terms of real partial derivatives, we can then solve for $\Delta_{s,\tau}$; this is how (1.9) arises. In the present Euclidean setting, we have

$$\Delta_{s,\tau} = \sum_{j=1}^d \left[\left(s-\frac{t}{2}\right)\frac{\partial^2}{\partial x_j^2} + \frac{t}{2}\frac{\partial^2}{\partial y_j^2} - u\frac{\partial^2}{\partial x_j\partial y_j}\right]. \tag{1.16}$$

As in Remark 1.5, it is easily verified that $\Delta_{s,\tau}$ is elliptic precisely when $\tau \in \mathbb{D}(s,s)$. Moreover, by a standard Fourier transform argument, one shows $e^{\frac{1}{2}\bar{\Delta}_{s,\tau}}f = f * \mu_{s,\tau}$ where

$$\mu_{s,\tau}(z) = (2\pi\sqrt{\alpha})^{-d}\left(-\frac{t/2}{2\alpha}|x|^2 - \frac{s-t/2}{2\alpha}|y|^2 - \frac{u}{2\alpha}x\cdot y\right), \tag{1.17}$$

where $z = x + iy \in \mathbb{R}^d + i\mathbb{R}^d = \mathbb{C}^d$, and $\alpha := \alpha(s, \tau)$ as in Eq. (1.10).

When $u = 0$, the density $\mu_{s,\tau}$ becomes a product of a Gaussian in the x variable and a Gaussian in the y variable, but with typically unequal variances. If $u = 0$ and $s = t$, the formula for $\mu_{s,\tau}$ reduces to

$$\mu_{t,t}(z) = (\pi t)^{-d} e^{-|z|^2/t},$$

which is the density for the standard Segal–Bargmann space over $\mathbb{C}^d$.

For a general Lie group K of compact type, we replace the partial derivatives in the preceding argument with left-invariant vector fields. The heuristic argument then goes through unchanged, *except* that we must remember that left-invariant vector fields do not, in general, commute. Thus, we must also verify that the particular operators involved in the calculation do, in fact, commute, allowing us to combine the exponents as above. For this, we need to use an inner product on the Lie algebra of K that is Ad-invariant; this is the reason for the assumption that K be of compact type.

Most of this paper is devoted to making the above argument rigorous. The key is to introduce a dense subspace (consisting of matrix entries; cf. Section 3.3) of the domain Hilbert space on which integration against the heat kernel can be computed rigorously by a power series in the relevant Laplacian. This argument can be found in Section 5, with the necessary background about heat kernel analysis on Lie groups in Section 3, and the analysis of the heat kernel $\mu_{s,\tau}$ and its generator $\Delta_{s,\tau}$ in Section 4.

The operator $\Delta_{s,\tau}$ was the starting point for the current investigation. It is the Laplacian for a left-invariant Riemannian metric on $K_{\mathbb{C}}$ for which the corresponding inner product on the Lie algebra is invariant under the Adjoint action of K. While the Lie algebra of the complexified Lie group $K_{\mathbb{C}}$ does not possess a fully Ad-invariant inner product, it does possess many inner products that are invariant under the adjoint action of K. These are the most natural from the perspective of diffusion processes, particularly in high dimension (cf. [29]). In fact, there is a natural three (real) parameter family of $\mathrm{Ad}(K)$-invariant inner products on $\mathrm{Lie}(K_{\mathbb{C}})$ (see (4.6) for the relation to the Segal–Bargmann transform parameters s and $\tau = t + iu$). In the case that K is simple, this is a complete characterization of all such invariant inner products; this is the statement of Theorem 4.2 below. It was this fact that led the authors backward to discover the complex-time Segal–Bargmann transform, which is therefore a natural completion of the versions of the transform previously introduced by Segal, Bargmann, and the first two authors of the present paper.

1.5. **Motivation.** The Segal–Bargmann transform $(B_\tau f)(z)$ is computed by integration of f against the function

$$\chi_\tau^z(x) := \rho_{\mathbb{C}}(\tau, x^{-1}z). \tag{1.18}$$

These functions may be thought of as "coherent states" on K. In the $\mathbb{R}^1$ case, coherent states are often defined as minimum uncertainty states, namely those giving equality in the classic Heisenberg uncertainty principle. There is, however, a stronger form of the uncertainty principle, due to Schrödinger [37], which says that

$$(\Delta_\chi X)^2 (\Delta_\chi P)^2 \geq \frac{\hbar^2}{4} + |\mathrm{Cov}_\chi(X, P)|^2, \tag{1.19}$$

where $\Delta_\chi X$ is the uncertainty of the observable X in state χ, and

$$\mathrm{Cov}_\chi(X, P) := \langle (XP + PX)/2 \rangle_\chi - \langle X \rangle_\chi \langle P \rangle_\chi$$

is the quantum covariance. (The classic Heisenberg principle omits the covariance term on the right-hand side of (1.19).)

States that give equality in (1.19) are Gaussian wave packets, but where the quadratic term in the exponent can be complex, as follows:

$$\chi(x) = C\exp\{iax^2 - b(x-c)^2 + idx\} \tag{1.20}$$

with $a, b, c, d \in \mathbb{R}$ and $b > 0$. This class of states is actually more natural than the usual ones with $a = 0$, because the collection of states of the form (1.20) is invariant under the metaplectic representation; that is, the natural (projective) unitary action of the group of symplectic linear transformations of $\mathbb{R}^2$.

If we specialize the states in (1.18) to the $\mathbb{R}^d$ case, we find that they are Gaussian wave packets, and that if $\operatorname{Im}\tau \neq 0$ then the quadratic part of the exponent is complex. We see, then, that allowing the time-parameter in the Segal–Bargmann transform to be complex amounts to considering a larger and more natural family of coherent states.

In the $s \to \infty$ version of the transform $B_{\infty,t+iu}$ of Theorem 1.9, the domain Hilbert space is $L^2(K)$. Since $e^{iu\Delta/2}$ is a unitary map of $L^2(K)$ to itself, in this case it is possible to derive the complex-time transform from the real one $B_{\infty,t}$ (denoted as the C-version of the transform C_t in [18]) by the decomposition $e^{\frac{1}{2}(t+iu)\Delta} = e^{t\Delta/2}e^{iu\Delta/2}$. This possibility has been exploited, for example, in the papers [13, 14] of C. Florentino, J. Mourão, and J. Nunes on the quantization of nonabelian theta functions on $\mathrm{SL}(n,\mathbb{C}) = \mathrm{SU}(n)_{\mathbb{C}}$. The authors show that these functions arise as the image of certain distributions on $\mathrm{SU}(n)$ under the heat operator, evaluated at a complex time, and use the Segal–Bargmann transform in the complexification process. These papers, then, show the utility of introducing a complex time-parameter into the (C-version) Segal–Bargmann transform. The present paper extends this complex time-parameter to the two-parameter transform.

Meanwhile, the Segal–Bargmann transform for K is related to the study of complex structures on the cotangent bundle $T^*(K)$. There is a natural one-parameter family of "adapted complex structures" on $T^*(K)$ arising from a general construction of Guillemen–Stenzel [16, 17] and Lempert–Szőke [31, 42]. Motivated by ideas of Thiemann [43], the second author and W. Kirwin in [25] showed that these structures arise from the "imaginary-time geodesic flow" on $T^*(K)$. The Segal–Bargmann transform can then be understood [11, 12, 22] as a quantum counterpart of the construction in [25].

As observed in [32], the adapted complex structures on $T^*(K)$ extend to a two-parameter family, by including both a real and an imaginary part to the time-parameter in the geodesic flow in [25]. The corresponding quantum construction has been done in [33] and can be thought of as adding a complex parameter to the C-version of the Segal–Bargmann transform for K. (Compare work of Kirwin and Wu [30] in the $\mathbb{R}^d$ case.) The present paper then extends the complex-time transform to its most natural range, in which the domain Hilbert space is taken to be $L^2(K)$ with respect to a heat kernel measure.

2. Compact-Type Lie Groups and their Complexifications

Let G be a real Lie group. Let e denote the identity element of G; let $\iota : G \to G$ be the inversion map, $\iota(g) = g^{-1}$ for all $g \in G$; and for any $g \in G$ let $L_g, R_g : G \to G$ be the left and right translation by g maps defined by $L_g(x) = gx$ and $R_g(x) = xg$ for all $x \in G$. We now choose once and for all a right Haar measure $\lambda = \lambda_G$ on G and usually simply write dx for $\lambda(dx)$ and $L^2(G)$ for $L^2(G,\lambda)$. The Lie algebra of G is taken to be $\mathfrak{g} := T_eG$ and to each $X \in \mathfrak{g}$ we let $\widetilde{X}$ denote the unique left-invariant vector field on G such that $\widetilde{X}(e) = X$, i.e., $\widetilde{X}(g) = L_{g*}X$ for all $g \in G$. As usual, for $g \in G$, Ad_g denotes the endomorphism of $\mathfrak{g}$ given by $\mathrm{Ad}_g = (C_g)_*$, where $C_g = L_gR_{g^{-1}}$ is the conjugation map on G. Then $\mathrm{Ad}\colon G \to \mathrm{GL}(\mathfrak{g})$ given by $g \mapsto \mathrm{Ad}_g$ is a Lie group homomorphism. Its derivative is $\mathrm{ad} = \mathrm{Ad}_*$, which is a Lie algebra homomorphism from $\mathfrak{g}$ to $\mathrm{End}(\mathfrak{g})$. It is given explicitly by $\mathrm{ad}_X(Y) = [X, Y]$.

2.1. **Complex Lie Groups and Complexificiation.** Suppose G is a complex Lie group, so that the Lie algebra $\mathfrak{g}$ of G is a complex Lie algebra. It is convenient, for reasons that will be apparent shortly, to write the "multiplication by i" map on $\mathfrak{g}$ as $J : \mathfrak{g} \to \mathfrak{g}$. (Thus, $J^2 = -I$.) Since $\mathfrak{g}$ is a complex Lie algebra, the bracket on $\mathfrak{g}$ is bilinear over $\mathbb{C}$, and in particular

$$[JX, Y] = J[X, Y] \tag{2.1}$$

for all $X, Y \in \mathfrak{g}$.

For any $X \in \mathfrak{g}$, the left-invariant vector field $\widetilde{X}$ is given by

$$(\widetilde{X}f)(g) = \left.\frac{d}{dt} f(ge^{tX})\right|_{t=0} \tag{2.2}$$

for any smooth real- or complex-valued function f on G. We may now appreciate the utility of the notion J for the "multiplication by i" map on $\mathfrak{g}$: in general, $\widetilde{JX}f \neq i\,\widetilde{X}f$ (for example, if f is real valued). On the other hand, a complex-valued function f on G is holomorphic if and only if the differential of f at each point $g \in G$ is a complex-linear map from $T_g(G)$ to $\mathbb{C}$. Thus, if f is holomorphic, then for all $X \in \mathfrak{g}$ and $g \in G$, we have

$$\widetilde{JX}f(g) = i\widetilde{X}f(g) \quad (f \text{ holomorphic}). \tag{2.3}$$

We note that if f is holomorphic then $\widetilde{X}f$ is again holomorphic for all $X \in \mathfrak{g}$, because the map $g \mapsto f(ge^{tX})$ is holomorphic for all t. Furthermore, for all $g \in G$ and $X \in \mathfrak{g}$, the maps $\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}$ and $\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}$ are complex linear:

$$\mathrm{Ad}_g\, J = J\mathrm{Ad}_g \quad \text{and} \quad \mathrm{ad}_X\, J = J\mathrm{ad}_X. \tag{2.4}$$

Let K be a connected real Lie group. We say that a pair (G, ϕ) is a **complexification** of K if G is a complex Lie group, $\phi\colon K \to G$ is a real Lie group homomorphism, and the following universal property holds: if H is any complex Lie group and $\Phi\colon K \to H$ is a real Lie group homomorphism, there exists a unique holomorphic homomorphism $\Phi_{\mathbb{C}}\colon G \to H$ through which ϕ factors as $\Phi = \Phi_{\mathbb{C}} \circ \phi$:

$$\begin{tikzcd} K \arrow[rr, "\phi"] \arrow[dr, "\Phi"'] & & G \arrow[dl, "\Phi_{\mathbb{C}}"] \\ & H & \end{tikzcd}$$

Proposition 2.1. *Let K be a connected Lie group with Lie algebra $\mathfrak{k}$, and assume that K is isomorphic to the direct product of a compact group K_0 and $\mathbb{R}^d$ for some non-negative integer d. Then there exists a complexification of K, and it is unique up to isomorphism; we refer to it as $K_{\mathbb{C}}$. Moreover, $K_{\mathbb{C}}$ is connected, the homomorphism $\phi\colon K \to K_{\mathbb{C}}$ is injective, and $\phi(K)$ is a closed subgroup of $K_{\mathbb{C}}$. The Lie algebra $\mathfrak{k}_{\mathbb{C}}$ of $K_{\mathbb{C}}$ is the complexification of the Lie algebra of K: $\mathfrak{k}_{\mathbb{C}} = \mathfrak{k} \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{k} \oplus J\mathfrak{k}$. Finally, $K_{\mathbb{C}}$ is isomorphic to $(K_0)_{\mathbb{C}} \times \mathbb{C}^d$.*

This is a standard result; see, for example, [26, XVII Theorem 5.1] and [4, Theorem 4.1, Propositions 8.4 and 8.6].

Remark 2.2. Actually, every connected Lie group K has a unique complexification. (See Section 6.10 of Chapter III in [3].) However, in general (if K is not of the special type in the proposition) the homomorphism ϕ may not be injective and the Lie algebra of the complexification may not be isomorphic to $\mathfrak{k} \otimes_{\mathbb{R}} \mathbb{C}$.

Example 2.3. The complexification of $\mathbb{R}^n$ is $\mathbb{C}^n$, with ϕ being the inclusion of $\mathbb{R}^n$ into $\mathbb{C}^n$. The compact Lie groups $\mathrm{SO}(n,\mathbb{R})$, $\mathrm{SU}(n)$, and $\mathrm{U}(n)$ have the following complexifications:

$$\mathrm{SO}(n,\mathbb{R})_{\mathbb{C}} = \mathrm{SO}(n,\mathbb{C}), \quad \mathrm{SU}(n)_{\mathbb{C}} = \mathrm{SL}(n,\mathbb{C}), \quad \mathrm{U}(n)_{\mathbb{C}} = \mathrm{GL}(n,\mathbb{C}).$$

In each case, the homomorphism ϕ is the standard inclusion of K into $K_{\mathbb{C}}$.

2.2. Lie Groups of Compact Type.

Definition 2.4. *Let G be a Lie group, and let $K \subseteq G$ be a Lie subgroup. An inner product $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ on $\mathfrak{g}$ is $\mathrm{Ad}(K)$-**invariant** if, for all $X_1, X_2 \in \mathfrak{g}$ and all $k \in K$,*

$$\langle \mathrm{Ad}_k X_1, \mathrm{Ad}_k X_2\rangle_{\mathfrak{g}} = \langle X_1, X_2\rangle_{\mathfrak{g}}.$$

*If the inner product is $\mathrm{Ad}(G)$-invariant, we simply call it Ad-**invariant**. A group whose Lie algebra possesses an Ad-invariant inner product is called **compact type**.*

Every compact Lie group possesses Ad-invariant inner products — simply average any inner product over the Haar measure on the group — thus explaining the terminology "compact type". The simplest examples are closed subgroups of $\mathrm{U}(n)$; for any such group K, the Hilbert–Schmidt inner product $\langle X_1, X_2\rangle = \mathrm{Tr}(X_1 X_2^*)$ is Ad-invariant. (In the case of the simple group $\mathrm{SU}(n)$, this is, up to scale, the only Ad-invariant inner product.)

Note that the existence of an Ad-invariant inner product means that there is a basis in which Ad_g is orthogonal for all g; if G is connected, this means $\det(\mathrm{Ad}_g) = 1$ for all g in this case. It follows (taking $g = e^{tX}$ and taking $\frac{d}{dt}\big|_{t=0}$) that ad_X is skew-symmetric, and thus that $\mathrm{Tr}(\mathrm{ad}_X) = 0$ in this case.

It turns out that the presence of an Ad-invariant inner product nearly forces the group to be compact.

Proposition 2.5 ([34], Lemma 7.5)**.** *If K is a compact-type Lie group with a specified Ad-invariant inner product, then K is isometrically isomorphic to a direct product group: $K \cong K_0 \times \mathbb{R}^d$ for some compact Lie group K_0 and some non-negative integer d.*

This result shows that every Lie group of compact type has a nice complexification, as described in Proposition 2.1.

2.3. **The Modular Function.** Recall that the modular function, $m : G \to (0,\infty)$, is the continuous (in fact smooth) group homomorphism determined by $(L_g)_* \lambda = m(g)\lambda$ for all $g \in G$. It is easy to verify that both $\iota_*\lambda$ and $m\lambda$ (where $m\lambda$ denotes the measure $d(m\lambda) := m\, d\lambda$) are left-invariant Haar measures on G, and hence $\iota_*\lambda = Cm\lambda$ for some $C > 0$. Applying ι_* to the equation $\iota_*\lambda = Cm\lambda$ using $\iota^{-1} = \iota$, $\iota_*(m\lambda) = m \circ \iota^{-1} \cdot \iota_*\lambda$, and $m \cdot m \circ \iota = 1$ by the homomorphism property of ι, one easily deduces that $\lambda = \iota_*^2\lambda = C^2\lambda$ from which it follows that $C = 1$, i.e., $\iota_*\lambda = m\lambda$. The above remarks may be summarized by the following identities:

$$\begin{aligned} &\int_G f(gx)\,dx = m(g)\int_G f(x)\,dx \quad \forall\, g \in G \text{ and} \\ &\int_G f(x^{-1})\,dx = \int_G f(x)\,m(x)\,dx \end{aligned} \tag{2.5}$$

which hold for all $f \in C_c(G)$.

A Lie group whose modular function is constantly equal to 1 is called *unimodular*. For example, compact Lie groups are always unimodular; in fact, $m|_K \equiv 1$ for any compact subgroup, $K \subseteq G$. This follows from the fact that $m(K)$ has to be a compact subgroup of $(0,\infty)$ and there is only one such subgroup, namely $\{1\}$.

Proposition 2.6. *If K is a connected Lie group of compact type, both K and $K_{\mathbb{C}}$ are unimodular.*

Proof. It is well known that a connected Lie group G is unimodular if and only if Ad_g has determinant one for all $g \in G$. (See, for example, Exercise 26 in Chapter 2 of [44].) There is, however, a subtlety: even if G happens to be a complex Lie group, so that Ad_g is a complex-linear map on the the Lie algebra, the determinant of Ad_g should be taken over $\mathbb{R}$ not $\mathbb{C}$. (After all, the complex structure plays no role in the definition of the modular function.)

Fix an Ad-invariant inner product on $\mathfrak{k}$ and identify $\mathfrak{k}$ with $\mathbb{R}^n$ using an orthonormal basis. Then the Adjoint representation of K maps into $\mathrm{O}(n, \mathbb{R})$, and actually into $\mathrm{SO}(n, \mathbb{R})$, since K is connected. Thus, Ad_x has determinant one for all $x \in K$, showing that K is unimodular.

Now, since the Adjoint representation of K maps into $\mathrm{SO}(n, \mathbb{R})$, the Adjoint representation of $K_{\mathbb{C}}$ maps into the complexification of $\mathrm{SO}(n, \mathbb{R})$, namely $\mathrm{SO}(n, \mathbb{C})$. (That is to say, the adjoint action of $K_{\mathbb{C}}$ preserves the complex-bilinear extension of the chosen inner product on $\mathfrak{k}$.) It follows that for all $g \in K_{\mathbb{C}}$, the determinant of Ad_g, computed over $\mathbb{C}$, is 1. It is then easily verified that the determinant over $\mathbb{R}$ of a complex-linear transformation, viewed as a real linear transformation, is the square of the absolute value of the determinant over $\mathbb{C}$. (Work in a basis over $\mathbb{C}$ in which the operator is upper triangular.) Thus, the determinant of Ad_g, computed over $\mathbb{R}$, is also 1, showing that $K_{\mathbb{C}}$ is unimodular. $\square$

3. Heat Kernels on Lie Groups

3.1. **Laplacians.** We now introduce left-invariant Laplacian operators, which in this context will mean any sum of squares of left-invariant vector fields. The results presented in this section are well-known; for an excellent presentation of many of them, see [34].

Notation 3.1. *To each real subspace $V \subseteq \mathfrak{g}$ equipped with a real inner product $\langle \cdot, \cdot \rangle_V$, let*

$$\Delta_V = \sum_{a=1}^{d_V} \widetilde{Z}_a^2$$

where $d_V = \dim_{\mathbb{R}} V$ and $\{Z_a\}_{a=1}^{d_V}$ is an orthonormal basis for $(V, \langle \cdot, \cdot \rangle_V)$.

By construction Δ_V is a left-invariant differential operator on G which (as the next lemma shows) is well defined, independent of basis.

Remark 3.2. In the case that G is unimodular and $V = \mathfrak{g}$, the operator Δ_V is the (negative) Laplace–Beltrami operator on G with respect to the left-invariant Riemannian metric induced by the given inner product; cf. [8, Remark 2.2].

Lemma 3.3 (Left-Invariant Laplacians). *Continuing the notation above, let $\{X_j\}_{j=1}^{d_V}$ be any basis for V, and define $q_{ij} := \langle X_i, X_j \rangle_V$ (the Gram matrix). If q^{-1} is the matrix inverse to q, then (as differential operators on $C^2(G)$)*

$$\sum_{i,j} q_{ij}^{-1} \widetilde{X}_i \widetilde{X}_j = \sum_a \widetilde{Z}_a^2$$

where i, j, a all run over $\{1, 2, \ldots, d_V\}$. As a corollary we see that these expressions are basis independent, i.e., the operators above are associated purely to the inner product $\langle \cdot, \cdot \rangle_V$.

Proof. If we let $A_{\ell j} := \langle Z_\ell, X_j\rangle_V$, then

$$\begin{aligned} q_{ij} = \langle X_i, X_j\rangle_V &= \sum_\ell \langle X_i, Z_\ell\rangle_V \langle Z_\ell, X_j\rangle_V \\ &= \sum_\ell A_{\ell j} A_{\ell i} = [A^\top A]_{ij} \end{aligned}$$

from which we easily conclude that $Aq^{-1}A^\top = I$. Using this identity, we find

$$\begin{aligned} \sum_{i,j} q_{ij}^{-1} \widetilde{X}_i \widetilde{X}_j &= \sum_{i,j,a,b} q_{ij}^{-1} \langle X_i, Z_a\rangle_V \langle X_j, Z_b\rangle_V \widetilde{Z}_a \widetilde{Z}_b \\ &= \sum_{i,j,a,b} q_{ij}^{-1} A_{ai} A_{bj} \widetilde{Z}_a \widetilde{Z}_b = \sum_{i,j,a,b} A_{ai} q_{ij}^{-1} A_{jb}^\top \widetilde{Z}_a \widetilde{Z}_b \\ &= \sum_{a,b} \left[Aq^{-1}A^\top\right]_{a,b} \widetilde{Z}_a \widetilde{Z}_b = \sum_a \widetilde{Z}_a^2, \end{aligned}$$

as claimed.

□

Notation 3.4. *For $g \in G$ and any function $f\colon G \to \mathbb{C}$, let $\hat{R}_g f$ and $\hat{L}_g f$ be the functions on G defined respectively by*

$$\left(\hat{R}_g f\right)(x) = f(xg) \ \ \textit{and} \ \ \left(\hat{L}_g f\right)(x) = f(gx) \quad \textit{for all} \ \ x \in G.$$

Note that $\hat{R}_g$ and $\hat{L}_g$ leave $C^\infty(G)$, $C_c^\infty(G)$, and $L^2(G)$ invariant, and and $\hat{R}_g$ acts as a unitary operator on $L^2(G)$. Moreover $\hat{L}_g$ acts unitarily on $L^2(G)$ whenever $m(g) = 1$ where m is the modular function on G. Since Δ_V is a linear combination of left-invariant differential operators, Δ_V is left invariant over G: $\Delta_V \hat{L}_g = \hat{L}_g \Delta_V$ on $C^\infty(G)$ for all $g \in G$. On the other hand, Δ_V need not be right invariant without further assumptions.

Lemma 3.5. *Let K be a Lie subgroup of G, let $V \subseteq \mathfrak{g}$ be an $\mathrm{Ad}(K)$-invariant subspace and let $\langle\cdot,\cdot\rangle_V$ be an $\mathrm{Ad}(K)$-invariant inner product on V. Then Δ_V is right-K-invariant; i.e., for all $k \in K$,*

$$\Delta_V \hat{R}_k = \hat{R}_k \Delta_V \ \textit{on} \ C^\infty(G). \tag{3.1}$$

Proof. For the desired right invariance, we first note that for any $g, x \in G$ and $Z \in \mathfrak{g}$,

$$\begin{aligned} \widetilde{Z}\left(\hat{R}_g f\right)(x) &= \left.\frac{d}{dt}\right|_{t=0} \left(\hat{R}_g f\right)(xe^{tZ}) = \left.\frac{d}{dt}\right|_{t=0} f(xe^{tZ}g) \\ &= \left.\frac{d}{dt}\right|_{t=0} f(xgg^{-1}e^{tZ}g) = \left.\frac{d}{dt}\right|_{t=0} f(xge^{t\mathrm{Ad}_{g^{-1}}Z}) = \left(\widetilde{Z^g} f\right)(xg) \end{aligned}$$

where $Z^g = \mathrm{Ad}_{g^{-1}} Z$. In other words, we have shown that

$$\widetilde{Z}\left(\hat{R}_g f\right) = \hat{R}_g \left(\widetilde{Z^g} f\right).$$

Thus, if $k \in K$, we have

$$\Delta_V \hat{R}_k f = \sum_{a=1}^{d_V} \widetilde{Z}_a^2 \hat{R}_k f = \sum_{a=1}^{d_V} \hat{R}_k \left(\widetilde{Z_a^k}\right)^2 f. \tag{3.2}$$

As $\langle\cdot,\cdot\rangle_V$ is $\mathrm{Ad}(K)$-invariant, we know that $\{Z_a^k\}_{a=1}^{d_V} = \{\mathrm{Ad}_{k^{-1}} Z_a\}_{a=1}^{d_V}$ is still an orthonormal basis for $(V, \langle\cdot,\cdot\rangle_V)$. By Lemma 3.3,

$$\sum_{a=1}^{d_V} (\widetilde{Z_a^k})^2 = \Delta_V,$$

and thus (3.2) shows $\Delta_V \hat{R}_k f = \hat{R}_k \Delta_V f$ as desired. □

Corollary 3.6. *Let K be a Lie subgroup of G, let $V \subseteq \mathfrak{g}$ be an $\mathrm{Ad}(K)$-invariant subspace and let $\langle\cdot,\cdot\rangle_V$ be an $\mathrm{Ad}(K)$-invariant inner product on V. Then $[\Delta_V, \widetilde{A}] = 0$ for all $A \in \mathfrak{k}$.*

Proof. Taking $k = e^{tA}$ in (3.1) and then differentiating the result at $t = 0$ shows the desired equality $\Delta_V \tilde{A} f = \tilde{A} \Delta_V f$. □

3.2. **Heat Operators and Heat Kernels.** We now come to the central objects used in this paper: heat operators (i.e., heat semigroups) and their integral kernels. The first important fact is that left-invariant Laplacians are always essentially self-adjoint. If G is unimodular and $V = \mathfrak{g}$, the result follows from the well-known essential self-adjointness of the Laplacian on a complete Riemannian manifold (e.g., Section 2 of [41]). If elements of V generate $\mathfrak{g}$ as a Lie algebra, then essential self-adjointness can be proved using methods of hypoellipticity, as in [27]. We provide a self-contained proof of the general result in Appendix A which expands on the brief outline given in [9, p. 950]. (The details of this argument were communicated to the first author by L. Gross.)

Theorem 3.7. *For any subspace $V \subseteq \mathfrak{g}$, the left-invariant Laplacian Δ_V, with domain $\mathcal{D}(\Delta_V) = C_c^\infty(G)$, is essentially self-adjoint as an unbounded operator on $L^2(G)$. Moreover, its closure $\bar{\Delta}_V$ is non-positive, and the associated heat operators $e^{\frac{t}{2}\bar{\Delta}_V}$ are left-invariant for each $t > 0$.*

As in Lemma 3.5, the assumption of $\mathrm{Ad}(K)$-invariance makes $e^{\frac{t}{2}\bar{\Delta}_V}$ right-K-invariant as well. To properly prove this (in Corollary 3.9 below), we first need the following technical functional analysis lemma, which is also used in the proof of Lemma 4.11.

Lemma 3.8. *Let $\mathscr{H}$ be a separable Hilbert space, let A and B be two essentially self-adjoint non-positive operators on $\mathscr{H}$, and suppose $Q\colon \mathscr{H} \to \mathscr{H}$ is a bounded operator such $QB \subseteq AQ$; i.e., $Q(\mathcal{D}(B)) \subseteq \mathcal{D}(A)$ and $QB = AQ$ on $\mathcal{D}(B)$. Then $Qe^{t\bar{B}} = e^{t\bar{A}}Q$ for all $t \geq 0$.*

Proof. If $f \in \mathcal{D}(\bar{B})$ and $f_n \in \mathcal{D}(B)$ such that $f_n \to f$ and $Bf_n \to \bar{B}f$, then $Qf_n \to Qf$ and $AQf_n = QBf_n \to Q\bar{B}f$ as $n \to \infty$. Therefore it follows that $Qf \in \mathcal{D}(\bar{A})$ and $\bar{A}Qf = Q\bar{B}f$ for all $f \in \mathcal{D}(\bar{B})$; i.e., $Q\bar{B} \subseteq \bar{A}Q$. So for any $\lambda \in \mathbb{C}$ we may conclude that $(\lambda I - \bar{A})Qf = Q(\lambda I - \bar{B})f$ for all $f \in \mathcal{D}(\bar{B})$. If we assume $\lambda > 0$ and $g \in \mathscr{H}$, we may take $f = (\lambda I - \bar{B})^{-1} g \in \mathcal{D}(\bar{B})$ in the previous identity to find

$$(\lambda I - \bar{A})Q(\lambda I - \bar{B})^{-1} g = Qg.$$

Multiplying this equation by $(\lambda I - \bar{A})^{-1}$ and using the fact that g was arbitrary shows that $Q(\lambda I - \bar{B})^{-1} = (\lambda I - \bar{A})^{-1} Q$ or, equivalently,

$$Q(I - \lambda^{-1}\bar{B})^{-1} = (I - \lambda^{-1}\bar{A})^{-1} Q \quad \text{for all } \lambda > 0.$$

A simple induction argument then shows that

$$Q(I - \lambda^{-1}\bar{B})^{-n} = (I - \lambda^{-1}\bar{A})^{-n} Q \quad \text{for all } \lambda > 0. \tag{3.3}$$

Now, note that $\lim_{n\to\infty}(1-\frac{y}{n})^{-n}=e^y$ and $0\le(1-\frac{y}{n})^{-n}\le 1$ for $y\le 0$; it thus follows from the spectral theorem and the dominated convergence theorem that

$$e^{t\bar{B}}=\text{strong}-\lim_{n\to\infty}\left(I-\frac{t}{n}\bar{B}\right)^{-n}\quad\text{and}\quad e^{t\bar{A}}=\text{strong}-\lim_{n\to\infty}\left(I-\frac{t}{n}\bar{A}\right)^{-n}.$$

Therefore, taking $\lambda=n/t$ in (3.3) and then letting $n\to\infty$ shows $Qe^{t\bar{B}}=e^{t\bar{A}}Q$ for all $t>0$. This completes the proof for $t>0$, and the $t=0$ case is immediate. □

Corollary 3.9. *If K is a Lie subgroup of G, $V\subseteq\mathfrak{g}$ is an $\mathrm{Ad}(K)$-invariant subspace, and $\langle\cdot,\cdot\rangle_V$ is an $\mathrm{Ad}(K)$-invariant inner product on V, then $[e^{\frac{t}{2}\bar{\Delta}_V},\hat{R}_k]=0$ for all $k\in K$ and $t\ge 0$.*

Proof. If $Q=\hat{R}_k$ and $A=B=\Delta_V$ with $\mathcal{D}(\Delta_V)=C_c^\infty(G)$, then by Lemma 3.5 $QB=AQ$, and Q preserves $\mathcal{D}(\Delta_V)$ in this case. The result now follow by an application of Lemma 3.8. □

Remark 3.10. We will be careful to always use the explicit closure $\bar{\Delta}_V$ when applying the heat operator defined through the spectral theorem for unbounded operators as above. In later sections, we will often work in a function space (sometimes nearly disjoint from L^2) on which the naïve power series definition of $e^{\frac{t}{2}\Delta_V}f$ converges for each f (cf. Section 3.3). In that case (and that case *only*), we use the notation $e^{\frac{t}{2}\Delta_V}$ without the closure.

Notation 3.11. *If $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ is an inner product on $\mathfrak{g}$, let $|x|_{\mathfrak{g}}$ denote the Riemannian distance from e to x in G relative to the unique left-invariant Riemannian metric on G which agrees with $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ on T_eG.*

The next theorem introduces the *heat kernel*: the integral kernel of $e^{\frac{t}{2}\bar{\Delta}_{\mathfrak{g}}}$. For proofs of the fundamental properties listed here, we refer the reader to [8, Proposition 3.1, Lemmas 4.2-4.3], [9, Section 3], and the references therein.

Theorem 3.12. *Let $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ be an inner product on $\mathfrak{g}$ and let $\Delta_{\mathfrak{g}}$ be the associated Laplacian as in Notation 3.1, with $\mathcal{D}(\Delta_{\mathfrak{g}}):=C_c^\infty(G)$. Then $\Delta_{\mathfrak{g}}$ is an elliptic differential operator and there is a smooth function $(0,\infty)\times G\ni(t,x)\mapsto\rho_t^{\Delta_{\mathfrak{g}}}(x)\in(0,\infty)$ so that*

$$e^{\frac{t}{2}\bar{\Delta}_{\mathfrak{g}}}=\int_G\rho_t^{\Delta_{\mathfrak{g}}}(x)\hat{R}_x\,dx\quad\forall\,t>0.\tag{3.4}$$

That is to say: for all $g\in G$ and $f\in f\in L^2(G)$,

$$\left(e^{\frac{t}{2}\bar{\Delta}_{\mathfrak{g}}}f\right)(g)=\int_G\rho_t^{\Delta_{\mathfrak{g}}}(x)f(gx)\,dx.$$

The function $\rho_t^{\Delta_{\mathfrak{g}}}$ is called the ***heat kernel****. It satisfies the following properties.*

(1) *The measures $\{\rho_t^{\Delta_{\mathfrak{g}}}(x)\,dx\}_{t>0}$ are invariant under the inversion map $\iota\colon x\mapsto x^{-1}$.*
(2) *$\rho_t^{\Delta_{\mathfrak{g}}}$ is conservative:*

$$\int_G\rho_t^{\Delta_{\mathfrak{g}}}(x)\,dx=1.\tag{3.5}$$

(3) *$\{\rho_t^{\Delta_{\mathfrak{g}}}\}_{t>0}$ satisfies the semigroup property:*

$$\rho_{s+t}^{\Delta_{\mathfrak{g}}}(x)=\int_G\rho_s^{\Delta_{\mathfrak{g}}}(xy^{-1})\rho_t^{\Delta_{\mathfrak{g}}}(y)\,dy\quad\forall\,s,t>0.\tag{3.6}$$

(4) $(t,x) \to \rho_t^{\Delta_{\mathfrak{g}}}(x)$ *satisfies the heat equation:*

$$\partial_t \rho_t^{\Delta_{\mathfrak{g}}}(x) = \frac{1}{2}\Delta_{\mathfrak{g}} \rho_t^{\Delta_{\mathfrak{g}}}(x) \quad \textit{for} \quad t > 0 \textit{ and } x \in G.$$

(5) $\{\rho_t^{\Delta_{\mathfrak{g}}}\}_{t>0}$ *is an approximate identity: for any* $f \in C_c(G)$ *and* $x \in G$*,*

$$\lim_{t\downarrow 0} \int_G f(xy^{-1})\rho_t^{\Delta_{\mathfrak{g}}}(y)\,dy = \lim_{t\downarrow 0} \int_G f(xy)\rho_t^{\Delta_{\mathfrak{g}}}(y)\,dy = f(x). \tag{3.7}$$

(6) *(Gaussian heat kernel bounds) There exists* $\nu \in \mathbb{R}$ *such that, for* $T > 0$ *and* $\epsilon \in (0,1]$*, there is a constant* $C(T,\epsilon)$ *such that, for* $0 < s \le T$ *and* $x \in G$*,*

$$\rho_s^{\Delta_{\mathfrak{g}}}(x) \le C(T,\epsilon)s^{-d}\exp\{-(|x| - \nu s)^2/(1+\epsilon)s\}. \tag{3.8}$$

Moreover, if G *is unimodular, these estimates hold with* $\nu = 0$*.*

(7) *(Exponential integrability) For all* $\kappa > 0$ *and compact intervals* $\mathfrak{J} \subset (0,\infty)$*,*

$$\int_G e^{\kappa|x|} \max_{s\in\mathfrak{J}} \rho_s^{\Delta_{\mathfrak{g}}}(x)\,dx < \infty.$$

(8) *(Concentration at the identity) For any* $s_1 > 0$*,*

$$\lim_{t\downarrow 0} \int_{|x|\ge 1} e^{|x|^2/s_1} \rho_t^{\Delta_{\mathfrak{g}}}(x)\,dx = 0.$$

3.3. **The Heat Operator on Matrix Entries.** In the case $G = \mathbb{R}^d$, it is convenient to do computations with the heat operator on polynomials. Although these functions are not in $L^2(\mathbb{R}^d)$, one can naïvely make sense of $e^{\frac{t}{2}\Delta_{\mathbb{R}^d}} f$ as a terminating power series for any polynomial f. It is then an easy matter to verify that the integral formula (3.18) for the heat operator coincides with its Taylor series: if f is a polynomial on $\mathbb{R}^d$, then

$$\int_{\mathbb{R}^d} \rho_t^{\Delta_{\mathbb{R}^d}}(x-y)f(y)\,dy = \sum_{n=0}^{\infty} \frac{(t/2)^n}{n!}(\Delta_{\mathbb{R}^d})^n f(x). \tag{3.9}$$

Equation (3.9) is easy to prove directly; the result is also a special case of Proposition 3.18 below.

We will need a counterpart of polynomial functions on a general (compact-type) Lie group; these are *matrix entries*, which we define as follows.

Definition 3.13. *Let* G *be a Lie group. Let* (π, V_π) *be a finite-dimensional complex representation of* G*, and let* $A \in \mathrm{End}(V_\pi)$ *be a fixed endomorphism. The associated* ***matrix entry*** *function* $f_{\pi,A}$ *on* G *is the function*

$$f_{\pi,A}(x) = \mathrm{Tr}(\pi(x)A).$$

If G *is a complex Lie group and the representation* $\pi : G \to GL(V_\pi)$ *is holomorphic, then we refer to* $f_{\pi,A}$ *as a* ***holomorphic matrix entry****. In particular, every holomorphic matrix entry on a complex Lie group is a holomorphic function.*

Remark 3.14. A number of comments on matrix entries are in order.

(1) Although some authors might require π to be irreducible in order to call $f_{\pi,A}$ a *matrix entry*, we make no irreducibility assumption in our definition. If G is compact, every finite-dimensional representation of G decomposes as a direct sum of irreducibles, in which case every matrix entry is a linear combination of matrix entries for irreducible representations. In general, not every matrix entry (in the sense of Definition 3.13) will decompose as a sum of matrix entries of irreducible representations.

(2) Some authors require a matrix entry to be of the form $f(x) = \xi(\pi(x)v)$ for some $v \in V$ and $\xi \in V^*$. This is a special case of Definition 3.13 with $f = f_{\pi,A}$ where $A(w) = \xi(w)v$, i.e., $A = \xi \otimes v$. The more general matrix entries of Definition 3.13 are linear combinations of these more restricted "rank-1 type" entries.
(3) Matrix entries are smooth functions on G.
(4) If $G = \mathbb{R}^d$, all polynomials are matrix entries. Indeed: if q is a polynomial of degree $\leq n$, take the representation space V to be all polynomials p of degree $\leq n$, where $\pi(x)p = p(\,\cdot + x)$. If $\xi_0(p) = p(0)$ is the evaluation linear functional, then $\xi_0(\pi(x)q) = q(x)$, so q is a matrix entry.
(5) Even if G is complex, we will have reason to consider matrix entries associated to representations of G that are not holomorphic.

Lemma 3.15. *For any Lie group G, the set of matrix entries on G forms a self-adjoint complex algebra.*

Proof. It is straightforward to compute that, for $\lambda \in \mathbb{C}$, $\lambda f_{\pi,A} = f_{\pi,\lambda A}$, while sums and products satisfy $f_{\pi,A} + f_{\sigma,B} = f_{\pi\oplus\sigma,A\oplus B}$ and $f_{\pi,A} f_{\sigma,B} = f_{\pi\otimes\sigma,A\otimes B}$. For complex conjugation, we must define the complex conjugate of a representation and an endomorphism. This can be done invariantly, but for our purposes there is no reason not to simply choose a basis. Given a representation (π, V_π) of dimension d, choose a complex-linear isomorphism $\varphi\colon V_\pi \to \mathbb{C}^d$, and let $[\pi(x)] = \varphi \circ \pi(x) \circ \varphi^{-1}$ and $[A] = \varphi \circ A \circ \varphi^{-1}$. As $d \times d$ complex matrices, both $[\pi(x)]$ and $[A]$ have complex conjugates $\overline{[\pi(x)]}$ and $\overline{[A]}$, defined entry-wise. Then

$$\bar{f}_{\pi,A}(x) = \overline{\mathrm{Tr}(\pi(x)A)} = \overline{\mathrm{Tr}([\pi(x)][A])} = \mathrm{Tr}(\overline{[\pi(x)]}\,\overline{[A]}). \tag{3.10}$$

The map $\overline{[\pi]}\colon G \to \mathrm{GL}(\mathbb{C}^d)$ given by $\overline{[\pi]}(x) = \overline{[\pi(x)]}$ is a representation of G on $\mathbb{C}^d$, and (3.10) shows that

$$\bar{f}_{\pi,A} = f_{\overline{[\pi]},\overline{A}}$$

is also a matrix entry of G. This concludes the proof. $\square$

The algebra of matrix entries is also closed under the action of complex left-invariant differential operators on G; see Lemma 3.16. A generic complex left-invariant differential operator on G has the form $\mathcal{L} = P(\widetilde{X}_1, \ldots, \widetilde{X}_d)$ where $\{X_1, \ldots, X_d\}$ is a basis for $\mathfrak{g}$ and P is a noncommutative complex polynomial in d indeterminates. Note that $\mathcal{L}$ naturally acts as a linear operator $C^\infty(G, \mathbb{C})$. If G is a complex Lie group then $\mathcal{L}$ leaves the subspace $\mathcal{H}(G)$ of holomorphic functions on G invariant.

Lemma 3.16. *Let $\{X_1, \ldots, X_d\}$ be a basis of $\mathfrak{g}$, let P be a noncommutative complex polynomial in d indeterminates, and let $\mathcal{L} = P(\widetilde{X}_1, \ldots, \widetilde{X}_d)$ be a complex left-invariant differential operator on G. For any representation (π, V_π) of G, define*

$$L_\pi := P(\pi_*(X_1), \ldots, \pi_*(X_d)) \in \mathrm{End}(V_\pi).$$

Then for any endomorphism A of V_π,

$$\mathcal{L} f_{\pi,A} = f_{\pi,L_\pi A}.$$

Proof. Given any $X \in \mathfrak{g}$ and $x \in G$, we compute

$$\begin{aligned}\widetilde{X} f_{\pi,A}(x) = \left.\frac{d}{dt}\right|_{t=0} f_{\pi,A}(xe^{tX}) = \left.\frac{d}{dt}\right|_{t=0} \mathrm{Tr}(\pi(x)\pi(e^{tX})A) &= \mathrm{Tr}(\pi(x)\pi_*(X)A) \\ &= f_{\pi,\pi_*(X)A}(x).\end{aligned}$$

The result now follows for monomials P by induction, and then in general by linearity. $\square$

If $\mathcal{L}$ is any complex left-invariant differential operator as in Lemma 3.16, we may define the formal exponential $e^{\mathcal{L}}$ acting on matrix entries, by a power series

$$\begin{aligned}(e^{\mathcal{L}} f_{\pi,A})(x) &:= \sum_{n=0}^{\infty} \frac{1}{n!} \mathcal{L}^n f_{\pi,A} \\ &= \operatorname{Tr}\left(\pi(x) \sum_{n=0}^{\infty} (L_\pi)^n A\right) \\ &= f_{\pi, e^{L_\pi} A}. \end{aligned} \tag{3.11}$$

In particular, we have a "heat operator" $e^{\frac{\tau}{2}\Delta_{\mathfrak{g}}}$ acting on matrix entries $f_{\pi,A}$ by

$$e^{\frac{\tau}{2}\Delta_{\mathfrak{g}}} f_{\pi,A} = f_{\pi, e^{\tau C_\pi/2}A} \text{ for all } \tau \in \mathbb{C} \tag{3.12}$$

where

$$C_\pi = \sum_{j=1}^{d} \pi_*(X_j)^2 \in \operatorname{End}(V_\pi). \tag{3.13}$$

Remark 3.17. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two complex left-invariant differential operators on G as in Lemma 3.16. If $\mathcal{L}_1$ and $\mathcal{L}_2$ commute on $C^\infty(G)$, then by simple finite-dimensional matrix algebra considerations, $e^{\mathcal{L}_1}e^{\mathcal{L}_2} = e^{\mathcal{L}_1+\mathcal{L}_2}$ as operators on matrix entries.

Even for $\tau = t > 0$, we should not confuse (3.12) with the heat operator $e^{\frac{t}{2}\bar{\Delta}_{\mathfrak{g}}}$ of Theorem 3.7: if the Lie group G is not compact, then the matrix entry functions $f_{\pi,A}$ are not in $L^2(G)$, which is where $e^{\frac{t}{2}\bar{\Delta}_{\mathfrak{g}}}$ is defined. Nevertheless, there is a link between these two "heat operators": both are given by integration against the heat kernel $\rho_t^{\Delta_{\mathfrak{g}}}$. In the case of $e^{\frac{t}{2}\bar{\Delta}_{\mathfrak{g}}}$, this is part of Theorem 3.12; in the case of the power-series heat operator on matrix entries, we have the following result.

Proposition 3.18. *Let G be a Lie group, with Lie algebra $\mathfrak{g}$ with a fixed inner product, and let $\rho_t^{\Delta_{\mathfrak{g}}}$ denote the heat kernel of Theorem 3.12. Then for any matrix entry function $f_{\pi,A}$ on G,*

$$\int_G \rho_t^{\Delta_{\mathfrak{g}}}(y) f_{\pi,A}(xy)\, dy = \left(e^{\frac{t}{2}\Delta_{\mathfrak{g}}} f_{\pi,A}\right)(x) = f_{\pi,e^{tC_\pi/2}A}(x), \tag{3.14}$$

with absolute convergence of the integral. In particular, the integral of $f_{\pi,A}$ against the heat kernel can be computed as

$$\int_G \rho_t^{\Delta_{\mathfrak{g}}}(y) f_{\pi,A}(y)\, dy = \left(e^{\frac{t}{2}\Delta_{\mathfrak{g}}} f_{\pi,A}\right)(e) = f_{\pi,e^{tC_\pi/2}A}(e). \tag{3.15}$$

Proof. The proposition is an immediate consequence of Langland's theorem (cf. [36, Theorem 2.1]). See also [18, Lemma 8]. If one assumes it is valid to differentiate under the integral and to integrate by parts, one can prove the proposition easily; see the proof of [6, Theorem 2.13]. □

Remark 3.19. If f is a matrix entry on G, then by Lemma 3.15, $|f|^2$ is also a matrix entry. Thus, the absolute convergence of the integral in Proposition 3.18 tells us that f belongs to $L^2(G, \rho_t^{\Delta_{\mathfrak{g}}})$.

3.4. **Heat Kernels and Lie Subgroups.** Let us now suppose that K is a proper Lie subgroup of G, $\mathfrak{k} := \mathrm{Lie}(K)$ is the Lie algebra of K which we identify with $i_*\mathfrak{k} \subset \mathfrak{g}$ where $i\colon K \hookrightarrow G$ is the inclusion map. For $X \in \mathfrak{k} \subseteq \mathfrak{g}$, we continue to let $\widetilde{X}$ to be the associated left invariant vector field on G be as described in (2.2). However, we now also let $\widetilde{X}^K$ be the associated left-invariant vector field on K; i.e., $\widetilde{X}^K(k) := L_{k*}X \in T_kK$ for all $k \in K$. For the rest of this section we suppose that $\langle\cdot,\cdot\rangle_{\mathfrak{k}}$ is a given (real) inner product on $\mathfrak{k}$.

Definition 3.20. *Continuing the notation above let $\Delta_{\mathfrak{k}}$ be the differential operator acting on $C^\infty(G)$ described in Notation 3.1 with $V = \mathfrak{k}$. On the other hand, let Δ_K denote the associated **Laplacian** acting on $C^\infty(K)$ defined by $\Delta_K := \sum_{a=1}^{\dim K}(\tilde{X}_a^K)^2$ where $\{X_a\}_{a=1}^{\dim K}$ is an orthonormal basis for $(\mathfrak{k}, \langle\cdot,\cdot\rangle_{\mathfrak{k}})$.*

Explicitly the operators $\Delta_{\mathfrak{k}}$ and Δ_K in Definition 3.20 are given by

$$\begin{aligned}
(\Delta_{\mathfrak{k}} f)(g) &= \sum_{a=1}^{\dim K} \frac{d^2}{dt^2}\bigg|_{t=0} f(ge^{tX_a}) \quad \text{for } g \in G,\ f \in C^\infty(G), \\
(\Delta_K v)(k) &= \sum_{a=1}^{\dim K} \frac{d^2}{dt^2}\bigg|_{t=0} v(ke^{tX_a}) \quad \text{for } k \in K,\ v \in C^\infty(K).
\end{aligned}$$

The operators $\Delta_{\mathfrak{k}}$ and Δ_K are not the same; they act on very different spaces. They are, of course, closely connected, as the following integration-by-parts formula attests.

Lemma 3.21. *If K is a closed Lie subgroup of G, $v \in C^\infty(K)$, $f \in C_c^\infty(G)$, and $g \in G$, then*

$$\int_K (\Delta_K v)(k) f(gk)\, dk = \int_K v(k)\, (\Delta_{\mathfrak{k}} f)(gk)\, dk.$$

Proof. Since the translated function $K \ni k \to f(gk)$ has support equal to $K \cap \big(L_{g^{-1}}\mathrm{supp}(f)\big)$, and since K is closed, it follows that this support set is compact. Therefore

$$\begin{aligned}
\int_K (\Delta_K v)(k) f(gk)\, dk &= \sum_{a=1}^{\dim K} \frac{d^2}{dt^2}\bigg|_{t=0} \int_K v(ke^{tX_a}) f(gk)\, dk \\
&= \sum_{a=1}^{\dim K} \frac{d^2}{dt^2}\bigg|_{t=0} \int_K v(k) f(gke^{-tX_a})\, dk \\
&= \sum_{a=1}^{\dim K} \frac{d^2}{dt^2}\bigg|_{t=0} \int_K v(k) f(gke^{-tX_a})\, dk \\
&= \int_K v(k)\, (\Delta_{\mathfrak{k}} f)(gk)\, dk.
\end{aligned}$$

□

As guaranteed by Theorem 3.7, $\bar{\Delta}_{\mathfrak{k}} := \overline{\Delta_{\mathfrak{k}}|_{C_c^\infty(G)}}$ is a self-adjoint operator in $L^2(G)$ and $\bar{\Delta}_K = \overline{\Delta_K|_{C_c^\infty(K)}}$ is a self-adjoint operator in $L^2(K)$.

Definition 3.22 (The heat kernel on K)**.** *For $t > 0$, let $\rho_t^{\Delta_K} \in C^\infty(K, (0,\infty))$ be the heat kernel on K as constructed in Theorem 3.12 with G replaced by K everywhere.*

Notice that the operator Δ_K is elliptic and so the construction in Theorem 3.12 applies. On the other hand, unless $K = G$, $\Delta_{\mathfrak{k}}$ acting on $C^\infty(G)$ is not elliptic (or even hypoelliptic) and hence we can not apply Theorem 3.12 in order to find a heat kernel density $\rho_t^{\Delta_{\mathfrak{k}}}$ on G associated to $\Delta_{\mathfrak{k}}$. In fact, $e^{t\bar{\Delta}_{\mathfrak{k}}/2}$ is not representable as convolution against a smooth heat kernel density on G. Nevertheless, in Proposition 3.25 we will see that $e^{t\bar{\Delta}_{\mathfrak{k}}/2}$ may still be represented as a convolution operator associated to the measure $\rho_t^{\Delta_K}(k)\,dk$ (thought of as a measure on G supported on K).

Notation 3.23. *For a probability measure γ on G, denote $\Gamma_\gamma = \int_G \hat{R}_x\,\gamma(dx)$. If μ is another probability measure on G, let $\gamma * \mu$ (the convolution of the measures over G) denote the new probability measure determined by*

$$\int_G f\,d(\gamma * \mu) = \int_{G\times G} f(xy)\gamma(dx)\mu(dy), \qquad \forall\ f \in C_c(G).$$

Here, $\int_G \hat{R}_x\,\gamma(dx)$ is the unique bounded operator Γ_γ on $L^2(G)$ with the property that

$$\langle f_1, \Gamma_\gamma f_2\rangle_{L^2(G)} = \int_G \langle f_1, \hat{R}_x f_2\rangle\,\gamma(dx)$$

for all f_1, f_2 in $L^2(G)$. The following lemma is straightforward to verify from the definitions.

Lemma 3.24. *For any probability measure γ on G, Γ_γ acts as a bounded operator on $L^2(G)$. Its adjoint is given by*

$$\Gamma_\gamma^* = \int_G \hat{R}_{x^{-1}}\,\gamma(dx).$$

Moreover, Γ converts convolution into multiplication: if μ is another probability measure on G, then $\Gamma_\gamma\Gamma_\mu = \Gamma_{\gamma*\mu}$.

Proposition 3.25. *If K is a closed Lie subgroup of G and $\langle\cdot,\cdot\rangle_{\mathfrak{k}}$ is an inner product on $\mathfrak{k} = \mathrm{Lie}(K)$, then the semigroup $\{e^{\frac{t}{2}\bar{\Delta}_{\mathfrak{k}}}\}_{t\geq 0}$ on $L^2(G)$ may be expressed as*

$$e^{\frac{t}{2}\bar{\Delta}_{\mathfrak{k}}} = \int_K \rho_t^{\Delta_K}(k)\,\hat{R}_k\,dk.$$

Proof. Let $T_t := \int_K \rho_t^{\Delta_K}(k)\hat{R}_k\,dk$. It is straightforward to verify that $\{T_t\}_{t>0}$ is a self-adjoint contraction semigroup. Specifically, the self-adjointness of T_t follows from Point 1 of Theorem 3.12 (applied with G replaced by K), the contractivity of T_t follows from Point 2 of Theorem 3.12, and the semigroup property follows from Point 3 of Theorem 3.12 and Lemma 3.24. The strong continuity of T_t is easily verified on $C_c(G)$ and then follows in general by the contractivity of T_t.

Furthermore, if $f \in C_c^\infty(G)$, then for each fixed $x \in G$,

$$\begin{aligned}\frac{d}{dt}\left(T_t f\right)(x) &= \frac{d}{dt}\int_K \rho_t^{\Delta_K}(k) f(xk)\,dk \\ &= \int_K \left(\frac{d}{dt}\rho_t^{\Delta_K}\right)(k) f(xk)\,dk = \int_K \frac{1}{2}\left(\Delta_K \rho_t^{\Delta_K}\right)(k)\cdot f(xk)\,dk \\ &= \int_K \rho_t^{\bar{\Delta}_K}(k)\frac{1}{2}\left(\Delta_{\mathfrak{k}} f\right)(xk)\,dk = \frac{1}{2}T_t\left(\Delta_{\mathfrak{k}} f\right)(x),\end{aligned} \tag{3.16}$$

wherein we have used Lemma 3.21 in the fourth equality. Hence it follows that

$$\frac{\left(T_t f\right)(x) - f(x)}{t} - \frac{1}{2}\Delta_{\mathfrak{k}} f(x) = \frac{1}{2t}\int_0^t \left[T_s(\Delta_{\mathfrak{k}} f)(x) - (\Delta_{\mathfrak{k}} f)(x)\right]\,ds$$

and therefore

$$\left\|\frac{T_t f - f}{t} - \frac{1}{2}\Delta_{\mathfrak{k}} f\right\|_{L^2(G)} \leq \frac{1}{2t}\int_0^t \|T_s(\Delta_{\mathfrak{k}} f) - \Delta_{\mathfrak{k}} f\|_{L^2(G)}\, ds \to 0 \text{ as } t \downarrow 0.$$

Hence, if A is the self-adjoint generator of the semigroup $\{T_t\}_{t\geq 0}$ acting on $L^2(G)$, we have just shown that $\frac{1}{2}\Delta_{\mathfrak{k}} \subseteq A$ and hence $\frac{1}{2}\bar{\Delta}_{\mathfrak{k}} \subseteq A$. By Theorem 3.7, $\frac{1}{2}\bar{\Delta}_{\mathfrak{k}}$ is also self-adjoint and so taking adjoints of the inclusion, $\frac{1}{2}\bar{\Delta}_{\mathfrak{k}} \subseteq A$ shows $A = \frac{1}{2}\bar{\Delta}_{\mathfrak{k}}$. Thus $A = \frac{1}{2}\bar{\Delta}_{\mathfrak{k}}$, and consequently $T_t = e^{tA} = e^{\frac{t}{2}\bar{\Delta}_{\mathfrak{k}}}$, completing the proof. □

If K is compact, it is automatically closed in G; but if K is compact type but not compact (e.g., $K \cong \mathbb{R}^k$) then it may not be closed in G. Nevertheless, the result of Proposition 3.25 still holds in general. We include a proof of this fact below for completeness.

Proposition 3.26 (Stochastic proof of (3.16))**.** *Let K be a Lie subgroup of G and $\langle\cdot,\cdot\rangle_{\mathfrak{k}}$ be an inner product on $\mathfrak{k} = \mathrm{Lie}(K)$. If $f \in C_c^\infty(G)$ and $x \in G$ then*

$$\frac{d}{dt}\int_K \rho_t^{\Delta_K}(k) f(xk)\, dk = \int_K \rho_t^{\Delta_K}(k)\frac{1}{2}\left(\Delta_{\mathfrak{k}} f\right)(xk)\, dk.$$

That is, (3.16) *is valid without assuming that K is closed. Therefore, Proposition 3.25 holds for general $K \subseteq G$.*

Proof. First, note that the only place where the assumption that $K \subseteq G$ is closed is used in the proof of Proposition 3.25 is in the use of the integration by parts formula of Lemma 3.21 (which, as proved, requires K to be closed) to deduce (3.16). Hence, the following proof of (3.16), which does not require K to be closed, suffices to establish Proposition 3.25 in general.

Let $(b_t)_{t\geq 0}$ be a $\mathfrak{k}$-valued Brownian motion, $(k_t)_{t\geq 0}$ be the K-valued Brownian motion, which is the solution of the stochastic differential equation

$$dk_t = k_t \circ db_t \quad \text{with} \quad k_0 = e \in K.$$

Then by Itô's lemma,

$$f(xk_t) = f(x) + \int_0^t \left(\partial_{db_s} f\right)(xk_s) + \int_0^t \frac{1}{2}\left(\Delta_{\mathfrak{k}} f\right)(xk_s)\, ds.$$

Taking expectations of this identity implies that

$$\begin{aligned}\int_K \rho_t^{\Delta_K}(k) f(xk)\, dk = \mathbb{E}\left[f(xk_t)\right] &= f(x) + \mathbb{E}\int_0^t \frac{1}{2}\left(\Delta_{\mathfrak{k}} f\right)(xk_s)\, ds \\ &= f(x) + \int_0^t \mathbb{E}\left[\frac{1}{2}\left(\Delta_{\mathfrak{k}} f\right)(xk_s)\right] ds\end{aligned}$$

wherein we have used the fact that derivatives of f are bounded to see that $\int_0^t \left(\partial_{db_s} f\right)(xk_s)$ is an L^2-martingale and to justify the use of Fubini's theorem. Differentiating the last displayed equation with respect to t then gives the desired result:

$$\frac{d}{dt}\int_K \rho_t^{\Delta_K}(k) f(xk)\, dk = \mathbb{E}\left[\frac{1}{2}\left(\Delta_{\mathfrak{k}} f\right)(xk_t)\right] = \int_K \rho_t^{\Delta_K}(k)\frac{1}{2}\left(\Delta_{\mathfrak{k}} f\right)(xk)\, dk.$$

□

3.5. **Heat Kernels under** $\mathrm{Ad}(K)$**-invariance.** In this section we assume that K is a proper Lie subgroup of G, $\mathfrak{k} := \mathrm{Lie}(K)$ and $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ is an inner product on $\mathfrak{g}$ which is $\mathrm{Ad}(K)$-invariant. We will further assume that $m(k) = 1$ for all $k \in K$, where m is the modular function on G.

Remark 3.27. For our purposes later, G will be equal to $K_{\mathbb{C}}$ and is therefore unimodular by Proposition 2.6 in which case $m|_K = 1$ certainly holds. Moreover, if K is actually compact (not just compact type), then the condition that the modular function of G equals 1 on K is automatic for any G, as noted just above Proposition 2.6.

Under the above assumptions, $e^{\frac{t}{2}\bar{\Delta}_{\mathfrak{g}}}$ is a right-K-invariant operator and the heat kernel is K-conjugate invariant.

Corollary 3.28. *Let $K \subseteq G$ be a compact type subgroup, and suppose that $\langle\cdot,\cdot\rangle_{\mathfrak{g}}$ is $\mathrm{Ad}(K)$-invariant. Suppose further that the modular function m of G is identically equal to 1 on K. Then*

$$[e^{\frac{t}{2}\bar{\Delta}_{\mathfrak{g}}}, \hat{R}_k] = 0 \quad \textit{and} \quad \rho_t^{\Delta_{\mathfrak{g}}}\left(kxk^{-1}\right) = \rho_t^{\Delta_{\mathfrak{g}}}(x) \tag{3.17}$$

for all $t > 0$, $k \in K$, and $x \in G$.

Proof. The first identity in (3.17) follows from Corollary 3.9 with $V = \mathfrak{g}$. To prove the second identity in (3.17), let $f \in C_c(G)$ be given. Applying Theorem 3.12 to $\hat{R}_k f$ for $k \in K$, the first identity in (3.17) gives

$$\int_G \rho_t^{\Delta_{\mathfrak{g}}}(x)\hat{R}_x\hat{R}_k f\,dx = \hat{R}_k\int_G \rho_t^{\Delta_{\mathfrak{g}}}(x)\hat{R}_x f\,dx.$$

Both sides of this L^2-identity are continuous functions; hence we may evaluate both sides at $e \in G$ to find

$$\int_G \rho_t^{\Delta_{\mathfrak{g}}}(x)f(xk)\,dx = \int_G \rho_t^{\Delta_{\mathfrak{g}}}(x)f(kx)\,dx.$$

Making the change of variables $x \mapsto xk^{-1}$ on the left and $x \mapsto k^{-1}x$ on the right, and using the fact that the modular function is equal to 1 on K, we have

$$\int_G \rho_t^{\Delta_{\mathfrak{g}}}(xk^{-1})f(x)\,dx = \int_G \rho_t^{\Delta_{\mathfrak{g}}}(k^{-1}x)f(x)\,dx$$

for all $f \in C_c(G)$. It follows that $\rho_t^{\Delta_{\mathfrak{g}}}(xk^{-1}) = \rho_t^{\Delta_{\mathfrak{g}}}(k^{-1}x)$ for all $x \in G$ and $k \in K$; the result follows by substituting $x \mapsto kx$. □

Corollary 3.29. *Let K be a compact-type Lie group and fix an $\mathrm{Ad}(K)$-invariant inner product on $\mathfrak{k}$. The heat operator $e^{\frac{t}{2}\bar{\Delta}_K}$ on $L^2(K)$ is then given by the convolution formula*

$$\left(e^{\frac{t}{2}\bar{\Delta}_K}f\right)(x) = \int_K \rho_t^{\Delta_K}(xk^{-1})f(k)\,dk. \tag{3.18}$$

Proof. From Theorem 3.12, we have

$$\left(e^{\frac{t}{2}\bar{\Delta}_K}f\right)(x) = \int_K \rho_t^{\Delta_K}(k)f(xk)\,dk.$$

By item (1) of that theorem, the heat kernel measure $\rho_t^{\Delta_K}(k)\,dk$ is invariant under $k \mapsto k^{-1}$; hence the integral is equal to

$$\int_K \rho_t^{\Delta_K}(k)f(xk^{-1})\,dk.$$

Now make the change of variables $k \mapsto kx$; the integral is thus equal to

$$\int_K \rho_t^{\Delta_K}(kx) f(k^{-1})\, dk.$$

Since K is unimodular (Proposition 2.6), the further change of variables $k \mapsto k^{-1}$ then shows (via (2.5)) that

$$\left(e^{\frac{t}{2}\bar{\Delta}_K} f\right)(x) = \int_K \rho_k^{\Delta_K}(k^{-1}x) f(k)\, dk.$$

Finally, by Corollary 3.28, $\rho_t^{\Delta_K}(k^{-1}x) = \rho_t^{\Delta_K}(xk^{-1})$, concluding the proof. $\square$

The next result, which is the final theorem of this section, regards the interaction of $\Delta_{\mathfrak{g}}$ and $\Delta_{\mathfrak{k}}$. In the presence of $\mathrm{Ad}(K)$-invariance, these two Laplacians commute, as do their closures and heat operators. The precise statement and proof follows the second author's paper [18, pp. 124-125].

Theorem 3.30. *Suppose that K is a compact-type Lie subgroup of G, $\langle \cdot, \cdot \rangle_{\mathfrak{k}}$ is an inner product on $\mathfrak{k} = \mathrm{Lie}(K)$, and $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ is an $\mathrm{Ad}(K)$-invariant inner product on $\mathfrak{g}$. Then*

(1) $[e^{\frac{t}{2}\bar{\Delta}_{\mathfrak{k}}}, e^{\frac{s}{2}\bar{\Delta}_{\mathfrak{g}}}] = 0$ *for all* $s, t > 0$,
(2) $e^{\frac{t}{2}\bar{\Delta}_{\mathfrak{k}}} e^{\frac{t}{2}\bar{\Delta}_{\mathfrak{g}}} = e^{\frac{t}{2}\left(\overline{\Delta_{\mathfrak{k}} + \Delta_{\mathfrak{g}}}\right)}$, *and*
(3) *the heat kernel* $\rho_t^{\Delta_{\mathfrak{k}} + \Delta_{\mathfrak{g}}}$ *for* $\Delta_{\mathfrak{k}} + \Delta_{\mathfrak{g}}$ *may be expressed as*

$$\rho_t^{\Delta_{\mathfrak{k}} + \Delta_{\mathfrak{g}}}(x) = \int_K \rho_t^{\Delta_{\mathfrak{g}}}(xk^{-1})\, \rho_t^{\Delta_K}(k)\, dk \tag{3.19}$$

$$= \int_K \rho_t^{\Delta_{\mathfrak{g}}}(k^{-1}x)\, \rho_t^{\Delta_K}(k)\, dk. \tag{3.20}$$

Proof. We take each item in turn.

(1) By Corollary 3.28, $[e^{\frac{t}{2}\bar{\Delta}_{\mathfrak{g}}}, \hat{R}_k] = 0$ for all $k \in K$, and hence using Proposition 3.25,

$$[e^{\frac{t}{2}\bar{\Delta}_{\mathfrak{g}}}, e^{\frac{s}{2}\bar{\Delta}_{\mathfrak{k}}}] = \left[e^{\frac{t}{2}\bar{\Delta}_{\mathfrak{g}}}, \int_K \rho_s^{\Delta_{\mathfrak{k}}}(k)\, \hat{R}_k\, dk\right] = \int_K \rho_s^{\Delta_{\mathfrak{k}}}(k) [e^{\frac{t}{2}\bar{\Delta}_{\mathfrak{g}}}, \hat{R}_k]\, dk = 0.$$

(2) Since $\{e^{\frac{t}{2}\bar{\Delta}_{\mathfrak{k}}}\}_{t>0}$ and $\{e^{\frac{t}{2}\bar{\Delta}_{\mathfrak{g}}}\}_{t>0}$ are two commuting (strongly continuous) self-adjoint contraction semigroups, it follows that $T_t := e^{\frac{t}{2}\bar{\Delta}_{\mathfrak{k}}} e^{\frac{t}{2}\bar{\Delta}_{\mathfrak{g}}}$ for $t > 0$ is also a self-adjoint contraction semigroup. Moreover if $f \in \mathcal{D}(\bar{\Delta}_{\mathfrak{k}}) \cap \mathcal{D}(\bar{\Delta}_{\mathfrak{g}})$, then

$$\frac{T_t f - f}{t} = e^{\frac{t}{2}\bar{\Delta}_{\mathfrak{g}}} \frac{e^{\frac{t}{2}\bar{\Delta}_{\mathfrak{k}}} f - f}{t} + \frac{e^{\frac{t}{2}\bar{\Delta}_{\mathfrak{g}}} f - f}{t} \to \frac{1}{2}\left(\bar{\Delta}_{\mathfrak{k}} f + \bar{\Delta}_{\mathfrak{g}} f\right)$$

as $t \downarrow 0$. Therefore $\frac{1}{2}(\Delta_{\mathfrak{k}} + \Delta_{\mathfrak{g}}) \subseteq \frac{1}{2}\left(\bar{\Delta}_{\mathfrak{k}} + \bar{\Delta}_{\mathfrak{g}}\right) \subseteq A$, where A is the generator of $\{T_t\}_{t>0}$. Hence from Theorem 3.7 (as in the proof of Proposition 3.25), we have $A = \frac{1}{2}\overline{\Delta_{\mathfrak{k}} + \Delta_{\mathfrak{g}}}$. Item (2) follows.

(3) Theorem 3.12, Lemma 3.24, and Proposition 3.25 show that the statement of item (2) is equivalent to

$$\Gamma_{\rho_t^{\Delta_{\mathfrak{k}} + \Delta_{\mathfrak{g}}} \lambda_G} = \Gamma_{\rho_t^{\Delta_{\mathfrak{g}}} \lambda_G} \Gamma_{\rho_t^{\Delta_K} \lambda_K} = \Gamma_{(\rho_t^{\Delta_{\mathfrak{g}}} \lambda_G) * (\rho_t^{\Delta_K} \lambda_K)} \tag{3.21}$$

where λ_G is the right-invariant Haar measures on G and λ_K is the right Haar measure on K thought of as a measure on G which is supported on K. The identity (3.21) is, in turn, equivalent to (3.19) since, for all $f \in C_c(G)$, we have

$$\begin{aligned}\int f\, d((\rho_t^{\Delta_{\mathfrak{g}}}\lambda_G) * (\rho_t^{\Delta_K}\lambda_K)) &= \int_{G\times K} f(xk)\rho_t^{\Delta_{\mathfrak{g}}}(x)\,\rho_t^{\Delta_K}(k)\,dx\,dk \\ &= \int_{G\times K} f(x)\rho_t^{\Delta_{\mathfrak{g}}}(xk^{-1})\,\rho_t^{\Delta_K}(k)\,dx\,dk \\ &= \int_{G\times K} f(x)\left[\int_K \rho_t^{\Delta_{\mathfrak{g}}}(xk^{-1})\,\rho_t^{\Delta_K}(k)\,dk\right]\,dx.\end{aligned}$$

Equation (3.20) follows directly from (3.19) and Corollary 3.28.

□

3.6. **An Averaging Theorem.** In this section, we prove a regularity property of heat kernels on G associated to $\mathrm{Ad}(K)$-invariant Laplacians, for $K \subseteq G$ compact. A version of this theorem was proved in [18], in a context that applied to the case $G = K_{\mathbb{C}}$. We give a proof here that shows explicitly that the result (which is of independent interest) does not depend on any complex structure on G, or any special relationship between K and G other than that the inner product on $\mathfrak{g}$ is $\mathrm{Ad}(K)$-invariant. Throughout the section, we assume that K is actually compact, not just compact type.

To begin, we need the following lemma, which can also be found as [34, Lemma 7.4].

Lemma 3.31. *Let G be a Lie group with Lie algebra $\mathfrak{g}$, and let $K \subseteq G$ be a compact Lie subgroup with Lie algebra $\mathfrak{k} \subseteq \mathfrak{g}$. Suppose $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_{\mathfrak{g}}$ is $\mathrm{Ad}(K)$-invariant (such inner products always exists by averaging since K is compact), and let $\mathfrak{k}^\perp \subseteq \mathfrak{g}$ be the orthogonal complement of $\mathfrak{k}$. Then both $\mathfrak{k}$ and $\mathfrak{k}^\perp$ are invariant subspaces for Ad_k for each $k \in K$.*

Proof. Since $\mathfrak{k}$ is the Lie algebra of K, it is automatically invariant under Ad_k for each $k \in K$ (since K is invariant under conjugation by $k \in K$). Now, let Ad_k^* denote the adjoint of the operator Ad_k with respect to the given inner product. For $X \in \mathfrak{k}$ and $Y \in \mathfrak{k}^\perp$,

$$\langle \mathrm{Ad}_k^*(Y), X\rangle = \langle Y, \mathrm{Ad}_k(X)\rangle = 0$$

since $\mathrm{Ad}_k(X) \in \mathfrak{k}$. This shows that $\mathrm{Ad}_k(Y) \in \mathfrak{k}^\perp$, so $\mathfrak{k}^\perp$ is invariant under Ad_k^* for each $k \in K$. Since the inner product is $\mathrm{Ad}(K)$-invariant, Ad_k is unitary, and so $\mathrm{Ad}_k^* = \mathrm{Ad}_k^{-1} = \mathrm{Ad}_{k^{-1}}$. As this holds for all $k \in K$, it follows that $\mathfrak{k}^\perp$ is invariant under Ad_k for each $k \in K$, as desired. □

Notation 3.32. *If v_1 and v_2 are two positive functions on G and $C \geq 1$, let us write $v_1 \asymp_C v_2$ as short hand for $C^{-1}v_2 \leq v_1 \leq Cv_2$, i.e.,*

$$C^{-1}v_2(x) \leq v_1(x) \leq Cv_2(x) \qquad \textit{for } x \in G.$$

The reader may easily verify that $v_1 \asymp_C v_2$ iff $v_2 \asymp_C v_1$, and if v_3 is another positive function on G and $K \geq 1$ such that $v_2 \asymp_K v_3$, then $v_1 \asymp_{CK} v_3$.

Theorem 3.33 (Averaging Theorem)**.** *Let G be a connected Lie group with Lie algebra $\mathfrak{g}$, and let K be a compact connected Lie subgroup of G. Fix an $\mathrm{Ad}(K)$-invariant inner product on $\mathfrak{g}$, let $\Delta_{\mathfrak{g}}$ be the associated Laplacian, and let $\rho_t^{\Delta_{\mathfrak{g}}}$ be the associated heat kernel. Given a probability measure γ on K, let $\gamma_t : G \to (0,\infty)$ be the γ-**averaged heat kernel** defined by*

$$\gamma_t(x) = \int_K \rho_t^{\Delta_{\mathfrak{g}}}(xk)\,\gamma(dk).$$

Then for each $t > 0$, there is a constant $C(t) \in (1, \infty)$ (see (3.24) below) such that $\gamma_t \asymp_{C(t)} \rho_t^{\Delta_{\mathfrak{g}}}$, i.e.,

$$C(t)^{-1}\gamma_t(x) \leq \rho_t^{\Delta_{\mathfrak{g}}}(x) \leq C(t)\gamma_t(x), \qquad \textit{for all } x \in G. \tag{3.22}$$

Proof. Let $\mathfrak{k} \subseteq \mathfrak{g}$ denote the Lie algebra of K. Denote by $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ the given $\mathrm{Ad}(K)$-invariant inner product on $\mathfrak{g}$. Denote the dimensions of G and K as d_G and d_K respectively. Fix an orthonormal basis $\{X_1, \ldots, X_{d_G}\}$ for $\mathfrak{g}$ with the property that $\{X_1, \ldots, X_{d_K}\}$ is an orthonormal basis for $\mathfrak{k}$. Define four operators

$$\Delta_{\mathfrak{k}} := \sum_{j=1}^{d_K} \widetilde{X}_j^2, \qquad \Delta_{\mathfrak{k}^\perp} := \sum_{j=d_K+1}^{d_G} \widetilde{X}_j^2, \qquad \begin{aligned} \Delta_{\mathfrak{g}} &:= \Delta_{\mathfrak{k}} + \Delta_{\mathfrak{k}^\perp} \\ \Delta'_{\mathfrak{g}} &:= \tfrac{1}{2}\Delta_{\mathfrak{k}} + \Delta_{\mathfrak{k}^\perp}. \end{aligned}$$

By Theorem 3.7, all four operators are essentially self-adjoint in $L^2(G)$, with $C_c^\infty(G)$ as a common core. The operator $\Delta_{\mathfrak{g}}$ is the Laplacian on G determined by $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$. Also notice that $\Delta_{\mathfrak{g}} = \frac{1}{2}\Delta_{\mathfrak{k}} + \Delta'_{\mathfrak{g}}$, and that $\Delta'_{\mathfrak{g}}$ is the Laplacian associated to the modified inner product $\langle \cdot, \cdot \rangle'_{\mathfrak{g}}$ defined as follows: for $X_1, X_2 \in \mathfrak{k}$ an $Y_1, Y_2 \in \mathfrak{k}^\perp$,

$$\langle X_1 + Y_1, X_2 + Y_2 \rangle'_{\mathfrak{g}} := \frac{1}{2}\langle X_1, X_2 \rangle_{\mathfrak{g}} + \langle Y_1, Y_2 \rangle_{\mathfrak{g}}.$$

Using Lemma 3.31, it is straightforward to verify that $\langle \cdot, \cdot \rangle'_{\mathfrak{g}}$ is also $\mathrm{Ad}(K)$-invariant. Its restriction to $\mathfrak{k}$ is just $\frac{1}{2}$ times the original inner product, and $\frac{1}{2}\Delta_{\mathfrak{k}}$ is the Laplacian associated to this scaled inner product on $\mathfrak{k}$ (considered here as an operator on G; cf. Proposition 3.25). Thus, by Theorem 3.30 in this context,

$$\rho_t^{\Delta_{\mathfrak{g}}}(x) = \int_K \rho_t^{\Delta'_{\mathfrak{g}}}(xk^{-1})\, \rho_t^{\frac{1}{2}\Delta_K}(k)\, dk \quad \text{for all } t > 0, x \in G. \tag{3.23}$$

Let

$$\nu_t(x) := \int_K \rho_t^{\Delta'_{\mathfrak{g}}}(xk^{-1})\, dk \qquad \text{for all } x \in G$$

and

$$C(t) := \left[\max_{k \in K} \rho_t^{\frac{1}{2}\Delta_K}(k)\right]^2 \vee \left[\min_{k \in K} \rho_t^{\frac{1}{2}\Delta_K}(k)\right]^{-2}. \tag{3.24}$$

The constant $C(t)$ is finite and locally bounded in t because the heat kernel $\rho_t^{\frac{1}{2}\Delta_K}(k)$ is a continuous positive function of $(t, k) \in (0, \infty) \times K$; cf. Theorem 3.12. Using (3.23) and the bi-invariance of the Haar measure on K, it is readily verified that

$$\rho_t^{\Delta_{\mathfrak{g}}} \asymp_{\sqrt{C(t)}} \nu_t, \qquad \text{and} \qquad \nu_t \circ R_k = \nu_t \qquad \text{for all } k \in K.$$

Hence, for any $k \in K$, we have both $\rho_t^{\Delta_{\mathfrak{g}}} \asymp_{\sqrt{C(t)}} \nu_t$ and $\rho_t^{\Delta_{\mathfrak{g}}} \circ R_k \asymp_{\sqrt{C(t)}} \nu_t \circ R_k = \nu_t$. Thus

$$\rho_t^{\Delta_{\mathfrak{g}}} \circ R_k \asymp_{C(t)} \rho_t^{\Delta_{\mathfrak{g}}}. \tag{3.25}$$

The result follows now by integrating (3.25) against $\gamma(dk)$. □

Remark 3.34. The constant $C(t)$ in Theorem 3.33 (in (3.24)) depends on the restriction of the inner product to K. As noted in the proof, $C(t)$ is bounded for t in compact subsets of $(0, \infty)$.

Remark 3.35. Equation (3.25), used in the proof of Theorem 3.33, is in fact equivalent to the statement of the theorem. Indeed, (3.25) follows from Eq. (3.22) with $\gamma = \delta_k$ where δ_k is the unit point-mass measure concentrated at $k \in K$. In light of this observation, Theorem 3.33 may be interpreted as saying that the heat kernel measure $\rho_t^{\Delta_\mathfrak{g}}(x)\,dx$ is "*uniformly* quasi-invariant" under right multiplication by K. That is,

$$C(t)^{-1}\rho_t^{\Delta_\mathfrak{g}}(xk) \le \rho_t^{\Delta_\mathfrak{g}}(x) \le C(t)\rho_t^{\Delta_\mathfrak{g}}(xk) \qquad \text{for all } k \in K \text{ and } x \in G.$$

The K-averaged heat kernel will be used to determine the range of the complex-time Segal–Bargmann transform in Section 4.3 and it will also play a role in Section 5.3.

4. Invariant Metrics and Measures on $K_\mathbb{C}$

If G is a Lie group and $K \subseteq G$ is a compact Lie subgroup with Lie algebra $\mathfrak{k}$, then one can produce an $\mathrm{Ad}(K)$-invariant inner product on $\mathrm{Lie}(G)$ by averaging any inner product over the Adjoint representation of K, as above. This raises the question: how many $\mathrm{Ad}(K)$-invariant inner products does G possess? We now answer this question in the case that K is simple (and compact type), and $G = K_\mathbb{C}$ is the complexification of K.

4.1. Invariant Inner Products and Laplacians on $K_\mathbb{C}$.

Fix a compact-type Lie group K, and an $\mathrm{Ad}(K)$-invariant inner product $\langle\cdot,\cdot\rangle_\mathfrak{k}$ on its Lie algebra $\mathfrak{k}$. Let $K_\mathbb{C}$ denote the complexification of K (cf. Section 2.1); in particular $\mathfrak{k}_\mathbb{C} \equiv \mathrm{Lie}(K_\mathbb{C}) = \mathfrak{k} \oplus J\mathfrak{k}$. So every vector $Z \in \mathfrak{k}_\mathbb{C}$ has a unique decomposition $Z = X + JY$ for $X, Y \in \mathfrak{k}$.

Consider the following three-parameter family of inner products on $K_\mathbb{C}$:

$$\langle X_1 + JY_1, X_2 + JY_2\rangle_{a,b,c} := a\langle X_1, X_2\rangle_\mathfrak{k} + b\langle Y_1, Y_2\rangle_\mathfrak{k} + c(\langle X_1, Y_2\rangle_\mathfrak{k} + \langle X_2, Y_1\rangle_\mathfrak{k}) \quad (4.1)$$

for $X_1, X_2, Y_1, Y_2 \in \mathfrak{k}$, where $a, b > 0$ and $c^2 < ab$. It is straightforward to verify that the symmetric bilinear forms in (4.1) are real inner products on $\mathfrak{k}_\mathbb{C}$ (precisely under the conditions on a, b, c stated below the equation), and are all $\mathrm{Ad}(K)$-invariant. The main theorem of this section is that, in the case that K is simple, this is a complete characterization of all $\mathrm{Ad}(K)$-invariant inner products on $K_\mathbb{C}$. First, recall:

Definition 4.1. *A Lie group K is called* simple *if* $\dim K \ge 2$*, and the Lie algebra $\mathfrak{k}$ of K has no nontrivial ideals.*

Theorem 4.2. *If K is a simple (or 1-dimensional) compact type real Lie group, then $\mathfrak{k}$ has a unique (up to scale) Ad-invariant real inner product $\langle\cdot,\cdot\rangle_\mathfrak{k}$, and* all *$\mathrm{Ad}(K)$-invariant real inner products on $\mathfrak{k}_\mathbb{C}$ have the form* (4.1)*.*

Remark 4.3. For example, $K = \mathrm{SU}(n)$ is simple, with complexification $K_\mathbb{C} = \mathrm{SL}(n,\mathbb{C})$. Hence (4.1) characterizes all $\mathrm{Ad}(\mathrm{SU}(n))$-invariant inner products on $\mathrm{SL}(n,\mathbb{C})$, where $\langle X, Y\rangle_{\mathfrak{su}(n)} = \mathrm{Tr}(XY^*) = -\mathrm{Tr}(XY)$ is the unique (up to scale) Ad-invariant inner product on $\mathfrak{su}(n)$. In that case, the family can be written explicitly in terms of the trace as

$$\langle A, B\rangle_{a,b,c} = \frac{1}{2}(b+a)\mathrm{Re}\,\mathrm{Tr}(AB^*) + \frac{1}{2}\mathrm{Re}\left[(b - a + 2ic)\mathrm{Tr}(AB)\right]. \qquad (4.2)$$

Extending to $\mathrm{U}(n)$ and its complexification $\mathrm{GL}(n,\mathbb{C})$, it is easy to compute that all $\mathrm{Ad}(\mathrm{U}(n))$-invariant inner products on $\mathfrak{gl}(n,\mathbb{C})$ are of the form (4.1) plus one more term, involving the 1-dimensional subspace spanned by the identity matrix; extending (4.2), there is one more term involving $\mathrm{Tr}(A)\mathrm{Tr}(B)$. In [5, 28, 29], the third author studied the large-n limits of the diffusion processes on $\mathrm{GL}(n,\mathbb{C})$ invariant with respect to the inner products $\langle\ ,\ \rangle_{a,b,0}$. Part of the motivation for the present work was the question of whether those were the largest class of appropriately invariant diffusions; the answer provided by Theorem 4.2 is no, and the associated diffusions will be explored in a future publication.

Remark 4.4. The first statement of Theorem 4.2, that the Ad-invariant inner product on K is unique up to scale when K is simple, is well-known; it was proved, for example, in [34, Lemma 7.6].

We will use Schur's lemma as a tool in the proof of Theorem 4.2, but this is complicated by the fact that the inner products in question are *real*. We must therefore be careful about how and when we complexify.

Lemma 4.5. *If K is a simple (real) Lie group with Lie algebra $\mathfrak{k}$, then the (real) Adjoint representation of K on $\mathfrak{k}$ is irreducible. Moreover, if K is compact type, then the (complex) Adjoint representation of K on $\mathfrak{k}_{\mathbb{C}}$ is also irreducible.*

Proof. If $\mathcal{I} \subseteq \mathfrak{k}$ is an invariant real subspace for $\mathrm{Ad}(K)$, then $\mathrm{Ad}_{e^{tX}}(Y) \in \mathcal{I}$ for all $t \in \mathbb{R}$, $X \in \mathfrak{k}$, and $Y \in \mathcal{I}$. Taking the derivative at $t = 0$ shows that $\mathrm{ad}_X(Y) = [X, Y] \in \mathcal{I}$ for all $X \in \mathfrak{k}$ and $Y \in \mathcal{I}$, which means $\mathcal{I} \subseteq \mathfrak{k}$ is an ideal in $\mathfrak{k}$. Thus $\mathcal{I} \in \{0, \mathfrak{k}\}$, yielding the first statement of the lemma.

Now, [23, Theorem 7.32] states that the simplicity of $\mathfrak{k}$ implies that $\mathfrak{k}_{\mathbb{C}}$ is also simple as a complex Lie algebra. (The statement given there assumes K is compact, but the proof only uses the fact that it is compact type.) So, let $\mathcal{J} \subseteq \mathfrak{k}_{\mathbb{C}}$ be an invariant complex subspace for $\mathrm{Ad}(K)$. The same argument above shows that $[X, W] \in \mathcal{J}$ for all $X \in \mathfrak{k}$ and $W \in \mathcal{J}$. Any $Z \in \mathfrak{k}_{\mathbb{C}}$ has the form $Z = X + JY$ for $X, Y \in \mathfrak{k}$, and by (2.1), we therefore have

$$[Z, W] = [X + JY, W] = [X, W] + J[Y, W] \in \mathcal{J} + J\mathcal{J} = \mathcal{J}, \qquad \forall\, Z \in \mathfrak{k}_{\mathbb{C}}, W \in \mathcal{J}$$

where the final equality follows from the fact that $\mathcal{J}$ is a *complex* subspace. Hence $\mathcal{J}$ is a complex ideal in $\mathfrak{k}_{\mathbb{C}}$, and therefore $\mathcal{J} \in \{0, \mathfrak{k}_{\mathbb{C}}\}$. This concludes the proof of the second statement. □

We now prove the algebraic result that constitutes most of the proof of Theorem 4.2.

Proposition 4.6. *Let K be a simple (or 1-dimensional) real compact-type Lie group, and fix an Ad-invariant inner product $\langle\cdot,\cdot\rangle_{\mathfrak{k}}$ on its Lie algebra $\mathfrak{k}$. If $\mathcal{B}\colon \mathfrak{k}_{\mathbb{C}} \times \mathfrak{k}_{\mathbb{C}} \to \mathbb{R}$ is an $\mathrm{Ad}(K)$-invariant symmetric bilinear form, then $\mathcal{B}$ has the form (4.1) for some $a, b, c \in \mathbb{R}$.*

Proof. The result is straightforward when K is 1-dimensional, so we focus on the case that K is simple. We use the inner product $\langle\cdot,\cdot\rangle_{1,1,0}$ (cf. (4.1)) as a reference; there is then some endomorphism $M\colon \mathfrak{k}_{\mathbb{C}} \to \mathfrak{k}_{\mathbb{C}}$ such that

$$\mathcal{B}(Z, W) = \langle Z, M(W)\rangle_{1,1,0} \qquad \forall\, Z, W \in \mathfrak{k}_{\mathbb{C}}.$$

The symmetry of $\mathcal{B}$ forces M to be self-adjoint. By Proposition 2.1, $\mathfrak{k}_{\mathbb{C}} = \mathfrak{k} \oplus J\mathfrak{k}$; we view this as isomorphic to $\mathfrak{k} \oplus \mathfrak{k}$ (and so view the inner product on this space as well). Thus we can decompose the endomorphism M in block diagonal form

$$M = \begin{bmatrix} A & C \\ C^\top & B \end{bmatrix} \tag{4.3}$$

where A and B are symmetric matrices.

Now, by (2.4), the Adjoint representation of K commutes with J; it follows that, under the isomorphism $\mathfrak{k}_{\mathbb{C}} \cong \mathfrak{k} \oplus \mathfrak{k}$, Ad_k acts diagonally for all $k \in K$. Using the fact that both the inner product $\langle\cdot,\cdot\rangle_{1,1,0}$ and the bilinear form $\mathcal{B}$ are Ad_k-invariant, it is straightforward to compute that the matrices A, B, C, and $C^\top$ all commute with Ad_k for each $k \in K$. The same therefore applies to the complex-linear extensions of these endomorphisms to $\mathfrak{k}_{\mathbb{C}}$. It then follows from Lemma 4.5 and Schur's lemma that there are constants $a, b, c \in \mathbb{C}$ with $A = aI$, $B = bI$, and $C = C^\top = cI$. Since each of the endomorphisms preserves the real subspace $\mathfrak{k}$, it follows that $a, b, c \in \mathbb{R}$.

Hence, for $Z = X + JY \in \mathfrak{k}_{\mathbb{C}}$, (4.3) yields $M(Z) = (aX + cY) + J(cX + bY)$. From the definition of the inner product $\langle\cdot,\cdot\rangle_{1,1,0}$, we therefore have

$$\begin{aligned}\mathcal{B}(X_1 + JY_1, X_2 + JY_2) &= \langle X_1 + JY_1, aX_2 + cY_2 + J(cX_2 + bY_2)\rangle_{1,1,0} \\ &= \langle X_1, aX_2 + cY_2\rangle_{\mathfrak{k}} + \langle Y_1, cX_2 + bY_2\rangle_{\mathfrak{k}} \\ &= a\langle X_1, X_2\rangle_{\mathfrak{k}} + c\langle X_1, Y_2\rangle_{\mathfrak{k}} + c\langle Y_1, X_2\rangle_{\mathfrak{k}} + b\langle Y_1, Y_2\rangle_{\mathfrak{k}} \\ &= \langle X_1 + JY_1, X_2 + JY_2\rangle_{a,b,c}\end{aligned}$$

concluding the proof. □

The proof of Theorem 4.2 now follows quite easily.

Proof of Theorem 4.2. Let $\langle\cdot,\cdot\rangle_{\mathfrak{k}}$ and $\langle\cdot,\cdot\rangle'_{\mathfrak{k}}$ denote two Ad-invariant inner products on K. We may view the second inner product as a symmetric (degenerate) bilinear form on $\mathfrak{k}_{\mathbb{C}}$, which is $\mathrm{Ad}(K)$-invariant. By Proposition 4.6, it follows that $\langle\cdot,\cdot\rangle'_{\mathfrak{k}} = a\langle\cdot,\cdot\rangle_{\mathfrak{k}}$ for some $a \in \mathbb{R}$ (the other terms in (4.1) are 0); the fact that both are inner products forces $a > 0$. This proves the uniqueness, up to scale, of the Ad-invariant inner product on K.

Now, any real inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{k}_{\mathbb{C}}$ is a symmetric bilinear form on $\mathfrak{k}_{\mathbb{C}}$, and so by $\mathrm{Ad}(K)$-invariance, Proposition 4.6 shows that it has the form (4.1) for some $a, b, c \in \mathbb{R}$. Since it is an inner product, it follows that the matrix M of (4.3) is positive definite, and given its block diagonal form, this is equivalent to $a, b > 0$ and $ab - c^2 > 0$. This concludes the proof. □

We now turn to the Laplacian on $\mathfrak{k}_{\mathbb{C}}$ associated to the inner product (4.1). The notation $\Delta_{\mathfrak{k}_{\mathbb{C}}}$ for this Laplacian (cf. Section 3.1) is lacking, as it depends upon the specified inner product on $\mathfrak{k}_{\mathbb{C}}$ without notation to refer to it. Thus, we refer to the Laplacian in this case as $L_{a,b,c}$. Lemma 3.3 allows us to compute it quickly.

Proposition 4.7. *Let $L_{a,b,c}$ denote the Laplacian on $C^2(K_{\mathbb{C}})$ associated to the inner product $\langle\cdot,\cdot\rangle_{a,b,c}$ of* (4.1)*. Fix any basis $\{X_j\}_{j=1}^d$ of $\mathfrak{k}$ orthonormal with respect to the given $\mathrm{Ad}(K)$-invariant inner product on $\mathfrak{k}$, and let $Y_j = JX_j$. Then*

$$L_{a,b,c} = \frac{1}{ab - c^2}\sum_{j=1}^{d}\left[b\widetilde{X}_j^2 + a\widetilde{Y}_j^2 - 2c\widetilde{X}_j\widetilde{Y}_j\right]. \tag{4.4}$$

Proof. Since $\mathfrak{k}_{\mathbb{C}} = \mathfrak{k} \oplus J\mathfrak{k}$ (cf. Proposition 2.1), the set $\{X_j, Y_j\}_{j=1}^d$ is a basis for $\mathfrak{k}_{\mathbb{C}}$. Let $V_{2k-1} = X_k$ and $V_{2k} = Y_k$ for $1 \le k \le d$, and define $q_{ij} = \langle V_i, V_j\rangle_{a,b,c}$. By Lemma 3.3,

$$L_{a,b,c} = \sum_{i,j=1}^{d} q_{ij}^{-1}\widetilde{V}_i\widetilde{V}_j. \tag{4.5}$$

We can compute directly from (4.1) and the orthonormality of $\{X_j\}_{j=1}^d$ that

$$\langle X_i, X_j\rangle_{a,b,c} = a\delta_{ij}, \quad \langle Y_i, Y_j\rangle_{a,b,c} = b\delta_{ij}, \quad \langle X_i, Y_j\rangle_{a,b,c} = \langle Y_i, X_j\rangle_{a,b,c} = c\delta_{ij}.$$

It follows that the matrix q is block diagonal with 2×2 diagonal blocks all equal to the matrix B (below). Thus q^{-1} is also block diagonal with 2×2 diagonal blocks all equal to B^{-1} (below).

$$B = \begin{bmatrix} a & c \\ c & b \end{bmatrix}, \qquad B^{-1} = \frac{1}{ab - c^2}\begin{bmatrix} b & -c \\ -c & a \end{bmatrix}.$$

Combining this with (4.5) yields (4.4). □

To dispense with the cumbersome determinant in the denominator in (4.4), and match the parametrization relevant to the Segal–Bargmann transform, we make the following change of parametrization:

$$(s,t,u) = \Phi(a,b,c) := \frac{1}{ab-c^2}(a+b, 2a, 2c). \tag{4.6}$$

It is straightforward to verify that Φ is a diffeomorphism

$$\Phi\colon \{(a,b,c)\colon a,b>0, c^2<ab\} \to \{(s,t,u)\colon t>0, u\in\mathbb{R}, 2s>t+u^2/t\}$$

with inverse

$$(a,b,c) = \Phi^{-1}(s,t,u) = \frac{4}{2st-t^2-u^2}(\tfrac{t}{2}, s-\tfrac{t}{2}, \tfrac{u}{2}) = \tfrac{1}{\alpha}(\tfrac{t}{2}, s-\tfrac{t}{2}, \tfrac{u}{2}) \tag{4.7}$$

referring to the constant α of (1.10), which is positive precisely in range of Φ. From here on, we use the parameters (s,t,u) which leads to the notation used in Definition 1.4 of $\Delta_{s,\tau}$ on $K_{\mathbb{C}}$ in the introduction. In particular, this means that the Laplacian $\Delta_{s,\tau}$ corresponds to the inner product $\langle\cdot,\cdot\rangle_{a,b,c}$ where (a,b,c) are given as in (4.7). The fact that Φ is a bijections shows that there is a one-to-one correspondence between the Laplacians $\Delta_{s,\tau}$ and the inner products $\langle\cdot,\cdot\rangle_{a,b,c}$.

4.2. **Invariant Heat Kernels on K and $K_{\mathbb{C}}$.** Let us now fix notation for the heat kernels relevant to the Segal–Bargmann transform.

Definition 4.8. *Let K be a compact-type Lie group, with a fixed $\mathrm{Ad}(K)$-invariant inner product. Denote by Δ_K the associated Laplacian acting on $C^\infty(K)$; for $t\ge 0$, denote by $\rho_t = \rho_t^{\Delta_K}\colon K\to\mathbb{R}_+$ the associated heat kernel (cf. Theorem 3.12 applied with G replaced by K and $\Delta_{\mathfrak{g}}$ replaced by Δ_K). In addition, for $s>0$ and $\tau\in\mathbb{D}(s,s)$, denote by $\mu_{s,\tau}\colon K_{\mathbb{C}}\to\mathbb{R}_+$ the heat kernel (at time 1) associated to the Laplacian $\Delta_{s,\tau}$ of Definition 1.4.*

To be clear:

$$e^{\frac{t}{2}\bar{\Delta}_K} = \int_K \rho_t(k)\,\hat{R}_k\,dk \qquad \text{which is an operator on } L^2(K), \tag{4.8}$$

$$e^{\frac{t}{2}\bar{\Delta}_{\mathfrak{k}}} = \int_K \rho_t(k)\,\hat{R}_k\,dk \qquad \text{which is an operator on } L^2(K_{\mathbb{C}}), \text{ and}$$

$$e^{\frac{1}{2}\bar{\Delta}_{s,\tau}} = \int_{K_{\mathbb{C}}} \mu_{s,\tau}(z)\,\hat{R}_z\,dz \quad \text{which is an operator on } L^2(K_{\mathbb{C}}). \tag{4.9}$$

For the heat operator (4.9), we could include an additional time parameter $(e^{\frac{r}{2}\bar{\Delta}_{s,\tau}})_{r>0}$, but following Definition 1.4 we see that $r\Delta_{s,\tau} = \Delta_{rs,r\tau}$, so there is no loss in absorbing this extra parameter into ones already present.

Following the discussion after (4.7), $\Delta_{s,\tau}$ is the Laplacian for an $\mathrm{Ad}(K)$-invariant inner product. Therefore Corollary 3.28 applies:

$$\begin{aligned} [e^{\frac{1}{2}\bar{\Delta}_{s,\tau}}, \hat{R}_k] &= 0 \\ \mu_{s,\tau}(kzk^{-1}) &= \mu_{s,\tau}(z) \end{aligned} \qquad \text{for } k\in K,\ z\in K_{\mathbb{C}}. \tag{4.10}$$

Also Theorem 3.30 applies: $e^{\frac{r}{2}\bar{\Delta}_{\mathfrak{k}}}$ and $e^{\frac{1}{2}\Delta_{s,\tau}}$ commute for all r,s,τ.

Assumption 4.9. *For the remainder of this section, we assume K is compact.*

Definition 4.10 (K-averaging)**.** *Let P be the K**-averaging operator** defined on $L^1_{\mathrm{loc}}(K_{\mathbb{C}})$ by*

$$P = \Gamma_{\mathrm{Haar}} := \int_K \hat{R}_k \, dk,$$

or more explicitly by

$$(Pf)(z) = \int_K f(zk)\, dk$$

where dk denotes the Haar probability measure on K. (So K-averaging is γ-averaging, as in Theorem 3.33, in the case that γ is the Haar probability measure on K.)

Since the Haar measure on K is invariant under inversion and the convolution with itself is still Haar measure, it follows from Lemma 3.24 that $P\colon L^2(K_{\mathbb{C}}) \to L^2(K_{\mathbb{C}})$ is an orthogonal projection. The operator P also preserves the subspaces $C^\infty(K_{\mathbb{C}})$ and $C_c^\infty(K_{\mathbb{C}})$ and if $f \in C(K_{\mathbb{C}})$ we have $Pf(zk) = Pf(z)$ for all $k \in K$ and $z \in K_{\mathbb{C}}$. In short, $\hat{R}_k P = P$ for all $k \in K$. If $s > 0$ and $\tau = t + iu \in \mathbb{D}(s,s)$ we refer to $P\mu_{s,\tau}$ as a K**-averaged heat kernel** on $K_{\mathbb{C}}$; see Definition 1.8. Recall from Definition 1.8 that $\nu_t := P\mu_{t,t} \in C^\infty(K_{\mathbb{C}}, (0,\infty))$ is the K-averaged version of $\mu_{t,t}$. The next lemma shows that ν_t is also the K-averaged version of $\mu_{s,\tau}$ whenever $\operatorname{Re}\tau = t$.

Lemma 4.11 (K-averaged heat kernels)**.** *If $s > 0$ and $\tau = t + iu \in \mathbb{D}(s,s)$, then*

$$\int_K \mu_{s,\tau}(zk)\, dk = (P\mu_{s,\tau})(z) = \nu_t(z) \quad \text{for all } \ z \in K_{\mathbb{C}}.$$

Proof. If $X \in \mathfrak{k}$ and $f \in C_c^\infty(K_{\mathbb{C}})$, then $(Pf)(ze^{rX}) = (Pf)(z)$ for all $z \in K_{\mathbb{C}}$ and $r \in \mathbb{R}$. Differentiating at $r = 0$ shows that $\widetilde{X}Pf = 0$ for any $X \in \mathfrak{k}$. Using the fact that $\widetilde{X}_j\widetilde{Y}_j = \widetilde{Y}_j\widetilde{X}_j$, which follows from the definition $Y_j = JX_j$ and (2.1), it follows from Definition 1.4 that

$$\Delta_{s,\tau} P = \frac{t}{2}\Delta_{J\mathfrak{k}} P = P\frac{t}{2}\Delta_{J\mathfrak{k}} \ \text{ on } \ C_c^\infty(K_{\mathbb{C}}), \quad \text{ where } \Delta_{J\mathfrak{k}} := \sum_{j=1}^d \widetilde{Y}_j^2. \tag{4.11}$$

For the last equality we used Lemma 3.5 to conclude that $[P, \Delta_{J\mathfrak{k}}] = 0$ on $C_c^\infty(K_{\mathbb{C}})$. An application of Lemma 3.8 with $Q = P$, $A = \Delta_{s,\tau}$, and $B = \frac{t}{2}\Delta_{J\mathfrak{k}}$ gives, $Pe^{\frac{t}{2}\bar{\Delta}_{J\mathfrak{k}}} = e^{\bar{\Delta}_{s,\tau}}P$ for all $\tau \in \mathbb{D}\,(s,s)$ with $\operatorname{Re}\tau = t$. In particular we may conclude that

$$e^{\bar{\Delta}_{s,\tau}}P = e^{\bar{\Delta}_{s,t}}P \quad \forall\, \tau = t + iu \in \mathbb{D}\,(s,s) \tag{4.12}$$

or equivalently that

$$\langle e^{\bar{\Delta}_{s,\tau}}Pv, w\rangle_{L^2(K_{\mathbb{C}})} = \langle e^{\bar{\Delta}_{t,t}}Pv, w\rangle_{L^2(K_{\mathbb{C}})} \quad \forall\, u, v \in C_c(K_{\mathbb{C}}, \mathbb{R}). \tag{4.13}$$

For the rest of the proof let $\bar{\mu}_{s,\tau} = P\mu_{s,\tau}$ be the K-average of $\mu_{s,\tau}$. We may rewrite the left-hand-side of (4.13) as,

$$\begin{aligned}\langle e^{\bar{\Delta}_{s,\tau}} Pv, w\rangle_{L^2(K_{\mathbb{C}})} &= \int_{K_{\mathbb{C}}^2} \mu_{s,\tau}(g)(Pv)(zg)w(z)\,dg\,dz \\ &= \int_{K_{\mathbb{C}}^2\times K} \mu_{s,\tau}(g)v(zgk)w(z)\,dg\,dz\,dk \\ &= \int_{K_{\mathbb{C}}^2\times K} \mu_{s,\tau}(gk^{-1})v(zg)w(z)\,dg\,dz\,dk \\ &= \int_{K_{\mathbb{C}}^2\times K} \mu_{s,\tau}(gk)v(zg)w(z)\,dg\,dz\,dk \\ &= \int_{K_{\mathbb{C}}^2} \bar{\mu}_{s,\tau}(g)v(zg)w(z)\,dg\,dz.\end{aligned}$$

This equation with $\tau = t$ also shows the right-hand-side of (4.13) is given by

$$\langle e^{\bar{\Delta}_{t,t}} Pv, w\rangle_{L^2(K_{\mathbb{C}})} = \int_{K_{\mathbb{C}}^2} \nu_t(g)v(zg)w(z)\,dg\,dz.$$

Comparing the last two identities shows, for all $v, w \in C_c(K_{\mathbb{C}})$,

$$\int_{K_{\mathbb{C}}^2} \bar{\mu}_{s,\tau}(g)v(zg)w(z)\,dg\,dz = \int_{K_{\mathbb{C}}^2} \nu_t(g)v(zg)w(z)\,dg\,dz.$$

As $C_c(K_{\mathbb{C}})$ is dense in $L^2(K_{\mathbb{C}})$, we may conclude that for all $v \in C_c(K_{\mathbb{C}})$,

$$\int_{K_{\mathbb{C}}} \bar{\mu}_{s,\tau}(g)v(zg)\,dg = \int_{K_{\mathbb{C}}} \nu_t(g)v(zg)\,dg \quad \text{for a.e. } z$$

and hence for every $z \in K_{\mathbb{C}}$ as both sides of the previous equation are continuous in z. Thus, taking $z = e$, it follows that,

$$\int_{K_{\mathbb{C}}} \bar{\mu}_{s,\tau}(g)v(g)\,dg = \int_{K_{\mathbb{C}}} \nu_t(g)v(g)\,dg \;\forall\; v \in C_c(K_{\mathbb{C}}, \mathbb{R}).$$

So as above, the density of $C_c(K_{\mathbb{C}})$ in $L^2(K_{\mathbb{C}})$ along with the continuity of both $\bar{\mu}_{s,\tau}$ and ν_t, allows us to conclude that $\bar{\mu}_{s,\tau}(g) = \nu_t(g)$ for all $g \in K_{\mathbb{C}}$. □

Remark 4.12. By Theorem 3.33, $\mu_{s,\tau} \asymp_{C(s,\tau)} \nu_t$ for some constant $C(s,\tau)$. (Recall Notation 3.32.) As a consequence of Lemma 4.11, we also have $\mu_{t,t} \asymp_{C(t,t)} \nu_t$. It follows that $\mu_{s,\tau} \asymp_{C(t,t)C(s,\tau)} \mu_{t,t}$; i.e., all the densities $\mu_{s,t+iu}$ with t fixed are equivalent. It is worth commenting on the constant $C(s,\tau)$; it is bounded by the expression (3.24). In this context, we have the $\mathrm{Ad}(K)$-invariant inner product $\langle\cdot,\cdot\rangle_{a,b,c}$ with $(a,b,c) = \frac{1}{\alpha}(\frac{t}{2}, s-\frac{t}{2}, \frac{u}{2})$ (cf. (4.7)) which induces the heat kernel $\mu_{s,\tau}$. The restriction of this inner product to $\mathfrak{k}$ is

$$\langle X_1, X_2\rangle = \frac{t}{2\alpha}\langle X_1, X_2\rangle_{\mathfrak{k}},$$

i.e., it is a scalar multiple of the fixed background $\mathrm{Ad}(K)$-invariant inner product on $\mathfrak{k}$, where the scaling factor $\frac{t}{2\alpha} = \frac{2t}{2st-t^2-u^2}$ depends on all three parameters s, t, u continuously. The constant in (3.24) is determined by the absolute minimum and maximum values attained by the heat kernel $\rho_t^{s,\tau,K}$ on K corresponding to $\frac{1}{2}$ this Laplacian (at time 1, since

$\mu_{s,\tau}$ is the heat kernel for $\Delta_{s,\tau}$ at time 1). Let ρ^{Δ_K} denote the heat kernel on K determined by the reference metric $\langle\cdot,\cdot\rangle_{\mathfrak{k}}$. By definition, we have

$$\rho_1^{s,\tau,K} = \rho^{\Delta_K}_{t/(2st-t^2-u^2)}.$$

Since the heat kernel $\rho_t^{\Delta_K}$ is continuous (in fact smooth) in t, cf. Theorem 3.12, it follows that $C(s,\tau)$ depends continuously on (s,τ) in the domain $\tau\in\mathbb{D}(s,s)$ of interest. In particular, this means that $\mu_{s,\tau}$, $\mu_{t,t}$, and ν_t are all equivalent by constants that are locally uniformly bounded in (s,τ).

The K-averaged heat kernel, ν_t, will appear in an essential way in Section 5.3. It will also arise as a technical tool in Section 4.3 below.

4.3. **Two Density Results for Matrix Entries.** To show that the complex-time Segal–Bargmann transform maps *onto* $\mathcal{H}L^2(K_{\mathbb{C}},\mu_{s,\tau})$, we require a density result. Recall (Definition 3.13) the notion of a matrix entry on an arbitrary Lie group and the notion of a holomorphic matrix entry on a complex Lie group. The main result of this section is the following theorem

Theorem 4.13. *Let K be a real Lie group of compact type. For any $s>0$, the matrix entries on K are dense in $L^2(K,\rho_s)$. If $s>0$ and $\tau\in\mathbb{D}(s,s)$, then the holomorphic matrix entries on $K_{\mathbb{C}}$ are dense in $\mathcal{H}L^2(K_{\mathbb{C}},\mu_{s,\tau})$.*

Proof. We consider first the case that $K=\mathbb{R}^d$ and $K_{\mathbb{C}}=\mathbb{C}^d$. Then ρ_s is a Gaussian measure on K. Since every polynomial on $\mathbb{R}^d$ is a matrix entry, we may appeal to the classical result that polynomials are dense in L^2 of Gaussian measures on $\mathbb{R}^d$. (For a proof of a more general result, see [10, Theorem 3.6].) On the complex side, every holomorphic polynomial is a holomorphic matrix entry, and the measure $\mu_{s,\tau}$ on $\mathbb{C}^d$ is Gaussian. Thus, by [10, Proposition 3.5], matrix entries are dense in $\mathcal{H}L^2(\mathbb{C}^d,\mu_{s,\tau})$. (Note that, in general, the measure $\mu_{s,\tau}$ is not invariant under multiplication by $e^{i\theta}$ and monomials of different degrees are not necessarily orthogonal. Thus the proof of density of holomorphic polynomials in [1, Section 1b] does not apply.)

We consider next the case that K is compact. In that case, the heat kernel density ρ_s on K is bounded and bounded away from zero for each fixed $s>0$. Thus, the Hilbert space $L^2(K,\rho_s)$ is the same as the Hilbert space $L^2(K)$, with a different but equivalent norm. Hence, the density of matrix entries in $L^2(K,\rho_s)$ follows from the Peter–Weyl theorem. On the complex side, we appeal to the averaging result in Theorem 3.33, which tells us that the Hilbert space $\mathcal{H}L^2(K_{\mathbb{C}},\mu_{s,\tau})$ is the same as the Hilbert space $\mathcal{H}L^2(K_{\mathbb{C}},\nu_t)$, with a different but equivalent norm. Thus, it suffices to establish the density of matrix entries in $\mathcal{H}L^2(K_{\mathbb{C}},\nu_t)$; this claim follows verbatim from the proof of the "onto" part of Theorem 2 in [18, Section 8]; we give a very brief outline of the idea of this proof at the end of this section below.

We consider finally the case of a general compact-type group K. Recall (Proposition 2.1) that K is isometrically isomorphic to $K_0\times\mathbb{R}^d$ for some compact Lie group K_0 and some $d\geq 0$. Thus, the heat kernel measure ρ_s on K factors as a product of the heat kernel measures ρ_s^0 on K_0 and ρ_s^1 on $\mathbb{R}^d$. Now, a standard result from measure theory tells us that there is a unitary map U from $L^2(K_0,\rho_s^0)\otimes L^2(\mathbb{R}^d,\rho_s^1)$ onto $L^2(K,\rho_s)$ uniquely determined by the requirement that $U(f_1\otimes f_2)(x_1,x_2)=f_1(x_1)f_2(x_2)$. If f_1 and f_2 are matrix entries on K_0 and $\mathbb{R}^d$, respectively, then $f_1(x_1)f_2(x_2)$ is a matrix entry on K (by an argument very similar to the proof of Lemma 3.15). Using the density results for K_0 and for $\mathbb{R}^d$ and the unitary map U, we can easily show that linear combinations of matrix entries of this sort (which are again matrix entries) are dense in $L^2(K,\rho_s)$.

On the complex side, recall (Proposition 2.1) that $K_{\mathbb{C}}$ is isomorphic to $(K_0)_{\mathbb{C}} \times \mathbb{C}^d$. If we restrict our Ad-invariant inner product on $\mathfrak{k}$ to the Lie algebras of K_0 and of $\mathbb{R}^d$, these restrictions will also be Ad-invariant. We may then construct left-invariant metrics on $(K_0)_{\mathbb{C}}$ and $\mathbb{C}^d$ by the same procedure as for $K_{\mathbb{C}}$. In that case, it is easily verified that the isomorphism $K_{\mathbb{C}} \cong (K_0)_{\mathbb{C}} \times \mathbb{C}^d$ is isometric. Thus, the heat kernel measure $\mu_{s,\tau}$ on $K_{\mathbb{C}}$ is a product of the associated heat kernel measures $\mu^0_{s,\tau}$ on $(K_0)_{\mathbb{C}}$ and $\mu^1_{s,\tau}$ on $\mathbb{C}^d$.

Then, as on the real side, we have a unitary map V from $L^2((K_0)_{\mathbb{C}}, \mu^0_{s,\tau}) \otimes L^2(\mathbb{C}^d, \mu^1_{s,\tau})$ onto $L^2(K_{\mathbb{C}}, \mu_{s,\tau})$. According to the Appendix of [19], the restriction of V to the tensor product of the two $\mathcal{H}L^2$ spaces maps *onto* $\mathcal{H}L^2(K_{\mathbb{C}}, \mu_{s,\tau})$. (It is easy to see that V maps the tensor product of the two $\mathcal{H}L^2$ spaces *into* $\mathcal{H}L^2(K_{\mathbb{C}}, \mu_{s,\tau})$; it requires some small argument to show that it maps onto.) Thus, as on the real side, the density result for $K_{\mathbb{C}}$ reduces to the previously established results for $(K_0)_{\mathbb{C}}$ and for $\mathbb{C}^d$. □

For the convenience of the reader, we briefly outline the proof of the density of holomorphic matrix entries in $\mathcal{H}L^2(K_{\mathbb{C}}, \nu_t)$, in the case that K is compact. The proof is similar in spirit to the proof of the density of polynomials in the classical Segal–Bargmann space over $\mathbb{C}^d$ in [1, Section 1b]. Consider a fixed function F in $\mathcal{H}L^2(K_{\mathbb{C}}, \nu_t)$. By the Peter–Weyl theorem, the restriction of F to K can be expanded in a "Fourier series," that is, a series in terms of matrix entries for irreducible representations of K:

$$F|_K = \sum_{\pi} f_{\pi, A_\pi}$$

for some endomorphisms A_π on the (complex) representation spaces. By the universal property of the complexification, each representation π of K has a holomorphic extension to a holomorphc representation $\pi_{\mathbb{C}}$ of $K_{\mathbb{C}}$; thus, the matrix entry f_{π,A_π} has a holomorphic extension $f_{\pi_{\mathbb{C}},A_\pi}$ to $K_{\mathbb{C}}$.

Now, a fairly elementary argument shows that the resulting holomorphically extended Fourier series converges to F uniformly on compact subsets of $K_{\mathbb{C}}$:

$$F = \sum_{\pi} f_{\pi_{\mathbb{C}}, A_\pi}. \tag{4.14}$$

Let E_n be a nested sequence of K-bi-invariant compact sets with union equal to $K_{\mathbb{C}}$. The K-invariance of the measure ν_t implies that the functions $f_{\pi_{\mathbb{C}},A_\pi}$ are orthogonal, not just in $L^2(K_{\mathbb{C}}, \nu_t)$ but also in $L^2(E_n, \nu_t)$. Orthogonality and uniform convergence over E_n tells us that

$$\|F\|^2_{L^2(E_n,\nu_t)} = \sum_{\pi} \|f_{\pi_{\mathbb{C}},A_\pi}\|^2_{L^2(E_n,\nu_t)}.$$

Letting n tend to infinity and using monotone convergence, we obtain

$$\|F\|^2_{L^2(K_{\mathbb{C}},\nu_t)} = \sum_{\pi} \|f_{\pi_{\mathbb{C}},A_\pi}\|^2_{L^2(K_{\mathbb{C}},\nu_t)}.$$

It follows that the (orthogonal) series in (4.14) converges not only pointwise, but also in $L^2(K_{\mathbb{C}}, \nu_t)$.

5. The Segal–Bargmann Transform

We analyze the complex-time Segal–Bargmann transform for a connected Lie group of compact type in two stages. In the first stage, we consider a transform M_τ, defined on matrix entries using a power-series definition of the heat operator. Using the strategy outlined in Section 1.4 along with density results established in Section 4.3, we show that M_τ maps a dense subspace of $L^2(K,\rho_s)$ isometrically onto a dense subspace of $\mathcal{H}L^2(K_{\mathbb{C}},\mu_{s,\tau})$. Thus, M_τ extends to a unitary map $\overline{M}_{s,\tau}$ of $L^2(K,\rho_s)$ onto $\mathcal{H}L^2(K_{\mathbb{C}},\mu_{s,\tau})$.

In the second stage, we show that the heat kernel $\rho_t(x)$ on K has a holomorphic extension in both t and x, denoted $\rho_{\mathbb{C}}(\cdot,\cdot)$. We then prove that the unitary map $\overline{M}_{s,\tau}$ may be computed by "convolution" with the holomorphically extended heat kernel. That is to say,

$$(\overline{M}_{s,\tau}f)(z)=\int_K \rho_{\mathbb{C}}(\tau,zk^{-1})f(k)\,dk$$

for all $s>0$, $f\in L^2(K,\rho_s)$, $\tau\in\mathbb{D}(s,s)$, and $z\in K_{\mathbb{C}}$.

The advantage of the two-stage approach to the proof is that we can use the unitary map $\overline{M}_{s,\tau}$ to establish the existence of the holomorphic extension of the heat kernel, thus avoiding the representation-theoretic estimates used in [18].

5.1. **Constructing a Unitary Map.** As usual, we work on a connected Lie group K of compact type, with a fixed $\mathrm{Ad}(K)$-invariant inner product on its Lie algebra $\mathfrak{k}$. According to Theorem 4.13, the space of matrix entries is dense in $L^2(K,\rho_s)$ and the space of holomorphic matrix entries is dense in $\mathcal{H}L^2(K_{\mathbb{C}},\mu_{s,\tau})$.

We now define a transform M_τ directly by its action on matrix entries. Let $f_{\pi,A}$ be a matrix entry on K acting on a complex vector space V_π. By the universal property of complexifications, the representation π extends uniquely to a holomorphic representation $\pi_{\mathbb{C}}$ of $K_{\mathbb{C}}$ on V_π. Hence, the matrix entry $f_{\pi,A}$ has an analytic continuation as well,

$$(f_{\pi,A})_{\mathbb{C}}(g)=\mathrm{Tr}(\pi_{\mathbb{C}}(g)A)=f_{\pi_{\mathbb{C}},A}(g),\qquad g\in K_{\mathbb{C}}.$$

Definition 5.1. *For $\tau\in\mathbb{C}_+$, define M_τ on matrix entries on K as*

$$M_\tau f_{\pi,A}=[e^{\frac{\tau}{2}\Delta_{\mathfrak{k}}}(f_{\pi,A})]_{\mathbb{C}}=\left[\sum_{n=0}^{\infty}\frac{(\tau/2)^n}{n!}(\Delta_{\mathfrak{k}})^n f_{\pi,A}\right]_{\mathbb{C}}.$$

Note that, by (3.11), $e^{\frac{\tau}{2}\Delta_{\mathfrak{k}}}(f_{\pi,A})$ is again a matrix entry, and thus has a holomorphic extension. We can therefore write the action of M_τ on matrix entries more explicitly as

$$M_\tau f_{\pi,A}=f_{\pi_{\mathbb{C}},e^{\tau C_{\pi_{\mathbb{C}}}/2}\cdot A}. \tag{5.1}$$

5.1.1. *Complex Vector Fields and Commutation Relations.* We would now like to emulate the proof of the Segal–Bargmann isometry for the $\mathbb{R}^d$ case outlined in Section 1.4. To that end, we must introduce the complex vector fields generalizing the complex derivatives $\partial/\partial z_j$ and $\partial/\partial\bar{z}_j$ in the Euclidean context.

Definition 5.2. *Let G be a complex Lie group with Lie algebra $\mathfrak{g}$ and let X be an element of $\mathfrak{g}$. The **holomorphic** and **antiholomorphic vector fields** associated to X are complex vector fields ∂_X and $\bar\partial_X$ on G defined by*

$$\partial_X\equiv\frac{1}{2}\left(\widetilde{X}-i\,\widetilde{JX}\right)\quad\textit{and}\quad\bar\partial_X\equiv\frac{1}{2}\left(\widetilde{X}+i\,\widetilde{JX}\right). \tag{5.2}$$

In the special case $G=\mathbb{C}^d$, if $X=\partial/\partial x_j$ then $\partial_X=\partial/\partial z_j$ and $\bar\partial_X=\partial/\partial\bar z_j$. (The reader is warned, therefore, that the notation is somewhat counterintuitive when compared to the Euclidean context.) By (2.3), if $X\in\mathfrak{g}$ and F is holomorphic on G then

$$\partial_X F=\widetilde{X}F,\qquad \bar\partial_X F=0 \tag{5.3}$$

$$\partial_X\bar F=0,\qquad \bar\partial_X F=\widetilde{X}F. \tag{5.4}$$

Lemma 5.3. *If $X, V \in \mathfrak{g}$, then*

$$[\partial_V, \widetilde{JX}] = i[\partial_V, \widetilde{X}], \qquad \textit{and} \qquad [\bar{\partial}_V, \widetilde{JX}] = -i[\bar{\partial}_V, \widetilde{X}].$$

Proof. By (2.1), for any $W_1, W_2 \in \mathfrak{g}$, $[JW_1, W_2] = J[W_1, W_2] = [W_1, JW_2]$ and therefore by the definition of the Lie bracket,

$$[\widetilde{JW_1}, \widetilde{W_2}] = [\widetilde{JW_1, W_2}] = J[\widetilde{W_1, W_2}] = [\widetilde{W_1, JW_2}] = [\widetilde{W_1}, \widetilde{JW_2}].$$

We can then compute from the definition that

$$[\partial_V, \widetilde{JX}] = \frac{1}{2}[\widetilde{V} - i\widetilde{JV}, \widetilde{JX}] = \frac{1}{2}[\widetilde{JV} - i\widetilde{JJV}, \widetilde{X}] = \frac{1}{2}[\widetilde{JV} + i\widetilde{V}, \widetilde{X}] = i[\partial_V, \widetilde{X}].$$

The calculation for $\bar{\partial}_V$ is similar. □

We now specialize to the case $G = K_{\mathbb{C}}$ for a compact-type Lie group K.

Definition 5.4. *Fix an orthonormal basis $\{X_1, \ldots, X_d\}$ for $\mathfrak{k}$, and let $\partial_j := \partial_{X_j}$ as in* (5.2)*. Then set*

$$\partial^2 \equiv \sum_{j=1}^d \partial_j^2, \qquad \textit{and} \qquad \bar{\partial}^2 \equiv \sum_{j=1}^d \bar{\partial}_j^2. \tag{5.5}$$

A routine calculation shows that the operators ∂^2 and $\bar{\partial}^2$ are well-defined, independent of the choice of orthonormal basis.

Lemma 5.5. *For all $f \in C^\infty(K_{\mathbb{C}})$ and all $k \in K$,*

$$\partial^2(f \circ R_k) = (\partial^2 f) \circ R_k, \qquad \textit{and} \qquad \bar{\partial}^2(f \circ R_k) = (\bar{\partial}^2 f) \circ R_k. \tag{5.6}$$

The proof of Lemma 5.5 is very similar to the proof of Lemma 3.5, and is left to the reader.

This brings us to the main commutator result of this section.

Proposition 5.6. *For any $A \in \mathfrak{k}_{\mathbb{C}}$,*

$$[\partial^2, \widetilde{A}] = [\bar{\partial}^2, \widetilde{A}] = 0.$$

Proof. As any $A \in \mathfrak{k}_{\mathbb{C}}$ has the form $A = V + JW$ for some $V, W \in \mathfrak{k}$, it suffices by linearity to prove that ∂^2 and $\bar{\partial}^2$ commute with $\widetilde{V}$ and $\widetilde{JV}$ for any $V \in \mathfrak{k}$. For the former statement, apply Lemma 5.5 with $k = e^{tV}$, and differentiate at $t = 0$ to yield the result. For the second statement, we employ Lemma 5.3 and compute as follows.

$$\begin{aligned}
[\partial^2, \widetilde{JV}] = \sum_{j=1}^d [\partial_j\partial_j, \widetilde{JV}] &= \sum_{j=1}^d \Big(\partial_j[\partial_j, \widetilde{JV}] + [\partial_j, \widetilde{JV}]\partial_j\Big) \\
&= i\sum_{j=1}^d \Big(\partial_j[\partial_j, \widetilde{V}] + [\partial_j, \widetilde{V}]\partial_j\Big) = i\sum_{j=1}^d [\partial_j\partial_j, \widetilde{V}] = i[\partial^2, \widetilde{V}]
\end{aligned}$$

and we already showed that $[\partial^2, \widetilde{V}] = 0$. A similar calculation proves the result for $\bar{\partial}^2$. □

Corollary 5.7. *The operators ∂^2, $\bar{\partial}^2$, $\Delta_{\mathfrak{k}}$, and $\Delta_{s,\tau}$ all mutually commute.*

Proof. Since $\Delta_{\mathfrak{k}}$ and $\Delta_{s,\tau}$ are linear combinations of squares of left-invariant vector fields on $K_{\mathbb{C}}$, Proposition 5.6 shows that they both commute with ∂^2 and $\bar{\partial}^2$. Similarly, letting $Y_j = JX_j$, since ∂_j^2 and $\bar{\partial}_j^2$ are linear combinations of $\widetilde{X}_j^2$, $\widetilde{Y}_j^2$, and $\widetilde{X}_j\widetilde{Y}_j = \widetilde{Y}_j\widetilde{X}_j$ (cf. (2.1)), the commutator $[\partial^2, \bar{\partial}^2] = 0$ also follows from Proposition 5.6. Finally, $\Delta_{\mathfrak{k}} = \sum_j \widetilde{X}_j^2$ with $X_j \in \mathfrak{k}$, while $\Delta_{s,\tau}$ is the Laplacian associated to an $\mathrm{Ad}(K)$-invariant inner product on $\mathfrak{k}_{\mathbb{C}}$; it follows from Corollary 3.6 that $\Delta_{\mathfrak{k}}$ and $\Delta_{s,\tau}$ commute. □

Remark 5.8. The fact that $[\partial^2, \bar{\partial}^2] = 0$ holds quite generally. Indeed, on any complex manifold, if $Z = \sum_j a_j(z)\frac{\partial}{\partial z_j}$ and $W = \sum_j b_j(z)\frac{\partial}{\partial z_j}$ are two holomorphic vector fields, than a simple computation shows that $[Z, \bar{W}] = 0$.

5.1.2. *The Transform M_τ, and the Isomorphism $\overline{M}_{s,\tau}$.* The usefulness of the ∂^2 and $\bar{\partial}^2$ operators and the commutation result in Corollary 5.7 in the present context lies in the following result.

Lemma 5.9. *Let $s > 0$ and $\tau \in \mathbb{D}(s,s)$. Let $\Delta_{s,\tau}$ denote the $K_{\mathbb{C}}$ Laplacian of Definition 1.4, and let $\Delta_{\mathfrak{k}}$ denote the Laplacian of K acting on $C^\infty(K_{\mathbb{C}})$ as usual. Then*

$$s\Delta_{\mathfrak{k}} = \Delta_{s,\tau} + \tau\partial^2 + \bar{\tau}\bar{\partial}^2$$

where all operators appearing in this identity are mutually commuting.

Proof. Fix an orthonormal basis $\{X_1, \ldots, X_d\}$ of $\mathfrak{k}$. For ease of reading, let $Y_j = JX_j$. To begin, we compute that, for each j,

$$\partial_j^2 + \bar{\partial}_j^2 = \frac{1}{4}(\widetilde{X}_j - i\widetilde{Y}_j)^2 + \frac{1}{4}(\widetilde{X}_j + i\widetilde{Y}_j)^2 = \frac{1}{2}(\widetilde{X}_j^2 - \widetilde{Y}_j^2), \tag{5.7}$$

$$\partial_j^2 - \bar{\partial}_j^2 = \frac{1}{4}(\widetilde{X}_j - i\widetilde{Y}_j)^2 - \frac{1}{4}(\widetilde{X}_j + i\widetilde{Y}_j)^2 = -i\widetilde{X}_j\widetilde{Y}_j \tag{5.8}$$

where we have used the fact that $[\widetilde{X}_j, \widetilde{Y}_j] = 0$ (cf. (2.1)).

Now, let $\tau = t + iu$. Then for each j,

$$\tau\partial_j^2 + \bar{\tau}\bar{\partial}_j^2 = t(\partial_j^2 + \bar{\partial}_j^2) + iu(\partial_j^2 - \bar{\partial}_j^2) = \frac{t}{2}(\widetilde{X}_j^2 - \widetilde{Y}_j^2) + u\widetilde{X}_j\widetilde{Y}_j.$$

Thus, we have

$$\left[\left(s - \frac{t}{2}\right)\widetilde{X}_j^2 + \frac{t}{2}\widetilde{Y}_j^2 - u\widetilde{X}_j\widetilde{Y}_j\right] + \tau\partial_j^2 + \bar{\tau}\bar{\partial}_j^2 = s\widetilde{X}_j^2. \tag{5.9}$$

Summing (5.9) on j proves the lemma. □

We can now prove that M_τ is a bijection from the space of matrix entries on K to the space of holomorphic matrix entries on $K_{\mathbb{C}}$, isometric from $L^2(K, \rho_s)$ into $L^2(K_{\mathbb{C}}, \mu_{s,\tau})$.

Theorem 5.10. *Let f be a matrix entry function on K. Then for $s > 0$ and $\tau \in \mathbb{D}(s,s)$,*

$$\|M_\tau f\|_{L^2(K_{\mathbb{C}}, \mu_{s,\tau})} = \|f\|_{L^2(K,\rho_s)}. \tag{5.10}$$

Moreover, every holomorphic matrix entry F on $K_{\mathbb{C}}$ has the form $F = M_\tau f$ for some matrix entry f on K.

The proof of (5.10) follows the strategy outlined in Section 1.4, using left-invariant vector fields in place of the partial derivatives in the Euclidean case. A key step in the argument requires us to combine exponentials, which is possible only if the operators in the exponent commute. It is at this point that we use the commutativity result in Corollary 5.7.

Proof. Let $F = M_\tau f$. The matrix entry f on K has a holomorphic extension $f_{\mathbb{C}}$ to $K_{\mathbb{C}}$. Now, $\Delta_{\mathfrak{k}}$ is a left-invariant differential operator on $K_{\mathbb{C}}$, and this operator—being a sum of squares of left-invariant vector fields—preserves the space of holomorphic functions. Thus, we have that $((\Delta_{\mathfrak{k}})^n f)_{\mathbb{C}} = (\Delta_{\mathfrak{k}})^n(f_{\mathbb{C}})$ for all $n \geq 0$. It follows that F may be computed as $F = e^{\tau\Delta_{\mathfrak{k}}/2}(f_{\mathbb{C}})$. Since $f_{\mathbb{C}}$ is holomorphic, we may use (5.3) to rewrite this relation as

$$F = e^{\tau\partial^2/2}(f_{\mathbb{C}}).$$

It is then straightforward, using (5.3) and (5.4), to see that

$$|F|^2 = e^{\tau\partial^2/2} e^{\bar{\tau}\bar{\partial}^2/2}(f_{\mathbb{C}}\bar{f}_{\mathbb{C}}).$$

Thus, using Proposition 3.18, we may compute the norm of F as

$$\begin{aligned}\|F\|^2_{L^2(K_{\mathbb{C}},\mu_{s,\tau})} &= \left(e^{\Delta_{s,\tau}/2}|F|^2\right)(e) \\ &= \left(e^{\Delta_{s,\tau}/2} e^{\tau\partial^2/2} e^{\bar{\tau}\bar{\partial}^2/2}(f_{\mathbb{C}}\bar{f}_{\mathbb{C}})\right)(e). \qquad (5.11)\end{aligned}$$

By the commutativity result in Corollary 5.7 and Remark 3.17 we may combine the exponents in the last expression in (5.11). Note that there are no domain issues to worry about here: All the exponentials in (5.11) are defined by power series and since $f_{\mathbb{C}}\bar{f}_{\mathbb{C}}$ is a matrix entry (cf. Lemma 3.15), all exponentials are acting in a fixed finite-dimensional subspace of functions on $K_{\mathbb{C}}$. Using Lemma 5.9, (5.11) therefore becomes

$$\|F\|^2_{L^2(K_{\mathbb{C}},\mu_{s,\tau})} = (e^{s\Delta_{\mathfrak{k}}/2}\,|f_{\mathbb{C}}|^2)(e) = (e^{s\Delta_{\mathfrak{k}}/2}\,|f|^2)(e).$$

The last equality holds because e belongs to K and $\Delta_{\mathfrak{k}}$ is a sum of squares of left-invariant vector fields associated to elements of $\mathfrak{k}$. Using Proposition 3.18 again, we finally conclude that

$$\|F\|^2_{L^2(K_{\mathbb{C}},\mu_{s,\tau})} = \|f\|^2_{L^2(K,\rho_s)}$$

establishing (5.10).

Suppose now that F is a holomorphic matrix entry on $K_{\mathbb{C}}$; that is, $F = f_{\pi_{\mathbb{C}},A}$ for some finite-dimensional holomorphic representation $\pi_{\mathbb{C}}$ of $K_{\mathbb{C}}$. Then $F|_K = f_{\pi,A}$, where π is the restriction of $\pi_{\mathbb{C}}$ to K. We may then define

$$f = e^{-\frac{\tau}{2}\Delta_{\mathfrak{k}}}(F|_K) = f_{\pi,e^{-\frac{\tau}{2}C_\pi}A}$$

as in (3.12). Then f is a matrix entry and we have, using (3.12) again, $M_\tau f = (e^{\frac{\tau}{2}\Delta_{\mathfrak{k}}}f)_{\mathbb{C}} = F$. □

Theorem 5.11. *The map M_τ has a unique continuous extension to $L^2(K,\rho_s)$, denoted $\overline{M}_{s,\tau}$, and this extension is a unitary map from $L^2(K,\rho_s)$ onto $\mathcal{H}L^2(K_{\mathbb{C}},\mu_{s,\tau})$.*

Proof. Theorem 4.13 tells us that M_τ is defined on a dense subspace of $L^2(K,\rho_s)$. Since M_τ is isometric, the bounded linear transformation theorem (e.g., Theorem I.7 in [35]) tells us that M_τ has a unique continuous extension to a map $\overline{M}_{s,\tau}$ of $L^2(K,\rho_s)$ into $\mathcal{H}L^2(K_{\mathbb{C}},\mu_{s,\tau})$. This extension is easily seen to be isometric, and since (by Theorem 4.13 again) the image of M_τ is dense, the extension is actually a unitary map. □

For a general $f \in L^2(K,\rho_s)$, the value of $\overline{M}_{s,\tau}$ may be computed by approximating f by a sequence f_n of matrix entries and setting

$$\overline{M}_{s,\tau}f = \lim_{n\to\infty} M_\tau f_n. \qquad (5.12)$$

(The bounded linear transformation theorem guarantees that the limit exists and that the value of $\overline{M}_{s,\tau}$ is independent of the choice of approximating sequence.) Now, (5.12) is not a very convenient way to computing. In the next section, we will seek a direct way of computing $\overline{M}_{s,\tau}$, which will also demonstrate that $\overline{M}_{s,\tau}$ coincides with the way we defined the complex-time Segal–Bargmann transform in the introduction; cf. (1.7). A first step in that direction is proving that $(\overline{M}_{s,\tau}f)(z)$ is holomorphic in both τ and z.

Lemma 5.12. *Fix $s > 0$. For each $f \in L^2(K,\rho_s)$, the function $(\tau,z) \mapsto (\overline{M}_{s,\tau}f)(z)$ is a holomorphic function on $\mathbb{D}(s,s)\times K_{\mathbb{C}}$.*

Proof. If $f = f_{\pi,A}$ is a matrix entry, then

$$(\overline{M}_{s,\tau} f_{\pi,A})(z) = (M_\tau f_{\pi,A})(z) = \mathrm{Tr}(\pi_{\mathbb{C}}(g) e^{\tau C_\pi/2} A)$$

which is easily seen to depend holomorphically on τ and z.

We then approximate an arbitrary $f \in L^2(K,\rho_s)$ by a sequence f_n of matrix entries. Then $M_\tau f_n = \overline{M}_{s,\tau} f_n$ will converge to $\overline{M}_{s,\tau} f$ in $\mathcal{H}L^2(K_{\mathbb{C}},\mu_{s,\tau})$. It is well known that the evaluation map $F \mapsto F(z)$ on $\mathcal{H}L^2(K_{\mathbb{C}},\mu_{s,\tau})$ is a bounded linear functional; this is due to the ubiquitous pointwise L^2 estimates in this holomorphic space (cf. [7, 20]). We claim that we can actually find locally uniform bounds on this functional. That is to say: for each precompact open subset U of $K_{\mathbb{C}}$ and $r \in (0,s)$, there exists $C = C(r,U) < \infty$ such that, for all $\tau \in \mathbb{D}(s,r)$ and $F \in \mathcal{H}L^2(K_{\mathbb{C}},\mu_{s,\tau})$,

$$\sup_{z\in U} |F(z)| \le C(r,U)\, \|F\|_{L^2(K_{\mathbb{C}},\mu_{s,\tau})}. \tag{5.13}$$

Assuming this result for the moment, we can conclude that the convergence of $(\overline{M}_{s,\tau} f_n)(z)$ to $(\overline{M}_{s,\tau} f)(z)$ is locally uniform jointly in (τ,z), and since each function in the sequence is holomorphic, it follows that the limit $(\overline{M}_{s,\tau} f)(z)$ is jointly holomorphic in (τ,z) as claimed.

To establish the bound in (5.13), we observe that the norm of the pointwise evaluation functional can be estimated in terms of lower bounds on the density $\mu_{s,\tau}$. For example, [7, Theorem 3.6] shows (in our context) that, for any precompact neighborhood V of the identity e, there is a constant $C(V)$ so that, for all holomorphic F and $z \in K_{\mathbb{C}}$,

$$|F(z)| \le \frac{C(V)}{\inf_{v\in V} \sqrt{\mu_{s,\tau}(vz)}} \|F\|_{L^2(K_{\mathbb{C}},\mu_{s,\tau})}.$$

The constant $C(V)$ is determined only by the holomorphic structure of the group (given by averaging a symmetrized bump function on V, applying the Cauchy integral formula); hence, $C(V)$ is independent of s and τ. Hence, it suffices to show that $\mu_{s,\tau}(z)$ is bounded strictly above 0 locally uniformly in τ and z.

The group, $K_{\mathbb{C}}$, factors as $(K_0)_{\mathbb{C}} \times \mathbb{C}^d$ as in Proposition 2.1, and the heat kernel $\mu_{s,\tau}$ also factors over this product. On the $\mathbb{C}^d$ side, there is an explicit formula for $\mu_{s,\tau}(z)$ (given in (1.17)) which is manifestly bounded away from zero locally in both τ and z. Thus, it suffices to assume that K is compact, which we do from now on.

Denote $t = \mathrm{Re}\,\tau$. From the averaging Theorem 3.33 and Lemma 4.11, we see that there is a strictly positive constant $C'(s,\tau)$ such that $\mu_{s,\tau} \asymp_{C'(s,\tau)} \mu_{t,t}$; moreover, as detailed in Remark 4.12, the constant $C'(s,\tau)$ is a continuous function of (s,τ). Note that $\mu_{t,t} = \rho_t^{\Delta_{1,1}}$ is the heat kernel for a single metric (cf. (4.9) and the remark following it) and by Theorem 3.12 it follows that $\mu_{t,t}(z) = \rho_t^{\Delta_{1,1}}(z)$ is a continuous positive function of $(t,z) \in (0,\infty) \times K_{\mathbb{C}}$. In particular, $\mu_{t,t}(z)$ is bounded strictly away from 0 for (t,z) in compact subsets of $(0,\infty) \times K_{\mathbb{C}}$. It follows from the continuity of the function $(s,\tau) \mapsto C'(s,\tau)$ that the same holds true for $\mu_{s,\tau}(z)$, establishing (5.13) and completing the proof. □

5.2. **The Analytic Continuation of the Heat Kernel.** In this section, we show that the unitary map $\overline{M}_{s,\tau}\colon L^2(K,\rho_s) \to \mathcal{H}L^2(K_{\mathbb{C}},\mu_{s,\tau})$ constructed in Section 5.1 may be computed as a "convolution" against a holomorphic extension of the heat kernel ρ_t on K. The following theorem makes this precise.

Theorem 5.13. *Let K be a compact-type Lie group.*

(1) *There exists a unique holomorphic function* $\rho_{\mathbb{C}} : \mathbb{C}_+ \times K_{\mathbb{C}} \to \mathbb{C}$ *such that for* $t > 0$ *and* $x \in K$ *we have*

$$\rho_{\mathbb{C}}(t, x) = \rho_t(x).$$

(2) *If* $s > 0$ *and* $\tau \in \mathbb{D}(s, s)$*, then for each* $z \in K_{\mathbb{C}}$*, the function*

$$x \mapsto \frac{\rho_{\mathbb{C}}(\tau, zx^{-1})}{\rho_s(x)}$$

belongs to $L^2(K, \rho_s)$*.*

(3) *The unitary map* $\overline{M}_{s,\tau}$ *may be computed as*

$$(\overline{M}_{s,\tau} f)(z) = \int_K \rho_{\mathbb{C}}(\tau, zk^{-1}) f(k)\, dk$$

for all $f \in L^2(K, \rho_s)$ *and all* $z \in K_{\mathbb{C}}$*.*

Since

$$\rho_{\mathbb{C}}(\tau, zk^{-1}) f(k)\, dk = \frac{\rho_{\mathbb{C}}(\tau, zk^{-1})}{\rho_s(k)} f(k)\ \rho_s(k) dk$$

it follows by the Cauchy–Schwarz inequality and Theorem 5.13(2) that the function $k \mapsto \rho_{\mathbb{C}}(\tau, zk^{-1}) f(k)$ is integrable. Using the decomposition of K as $K_0 \times \mathbb{R}^d$, where K_0 is compact (Proposition 2.5), we may easily reduce the general case to the compact case and the Euclidean case, which we now address separately.

5.2.1. *The Compact Case.* It is possible to construct the holomorphic extension of the heat kernel on K using the method of [18, Section 4], which is based on a term-by-term analytic continuation of the expansion of the heat kernel in terms of characters. Indeed, replacing t by $t + iu$ in the heat kernel makes no change to the (absolute) convergence estimates in [18]. (The time-parameter occurs only linearly in the exponent there, so the absolute value of each term would be independent of u.) On the other hand, the argument in [18] requires detailed knowledge of the representation theory of K. We present here a different argument (similar to the proof of Corollary 4.6 in [6]) that uses the unitary map $\overline{M}_{s,\tau}$ of Theorem 5.11 to construct the desired analytic continuation.

Lemma 5.14. *If* K *is compact,* $s > 0, 0 < t < 2s$*, and* $\overline{M}_{s,t}$ *are the unitary maps as in Theorem 5.11, then for any* $f \in L^2(K, \rho_s)$*,*

$$\left(\overline{M}_{s,t} f\right)(x) = (\rho_t * f)(x) = \int_K \rho_t(xk^{-1}) f(k)\, dk\ \forall\, x \in K \subset K_{\mathbb{C}}. \tag{5.14}$$

(Note: for K *compact,* $L^2(K) = L^2(K, \rho_s)$ *independent of* $s > 0$ *and hence* $\overline{M}_{s,t} f$ *does not really depend on* s*.)*

Proof. Recall from Eq. (5.1) that

$$\overline{M}_{s,t} f_{\pi,A} = M_t f_{\pi,A} = f_{\pi_{\mathbb{C}}, e^{tC_{\pi_{\mathbb{C}}}/2} \cdot A}$$

and so

$$\left(\overline{M}_{s,t} f_{\pi,A}\right)|_K = f_{\pi_{\mathbb{C}}, e^{tC_{\pi_{\mathbb{C}}}/2} \cdot A}|_K = f_{\pi, [e^{tC_\pi/2} A]}.$$

By Proposition 3.18, if $x \in K$, then

$$\begin{aligned} f_{\pi,[e^{tC_\pi/2}A]}(x) &= \int_K \rho_t(k) f_{\pi,A}(xk)\, dk \\ &= \int_K \rho_t(xk^{-1}) f_{\pi,A}(k)\, dk = (\rho_t * f_{\pi,A})(x). \end{aligned}$$

This suffices to complete the proof as matrix entries are dense in $L^2(K)$ and both $L^2(K) \ni f \to (\overline{M}_{s,t} f)(x) \in \mathbb{C}$ and $L^2(K) \ni f \to (\rho_t * f)(x) \in \mathbb{C}$ are continuous linear functionals on $L^2(K)$ for each fixed $x \in K$. The first assertion holds since $\overline{M}_{s,\tau} : L^2(K, \rho_s) \to \mathcal{H}L^2(K_{\mathbb{C}}, \mu_{s,\tau})$ is unitary and pointwise evaluation on $\mathcal{H}L^2(K_{\mathbb{C}}, \mu_{s,\tau})$ is continuous and the second follows by Hölder's inequality. □

Proof of Theorem 5.13 in the compact group case. We begin with point (1): the space-time analytic continuation of the heat kernel. Let $0 < \delta < r < \infty$, and consider the vertically symmetric rectangle $U_{\delta,r} = \{\tau \in \mathbb{C}_+ : \delta < \operatorname{Re}\tau < r,\ |\operatorname{Im}\tau| < r\}$. Let $0 < \epsilon < \delta$, and fix $s > 0$ large enough that $U_{\delta,r} - \epsilon \subset \mathbb{D}(s,s)$. The function ρ_ϵ is continuous and hence in $L^2(K, \rho_s)$. We then define $\rho_{\mathbb{C}} : U_{\delta,r} \times K_{\mathbb{C}} \to \mathbb{C}$ by

$$\rho_{\mathbb{C}}(\tau, z) = \left(\overline{M}_{s,\tau-\epsilon}\rho_\epsilon\right)(z). \tag{5.15}$$

By Lemma 5.12, $\rho_{\mathbb{C}}$ is analytic in both variables so long as $\tau - \epsilon \in \mathbb{D}(s,s)$; in particular, $\rho_{\mathbb{C}}$ is analytic on $U_{\delta,r} \times K_{\mathbb{C}}$. For the moment, it appears a priori that the value of $\rho_{\mathbb{C}}$ depends on s and ϵ.

Now consider the restriction of $\rho_{\mathbb{C}}$ to $(t,x) \in (U_{\delta,r} \cap \mathbb{R}) \times K$. By lemma 5.14 and the semigroup property of the heat kernel (Theorem 3.12(3)),

$$\rho_{\mathbb{C}}(t,x) = \left(\overline{M}_{s,t-\epsilon}\rho_\epsilon\right)(x) = (\rho_{t-\epsilon} * \rho_\epsilon)(x) = \rho_t(x)\ \forall\, x \in K. \tag{5.16}$$

Thus, $\rho_{\mathbb{C}}$ is a holomorphic extension of the heat kernel $\rho_t(x)$ in t and x. Analytic continuation from K to $K_{\mathbb{C}}$ is unique (cf. [44, Lemma 4.11.13]), and also from $U_{\delta,r} \cap \mathbb{R}$ to $U_{\delta,r}$ by elementary complex analysis. In particular, since $\rho_t(x)$ does not depend on s or ϵ, neither does the function $\rho_{\mathbb{C}}$.

Thus, for each rectangle $U_{\delta,r}$, there is a unique analytic continuation of the heat kernel to a holomorphic function $\rho_{\mathbb{C}}$ on $U_{\delta,r} \times K_{\mathbb{C}}$. Let δ_n and r_n be sequences with $\delta_n \downarrow 0$ and $r_n \uparrow \infty$, let $U_n = U_{\delta_n, r_n}$, and let $\rho_{\mathbb{C}}^n$ be the analytic continuation of $\rho_t(x)$ to U_n. The rectangles U_n are nested with union $\mathbb{C}_+$; since $\rho_{\mathbb{C}}^n$ and $\rho_{\mathbb{C}}^m$ agree on $(U_{n\wedge m} \cap \mathbb{R}) \times K$, uniqueness of analytic continuation shows that they agree on their common domain $U_{n\wedge m} \times K_{\mathbb{C}}$. Thus, there is a globally defined holomorphic function $\rho_{\mathbb{C}}$ whose value in $U_n \times K_{\mathbb{C}}$ is $\rho_{\mathbb{C}}^n$, and thus restricts to $\rho_t(x)$ on $(U_n \cap \mathbb{R}) \times K$; ergo $\rho_{\mathbb{C}}(t,x) = \rho_t(x)$ for $t > 0$ and $x \in K$, as desired. Uniqueness again follows from [44, Lemma 4.11.13]. This establishes point (1).

Point (2) is immediate since K is compact and the function in question is continuous. For point (3), we first note that, by Lemma 5.12, $(\overline{M}_{s,\tau} f)(z)$ is holomorphic in τ and z. Meanwhile, since $\rho_{\mathbb{C}}(\tau, zk^{-1})$ is holomorphic in τ and z for each fixed $k \in K$, we may use Fubini's theorem and Morera's theorem to verify that $\int_K \rho_{\mathbb{C}}(\tau, zk^{-1}) f(k)\, dx$ is also holomorphic in τ and z. Since both sides of the desired equality are holomorphic in τ and z, it suffices by uniqueness of analytic continuation to verify the result when $\tau = t \in (0, 2s)$ and $z = x$ belongs to K. Using Lemma 5.14 and the defining property of $\rho_{\mathbb{C}}$, the desired equality thus becomes

$$(e^{\frac{t}{2}\bar{\Delta}_K} f)(x) = \int_K \rho_t(xk^{-1}) f(k)\, dk,$$

which is true by Corollary 3.29. This concludes the proof. □

5.2.2. *The Euclidean Case.* The heat kernel ρ_s on $\mathbb{R}^d$ is explicitly known to be the Gaussian density mentioned in the introduction:

$$\rho_s(x) = (2\pi s)^{-d/2} \exp\left(-\frac{|x|^2}{2s}\right)$$

and the density $\mu_{s,\tau}(z)$ in this case has been described in (1.17) in the introduction.

Proof of Theorem 5.13 in the Euclidean case. For point (1), the desired holomorphic extension is given by

$$\rho_{\mathbb{C}}(\tau, z) := \left(\sqrt{2\pi\tau}\right)^{-d} \exp\left(-\frac{z \cdot z}{2\tau}\right) \tag{5.17}$$

where $z \cdot z = \sum_{j=1}^{d} z_j^2$ and where $\sqrt{2\pi\tau}$ is defined by the standard branch of the square root (with branch cut along the negative real axis).

Point (2) of the theorem is an elementary computation. Using additive notation for the group operation, we need to verify that

$$\int_{\mathbb{R}^d} \frac{|\rho_{\mathbb{C}}(\tau, z-x)|^2}{\rho_s(x)^2} \rho_s(x)\, dx < \infty \tag{5.18}$$

for all $z \in \mathbb{C}^d$, provided that $s > 0$ and $\tau \in \mathbb{D}(s,s)$ (or, equivalently, provided that $\alpha > 0$; cf. (1.10)). Equation (5.18) is a Gaussian integral whose computation is tedious but straightforward. (The integral factors into separate integrals over each copy of $\mathbb{R}$, which may then be evaluated in a computer algebra system.) We record the result here: if $z = \xi + i\eta$ and $\tau = t + iu$, then

$$\int_{\mathbb{R}^d} \frac{|\rho_{\mathbb{C}}(\tau, z-x)|^2}{\rho_s(x)^2} \rho_s(x)\, dx = \left(\frac{\pi s}{\sqrt{\alpha}}\right)^d \exp\left(\frac{t/2}{2\alpha}|\xi|^2 + \frac{s-t/2}{2\alpha}|\eta|^2 + \frac{u}{2\alpha}\xi \cdot \eta\right) \tag{5.19}$$

where, as in (1.10), $\alpha = (2st - t^2 - u^2)/4$.

For point (3), we must show that $(\overline{M}_{s,\tau} f)(z)$ may be computed as

$$(\overline{M}_{s,\tau} f)(z) = \int_{\mathbb{R}^d} \rho_{\mathbb{C}}(\tau, z-x) f(x)\, dx \tag{5.20}$$

for all $f \in L^2(\mathbb{R}^d, \rho_s)$. If f is a polynomial (and thus a matrix entry) and $\tau \in \mathbb{R}$ and $z \in \mathbb{R}^d$, (5.20) follows from Proposition 3.18. Furthermore, when f is a polynomial, both sides of (5.20) are holomorphic in τ and z, so the result continues to hold when $\tau \in \mathbb{C}_+$ and $z \in \mathbb{C}^d$. Now, both sides of (5.20) depend continuously on $f \in L^2(\mathbb{R}^d, \rho_s)$—the left-hand side by the unitarity of $\overline{M}_{s,\tau}$ and the continuity of pointwise evaluation, and the right-hand side by the fact that $\rho_{\mathbb{C}}(t, z-x)$ is square-integrable in x. Thus, we may pass to the limit starting from polynomials to obtain the result for all $f \in L^2(\mathbb{R}^d, \rho_s)$, thus completing the proof of Theorem 5.13 in the $\mathbb{R}^d$ case. □

We note that, by (5.19), we have bounds on the value of $(\overline{M}_{s,\tau} f)(z)$ in terms of the L^2 norm of f. Since $\overline{M}_{s,\tau}$ maps isometrically onto $\mathcal{H}L^2(\mathbb{C}^d, \mu_{s,\tau})$, these bounds translate into pointwise bounds in $\mathcal{H}L^2(\mathbb{C}^d, \mu_{s,\tau})$ as follows:

$$|F(\xi + i\eta)|^2 \leq \left(\frac{\pi s}{\sqrt{\alpha}}\right)^d \exp\left(\frac{t/2}{2\alpha}|\xi|^2 + \frac{s-t/2}{2\alpha}|\eta|^2 + \frac{u}{2\alpha}\xi \cdot \eta\right) \|F\|^2_{L^2(\mathbb{C}^d, \mu_{s,\tau})}, \tag{5.21}$$

where $\mu_{s,\tau}$ is given as in (1.17). Note that the bounds on $|F(z)|^2$ are, up to a constant, just the reciprocal of the density $\mu_{s,\tau}$. This is typical behavior for $\mathcal{H}L^2$ spaces over $\mathbb{C}^d$ with respect to a Gaussian measure.

5.3. **The $s \to \infty$ Limit.** Throughout this section, we assume that the compact-type group K is actually compact and we normalize the Haar measure dk on K to be a probability measure. Recall that $\nu_t \in C^\infty(K_{\mathbb{C}}, (0, \infty))$ is the K-averaged heat kernel measure as in

Definition 1.8 which, by Lemma 4.11, also satisfies,

$$\nu_t(z) = \int_K \mu_{s,\tau}(zk)\, dk \quad \text{for } s > 0,\ \tau = t + iu \in \mathbb{D}(s,s), \text{ and } z \in K_{\mathbb{C}}. \tag{5.22}$$

We will use the following well-known result for the heat kernel measure on a compact Lie group at large time.

Lemma 5.15. *If K is a compact Lie group, the heat kernel ρ_s converges to the constant 1 uniformly over K (exponentially fast) as $s \to \infty$.*

For completeness, we include a complete proof of Lemma 5.15 in Appendix B. (The proof given there works essentially unchanged for compact Riemannian manifolds.)

Corollary 5.16. *Let K be a compact Lie group. Then for each $s > 0$, there are positive constants β_s and γ_s such that, for all $\tau = t + iu \in \mathbb{D}(s,s)$,*

$$\beta_s \nu_t(z) \le \mu_{s,\tau}(z) \le \gamma_s \nu_t(z) \tag{5.23}$$

for all $z \in K_{\mathbb{C}}$. Furthermore, the constants may be chosen so that $\beta_s \to 1$ and $\gamma_s \to 1$ as $s \to \infty$ with τ fixed.

Proof. By a small modification of the proof of Theorem 3.33, we have the following formula for $\mu_{s,\tau}$:

$$\mu_{s,\tau}(z) = \int_K \mu_{s_0,\tau}(zk^{-1})\rho_{s-s_0}(k)\, dk,$$

for any $s_0 \in (0,s)$ and $\tau \in \mathbb{D}(s_0,s_0) \supset \mathbb{D}(s,s)$. (Write $\Delta_{s,\tau} = (s-s_0)\Delta_{\mathfrak{k}} + \Delta_{s_0,\tau}$ and then follow the proof of Theorem 3.33 up to (3.23).) If we fix such an s_0, then (5.23) holds with $\beta_s = \inf_{k\in K} \rho_{s-s_0}(k)$ and $\gamma_s = \sup_{k\in K} \rho_{s-s_0}(k)$. With that choice of β_s and γ_s, Lemma 5.15 tells us that β_s and γ_s tend to 1 as s tends to infinity. $\square$

With these results in hand, we may now prove Theorem 1.9, describing the large-s limit of the transform $B_{s,\tau}$.

Proof of Theorem 1.9. Since K is compact, the function ρ_s is bounded and bounded away from zero, showing that $L^2(K) = L^2(K,\rho_s)$ as sets. The equality of $L^2(K_{\mathbb{C}},\nu_t)$ and $L^2(K_{\mathbb{C}},\mu_{s,\tau})$ as sets follows from Corollary 5.16. The claimed convergence of norms then follows easily from Lemma 5.15 and the fact that the constants β_s and γ_s in Corollary 5.16 may be chosen to converge to 1 as s tends to infinity. The equalities of the various Hilbert spaces as sets and the convergence of the norms allows us to deduce the unitarity of $B_{\infty,\tau}$ from the unitarity of the maps $B_{s,\tau}$. $\square$

Appendix A. Essential Self-Adjointness of the Laplacian

This section provides a self-contained proof that, on any Lie group, any "sum of squares" Laplacian is essentially self-adjoint, with $C_c^\infty(G)$ as a core. This proof is adapted from notes due to L. Gross.

Let G be a real Lie group with Lie algebra $\mathfrak{g}$, on which we fix an inner product throughout. Let $\{X_j\}_{j=1}^k$ be a collection of left-invariant vector fields on G, and define

$$L_0 := \sum_{j=1}^k X_j^2$$

acting on $C^2(G)$ and let $L := L_0|_{C_c^\infty(G)}$, i.e., $L = L_0$ on $\mathcal{D}(L) := C_c^\infty(G)$. Further let λ denote a right-invariant Haar measure on G.

Theorem A.1. *The second-order differential operator L is essentially self-adjoint as an unbounded operator on $L^2(G,\lambda)$ with domain $C_c^\infty(G)$.*

It is important to note that we do not assume that the vector fields $\{X_j\}_{j=1}^k$ span $\mathfrak{g}$, nor that they generate $\mathfrak{g}$ as a Lie algebra. Thus, L_0 is not necessarily elliptic or even hypoelliptic. Thus, proofs that rely on hypoellipticity, such as the one in [27], do not apply in this setting.

Before giving the proof we will need a little notation and a few preparatory results.

Notation A.2. *For $u \in C_c^\infty(G)$ and $f \in L^2(G)$, define the **convolution** of u and f by*

$$(u * f)(x) = \int_G u(xy^{-1}) f(y)\, \lambda(dy). \tag{A.1}$$

We will also let

$$\tilde{u}(x) = \bar{u}(x^{-1})$$

which should not be confused with the notation $\widetilde{\xi}$ for the left-invariant vector field determined by an element $\xi \in \mathfrak{g}$.

Proposition A.3. *For all $f, g \in L^2(G,\lambda)$ and $X \in \mathfrak{g}$, the following results hold.*

(1) *If $u \in C_c^\infty(G)$, then $u * f \in L^2(G,\lambda)$ and*

$$\|u * f\|_2 \le \left(\int_G |u| \sqrt{m}\, d\lambda \right) \|f\|_2$$

where m is the modular function of G.

(2) $\langle u * f, g\rangle_{L^2(G,\lambda)} = \langle f, \tilde{u} * g\rangle_{L^2(G,\lambda)}$ *for all $u \in C_c^\infty(G)$.*

(3) $\widetilde{X}(u * v) = u * (\widetilde{X} v)$ *for all $u \in C_c^\infty(G)$ and $v \in C^\infty(G)$.*

(4) $\langle \widetilde{X} u, v\rangle_{L^2(G,\lambda)} = -\langle u, \widetilde{X} v\rangle_{L^2(G,\lambda)}$ *for all $u \in C^\infty(G)$ and $v \in C_c^\infty(G)$.*

(5) *There exist $u_n \in C_c^\infty(G,\mathbb{R})$ such that $u_n * \cdot \to I$ strongly on $L^2(G,\lambda)$.*

Proof. In the following argument, we will use the right invariance of Haar measure, the definition of convolution in (A.1), and the left invariance of $\widetilde{X}$ without further mention. Using the definition of the modular function, $\|g \circ L_x\|_2^2 = m(x) \|g\|_2^2$ for all $x \in G$. Therefore,

$$\begin{aligned}
\int_{G^2} |u(xy^{-1})||f(y)||g(x)|\, \lambda(dx)\, \lambda(dy) &= \int_{G^2} |u(x)||f(y)||g(xy)|\, \lambda(dx)\, \lambda(dy) \\
&\le \int_G |u(x)| \|f\|_2 \cdot \|g \circ L_x\|_2\, \lambda(dx) \\
&\le \int_G |u(x)| \sqrt{m(x)}\, \lambda(dx) \cdot \|f\|_2 \cdot \|g\|_2.
\end{aligned}$$

This proves item (1) as a consequence of the converse to Hölder's inequality. It also justifies the use of Fubini's theorem used to prove item (2):

$$\begin{aligned}
\langle u * f, g\rangle_{L^2(G,\lambda)} &= \int_{G^2} u(xy^{-1}) f(t) \bar{g}(x)\, \lambda(dx)\, \lambda(dy) \\
&= \int_{G^2} f(t) \overline{\tilde{u}(yx^{-1}) g(x)}\, \lambda(dx)\, \lambda(dy) = \langle f, \tilde{u} * g\rangle_{L^2(G,\lambda)}.
\end{aligned}$$

For items (3) and (4), we have

$$\begin{aligned}\widetilde{X}(u*v)(x) &= \frac{d}{dt}\bigg|_{t=0}\int_G u\left(xe^{tX}y^{-1}\right)v(y)\,\lambda(dy)\\ &= \frac{d}{dt}\bigg|_{t=0}\int_G u(xy^{-1})v\left(ye^{tX}\right)\,\lambda(dy) = u*\widetilde{X}v\end{aligned}$$

and

$$\begin{aligned}\langle Xu, v\rangle_{L^2(G,\lambda)} &= \frac{d}{dt}\bigg|_{t=0}\int_G u\left(xe^{tX}\right)\bar{v}(x)\,\lambda(dx)\\ &= \frac{d}{dt}\bigg|_{t=0}\int_G u(x)\bar{v}\left(xe^{-tX}\right)\,\lambda(dx) = -\langle u, \widetilde{X}v\rangle_{L^2(G,\lambda)}.\end{aligned}$$

For item (5) we apply the usual approximate identity sequence arguments to any sequence of functions $\{u_n\}_{n=1}^{\infty}\subset C_c^{\infty}(G,[0,\infty))$ with the following properties: 1) $\int_G u_n\,d\lambda = 1$ for all n and 2) $\operatorname{supp}(u_n)\downarrow\{e\}$ as $n\to\infty$. □

Lemma A.4. *For $f\in C^{\infty}(G)\cap\mathcal{D}(L^*)$, $L^*f = L_0 f$ and moreover $C^{\infty}(G)\cap\mathcal{D}(L^*)$ is core for L^*.*

Proof. If $f\in C^{\infty}(G)\cap\mathcal{D}(L^*)$ and $v\in C_c^{\infty}(G)$, then by the definition of L^* and repeated use of Proposition A.3,

$$\langle L^*f, v\rangle_{L^2(G,\lambda)} = \langle f, Lv\rangle_{L^2(G,\lambda)} = \langle L_0 f, v\rangle_{L^2(G,\lambda)}.$$

Since $v\in C_c^{\infty}(G)$ is arbitrary, it follows that $L^*f = L_0 f$.

Now suppose that $f\in\mathcal{D}(L^*)$ and that $u,v\in C_c^{\infty}(G)$. Then $\tilde{u}*v\in C_c^{\infty}(G)$ and therefore

$$\begin{aligned}\langle u*L^*f, v\rangle_{L^2(G,\lambda)} &= \langle L^*f, \tilde{u}*v\rangle_{L^2(G,\lambda)} = \langle f, L(\tilde{u}*v)\rangle_{L^2(G,\lambda)}\\ &= \langle f, \tilde{u}*Lv\rangle_{L^2(G,\lambda)} = \langle u*f, Lv\rangle_{L^2(G,\lambda)}.\end{aligned}$$

It follows from this equation that $u*f\in\mathcal{D}(L^*)$ and that

$$L^*(u*f) = u*L^*f \text{ for all } u\in C_c^{\infty}(G). \tag{A.2}$$

Now choose u_n as in Proposition A.3(5). Since each $u_n*f\in C^{\infty}(G)\cap\mathcal{D}(L^*)$, the lemma follows from (A.2). □

Notation A.5. *The tensor $D^n f(x)\in(\mathfrak{g}^{\otimes n})^*$ of n^{th}-order derivatives of f at x is defined by*

$$\langle\left(D^n f\right)(x), \xi_1\otimes\cdots\otimes\xi_n\rangle = \left(\widetilde{\xi}_1\ldots\widetilde{\xi}_n f\right)(x) \tag{A.3}$$

where $\xi_j\in\mathfrak{g}$ for $1\le j\le n$, and the inner product is the standard one induced by the inner product on $\mathfrak{g}$.

Let us now recall from [8, Lemma 3.6] that there exists $h_n\in C_c^{\infty}(G,[0,1])$ such that h_n is increasing, $h_n^{-1}(\{1\})\uparrow G$ as $n\uparrow\infty$, and $\sup_n\sup_{x\in G}|D^k h_n(x)|<\infty$ for any $k\in\mathbb{N}$ where $D^k h_n$ is as defined in (A.3).

Lemma A.6. *If f is a real-valued function in $C^{\infty}(G)\cap\mathcal{D}(L^*)$, then*

$$\int_G\sum_{j=1}^{k}(\widetilde{X}_j f)^2\,d\lambda = -\langle L^*f, f\rangle_{L^2(G,\lambda)}. \tag{A.4}$$

For any f in $C^{\infty}(G)\cap\mathcal{D}(L^*)$, the function $\widetilde{X}_j f$ belongs to $L^2(G,\lambda)$ for all j.

Proof. Let f be a real-valued element of $C^\infty(G)\cap\mathcal{D}(L^*)$. For any $h\in C_c^\infty(G,[0,\infty))$, we have

$$\begin{aligned}\sum_{j=1}^k\int_G h(x)(\widetilde{X}_jf)^2\,\lambda(dx) &= -\sum_{j=1}^k\int_G[h(x)(\widetilde{X}_j^2f)f+(\widetilde{X}_jh)(\widetilde{X}_jf)f]\,d\lambda\\ &= -\int_G h(x)(L^*f)f\,d\lambda-\frac{1}{2}\sum_{j=1}^k\int_G(\widetilde{X}_jh)\widetilde{X}_j(f^2)\,d\lambda\\ &= -\langle hL^*f,f\rangle_{L^2(G,\lambda)}+\frac{1}{2}\int_G(Lh)f^2\,d\lambda. \qquad \text{(A.5)}\end{aligned}$$

Now replace h in the above identity by h_n as in [8, Lemma 3.6], so in particular $h_n\uparrow 1$ as $n\to\infty$ and $Lh_n\to 0$ boundedly. We now use the monotone convergence theorem on the left-hand side of (A.5) and the dominated convergence theorem on both terms on the right-hand side to verify (A.4). It follows that $\widetilde{X}_jf$ belongs to $L^2(G,\lambda)$ for each j.

Now, since the domain of L is invariant under complex conjugation and since L commutes with complex conjugation, the domains of both L and L^* are invariant under complex conjugation. Thus, if $f\in C^\infty(G)\cap\mathcal{D}(L^*)$, the real and imaginary parts, f_1 and f_2, of f are also in $C^\infty(G)\cap\mathcal{D}(L^*)$. Equation (A.4) then shows that $\widetilde{X}_jf_1$ and $\widetilde{X}_jf_2$ are in $L^2(G,\lambda)$, so that $\widetilde{X}_jf$ is also in $L^2(G,\lambda)$. □

Proof of Theorem A.1. Let $\overline{L}$ denote the closure of L. By Proposition A.3, each X_j is skew symmetric on $C_c^\infty(G)$ and as a consequence L is symmetric on $C_c^\infty(G)$. That is $L\subseteq L^*$ and therefore $\overline{L}\subseteq L^*=\overline{L}^*$. So it remains only to show $L^*\subseteq\overline{L}$, or equivalently that $C_c^\infty(G)$ is a core for L^*.

Using Lemma A.4, it suffices to prove the following: For every $f\in C^\infty(G)\cap\mathcal{D}(L^*)$, there exists $f_n\in C_c^\infty(G)$ such that f_n converges to f in the L^*-graph norm. Choose $0\le h_n\le 1$ with $h_n\in C^\infty(G)$ as in [8, Lemma 3.6] and let $f_n(x)=h_n(x)f(x)$. Then $f_n\in C^\infty(G)$ and $f_n\to f$ in L^2 since $h_n\uparrow 1$. Moreover,

$$Lf_n=(Lh_n)f+h_n(L^*f)+2\sum_{j=1}^k(\widetilde{X}_jh_n)(\widetilde{X}_jf) \qquad \text{(A.6)}$$

and $Lh_n\to 0$ boundedly by [8, Lemma 3.6]. The first two terms on the right-hand side of (A.6) therefore together converge to L^*f. Now, Lemma A.6 tells us that each $\widetilde{X}_jf$ belongs to $L^2(G,\lambda)$. Since also $X_jh_n\to 0$ pointwise and boundedly, dominated convergence tells us that the third term on the right-hand side of (A.6) converges to zero in $L^2(G,\lambda)$. Thus $Lf_n\to L^*f$ in $L^2(G,\lambda)$, concluding the proof. □

Appendix B. Uniform Convergence of $\rho_s\to 1$ as $s\to\infty$

Theorem B.1. *Let λ_1 be the first nonzero eigenvalue of $-\Delta_K/2$. Then for every $\epsilon>0$,*

$$C_\epsilon:=\sup_{s\ge\epsilon}\max_{x\in K}|\rho_s(x)-1|\,e^{s\lambda_1/2}<\infty. \qquad \text{(B.1)}$$

In particular, $\|\rho_s-1\|_{L^\infty(K)}\le C_\epsilon e^{-s\lambda_1/2}$ for all $s\ge\epsilon$.

Proof. The Laplacian Δ_K for K has discrete, nonpositive spectrum, with zero being an eigenvalue of multiplicity 1 with the corresponding eigenfunction $\varphi_0\equiv 1$ (as we are using normalized Haar measure on K). Denote the eigenvalues by $-\lambda_n$ (so that λ_n are ≥ 0), and arrange them so that

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \cdots,$$

with $\lambda_n \to \infty$ as $n \to \infty$. Let φ_n be the corresponding L^2-normalized eigenfunctions, chosen to be real-valued. (See [45, Exercise 16, p. 254] and [15, Section 1.6].) Since, for any $s > 0$,

$$\int_K \rho_s(x)\varphi_n(x)\,dx = \left(e^{s\bar{\Delta}_K/2}\varphi_n\right)(e) = e^{-s\lambda_n/2}\varphi_n(e)$$

we conclude (using $\varphi_0 \equiv 1$) that the eigenfunction expansion of $\rho_s - 1$ is given by

$$\rho_s(x) - 1 = \sum_{n=1}^{\infty} \varphi_n(e)e^{-s\lambda_n/2}\varphi_n(x),$$

where the sum is convergent in $L^2(K)$.

Convolving this identity with ρ_1 then gives the pointwise identity

$$\rho_{s+1}(x) - 1 = ((\rho_s - 1) * \rho_1)(x) = \int_K (\rho_s(y) - 1)\rho_1(xy^{-1})\,dy.$$

Hence by the Cauchy–Schwarz inequality,

$$|\rho_{s+1}(x) - 1|^2 \leq K \cdot \|\rho_s - 1\|^2_{L^2(K)} = K \cdot \sum_{n=1}^{\infty} |\varphi_n(e)|^2 e^{-s\lambda_n}$$

where

$$K = \int_K \left|\rho_1(xy^{-1})\right|^2 dy = \rho_2(e) < \infty.$$

Combing these results shows

$$e^{\lambda_1 s} \max_{x\in K} |\rho_{s+1}(x) - 1|^2 \leq \rho_2(e) \sum_{n=1}^{\infty} |\varphi_n(e)|^2 e^{-s(\lambda_n - \lambda_1)}. \tag{B.2}$$

Let $m = \min\{n \in \mathbb{N} : \lambda_n > \lambda_1\}$ and then define $\delta := \lambda_1/\lambda_m < 1$ so that $\lambda_1 = \delta\lambda_m \leq \delta\lambda_n$ and hence $\lambda_n - \lambda_1 \geq (1-\delta)\lambda_n$ for all $n \geq m$. Therefore, for $s \geq 1$,

$$\begin{aligned}
\sum_{n=1}^{\infty} |\varphi_n(e)|^2 e^{-s(\lambda_n-\lambda_1)} &= \sum_{n=1}^{m-1} |\varphi_n(e)|^2 + \sum_{n=m}^{\infty} |\varphi_n(e)|^2 e^{-s(\lambda_n-\lambda_1)} \\
&\leq \sum_{n=1}^{m-1} |\varphi_n(e)|^2 + \sum_{n=m}^{\infty} |\varphi_n(e)|^2 e^{-(1-\delta)\lambda_n} \\
&\leq \sum_{n=1}^{m-1} |\varphi_n(e)|^2 + \|\rho_{1-\delta} - 1\|^2_{L^2(K)}.
\end{aligned}$$

This estimate combined with (B.2) shows the constant C_ϵ as in (B.1) is finite when $\epsilon = 2$. This is sufficient to show $C_\epsilon < \infty$ in general because $(s,x) \to |\rho_s(x) - 1|\, e^{s\lambda_1/2}$ is continuous and hence bounded on $[\epsilon, 2] \times K$. □

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