Three proofs of the Makeenko–Migdal equation for Yang–Mills theory on the plane

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March 29, 2016

Abstract

We give three short proofs of the Makeenko–Migdal equation for the Yang–Mills measure on the plane, two using the edge variables and one using the loop or lasso variables. Our proofs are significantly simpler than the earlier pioneering rigorous proofs given by T. Lévy and by A. Dahlqvist. In particular, our proofs are "local" in nature, in that they involve only derivatives with respect to variables adjacent to the crossing in question. In an accompanying paper with F. Gabriel, we show that two of our proofs can be adapted to the case of Yang–Mills theory on any compact surface.

*Supported in part by NSF grant DMS-1301534
†Supported in part by NSF CAREER award DMS-1254807

1 Introduction

The (Euclidean) Yang–Mills field theory describes a random connection on a principal bundle for a compact Lie group K, known as the structure group. In two dimensions, the theory is tractable and has been studied extensively. In particular, for Yang–Mills theory on the plane, it is possible to use a gauge fixing to make the measure Gaussian, opening the door to rigorous calculations. This approach was developed simultaneously in two papers: [GKS] by L. Gross, C. King, and A. Sengupta; and [Dr] by B. Driver.

The typical objects of study in the theory are the Wilson loop functionals, given by

$$\mathbb{E}\{\text{trace}(\text{hol}(L))\}, \tag{1}$$

where $\mathbb{E}$ denotes the expectation value with respect to the Yang–Mills measure, $\text{hol}(L)$ denotes the holonomy of the random connection around a loop L, and the trace is taken in some fixed representation of the structure group K. If L is traced out on a graph in the plane, work of Driver [Dr] gives a formula [Dr, Theorem 6.4] for the Wilson loop functional in terms of the *heat kernel measure* on K. (See (7) below.) One noteworthy feature of the two-dimensional Yang–Mills theory is its invariance under area-preserving diffeomorphisms. This invariance is reflected in Driver's formula: the expectation (1) may be expressed as a function (determined by the topology of the graph) of all the areas of the faces of the graph.

The *Makeenko–Migdal equation* relates variations of a Wilson loop functional in the neighborhood of a simple crossing to the associated Wilson loops on either side of the crossing, in the case $K = U(N)$. The original equations, in any dimension, were the subject of [MM]. In [KK, Section 4], V. A. Kazakov and I. K. Kostov show that in two dimensions, the "keyboard-type" variation in Eq. (3) of [MM] can be interpreted as the alternating sum of derivatives of the Wilson loop functional with respect to the areas of the faces surrounding a simple crossing. (See [KK, Equation 24], [K, Equation 9], and [GG, Equation 6.4].) Lévy [Lévy2] then provided a rigorous proof of the planar Makeenko–Migdal equation, using Driver's formula. A different proof was subsequently given by A. Dahlqvist in [Dahl]. In this paper, we offer three new, short proofs of the equation. As we show in an accompanying paper with F. Gabriel [DGHK], two of these proofs can be adapted to give a new result, namely the Makeenko–Migdal equation for Yang–Mills theory on an arbitrary compact surface.

We use the bi-invariant metric on $U(N)$ whose value on the Lie algebra $\mathfrak{u}(N) = T_e(U(N))$ is a scaled version of the Hilbert–Schmidt inner product:

$$\langle X, Y\rangle = N\text{trace}(X^*Y). \tag{2}$$

It is then convenient to express the Wilson loop functionals using the *normalized* trace,

$$\text{tr}(A) := \frac{1}{N}\text{trace}(A).$$

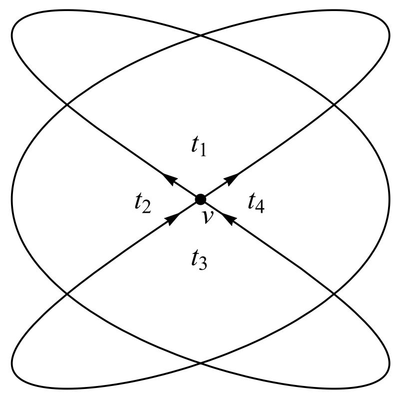

Figure 1: A typical loop L for the Makeenko–Migdal equation

We now consider a loop L with simple crossings, and we let v be one such crossing. We label the four faces of L adjacent to the crossing in cyclic order as F_1, F_2, F_3, and F_4, with F_1 denoting the face whose boundary contains the two outgoing edges of L. We choose the cyclic ordering of the faces so that the first edge traversed by L lies between F_1 and F_4. We then let t_1, t_2, t_3, and t_4 denote the areas of these faces. (See Figure 1.) We also let L_1 denote the loop from the beginning to the first return to v and let L_2 denote the loop from the first return to the end. (See Figure 2.) The Makeenko–Migdal equation, in the plane case, then gives a formula for the alternating sum of the derivatives of the Wilson loop functional with respect to these areas.

Theorem 1 (Makeenko–Migdal equation for $U(N)$) *Let L be a closed curve with simple crossings and let v be a crossing. Parametrize L over the time interval $[0,1]$ with $L(0) = L(1) = v$, and let s_0 be the unique time with $0 < s_0 < 1$ such that $L(s_0) = v$. Let L_1 be the restriction of L to $[0, s_0]$ and let L_2 be the restriction of L to $[s_0, 1]$. Then*

$$\left(\frac{\partial}{\partial t_1} - \frac{\partial}{\partial t_2} + \frac{\partial}{\partial t_3} - \frac{\partial}{\partial t_4}\right) \mathbb{E}\{\mathrm{tr}(\mathrm{hol}(L))\} = \mathbb{E}\{\mathrm{tr}(\mathrm{hol}(L_1))\mathrm{tr}(\mathrm{hol}(L_2))\}. \quad (3)$$

We follow the convention that if any of the adjacent faces is the unbounded face, the corresponding derivative on the left-hand side of (3) is omitted. Note also that the faces F_1, F_2, F_3, and F_4 are not necessarily distinct, so that the same derivative may occur more than once on the left-hand side of (3). We will actually prove (following Lévy) an abstract Makeenko–Migdal equation that allows one to compute alternating sums of derivatives of more general functions; see Section 2.5 for additional examples.

The original argument of Makeenko and Migdal for the equation that bears their names was based on heuristic calculations with a path integral and is far from rigorous. (See Section 0.6 of [Lévy2].) Rigorous proofs have been given

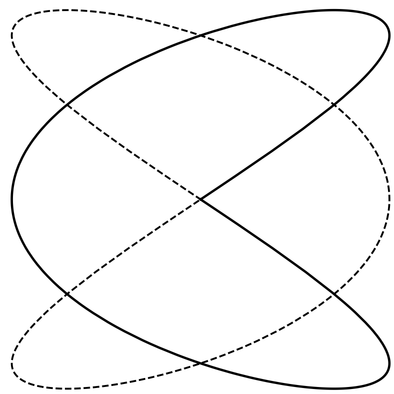

Figure 2: The loops L_1 (black) and L_2 (dashed)

by Lévy [Lévy2] and Dahlqvist [Dahl]. The goal of the current paper is to provide three short proofs of the result, each of which is substantially simpler than the proofs in [Lévy2] and [Dahl]. Our proofs are "local" in the sense that the key calculations involve only the edges and faces adjacent to the crossing v. This local nature of the proofs allows two of them, the proofs based on the edge variables, to be extended to the case of Yang–Mills theory over arbitrary compact surfaces; cf. [DGHK]. In particular, our proofs, in contrast to those of Lévy and Dahlqvist, make no reference to the unbounded face.

The significance of (3) is that the two loops L_1 and L_2 on the right-hand side are simpler than the loop L. On the other hand, if one is attempting to compute Wilson loop expectations recursively, the right-hand side of (3) cannot be considered as a "known" quantity, because it involves the *expectation of the product* of traces, rather than the *product of the expectations*. Thus, Theorem 1 is not especially useful in computing Wilson loop expectations for a fixed rank N.

Nevertheless, it has been suggested at least since the work of 't Hooft ['t Hooft] that quantum gauge theories with structure group $U(N)$ simplify in the $N \to \infty$ limit. Specifically, it has been suggested that in this limit, the Euclidean Yang–Mills path-integral concentrates onto a *single* connection (modulo gauge transformations), known as the *master field*. In the plane case, the structure of the master field has been described by I. M. Singer [Sing] and R. Gopakumar and D. Gross [GG, Gop]; see also A. N. Sengupta's paper [Sen]. Recently, rigorous analyses of the master field on the plane have been given by M. Anshelevich and A. N. Sengupta [AS] and T. Lévy [Lévy2]. Lévy, in particular, shows in detail that the Wilson loop functionals become deterministic in the large-N limit.

In the large-N limit, then, all variances and covariances go to zero, meaning that there is no difference between an expectation of a product and a product of the expectations. For the master field on the plane, the Makeenko–Migdal equation takes the form

$$\left(\frac{\partial}{\partial t_1} - \frac{\partial}{\partial t_2} + \frac{\partial}{\partial t_3} - \frac{\partial}{\partial t_4}\right) \tau(\mathrm{hol}(L)) = \tau(\mathrm{hol}(L_1))\tau(\mathrm{hol}(L_2)), \tag{4}$$

where $\tau(\cdot)$ is the limiting value of $\mathbb{E}\{\mathrm{tr}(\cdot)\}$. Lévy shows (Section 9.4 of [Lévy2]) that by using the Makeenko–Migdal equation at each crossing of the loop (along with a simpler relation that we describe in Theorem 2), one can recover the derivative of a Wilson loop functional with respect to the area of any one face. This result leads to an effective procedure for (recursively) computing the Wilson loop functionals for the master field.

S. Chatterjee has given a rigorous version of the Makeenko–Migdal equation for lattice gauge theories in any dimension (Theorem 3.6 of [Chatt]). This equation takes a somewhat different form from the two-dimensional continuum result in Theorem 1.

We thank Franck Gabriel for comments on and corrections to an earlier version of this article. We also thank Ambar Sengupta for many useful discussions about Yang–Mills theory and the Master Field.

2 Two proofs using edge variables

2.1 The set-up

Beginning at least from the work of Migdal, it has been apparent that the heat kernel on K should play a role in the computation of expected traces of holonomies in the Yang–Mills field theory on the plane. (Equation (27) in [Mig], for example, can be understood as the expansion of the heat kernel on K in terms of characters.) A rigorous approach to such computations was developed simultaneously by L. Gross, C. King, and A. Sengupta in [GKS] and B. Driver in [Dr]. In particular, Driver gave a formula [Dr, Theorem 6.4] for the Yang–Mills measure on a graph $\mathbb{G}$ on the plane, under mild restrictions on the nature of the edges involved.

Driver's formula involves the heat kernel ρ_t on the structure group K with respect to a fixed bi-invariant metric. That is to say, ρ_t satisfies the heat equation

$$\frac{\partial \rho_t}{\partial t} = \frac{1}{2}\Delta \rho_t, \tag{5}$$

where Δ is the Laplacian associated to the given metric, and for any continuous function f on K, we have

$$\lim_{t\to 0} \int_K f(x)\rho_t(x)\, dx = f(\mathrm{id}),$$

where id is the identity element of K and where dx is the normalized Haar measure on K. It will be important in our computations to note that the heat kernel with respect to a bi-invariant metric on a compact Lie group is conjugation invariant:

$$\rho_t(uxu^{-1}) = \rho_t(x), \qquad \forall\, x, u \in K. \tag{6}$$

(This identity holds because the adjoint action of K on itself is isometric and fixes the origin.)

We now consider the appropriate notion of a graph in the plane. By an edge we will mean a continuous map $\gamma : [0,1] \to \mathbb{R}^2$, assumed to be injective except possibly that $\gamma(0) = \gamma(1)$. We identify two edges if they differ by an orientation-preserving reparametrization. Two edges that differ by an orientation-reversing reparametrization are said to be inverses of each other. A graph is then a finite collection of edges (and their inverses) that meet only at their endpoints. Given a graph $\mathbb{G}$, we choose arbitrarily one element out of each pair consisting of an edge and its inverse. We then refer to the chosen edges as the positively oriented edges.

If n denotes the number of edges in $\mathbb{G}$, Driver's result then says that the expectation value of any gauge-invariant function of the parallel transport along the edges of $\mathbb{G}$ may be computed as integration against a measure μ on K^n. To compute μ, we associate to each positively oriented edge e in $\mathbb{G}$ an *edge variable* $x \in K$, and correspondingly associate x^{-1} to the inverse of e. We interpret the edge variable as the parallel transport of a connection along the edge. Then μ is given by

$$d\mu = \left(\prod \rho_{|F_i|}(h_i) \right) \, d\mathbf{x}, \tag{7}$$

where the product is over all the bounded faces F_i of the graph, that is, over all the bounded components of the complement of the graph in the plane. Here $d\mathbf{x}$ denotes the product of normalized Haar measures in all the edge variables, $|F_i|$ denotes the area of F_i, and h_i denotes the "holonomy" around F_i, that is, the product of edge variables and their inverses going around the boundary of F_i; in [MM], these discrete holonomies were referred to as plaquettes. Note: since the Haar measure on K is symmetric (i.e. invariant under $x \mapsto x^{-1}$), the measure μ is independent of the choice of which edged in $\mathbb{G}$ are positively oriented.

It is harmless to assume that the boundary of each face F_i of $\mathbb{G}$ is connected, although as shown in [Dr, pp. 591–592] this assumption is not actually necessary. If the boundary of F_i is connected, it is easy to see that the value of $\rho_{|F_i|}(h_i)$ does not depend on where one starts on the boundary of F_i or on the direction one proceeds.

In particular, each Wilson loop functional as in (1) may be computed by means of a *finite-dimensional* integral over K^n, with respect to the measure μ. Of course, once the measure μ has been defined, it makes sense integrate *any* function f of the edge variables, whether or not f has any special invariance property.

2.2 A simple area-derivative formula

Before coming the Makeenko–Migdal equation itself, we record a simple result that can be proven much more easily. This result was stated by Kazakov in [K, Equation 10]; it is also a special case of Corollary 6.5 of [Lévy2], but in this case, the proof simplifies dramatically. We include a proof here for completeness and to give an indication of the difficulties in computing area derivatives in general.

In [Lévy2], Lévy shows that the master field (i.e. the large-N limit of Yang–Mills for $U(N)$) is completely characterized by the limiting Makeenko–Migdal equation (4) and the large-N limit of (9).

Theorem 2 (Unbounded Face Condition) *Suppose f is a smooth function of the edge variables associated to a graph $\mathbb{G}$ and that F is a bounded face of $\mathbb{G}$ that shares an edge e with the unbounded face. Let $a \in K$ denote the edge variable associated to the edge e and let t denote the area of F. Then we have*

$$\frac{d}{dt}\int f \; d\mu = \frac{1}{2}\int \Delta^a f \; d\mu, \tag{8}$$

where Δ^a denotes the Laplacian with respect to a with the other edge variables held constant.

In particular, suppose that $K = U(N)$, that L is a loop traced out on $\mathbb{G}$ in which the edge e is traversed exactly once, and that $f = \mathrm{tr}(\mathrm{hol}(L))$. Then (8) reduces to

$$\frac{d}{dt}\mathbb{E}\{\mathrm{tr}(\mathrm{hol}(L))\} = -\frac{1}{2}\mathbb{E}\{\mathrm{tr}(\mathrm{hol}(L))\}. \tag{9}$$

Finally, if $K = U(N)$ and L is a simple closed curve enclosing area t, we have

$$\mathbb{E}\{\mathrm{tr}(\mathrm{hol}(L))\} = e^{-t/2}.$$

The key idea in the proof of (8) is that because the edge e lies on the boundary of only one bounded face, the edge variable a occurs in only one of the heat kernels in Driver's formula. By contrast, a generic edge variable lies in two different heat kernels, which is a substantial complicating factor in the proof of the Makeenko–Migdal formula.

Proof. We may choose the orientation of the boundary of F so that it contains the edge e (as opposed to e^{-1}) exactly once. (For example, referring to Figure 5 below, we may take $F = F_5$ and $e = e_7$.) It is harmless to assume that e is the first edge traversed, in which case, since parallel transport is order-reversing, the holonomy h around ∂F will have the form

$$h = \alpha a,$$

where α is a word in edge variables other than a. We then note that $(\Delta\rho_t)(h)$ may be computed as

$$(\Delta\rho_t)(h) = \Delta^a(\rho_t(\alpha a)).$$

Thus, using Driver's formula (7) and differentiating under the integral, we obtain

$$\frac{d}{dt}\int f \; d\mu = \frac{1}{2}\int f \; [\Delta^a \rho_t(\alpha a)] \prod_{F_i \neq F} \rho_{|F_i|}(h_i).$$

Now, since e lies between F and the unbounded face, the edge variable a does not occur in any other heat kernel besides $\rho_t(\alpha a)$. Thus, if we integrate by parts, the Laplacian does not hit any other heat kernel, but hits only f, giving (8).

Meanwhile, suppose $K = U(N)$ and f is the normalized trace of the holonomy of L, where L traverses e exactly once. If L traverses e in the positive direction, then f will have the form

$$f = \mathrm{tr}(\beta a \gamma),$$

where β and γ are words in edge variables distinct from a. Then

$$\Delta^a f = \sum_X \mathrm{tr}(\beta a X^2 \gamma),$$

where X ranges over an orthonormal basis for the Lie algebra $\mathfrak{k} = \mathfrak{u}(N)$. But a simple argument (e.g., Proposition 3.1 in [DHK]) shows that if the inner product on $\mathfrak{u}(N)$ is normalized as in (2) we have

$$\sum_X X^2 = -I,$$

in which case (8) reduces to (9). If L traverses e negatively, the argument is almost identical. Finally, if L has only one bounded face with area t, Driver's formula (7) tells us that at $t = 0$, the holonomy concentrates at the identity, so that the normalized trace of the holonomy is 1. ■

2.3 An abstract Makeenko–Migdal equation

Suppose now that $\mathbb{G}$ is a graph in the plane and v is a vertex of $\mathbb{G}$ with four incident edges. We assume for now that these edges are distinct; this assumption is removed in Section 4. We label the four (not necessarily distinct) faces of $\mathbb{G}$ adjacent to v in cyclic order as F_1, F_2, F_3, and F_4. We then let e_1, e_2, e_3, and e_4 be the *outgoing* edges at v, labeled so that e_1 lies between F_4 and F_1 and e_2 lies between F_1 and F_2, etc. (See Figure 3.) We also let a_i denote the edge variable, with values in K, associated to e_i. We write the collection $\mathbf{x}$ of all edge variables in our graph as

$$\mathbf{x} = (a_1, a_2, a_3, a_4, \mathbf{b}),$$

where $\mathbf{b}$ is the tuple of all edge variables other than a_1, a_2, a_3, and a_4.

In [Lévy2], Lévy isolates a version of the Makeenko–Migdal equation that is valid for an arbitrary compact structure group K, and in which the function does not have to be the trace of a holonomy.

Definition 3 *If the edges $e_1, \ldots, e_4$ are distinct, a function $f(a_1, a_2, a_3, a_4, \mathbf{b})$ of the edge variables has* extended gauge invariance at v *if, for all $x \in K$,*

$$f(a_1, a_2, a_3, a_4, \mathbf{b}) = f(a_1 x, a_2, a_3 x, a_4, \mathbf{b}) = f(a_1, a_2 x, a_3, a_4 x, \mathbf{b}). \tag{10}$$

By contrast, if the edges are distinct, f has *ordinary gauge invariance* at v (i.e., invariance under a gauge transformation supported at the vertex v) if

$$f(a_1, a_2, a_3, a_4, \mathbf{b}) = f(a_1 x, a_2 x, a_3 x, a_4 x, \mathbf{b}) \tag{11}$$

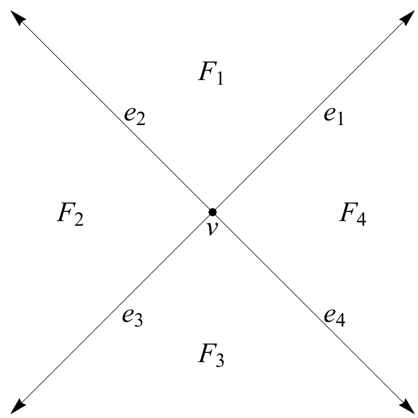

Figure 3: Labeling of faces and edges adjacent to v

for all $x \in K$. Clearly, extended gauge invariance at v implies ordinary gauge invariance at v, but not vice versa.

Suppose, for example, that f is the trace of the holonomy around a loop L traced out on $\mathbb{G}$, as in Figure 1. We label things so that L first traverses the edge between F_4 and F_1, namely e_1. The first return to v must then be along the only incoming edge of L at v that is not "straight across" from e_1, namely e_4^{-1}. Thus, L will have the form

$$L = e_1 A e_4^{-1} e_2 B e_3^{-1},$$

where A and B are sequences of edges not belonging to $\{e_1, e_2, e_3, e_4\}$. Since parallel transport is order-reversing, the trace of the holonomy around L is then represented by a function of the form

$$f(a_1, a_2, a_3, a_4, \mathbf{b}) := \operatorname{tr}(a_3^{-1} \beta a_2 a_4^{-1} \alpha a_1), \tag{12}$$

where α and β are words the $\mathbf{b}$ variables. This function is easily seen to have extended gauge invariance at v.

Definition 4 *If f is a smooth function on K, the* left-invariant gradient *of f, denoted ∇f, is the function with values in the Lie algebra $\mathfrak{k}$ of K given by*

$$(\nabla f)(x) = \sum_X \left(\frac{d}{ds} f(xe^{sX}) \bigg|_{s=0} \right) X,$$

where the sum is over any orthonormal basis of $\mathfrak{k}$. More generally, if f is a smooth function of the edge variables and a is one of the edge variables, we let $\nabla^a f$ denote the left-invariant gradient of f with respect to a with the other variables fixed. Finally, if a and b are two distinct edge variables, $\nabla^a \cdot \nabla^b f$ is the scalar-valued function defined by

$$(\nabla^a \cdot \nabla^b f)(a, b, \mathbf{c}) = \sum_X \frac{\partial^2}{\partial s \partial t} f(ae^{sX}, be^{tX}, \mathbf{c}) \bigg|_{s=t=0}$$

where **c** *is the tuple of edge variables other than a and b.*

If f is smooth and has extended gauge invariance at v, then by differentiating (10), we obtain

$$\nabla^{a_i} f = -\nabla^{a_{i+2}} f,$$

where $i+2$ is computed mod 4. Since, also, ∇^{a_i} commutes with ∇^{a_j}, we have

$$\begin{aligned}\nabla^{a_i} \cdot \nabla^{a_j} f &= -\nabla^{a_i} \cdot \nabla^{a_{j+2}} f = -\nabla^{a_{j+2}} \cdot \nabla^{a_i} f \\ &= \nabla^{a_{j+2}} \cdot \nabla^{a_{i+2}} f = \nabla^{a_{i+2}} \cdot \nabla^{a_{j+2}} f,\end{aligned} \tag{13}$$

even though $\nabla^{a_j} f$ does not necessarily have extended gauge invariance.

We are now ready to state (a special case of) Lévy's abstract form of the Makeenko–Migdal equation.

Theorem 5 (T. Lévy) *Suppose $\mathbb{G}$ is a graph in the plane and v is a vertex of $\mathbb{G}$ with four distinct edges emanating from v. Label the four faces of $\mathbb{G}$ adjacent to v in cyclic order as $F_1, \ldots, F_4$ and label the outgoing edges in cyclic order as $e_1, \ldots, e_4$, with e_1 lying between F_4 and F_1. Then if f is a smooth function of the edge variables of $\mathbb{G}$ having extended gauge invariance at v, we have*

$$\left(\frac{\partial}{\partial t_1} - \frac{\partial}{\partial t_2} + \frac{\partial}{\partial t_3} - \frac{\partial}{\partial t_4}\right) \int f \; d\mu = -\int \nabla^{a_1} \cdot \nabla^{a_2} f \; d\mu, \tag{14}$$

where t_i is the area of F_i, $i = 1, \ldots, 4$.

As usual, we set $\partial/\partial t_i$ equal to zero if F_i is the unbounded face. A version of the theorem still holds even if the edges $e_1, \ldots, e_4$ are not distinct; see Section 4. Note that f is not assumed to have any special invariance property at any vertex other than v.

Theorem 5 is a special case of Proposition 6.22 in [Lévy2]. Specifically, since the Yang–Mills measure does not depend on the orientation of the plane, it is harmless to assume that the faces F_1, F_2, F_3, F_4 in our labeling scheme occur in counterclockwise order, as in Figure 1. We may take the set I in Levy's Proposition 6.22 to be $\{e_1, e_3\}$, as in Figure 25 in [Lévy2]. Then the left-hand side of Proposition 6.22 is actually the negative of the usual alternating sum of area-derivatives. On the right-hand side of Proposition 6.22, meanwhile, there is only one term in the sum, namely $\int \Delta^{e_1;e_2} f \; d\mu$, which corresponds to $\int \nabla^{a_1} \cdot \nabla^{a_2} f \; d\mu$ in our notation.

Note that since f is assumed to have extended gauge invariance at v, we have, as in (13),

$$\nabla^{a_1} \cdot \nabla^{a_2} f = -\nabla^{a_2} \cdot \nabla^{a_3} f = \nabla^{a_3} \cdot \nabla^{a_4} f = -\nabla^{a_4} \cdot \nabla^{a_1} f. \tag{15}$$

If we specialize Theorem 5 to the case in which $K = U(N)$ and f is as in (12), we find that

$$\begin{aligned}&\left(\frac{\partial}{\partial t_1} - \frac{\partial}{\partial t_2} + \frac{\partial}{\partial t_3} - \frac{\partial}{\partial t_4}\right) \int \mathrm{tr}(a_3^{-1} \beta a_2 a_4^{-1} \alpha a_1) \; d\mu \\ &= -\sum_X \int \mathrm{tr}(a_3^{-1} \beta a_2 X a_4^{-1} \alpha a_1 X) \; d\mu,\end{aligned} \tag{16}$$

where the sum is over any orthonormal basis $\{X\}$ for $\mathfrak{u}(N)$. But an elementary argument (e.g. [DHK, Proposition 3.1]) shows that if we normalize the inner product on $\mathfrak{u}(N)$ as in (2), then

$$\sum_X XCX = -\mathrm{tr}(C)I \tag{17}$$

for any $N \times N$ matrix C. Thus, (16) reduces to

$$\left(\frac{\partial}{\partial t_1} - \frac{\partial}{\partial t_2} + \frac{\partial}{\partial t_3} - \frac{\partial}{\partial t_4}\right) \int \mathrm{tr}(a_3^{-1}\beta a_2 a_4^{-1}\alpha a_1)\ d\mu$$
$$= \int \mathrm{tr}(a_4^{-1}\alpha a_1)\mathrm{tr}(a_3^{-1}\beta a_2)\ d\mu,$$

which is—in light of Driver's formula—just the Makeenko–Migdal equation for $U(N)$, as in Theorem 1.

The goal of this section is to give two short proofs of Theorem 5. In [Lévy2], Lévy develops a method of differentiating any function with respect to the area t_i of some face F_i. Specifically, if f is any smooth function of the edge variables—which need not have any special invariance property—Lévy shows that

$$\frac{\partial}{\partial t_i} \int f\ d\mu = \int Df\ d\mu, \tag{18}$$

where D is a certain differential operator. (See Corollary 6.5 in [Lévy2].) The formula for D involves the choice of a maximal tree in $\mathbb{G}$ and a sum over a sequence of adjacent faces proceeding from F_i to the unbounded face. Thus, D contains, in general, derivatives involving edges far from the vertex in question.

Lévy then specializes his result to the case where f has extended gauge invariance at v and takes the alternating sum of derivatives around a vertex. At that point, a substantial cancellation occurs: all derivatives of f drop out, except for derivatives involving edges coming out of the crossing, and Lévy then obtains the abstract Makeenko–Migdal equation of Theorem 5. (See the proof of Proposition 6.22 in [Lévy2].)

Our strategy for a simplified proof of Theorem 5 is to think that if the cancellation described in the previous paragraph actually occurs, it should be possible to see the cancellation "locally," that is, in such a way that derivatives involving far away edges never occur in the first place. Of course, Lévy's formula (18) may be useful for various computations, but we do not use it in our proofs of the Makeenko–Migdal equation (3). The local nature of our argument allows us to prove a new result, namely that the Makeenko–Migdal equation holds also for Yang–Mills theory on an arbitrary compact surface, as shown in our paper [DGHK].

2.4 Two "local" proofs of the theorem

We consider at first the "generic" case, in which the faces F_1, F_2, F_3, and F_4 are distinct and bounded, and the outgoing edges e_1, e_2, e_3, and e_4 from v are distinct. (These assumptions are lifted in Section 4.) In that case, the boundary of F_i may be represented by a loop of the form

$$\partial F_i = e_i A_i e_{i+1}^{-1}, \tag{19}$$

where A_i is a sequence of edges not belonging to $\{e_1, e_2, e_3, e_4\}$, where the index i is understood to be in $\mathbb{Z}/4$. Since parallel transport is order reversing, the holonomy h_i around ∂F_i is represented by an expression of the form

$$h_i = a_{i+1}^{-1} \alpha_i a_i, \quad i = 1, 2, 3, 4, \tag{20}$$

where α_i is a word in the $\mathbf{b}$ variables (i.e., the edge variables not belonging to $\{a_1, a_2, a_3, a_4\}$). Furthermore, none of the variables a_1, a_2, a_3, a_4 appears in any holonomy other than ones associated to F_1, F_2, F_3, F_4. Thus, the Yang–Mills measure μ takes the form

$$d\mu = \rho_{t_1}(a_2^{-1}\alpha_1 a_1)\rho_{t_2}(a_3^{-1}\alpha_2 a_2)\rho_{t_3}(a_4^{-1}\alpha_3 a_3)\rho_{t_4}(a_1^{-1}\alpha_4 a_4)\nu(\mathbf{b})\ d\mathbf{x}, \tag{21}$$

where $d\mathbf{x}$ is the product of the normalized Haar measures in all the edge variables, and $\nu(\mathbf{b})$ is a product of heat kernels in $\mathbf{b}$ variables.

Our proofs based on the edge variables are based on the following "local" version of the abstract Makeenko–Migdal equation. Since the local structure of the Yang–Mills measure on an arbitrary compact surface is the same as on the plane, Theorem 6 can be applied also on surfaces. This observation leads to a proof of the Makeenko–Migdal equation over surfaces, as worked out in [DGHK].

Theorem 6 (Local Abstract Makeenko–Migdal Equation) *Suppose $f : K^4 \to \mathbb{C}$ is a smooth function satisfying the following "extended gauge invariance" property:*

$$f(a_1, a_2, a_3, a_4) = f(a_1 x, a_2, a_3 x, a_4) = f(a_1, a_2 x, a_3, a_4 x)$$

for all $\mathbf{a} = (a_1, a_2, a_3, a_4)$ in K^4 and all x in K. For each fixed $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ in K^4 and $\mathbf{t} = (t_1, t_2, t_3, t_4)$ in $(\mathbb{R}^+)^4$, define a measure $\mu_{\alpha,\mathbf{t}}$ on K^4 by

$$d\mu_{\alpha,\mathbf{t}}(\mathbf{a}) = \rho_{t_1}(a_2^{-1}\alpha_1 a_1)\rho_{t_2}(a_3^{-1}\alpha_2 a_2)\rho_{t_3}(a_4^{-1}\alpha_3 a_3)\rho_{t_4}(a_1^{-1}\alpha_4 a_4)\ d\mathbf{a},$$

where $d\mathbf{a}$ is the normalized Haar measure on K^4. Then for all $\alpha \in K^4$, we have

$$\left(\frac{\partial}{\partial t_1} - \frac{\partial}{\partial t_2} + \frac{\partial}{\partial t_3} - \frac{\partial}{\partial t_4}\right) \int_{K^4} f\ d\mu_{\alpha,\mathbf{t}} = -\int_{K^4} \nabla^{a_1} \cdot \nabla^{a_2} f\ d\mu_{\alpha,\mathbf{t}}. \tag{22}$$

Our first proof of Theorem 6 proceeds by directly computing the alternating sum of area-derivatives, and integrating by parts twice. Our second proof, which is even shorter, proceeds from the right-hand-side of (22) and relies on the decomposition of the density of $\mu_{\alpha,\mathbf{t}}$ into the product of (t_1, t_2) heat kernels (both independent of edge variable a_4) and (t_3, t_4) heat kernels (both independent of edge variable a_2).

We also prove Theorem 5, for gauge invariant functions, in Section 3 using the loop or lasso variables. This third proof is also in a sense local.

We now observe that Theorem 6 easily implies the generic case of the abstract Makeenko–Migdal equation in Theorem 5.

Proof of Theorem 5 (Generic Case). If $f : K^n \to \mathbb{C}$ has extended gauge invariance at v, then $f(a_1, a_2, a_3, a_4, \mathbf{b})$ has extended gauge invariance as a function of $a_1, \ldots, a_4$ for each $\mathbf{b}$. In light of (20), we will have

$$\int_{K^n} f \; d\mu = \int_{K^{n-4}} \int_{K^4} f(\mathbf{a}, \mathbf{b}) \; d\mu_{\alpha,\mathbf{t}}(\mathbf{a}) \; \nu(\mathbf{b}) \; d\mathbf{b},$$

where $\nu(\mathbf{b})$ is a product of heat kernels in the $\mathbf{b}$ variables. Since the only dependence on (t_1, t_2, t_3, t_4) in the integral is in $\mu_{\alpha,\mathbf{t}}$, the time derivatives in Theorem 5 will pass over the outer integral and hit on the integral over K^4. Theorem 5 then follows from Theorem 6. ■

It remains to prove the local result in Theorem 6.

2.4.1 First proof of Theorem 6

Our strategy is to differentiate under the integral sign, use the heat equation satisfied by the heat kernel, and then integrate by parts. In this process, we will get "good terms" in which derivatives hit on the function f, and "bad terms" in which derivatives hit on other heat kernels. In each of the two stages of integration by parts, we obtain a cancellation of the bad terms, allowing all of the derivatives to move off of the heat kernels and onto f, at which point we easily obtain the local Makeenko–Migdal equation in (22).

Since the heat kernel on K is invariant under conjugation, we can compute $(\Delta\rho_{t_i})(a_{i+1}^{-1}\alpha_i a_i)$ by various combinations of derivatives with respect to a_i and derivatives with respect to a_{i+1}. It turns out that the most convenient way to do the computation is as follows:

$$\begin{aligned}(\Delta\rho_{t_i})(a_{i+1}^{-1}\alpha_i a_i) &= \frac{1}{4}\left(\nabla^{a_i} - \nabla^{a_{i+1}}\right)^2 \left[\rho_{t_i}(a_{i+1}^{-1}\alpha_i a_i)\right] \\ &= \frac{1}{4}\sum_X \left(\frac{\partial}{\partial s} - \frac{\partial}{\partial t}\right)^2 \rho_{t_i}(e^{sX} a_{i+1}^{-1}\alpha_i a_i e^{-tX})\bigg|_{s=t=0}.\end{aligned}$$

Here, ∇^{a_i} is the left-invariant gradient of Definition 4. Using the heat equation, we obtain

$$\frac{\partial}{\partial t_i}\int f \; d\mu_{\alpha,\mathbf{t}} = \frac{1}{8}\int f[(\nabla^{a_i} - \nabla^{a_{i+1}})^2 \rho_{t_i}]\rho_{t_1}\cdots\widehat{\rho_{t_i}}\cdots\rho_{t_4} \; d\mathbf{a}, \tag{23}$$

where $\widehat{\rho_{t_i}}$ indicates that the given heat kernel is omitted. Here and below, we frequently omit the arguments of the heat kernels, since they are always the same.

If we now integrate by parts a first time, we get some terms where $\nabla^{a_i} - \nabla^{a_{i+1}}$ hits on f and some terms where $\nabla^{a_i} - \nabla^{a_{i+1}}$ hits another heat kernel. But the only heat kernels besides ρ_{t_i} that contain the variables a_i or a_{i+1} are $\rho_{t_{i-1}}(a_i^{-1}\alpha_{i-1}a_{i-1})$ and $\rho_{t_{i+1}}(a_{i+2}^{-1}\alpha_{i+1}a_{i+1})$. Furthermore, using the conjugation invariance of ρ_t (cf. (6)), we find that

$$(\nabla^{a_i} - \nabla^{a_{i+1}})[\rho_{t_{i-1}}(a_i^{-1}\alpha_{i-1}a_{i-1})] = -(\nabla\rho_{t_{i-1}})(a_i^{-1}\alpha_{i-1}a_{i-1})$$
$$(\nabla^{a_i} - \nabla^{a_{i+1}})[\rho_{t_{i+1}}(a_{i+2}^{-1}\alpha_{i+1}a_{i+1})] = -(\nabla\rho_{t_{i+1}})(a_{i+2}^{-1}\alpha_{i+1}a_{i+1}).$$

Each time we encounter a gradient of a heat kernel, we multiply and divide by ρ_t and use the identity

$$\frac{\nabla\rho_t}{\rho_t} = \nabla\log\rho_t$$

to write the answer as an integral against the Yang–Mills measure, giving

$$\frac{\partial}{\partial t_i}\int f\ d\mu_{\alpha,\mathbf{t}} = -\frac{1}{8}\int[(\nabla^{a_i} - \nabla^{a_{i+1}})f]\cdot[(\nabla^{a_i} - \nabla^{a_{i+1}})\log\rho_{t_i}]\ d\mu_{\alpha,\mathbf{t}}$$
$$+\frac{1}{4}\int f[\nabla\log\rho_{t_i}\cdot\nabla\log\rho_{t_{i-1}} + \nabla\log\rho_{t_i}\cdot\nabla\log\rho_{t_{i+1}}]\ d\mu_{\alpha,\mathbf{t}}.$$

Upon taking the alternating sum, each term involving a product of two log-gradients of heat kernels will occur twice with opposite signs. Thus, even without assuming extended gauge invariance, we have a significant cancellation, yielding

$$\left(\frac{\partial}{\partial t_1} - \frac{\partial}{\partial t_2} + \frac{\partial}{\partial t_3} - \frac{\partial}{\partial t_4}\right)\int f\ d\mu_{\alpha,\mathbf{t}}$$
$$= -\frac{1}{8}\sum_i(-1)^{i+1}\int[(\nabla^{a_i} - \nabla^{a_{i+1}})f]\cdot[(\nabla^{a_i} - \nabla^{a_{i+1}})\rho_{t_i}]\rho_{t_1}\cdots\widehat{\rho_{t_i}}\cdots\rho_{t_4}\ d\mathbf{a}.$$

We now integrate by parts a second time, pushing the remaining derivatives off of ρ_{t_i} and onto all the other functions involved. By the same argument as in the first integration by parts, this gives

$$\left(\frac{\partial}{\partial t_1} - \frac{\partial}{\partial t_2} + \frac{\partial}{\partial t_3} - \frac{\partial}{\partial t_4}\right)\int f\ d\mu_{\alpha,\mathbf{t}} = \frac{1}{8}\sum_i(-1)^{i+1}\int(\nabla^{a_i} - \nabla^{a_{i+1}})^2 f\ d\mu_{\alpha,\mathbf{t}}$$
$$-\frac{1}{8}\sum_i(-1)^{i+1}\int[(\nabla^{a_i} - \nabla^{a_{i+1}})f(\mathbf{a})]\cdot[\nabla\log\rho_{t_{i-1}} + \nabla\log\rho_{t_{i+1}}]\ d\mu_{\alpha,\mathbf{t}}.$$

Now, since f has extended gauge invariance, we have

$$(\nabla^{a_i} - \nabla^{a_{i+1}})f = -(\nabla^{a_{i+2}} - \nabla^{a_{i+3}})f.$$

Meanwhile, if we replace i by $i+2$, then $\nabla\rho_{t_{i-1}}$ becomes $\nabla\rho_{t_{i+1}}$ and $\nabla\rho_{t_{i+1}}$ becomes $\nabla\rho_{t_{i+3}} = \nabla\rho_{t_{i-1}}$, since i is understood to be in $\mathbb{Z}/4$. Thus, in the alternating sum, all the derivatives of heat kernels cancel ($i=1$ with $i=3$ and $i=2$ with $i=4$), leaving us with

$$\left(\frac{\partial}{\partial t_1} - \frac{\partial}{\partial t_2} + \frac{\partial}{\partial t_3} - \frac{\partial}{\partial t_4}\right)\int f\ d\mu_{\alpha,\mathbf{t}}$$
$$= \frac{1}{8}\sum_i(-1)^{i+1}\int(\nabla^{a_i} - \nabla^{a_{i+1}})^2 f\ d\mu_{\alpha,\mathbf{t}}. \tag{24}$$

We now note that

$$(\nabla^{a_i} - \nabla^{a_{i+1}})^2 f = \nabla^{a_i} \cdot \nabla^{a_i} f - 2\nabla^{a_i} \cdot \nabla^{a_{i+1}} f + \nabla^{a_{i+1}} \cdot \nabla^{a_{i+1}} f.$$

All the "Laplacian" terms will cancel in the alternating sum, so that (24) becomes

$$\begin{aligned}\left(\frac{\partial}{\partial t_1} - \frac{\partial}{\partial t_2} + \frac{\partial}{\partial t_3} - \frac{\partial}{\partial t_4}\right) \int f \, d\mu_{\alpha,\mathbf{t}} &= -\frac{1}{4} \sum_i \int (-1)^{i+1} \nabla^{a_i} \cdot \nabla^{a_{i+1}} f \, d\mu_{\alpha,\mathbf{t}} \\ &= -\int \nabla^{a_1} \cdot \nabla^{a_2} f \, d\mu_{\alpha,\mathbf{t}},\end{aligned}$$

where we have used (15) in the second equality.

2.4.2 Second proof of Theorem 6

In our second proof, which is likely to be about as short as possible, we begin by writing the density of $\mu_{\alpha,\mathbf{t}}$ as a product of two terms: those corresponding to (t_1, t_2) and those to (t_3, t_4):

$$\mathcal{R}_{12} = \rho_{t_1} \rho_{t_2}, \quad \mathcal{R}_{34} = \rho_{t_3} \rho_{t_4}$$

where, as above, we suppress the explicit variable dependences. For clarification, $\mathcal{R}_{12}$ depends on a_1, a_2, a_3, while $\mathcal{R}_{34}$ depends on a_1, a_3, a_4. Then

$$d\mu_{\alpha,\mathbf{t}} = \mathcal{R}_{12}\mathcal{R}_{34} \, d\mathbf{a}.$$

For the remainder of the proof, we write integrals of functions g against $d\mathbf{a}$ simply as $\int g$.

Now, using extended gauge invariance as in (15), taking care to commute partial derivatives, we may write

$$\nabla^{a_1} \cdot \nabla^{a_2} f = \frac{1}{2}(\nabla^{a_1} - \nabla^{a_3}) \cdot \nabla^{a_2} f.$$

Then we integrate by parts once, and use the product rule.

$$\begin{aligned}-\int \nabla^{a_1} \cdot \nabla^{a_2} f \;\; d\mu_{\alpha,\mathbf{t}} &= -\frac{1}{2} \int \left[(\nabla^{a_1} - \nabla^{a_3}) \cdot (\nabla^{a_2} f)\right] \mathcal{R}_{12}\mathcal{R}_{34} \\ &= \frac{1}{2} \int [\nabla^{a_2} f \cdot (\nabla^{a_1} - \nabla^{a_3})(\mathcal{R}_{12}\mathcal{R}_{34})] \\ &= \frac{1}{2} \int \mathcal{R}_{34}[\nabla^{a_2} f \cdot (\nabla^{a_1} - \nabla^{a_3})(\mathcal{R}_{12})] \\ &\quad + \frac{1}{2} \int \mathcal{R}_{12}[\nabla^{a_2} f \cdot (\nabla^{a_1} - \nabla^{a_3})(\mathcal{R}_{34})].\end{aligned}$$

We now use extended gauge invariance once more, in the second term, writing $\nabla^{a_2} f = -\nabla^{a_4} f$, yielding

$$\frac{1}{2} \int \mathcal{R}_{34}[\nabla^{a_2} f \cdot (\nabla^{a_1} - \nabla^{a_3})(\mathcal{R}_{12})] - \frac{1}{2} \int \mathcal{R}_{12}[\nabla^{a_4} f \cdot (\nabla^{a_1} - \nabla^{a_3})(\mathcal{R}_{34})].$$

Since $\mathcal{R}_{12}$ does not depend on a_4, and $\mathcal{R}_{34}$ does not depend on a_2, we can integrate this by parts once more, and the ∇^{a_2} and ∇^{a_4} derivatives only hit the already differentiated factors. Thus,

$$-\int \nabla^{a_1} \cdot \nabla^{a_2} f \;\; d\mu_{\alpha,\mathbf{t}} = -\frac{1}{2}\int f\mathcal{R}_{34}[\nabla^{a_2} \cdot (\nabla^{a_1} - \nabla^{a_3})\mathcal{R}_{12}] \\ + \frac{1}{2}\int f\mathcal{R}_{12}[\nabla^{a_4} \cdot (\nabla^{a_1} - \nabla^{a_3})\mathcal{R}_{34}]. \tag{25}$$

Finally, we compute the second derivatives. Recalling that $\mathcal{R}_{12} = \rho_{t_1}\rho_{t_2}$ and recalling the arguments of the heat kernels from the definition of $\mu_{\alpha,\mathbf{t}}$, we have

$$\begin{aligned}(\nabla^{a_1} - \nabla^{a_3})\mathcal{R}_{12} &= (\nabla^{a_1} - \nabla^{a_3})(\rho_{t_1}(a_2^{-1}\alpha_1 a_1)\rho_{t_2}(a_3^{-1}\alpha_2 a_2)) \\ &= \rho_{t_2}(a_3^{-1}\alpha_2 a_2)(\nabla\rho_{t_1})(a_2^{-1}\alpha_1 a_1) \\ &+ \rho_{t_1}(a_2^{-1}\alpha_1 a_1)(\nabla\rho_{t_2})(a_3^{-1}\alpha_2 a_2).\end{aligned}$$

Applying ∇^{a_2} then yields

$$\nabla^{a_2} \cdot (\nabla^{a_1} - \nabla^{a_3})\mathcal{R}_{12} = \nabla\rho_{t_2} \cdot \nabla\rho_{t_1} - \rho_{t_2}\Delta\rho_{t_1} - \nabla\rho_{t_2} \cdot \nabla\rho_{t_1} + \rho_{t_1}\Delta\rho_{t_2}.$$

The first and third terms cancel, and we see that the first term on the right-hand side of (25) is equal to

$$\begin{aligned}-\frac{1}{2}\int f\mathcal{R}_{34}[\nabla^{a_2} \cdot (\nabla^{a_1} - \nabla^{a_3})\mathcal{R}_{12}] &= -\frac{1}{2}\int f\rho_{t_3}\rho_{t_4}(-\rho_{t_2}\Delta\rho_{t_1} + \rho_{t_1}\Delta\rho_{t_2}) \\ &= \left(\frac{\partial}{\partial t_1} - \frac{\partial}{\partial t_2}\right)\int f \;\; d\mu_{\alpha,\mathbf{t}},\end{aligned}$$

where we have used the heat equation (5) in the second equality.

An entirely analogous computation shows that the second term on the right-hand side of (25) is equal to $(\frac{\partial}{\partial t_3} - \frac{\partial}{\partial t_4})\int f\, d\mu_{\alpha,\mathbf{t}}$, and adding these up gives the left-hand-side of Theorem 5, concluding the proof.

2.5 Additional examples of the abstract Makeenko–Migdal equation

We have noted that the Makeenko–Migdal equation in Theorem 1 is a special case of the abstract Makeenko–Migdal equation. As Lévy has noted [Lévy2, Proposition 6.24], the abstract result can be specialized in many interesting ways; we mention a few of these here. We now take $K = U(N)$, with metric normalized as in (2).

First, suppose that $\mathbb{G}$ is a graph and $L_1, \ldots, L_r$ are loops traced out in $\mathbb{G}$. Assume that L_r has a simple crossing at v and that none of the remaining loops passes through v. Let L_r' and L_r'' be the splitting of L_r at v, as in the statement of Theorem 1. Then

$$\left(\frac{\partial}{\partial t_1} - \frac{\partial}{\partial t_2} + \frac{\partial}{\partial t_3} - \frac{\partial}{\partial t_4}\right)\mathbb{E}\{\mathrm{tr}(\mathrm{hol}(L_1)) \cdots \mathrm{tr}(\mathrm{hol}(L_{r-1}))\mathrm{tr}(\mathrm{hol}(L_r))\} \\ = \mathbb{E}\{\mathrm{tr}(\mathrm{hol}(L_1)) \cdots \mathrm{tr}(\mathrm{hol}(L_{r-1}))\mathrm{tr}(\mathrm{hol}(L_r'))\mathrm{tr}(\mathrm{hol}(L_r''))\}.$$

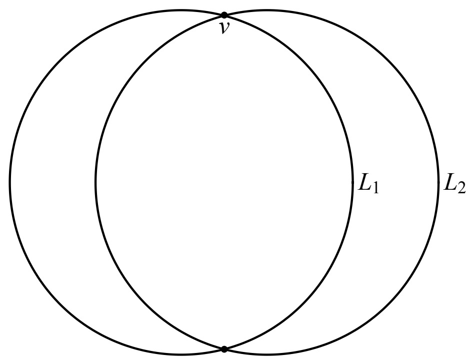

Figure 4: Two loops passing through v

The derivation of this example from Theorem 5 is precisely the same as in (16); the additional loop $L_1, \ldots, L_{r-1}$ simply tag along for the ride.

Second, suppose L_1 and L_2 are two loops traced out in a graph $\mathbb{G}$ and suppose L_1 starts at v, goes out along e_1, and then eventually returns to v along e_3^{-1}, but otherwise does not pass through any of $e_1, \ldots, e_4$. Suppose L_2 starts at v goes out along e_2 and then eventually returns to v along e_4^{-1}, but otherwise does not pass through any of $e_1, \ldots, e_4$. (See Figure 4 for a simple example.) Then

$$\begin{aligned}&\left(\frac{\partial}{\partial t_1} - \frac{\partial}{\partial t_2} + \frac{\partial}{\partial t_3} - \frac{\partial}{\partial t_4}\right) \mathbb{E}\left[\mathrm{tr}(\mathrm{hol}(L_1))\mathrm{tr}(\mathrm{hol}(L_2))\right] \\ &= \frac{1}{N^2}\mathbb{E}\left[\mathrm{tr}\{\mathrm{hol}(L_1)\mathrm{hol}(L_2)\}\right]. \end{aligned} \tag{26}$$

For this example, we note that the integrand on the left-hand side of (26) has the form

$$f(\mathbf{x}) = \mathrm{tr}(a_3^{-1}\alpha a_1)\mathrm{tr}(a_4^{-1}\beta a_2)$$

where α and β are words in the $\mathbf{b}$ variables. This function has extended gauge invariance at v, and we find that

$$\begin{aligned}(\nabla^{a_1} \cdot \nabla^{a_2} f)(\mathbf{x}) &= \sum_X \mathrm{tr}(a_3^{-1}\alpha a_1 X)\mathrm{tr}(a_4^{-1}\beta a_2 X) \\ &= -\frac{1}{N^2}\mathrm{tr}(a_3^{-1}\alpha a_1 a_4^{-1}\beta a_2) \\ &= -\frac{1}{N^2}\mathrm{tr}[\mathrm{hol}(L_1)\mathrm{hol}(L_2)],\end{aligned}$$

where in the second equality, we have used a simple identity (e.g., the last line of Proposition 3.1 in [DHK]). Then (26) follows from Theorem 5.

3 A proof using loop variables

In Section 7 of [Dahl], Dahlqvist gave a proof of the Makeenko–Migdal equation for $U(N)$ (Theorem 1) using "loop" or "lasso" variables. This proof stands in contrast to the proof in [Lévy2] of the abstract Makeenko–Migdal equation (which implies Theorem 1) using "edge" variables. Like Lévy's proof using edge variables, Dahlqvist's proof is based on a formula for the derivative with respect to an individual time variable: a formula which contains a large number of terms that must cancel upon taking the alternating sum. As in our proof using edge variables, we will work with the alternating sum from the beginning and obtain the necessary cancellations without ever encountering all the terms arising in [Dahl]. We actually prove an abstract form of Makeenko–Migdal, similar to Theorem 5, based on loop variables. This result contains, as a special case, the Makeenko–Migdal equation for $U(N)$.

3.1 The loop variables

It is well known that the fundamental group of any graph is free. Given a planar graph $\mathbb{G}$, a fixed vertex v of $\mathbb{G}$, and a maximal tree T in $\mathbb{G}$, Lévy gives a particular set of free generators for $\pi_1(\mathbb{G})$; the generators are in one-to-one correspondence with the bounded faces of $\mathbb{G}$. We refer to these generators as loops or lassos. Each generator is obtained by starting at v, traveling along a certain path p in T, then around the boundary of a particular bounded face, then back to v along the inverse of p. For the details of this construction, we refer to Section 4.3 of [Lévy2]. (See especially Proposition 4.2.)

We now introduce the loop variables, which are simply the products of the edge variables associated to the edges in the just-defined loops. The loop variables are almost the same as the holonomy variables h_i entering into Driver's formula (7), except that they contain a "tail" representing the path p in the previous paragraph. (Since ρ_t is conjugation invariant, the tail may be omitted from the heat kernel.)

We choose as our basepoint v the crossing involved in the Makeenko–Migdal equation. If F_1, F_2, F_3, and F_4 are the four faces surrounding v, then the associated loops just traverse the boundary of each face starting from v. Thus, as in (19), we have

$$L_i = \partial F_i = e_i A_i e_{i+1}^{-1}, \quad i = 1, 2, 3, 4,$$

where A_i is a word in edges not belonging to $\{e_1, e_2, e_3, e_4\}$. Since parallel transport is order-reversing, the corresponding loop *variables* (with values in K) will have the form

$$\ell_i = a_{i+1}^{-1} \alpha_i a_i, \quad i = 1, 2, 3, 4, \tag{27}$$

where α_i is a word in the $\mathbf{b}$ variables (that is, the edge variables not belonging to $\{a_1, a_2, a_3, a_4\}$); see Figure 5. Properly speaking, (27) applies only when F_i is a bounded face; if F_i is unbounded, there is no loop variable associated to F_i.

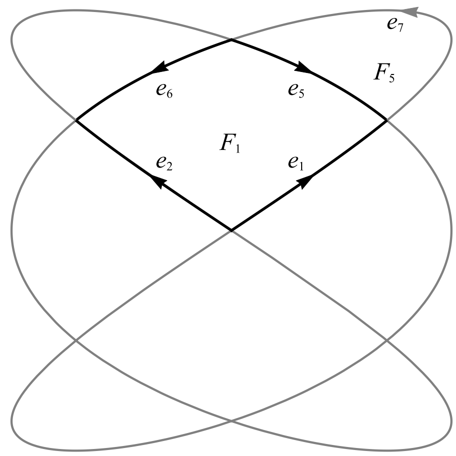

Figure 5: The loop L_1 associated to F_1 is $e_1 e_5^{-1} e_6 e_2^{-1}$

Meanwhile, for the loop L_j associated to any bounded face *other* than F_1, F_2, F_3, and F_4, the associated loop will have the form $L_j = e_{i_j} B_j e_{i_j}^{-1}$, where $e_{i_j} \in \{e_1, e_2, e_3, e_4\}$ is the first edge traversed by L_j and where B_j is a word in edge variables not in $\{e_1, e_2, e_3, e_4\}$. Thus, the corresponding loop *variable* will have the form

$$\ell_j = a_{i_j}^{-1} \beta_j a_{i_j}, \quad j \geq 5, \tag{28}$$

where β_j is a word in the $\mathbf{b}$ variables.

If n is the number of edges and m is the number of bounded faces, the assignments in (27) and (28) define a smooth map $\Psi : K^n \to K^m$, sending the edge variables to the loop variables. Typically, there will be fewer loop variables than edge variables. Thus, not every function of the edge variables (whether or not the function has extended gauge invariance at a particular vertex v) will be expressible as a function of the loop variables. On the other hand, since the loop variables generate the fundamental group of $\mathbb{G}$, the trace of the holonomy around any loop in $\mathbb{G}$ will be expressible as a function of the loop variables. More generally, according to Lemma 2.1.5 of [Lévy1], if f has ordinary gauge invariance (in the sense of (11)) at *every* vertex of $\mathbb{G}$, then f can be expressed as a function of the loop variables.

Meanwhile, using Driver's formula (7) for the Yang–Mills measure for $\mathbb{G}$, one can show that the loop variables are independent and heat kernel distributed. (See Proposition 4.4 in [Lévy2].) That is to say: the push-forward of the measure μ under the map Ψ is simply the product of heat kernel measures with time parameters equal to the areas of the bounded faces. If $\tilde{\mu}$ refers to this pushed forward measure, the measure-theoretic change of variables theorem says that

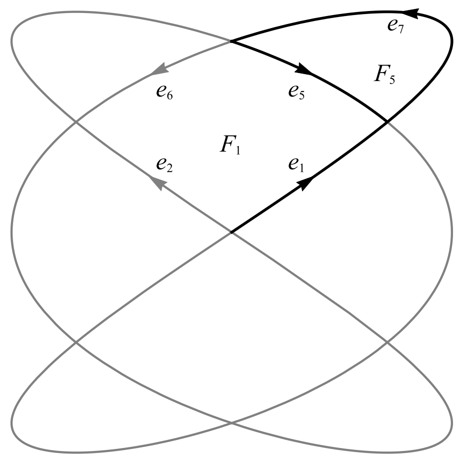

Figure 6: The loop L_5 associated to F_5 is $e_1e_7e_5e_1^{-1}$

if $f = g \circ \Psi$, then

$$\int_{K^n} f \; d\mu = \int_{K^m} g \; d\tilde{\mu}. \tag{29}$$

We now consider how changes in the four "adjacent" edge variables affect the loop variables. If we change, say, a_1 to a_1x, we can read off the corresponding change in the adjacent loop variables (as in (27)) as

$$(\ell_1, \ell_2, \ell_3, \ell_4) \mapsto (\ell_1 x, \ell_2, \ell_3, x^{-1}\ell_4).$$

Meanwhile, each nonadjacent loop variable ℓ_j with $j \geq 5$ (as in (28)) will either be conjugated by x or unchanged, depending on whether L_j goes out along e_1 or along e_2, e_3, or e_4. Similar transformation rules hold for changes in a_2, a_3, and a_4.

In particular, if we make the substitution

$$(a_1, a_2, a_3, a_4) \mapsto (a_1x, a_2y, a_3x, a_4y)$$

we have the following substitutions for the adjacent loop variables:

$$(\ell_1, \ell_2, \ell_3, \ell_4) \mapsto (y^{-1}\ell_1 x, x^{-1}\ell_2 y, y^{-1}\ell_3 x, x^{-1}\ell_4 y), \tag{30}$$

whereas each loop ℓ_j with $j \geq 5$ changes either as

$$\ell_j \mapsto x^{-1}\ell_j x \tag{31}$$

or as

$$\ell_j \mapsto y^{-1}\ell_j y \tag{32}$$

depending on the first outgoing edge traversed by the loop L_j.

The above calculation motivates the following definition.

Definition 7 *We say that a function on K^m has* extended gauge invariance *at v if it is invariant under every transformation of the sort in* (30), (31), *and* (32).

Since changes in the variables a_1, a_2, a_3, and a_4 translate into simple changes in the loop variables, we can translate the differential operators $\nabla^{a_i} \cdot \nabla^{a_{i+1}}$ on K^n into differential operators on K^m. In a slight abuse of notation, we will continue to refer to the operators on K^m as $\nabla^{a_i} \cdot \nabla^{a_{i+1}}$.

Definition 8 *Suppose $f : K^m \to \mathbb{C}$ is a smooth function of the loop variables. Then $\nabla^{a_i} \cdot \nabla^{a_j} f \colon K^m \to \mathbb{C}$, with $i \neq j$ in $\{1,2,3,4\}$, is the function computed as follows. Let $\boldsymbol{\ell} = (\ell_i)_{i=1}^m$ be the loop variables defined in* (27) *and* (28). *Define a parametrized surface $\boldsymbol{\ell}(s,t)$ in K^m by replacing a_i and a_j with $a_i \mapsto a_i e^{sX}$ and $a_j \mapsto a_j e^{tX}$ in those two equations. We then set*

$$(\nabla^{a_i} \cdot \nabla^{a_j} f)(\boldsymbol{\ell}) = \sum_X \frac{\partial^2}{\partial s \partial t} f(\boldsymbol{\ell}(s,t)) \bigg|_{s=t=0}.$$

We will compute these operators in the next subsection. We are now ready to state the abstract Makeenko–Migdal equation for the loop variables.

Theorem 9 *Let m be the number of loop variables and suppose $g : K^m \to \mathbb{C}$ is a smooth function having extended gauge invariance at v in the sense of Definition 7. Then*

$$\left(\frac{\partial}{\partial t_1} - \frac{\partial}{\partial t_2} + \frac{\partial}{\partial t_3} - \frac{\partial}{\partial t_4} \right) \int_{K^m} g \, d\tilde{\mu} = -\int_{K^m} \nabla^{a_1} \cdot \nabla^{a_2} g \, d\tilde{\mu}.$$

In light of (29), Theorem 9 implies Theorem 5 for any function f on K^n that has extended gauge invariance and that can be expressed in the form $f = g \circ \Psi$ for some function g on K^m. In particular, Theorem 9 implies the Makeenko–Migdal equation for $U(N)$, as in Theorem 1.

3.2 The proof

The advantage of working with the loop variables is that since each heat kernel is evaluated on a separate loop variable, when we differentiate and integrate by parts, none of the derivatives hits on any other heat kernel, but only on the function f. Thus,

$$\begin{aligned} &\left(\frac{\partial}{\partial t_1} - \frac{\partial}{\partial t_2} + \frac{\partial}{\partial t_3} - \frac{\partial}{\partial t_4} \right) \int_{K^m} f \, d\tilde{\mu} \\ &= \frac{1}{2} \int_{K^m} (\Delta_{\ell_1} - \Delta_{\ell_2} + \Delta_{\ell_3} - \Delta_{\ell_4}) f \, d\tilde{\mu}. \end{aligned} \tag{33}$$

On the other hand, even if f has extended gauge invariance, the integrand on the right-hand side of (33) will contain many other terms besides the one we want. We will have to show that these unwanted terms cancel out after integration.

We now consider the right-hand side of the Makeenko–Migdal equation in Theorem 9. A key point will be to exploit the invariance of the measure $\tilde{\mu}$ under conjugation in each variable. We consider both a left-invariant gradient ∇^L and right-invariant gradient ∇^R in the loop variables (similar to Definition 4):

$$\begin{aligned}
(\nabla^L f)(\ell) &= \sum_X \left(\frac{d}{ds} f(\ell e^{sX}) \bigg|_{s=0} \right) X \\
(\nabla^R f)(\ell) &= \sum_X \left(\frac{d}{ds} f(e^{sX}\ell) \bigg|_{s=0} \right) X.
\end{aligned}$$

Lemma 10 *If g is any smooth function on K^m, we have*

$$\begin{aligned}
\int_{K^m} \nabla^{\ell_j,L} \cdot \nabla^{\ell_k,L} g \; d\tilde{\mu} &= \int_{K^m} \nabla^{\ell_j,L} \cdot \nabla^{\ell_k,R} g \; d\tilde{\mu} \\
&= \int_{K^m} \nabla^{\ell_j,R} \cdot \nabla^{\ell_k,L} g \; d\tilde{\mu} = \int_{K^m} \nabla^{\ell_j,R} \cdot \nabla^{\ell_k,R} g \; d\tilde{\mu},
\end{aligned}$$

where $\nabla^{\ell_j,L}$ and $\nabla^{\ell_j,R}$ denote the left-invariant and right-invariant gradients in the variable ℓ_j, respectively, with the other variables fixed.

Proof. The invariance of each heat kernel under conjugation (cf. (6)) tells us that for any j, we have

$$\int_{K^m} g(\ell_1, \ldots, \ell_j x, \ldots, \ell_m) \; d\tilde{\mu} = \int_{K^m} g(\ell_1, \ldots, x\ell_j, \ldots, \ell_m) \; d\tilde{\mu}.$$

If follows that

$$\begin{aligned}
&\frac{\partial^2}{\partial s \partial t} \int_{K^m} g(\ell_1, \ldots, \ell_j e^{tX}, \ldots, \ell_k e^{sX}, \ldots, \ell_m) \; d\tilde{\mu} \bigg|_{s=t=0} \\
&= \frac{\partial^2}{\partial s \partial t} \int_{K^m} g(\ell_1, \ldots, e^{tX}\ell_j, \ldots, e^{sX}\ell_k, \ldots, \ell_m) \; d\tilde{\mu} \bigg|_{s=t=0},
\end{aligned}$$

from which it follows that $\int_{K^m} \nabla^{\ell_j,L} \cdot \nabla^{\ell_k,L} g \; d\tilde{\mu}$ coincides with $\int_{K^m} \nabla^{\ell_j,R} \cdot \nabla^{\ell_k,R} g \; d\tilde{\mu}$. (The argument works equally well whether $j = k$ or $j \neq k$.) All the other claimed equalities follow by analogous arguments. ■

In what follows, we will make repeated use, usually without mention, of Lemma 10. It is convenient, in this context, to use the notation

$$f \cong g$$

to indicate that f and g have the same integral.

If we make the substitutions $a_1 \mapsto a_1 e^{sX}$ and $a_2 \mapsto a_2 e^{tX}$, the loop variables change in some computable way. Recall under such substitutions, each ℓ_j with $j \geq 5$ merely gets conjugated (or not changed at all). Let $\mathbf{m}$ denote the tuple of variables $\ell_5, \ldots, \ell_m$ and let $\mathbf{m}'$ denote the new value of the these variables after changing a_1 and a_2 as above. Then, by the conjugation invariance of the measure, we have

$$\begin{aligned}
\int_{K^m} \nabla^{a_1}\cdot\nabla^{a_2} g\ d\tilde{\mu} &= \sum_X \frac{\partial^2}{\partial s\partial t}\int_{K^m} g(e^{-tX}\ell_1 e^{sX}, \ell_2 e^{tX}, \ell_3, e^{-sX}\ell_4, \mathbf{m}')\ d\tilde{\mu}\bigg|_{s=t=0} \\
&= \sum_X \frac{\partial^2}{\partial s\partial t}\int_{K^m} g(e^{(s-t)X}\ell_1, e^{tX}\ell_2, \ell_3, e^{-sX}\ell_4, \mathbf{m})\ d\tilde{\mu}\bigg|_{s=t=0} \\
&= \int_{K^m} (-\Delta_{\ell_1} + \nabla^{\ell_1}\cdot\nabla^{\ell_2} + \nabla^{\ell_1}\cdot\nabla^{\ell_4} - \nabla^{\ell_2}\cdot\nabla^{\ell_4}) g\ d\tilde{\mu}.
\end{aligned}$$

(Recall that $\mathbf{m}'$ differs from $\mathbf{m}$ only by conjugations in some of the variables, which does not affect the value of the integral.) Note that in light of Lemma 10, we do not have to specify whether the gradients are left-invariant or right-invariant.

After integrating and using the conjugation invariance of the measure, we are left with a "local" formula for $\int_{K^m} \nabla^{a_1}\cdot\nabla^{a_2} g\ d\mu$, that is, one in which only derivatives in the variables ℓ_1, ℓ_2, ℓ_3, and ℓ_4 enter. Since (33) is also local in this sense, there is no need to consider derivatives in any variables not belonging to $\{\ell_1, \ell_2, \ell_3, \ell_4\}$.

Using similar calculations for the other pairs of cyclically adjacent variables, we have

$$\begin{aligned}
\nabla^{a_1}\cdot\nabla^{a_2} g &\cong (-\Delta_{\ell_1} + \nabla^{\ell_1}\cdot\nabla^{\ell_2} + \nabla^{\ell_1}\cdot\nabla^{\ell_4} - \nabla^{\ell_2}\cdot\nabla^{\ell_4}) g \\
\nabla^{a_2}\cdot\nabla^{a_3} g &\cong (-\Delta_{\ell_2} + \nabla^{\ell_2}\cdot\nabla^{\ell_3} + \nabla^{\ell_1}\cdot\nabla^{\ell_2} - \nabla^{\ell_1}\cdot\nabla^{\ell_3}) g \\
\nabla^{a_3}\cdot\nabla^{a_4} g &\cong (-\Delta_{\ell_3} + \nabla^{\ell_3}\cdot\nabla^{\ell_4} + \nabla^{\ell_2}\cdot\nabla^{\ell_3} - \nabla^{\ell_2}\cdot\nabla^{\ell_4}) g \\
\nabla^{a_4}\cdot\nabla^{a_1} g &\cong (-\Delta_{\ell_4} + \nabla^{\ell_1}\cdot\nabla^{\ell_4} + \nabla^{\ell_3}\cdot\nabla^{\ell_4} - \nabla^{\ell_1}\cdot\nabla^{\ell_3}) g.
\end{aligned}$$

Now, if g has extended gauge invariance, each of these terms reduces to one of $\pm\nabla^{a_1}\cdot\nabla^{a_2} g$. Hence, we may take an alternating sum and divide by 4 to obtain

$$\begin{aligned}
\nabla^{a_1}\cdot\nabla^{a_2} g \cong& \left(-\frac{1}{2}\Delta_{\ell_1} + \frac{1}{2}\Delta_{\ell_2} - \frac{1}{2}\Delta_{\ell_3} + \frac{1}{2}\Delta_{\ell_4}\right) g \\
&+ \left(\frac{1}{4}\Delta_{\ell_1} - \frac{1}{4}\Delta_{\ell_2} + \frac{1}{4}\Delta_{\ell_3} - \frac{1}{4}\Delta_{\ell_4}\right) g \\
&+ \frac{1}{2}\left(\nabla^{\ell_1}\cdot\nabla^{\ell_3} - \nabla^{\ell_2}\cdot\nabla^{\ell_4}\right) g,
\end{aligned}$$

where we have written the "correct" Laplacian term (as in (33)) on the first line. To establish the Makeenko–Migdal equation, we need to prove that the last two lines disappear after integration:

$$\int_{K^m} \left(\left[\Delta_{\ell_1} - \Delta_{\ell_2} + \Delta_{\ell_3} - \Delta_{\ell_4} + 2\nabla^{\ell_1}\cdot\nabla^{\ell_3} - 2\nabla^{\ell_2}\cdot\nabla^{\ell_4}\right] g\right)\ d\tilde{\mu} = 0, \tag{34}$$

whenever g has extended gauge invariance at v.

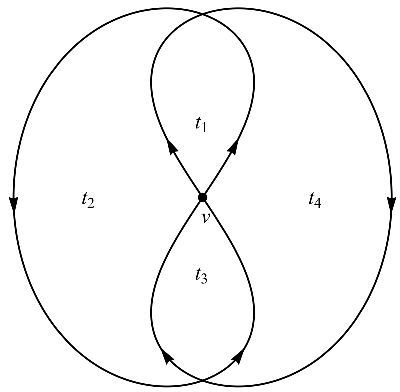

Figure 7: An example loop

To establish (34), we recall that extended gauge invariance means invariance under the transformations in (30), (31), and (32). Applying these transformations with $x = e^{tX}$ and $y = e$ and differentiating shows that

$$0 = (\nabla^{\ell_1,L} - \nabla^{\ell_2,R} + \nabla^{\ell_3,L} - \nabla^{\ell_4,R})g + \sum_{j\in I}(\nabla^{\ell_j,L} - \nabla^{\ell_j,R})g, \qquad (35)$$

where I refers to the set of indices $j \geq 5$ for which the loop goes out from the basepoint along a_1 or a_3. We now apply the operator $\nabla^{\ell_1,L}+\nabla^{\ell_2,L}+\nabla^{\ell_3,L}+\nabla^{\ell_4,L}$ to both sides of (35), integrate against $\tilde{\mu}$, and use Lemma 10. All terms involving derivatives with respect to ℓ_j, $j \geq 5$, will drop out, and we do not have to specify whether the remaining derivatives are left-invariant or right-invariant, giving

$$\int_{K^m} \left[(\nabla^{\ell_1} + \nabla^{\ell_2} + \nabla^{\ell_3} + \nabla^{\ell_4})(\nabla^{\ell_1} - \nabla^{\ell_2} + \nabla^{\ell_3} - \nabla^{\ell_4})g\right] \, d\tilde{\mu} = 0. \qquad (36)$$

If we expand out the product on the left-hand side of (36), we find that products of derivatives on cyclically adjacent variables (e.g., $\nabla^{\ell_1}\cdot\nabla^{\ell_2}$ or $\nabla^{\ell_4}\cdot\nabla^{\ell_1}$) cancel, while products of derivatives on "opposite" variables (i.e., $\nabla^{\ell_1} \cdot \nabla^{\ell_3}$ and $\nabla^{\ell_2} \cdot \nabla^{\ell_4}$) combine. Thus, (36) is precisely equivalent to the desired identity (34).

3.3 An example

We now illustrate the preceding proof of the Makeenko–Migdal equation for the loop in Figure 7. We begin by identifying the four generating loops of this graph, each of which runs counter-clockwise around one of the bounded faces of the graph, as in Figure 8. It is straightforward to check that the loop in Figure 7 decomposes as

$$(L_1L_2L_3)(L_1^{-1}L_4^{-1}L_3^{-1}),$$

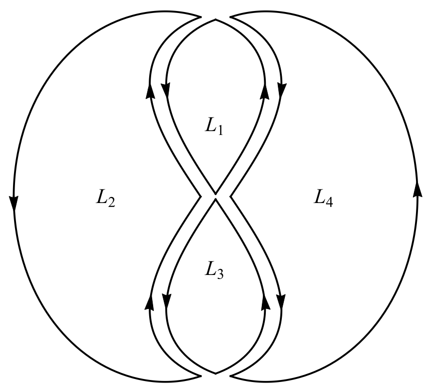

Figure 8: The generating loops for the example in Figure 7

where the notation means that we first traverse L_1, then L_2, and so on. The expressions in parentheses indicate the component loops in the $U(N)$ version of the Makeenko–Migdal equation. (We will carry out the calculation for a general compact group K and then indicate what happens when $K = U(N)$.) Since parallel transport is order-reversing, the holonomy around L is expressed in terms of the loop variables as

$$\mathrm{hol}(L) = \ell_3^{-1}\ell_4^{-1}\ell_1^{-1}\ell_3\ell_2\ell_1.$$

We start by computing $\nabla^{a_1} \cdot \nabla^{a_2} f$, with

$$f(\ell_1, \ell_2, \ell_3, \ell_4) = \mathrm{tr}(\ell_3^{-1}\ell_4^{-1}\ell_1^{-1}\ell_3\ell_2\ell_1),$$

with tr denoting the normalized trace in some representation of K. Recalling (27), we find that the substitutions $a_1 \mapsto a_1 e^{sX}$ and $a_2 \mapsto a_2 e^{tX}$ in the edge variables translates into the substitutions

$$(\ell_1, \ell_2, \ell_3, \ell_4) \mapsto (e^{-tX}\ell_1 e^{sX}, \ell_2 e^{tX}, \ell_3, e^{-sX}\ell_4)$$

in the loop variables. Thus,

$$\begin{aligned}
&- \nabla^{a_1} \cdot \nabla^{a_2} f \\
&= -\sum_X \frac{\partial^2}{\partial s\, \partial t} \mathrm{tr}[\ell_3^{-1}(\ell_4^{-1}e^{sX})(e^{-sX}\ell_1^{-1}e^{tX})\ell_3(\ell_2 e^{tX})(e^{-tX}\ell_1 e^{sX})]\Big|_{s=t=0} \\
&= -\sum_X \frac{\partial^2}{\partial s\, \partial t} \mathrm{tr}[\ell_3^{-1}\ell_4^{-1}\ell_1^{-1}e^{tX}\ell_3\ell_2\ell_1 e^{sX}]\Big|_{s=t=0} \\
&= -\sum_X \mathrm{tr}[\ell_3^{-1}\ell_4^{-1}\ell_1^{-1}X\ell_3\ell_2\ell_1 X], \qquad (37)
\end{aligned}$$

where in each term we sum X over an orthonormal basis of the Lie algebra $\mathfrak{k}$ of K. In the $U(N)$ case, the last line of (37) simplifies to $\mathrm{tr}(\ell_3^{-1}\ell_4^{-1}\ell_1^{-1})\mathrm{tr}(\ell_3\ell_2\ell_1)$, by (17).

We now compute the alternating sum of time derivatives of $\int f\,d\mu$, using (33). By Lemma 10, we are free to evaluate the Laplacians using any combination of derivatives on the left and on the right. The computations work out most simply if we compute each Laplacian as a product of a gradient on the left and a gradient on the right. With this convention, we easily obtain

$$\begin{aligned}\Delta_{\ell_1} f = \sum_X \big(&\mathrm{tr}(\ell_3^{-1}\ell_4^{-1}X\ell_1^{-1}X\ell_3\ell_2\ell_1) - \mathrm{tr}(\ell_3^{-1}\ell_4^{-1}X\ell_1^{-1}\ell_3\ell_2X\ell_1)\\ &- \mathrm{tr}(\ell_3^{-1}\ell_4^{-1}\ell_1^{-1}X\ell_3\ell_2\ell_1X) + \mathrm{tr}(\ell_3^{-1}\ell_4^{-1}\ell_1^{-1}\ell_3\ell_2X\ell_1X)\big)\end{aligned}$$

$$\Delta_{\ell_2} f = \sum_X \mathrm{tr}(\ell_3^{-1}\ell_4^{-1}\ell_1^{-1}\ell_3X\ell_2X\ell_1)$$

$$\begin{aligned}\Delta_{\ell_3} f = \sum_X \big(&\mathrm{tr}(X\ell_3^{-1}X\ell_4^{-1}\ell_1^{-1}\ell_3\ell_2\ell_1) - \mathrm{tr}(X\ell_3^{-1}\ell_4^{-1}\ell_1^{-1}X\ell_3\ell_2\ell_1)\\ &- \mathrm{tr}(\ell_3^{-1}X\ell_4^{-1}\ell_1^{-1}\ell_3X\ell_2\ell_1) + \mathrm{tr}(\ell_3^{-1}\ell_4^{-1}\ell_1^{-1}X\ell_3X\ell_2\ell_1)\big)\end{aligned}$$

$$\Delta_{\ell_4} f = \sum_X \mathrm{tr}(\ell_3^{-1}X\ell_4^{-1}X\ell_1^{-1}\ell_3\ell_2\ell_1),$$

where in each term, we sum X over an orthonormal basis for $\mathfrak{k}$.

After taking $\frac{1}{2}$ times the alternating sum of these Laplacians, we obtain two "good" terms (the third term in Δ_{ℓ_1} and the second term in Δ_{ℓ_3}), namely

$$-\frac{1}{2}\mathrm{tr}(\ell_3^{-1}\ell_4^{-1}\ell_1^{-1}X\ell_3\ell_2\ell_1X) - \frac{1}{2}\mathrm{tr}(X\ell_3^{-1}\ell_4^{-1}\ell_1^{-1}X\ell_3\ell_2\ell_1). \tag{38}$$

After using the cyclic invariance of the trace, these terms reduce precisely to (37). We are left with eight "bad" terms in the alternating sum that must cancel out after integration.

To verify this cancellation directly, we compute that

$$\begin{aligned}&(\nabla^{\ell_1,L}\cdot\nabla^{\ell_2,L} - \nabla^{\ell_1,R}\cdot\nabla^{\ell_2,R})f\\ &= \sum_X \big(- \mathrm{tr}(\ell_3^{-1}\ell_4^{-1}X\ell_1^{-1}\ell_3\ell_2X\ell_1) + \mathrm{tr}(\ell_3^{-1}\ell_4^{-1}\ell_1^{-1}\ell_3\ell_2X\ell_1X)\\ &+ \mathrm{tr}(\ell_3^{-1}\ell_4^{-1}\ell_1^{-1}X\ell_3X\ell_2\ell_1) - \mathrm{tr}(\ell_3^{-1}\ell_4^{-1}\ell_1^{-1}\ell_3X\ell_2X\ell_1)\big)\end{aligned} \tag{39}$$

and that

$$\begin{aligned}&(\nabla^{\ell_3,L}\cdot\nabla^{\ell_4,L} - \nabla^{\ell_3,R}\cdot\nabla^{\ell_4,R})f\\ &= \sum_X \big(\mathrm{tr}(X\ell_3^{-1}X\ell_4^{-1}\ell_1^{-1}\ell_3\ell_2\ell_1) - \mathrm{tr}(\ell_3^{-1}X\ell_4^{-1}\ell_1^{-1}\ell_3X\ell_2\ell_1)\\ &- \mathrm{tr}(\ell_3^{-1}X\ell_4^{-1}X\ell_1^{-1}\ell_3\ell_2\ell_1) + \mathrm{tr}(\ell_3^{-1}\ell_4^{-1}X\ell_1^{-1}X\ell_3\ell_2\ell_1)\big).\end{aligned} \tag{40}$$

One can easily check that the eight bad terms in the alternating sum of Laplacians are exactly the sum of the right-hand sides of (39) and (40). Thus, by Lemma 10, the bad terms integrate to zero.

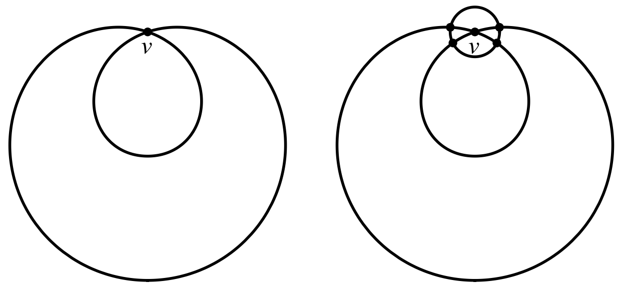

Figure 9: A nongeneric graph relative to v (left) and its generic counterpart (right)

4 Reduction to the generic case

In the preceding sections, we assumed that our graph $\mathbb{G}$ was "generic" relative to the given vertex v, meaning that the four adjacent faces are distinct and bounded and that the four edges emanating from v are distinct. (More precisely, nongeneric behavior of the edges is when one of the outgoing edges e_i emanating from v coincides with e_j^{-1} for some $j \neq i$.) In this section, we show that a version of Theorem 5 still holds even if the preceding assumptions are not satisfied.

If a graph $\mathbb{G}$ is not generic relative to a given vertex v, we construct a new graph $\mathbb{G}'$ by adding four new vertices and connecting them in a circular pattern as in Figures 9. Our strategy will be to "promote" a function of the edge variables of $\mathbb{G}$ to a function of the edge variables of $\mathbb{G}'$, apply the Makeenko–Migdal equation for $\mathbb{G}'$, and then deduce the Makeenko–Migdal equation for $\mathbb{G}$.

4.1 Consistency of the Yang–Mills measure

The first key point is to establish a consistency result, stating that the integral of the promoted function with respect to the Yang–Mills measure for $\mathbb{G}'$ is the same as the integral of the original function with respect to the Yang–Mills measure for $\mathbb{G}$.

Conceptually, this consistency result holds (at least for gauge-invariant functions) because both integrals are computing the same functional of the underlying white noise in the path-integral formulation of the Yang–Mills theory. It is also possible to establish consistency directly from the formulas for the integrals over the graphs, as we now explain. The consistency result has two aspects. First, suppose we add an extra vertex in the middle of an edge e, thus subdividing e into two new edges e' and e''. Thus, e is replaced by $e'e''$. Since parallel transport is order reversing, we will then replace the edge variable x associated with e by the product $x''x'$ of edge variables x' and x'' associated to e' and e''. Thus, if f is a function of the edge variables of the original graph, we form a function f' of the edge variables of the new graph by setting

$$f'(x', x'', \mathbf{y}) = f(x''x', \mathbf{y}), \tag{41}$$

where $\mathbf{y}$ is the collection of all edge variables different from x (in the original graph) or different from x' and x'' (in the new graph).

Consistency of the integrals under this change is easy to establish. If ν is the density of the Yang–Mills measure, it is easy to see from (7) that $\nu_{\mathbb{G}'}(x', x'', \mathbf{y}) = \nu_{\mathbb{G}}(x''x', \mathbf{y})$. Thus,

$$\begin{aligned}
\int f' \; d\mu_{\mathbb{G}'} &= \int \int \int f(x''x', \mathbf{y}) \nu_{\mathbb{G}}(x''x', \mathbf{y}) \; dx' \; dx'' \; d\mathbf{y} \\
&= \int \int f(z, \mathbf{y}) \nu_{\mathbb{G}}(z, \mathbf{y}) \; dz \; d\mathbf{y} \\
&= \int f \; d\mu_{\mathbb{G}},
\end{aligned}$$

where in the second equality, we have made the change of variable $z = x''x'$ in the x'-integral and used the normalization of the Haar measure in the x'' integral.

We also consider consistency under another type of change in the graph. Suppose we create a new graph $\mathbb{G}'$ from $\mathbb{G}$ by keeping the vertex set the same and adding some new edges. Then any function f of the edge variables of $\mathbb{G}$ can be promoted to a function of the edge variables of $\mathbb{G}'$ by making the f' independent of the new edge variables; that is,

$$f'(\mathbf{x}, \mathbf{y}) = f(\mathbf{y}), \tag{42}$$

where $\mathbf{x}$ represents the edge variables for the new edges and $\mathbf{y}$ represents the edges variables for the old edges. The consistency identity

$$\int f' \; d\mu_{\mathbb{G}'} = \int f \; d\mu_{\mathbb{G}} \tag{43}$$

is a special case of Theorem 1.22 in [Lévy1]. Note that adding an edge can divide a face with area t into two faces with areas s and s' satisfying $s + s' = t$. The key idea in verifying (43) is the convolution identity for heat kernels: $\rho_s * \rho_{s'} = \rho_t$.

4.2 Interpretation of the theorem

We now work toward establishing a version of the abstract Makeenko–Migdal equation (Theorem 5) in the nongeneric case. We must first describe the proper interpretation of the theorem in the nongeneric case. If one of the adjacent faces F_i is the unbounded face, the corresponding time derivative $\partial/\partial t_i$ should be interpreted as the zero operator. If $F_i = F_j$ for $i \neq j$, we simply have the same area-derivative twice on the left-hand side of the Makeenko–Migdal equation.

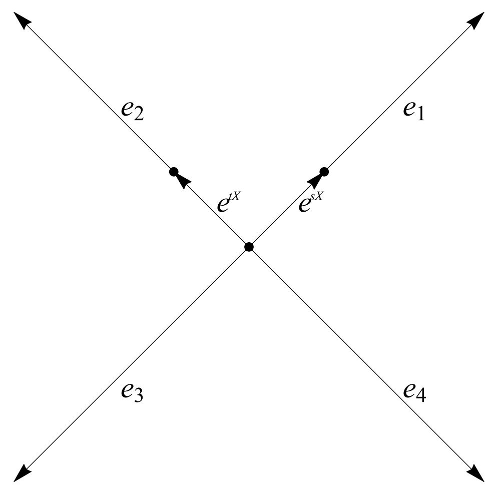

Figure 10: Geometric interpretation of $\nabla^{a_1} \cdot \nabla^{a_2} f$

Meanwhile, if for some $i \neq j$, the edges e_i and e_j are inverses of each other, we choose one of the indices (say, i) and we then no longer view a_j as an independent variable, but as simply another name for a_i^{-1}. Then to correctly interpret the expression $\nabla^{a_1} \cdot \nabla^{a_2} f$, we insert a factor of e^{sX} at the beginning of e_1 and a factor of e^{tX} at the beginning of e_2, as in Figure 10. We then determine the corresponding changes of the independent variables—keeping in mind that parallel transport is order reversing—differentiate at $s = t = 0$, and sum X over an orthonormal basis. Suppose, for example, that e_1 and e_2 are inverses of each other, but e_3 and e_4 are distinct, and suppose we take a_1, a_3, and a_4 as our independent variables. Then Figure 10 tells us that we should replace a_1 by $e^{-tX} a_1 e^{sX}$, so that

$$(\nabla^{a_1} \cdot \nabla^{a_2} f)(a_1, a_3, a_4, \mathbf{b}) = \sum_X \frac{\partial^2}{\partial s \partial t} f(e^{-tX} a_1 e^{sX}, a_3, a_4, \mathbf{b}) \Big|_{s=t=0}.$$

Finally, we describe the correct notion of extended gauge invariance at v in the case where the edges $e_1, \ldots, e_4$ are not necessarily distinct. We insert two extra factors of x as on either of the two sides of Figure 11. A function f has extended gauge invariance if the value of f is unchanged by this insertion. We may be more precise about this definition as follows. Let $\mathbb{G}$ be a graph and v a vertex of $\mathbb{G}$ with four edges, where we count an edge e twice if both ends of e are attached to v. Let f be a function of the edge variables of $\mathbb{G}$. Let $\mathbb{G}'$ be either of the two graphs in Figure 11, obtained by adding two new vertices to $\mathbb{G}$. Let $f'(x, y, \mathbf{z})$ be the function of the edge variables of $\mathbb{G}'$ formed by the method described earlier in this section, where x and y denote the edge variables associated to the two new edges emanating from v and $\mathbf{z}$ represents all the other

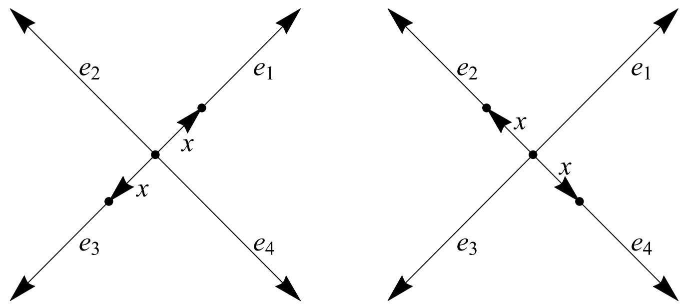

Figure 11: Geometric interpretation of extended gauge invariance

edge variables. We say that f has extended gauge invariance if

$$f'(x, x, \mathbf{z}) = f'(\mathrm{id}, \mathrm{id}, \mathbf{z}),$$

for all $x \in K$, where id is the identity element of K.

If the edges $e_1, \ldots, e_4$ are distinct, this new notion of extended gauge invariance agrees with the notion in Definition 3. After all, when the edges are distinct, $f'(x, x, \mathbf{z})$ will simply be (in the notation of Definition 3) either $f(a_1x, a_2, a_3x, a_4, \mathbf{b})$ or $f(a_1, a_2x, a_3, a_4x, \mathbf{b})$.

We consider two other examples of our new notion of extended invariance. Suppose that e_3 coincides with e_2^{-1} (but e_1 and e_4 are distinct), and that we choose a_1, a_2, and a_4 as our independent variables. Since parallel transport is order reversing, the transformation indicated by the left side of Figure 11 will change a_2 to $x^{-1}a_2$ and a_1 to a_1x, while leaving a_4 unchanged. Meanwhile, the transformation for the right-hand side of the figure changes a_2 and a_4 to a_2x and a_4x, as usual. Thus, when $e_3 = e_2^{-1}$, extended gauge invariance means that

$$f(a_1, a_2, a_4, \mathbf{b}) = f(a_1x, x^{-1}a_2, a_4, \mathbf{b}) = f(a_1, a_2x, a_4x, \mathbf{b}),$$

for all $x \in K$. Similarly, suppose e_1 and e_3 are inverses of each other (but e_2 and e_4 are distinct) and we take a_1, a_2, and a_4 as our independent variables. Then extended gauge invariance means that

$$f(a_1, a_2, a_4, \mathbf{b}) = f(x^{-1}a_1x, a_2, a_4, \mathbf{b}) = f(a_1, a_2x, a_4x, \mathbf{b}),$$

for all $x \in K$.

4.3 Proof of the theorem

We now consider a graph $\mathbb{G}$ that is not generic at v and another graph $\mathbb{G}'$ as in Figure 9 that is generic at v. We now promote any function f of the edge variables of $\mathbb{G}$ to a function f' of the edge variables of $\mathbb{G}'$, by the method described in Section 4.1. We label the edges and faces of $\mathbb{G}'$ around v as in Figure 12, and we let a_i' denote the edge variable associated to e_i'.

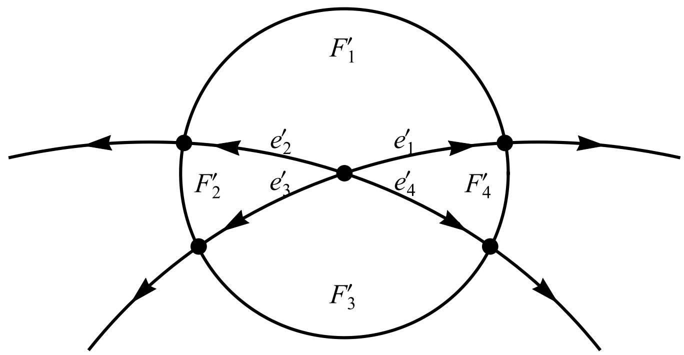

Figure 12: The adjacent faces and edge variables for the generic graph

Since the new graph is generic relative to v, all of our proofs of the Makeenko–Migdal equation apply to $\int f'\ d\mu_{\mathbb{G}'}$, where in the case of the loop-based proof, we would have to assume that f is expressible in terms of the loop variables for $\mathbb{G}$, in which case f' will expressible in terms of the loop variables for $\mathbb{G}'$. It now remains only to see that the Makeenko–Migdal equation for $\int f'\ d\mu_{\mathbb{G}'}$ reduces to the Makeenko–Migdal equation for $\int f\ d\mu_{\mathbb{G}}$.

We begin with the time derivatives and we consider first the possibility that one of the adjacent faces F_i in the original graph is the unbounded face. In that case, the face F_i' in the new graph will share a "circular" edge with the unbounded face. Since the circular edge lies between F_i' and the unbounded face, the corresponding edge variable c_i will occur in *only one* of the heat kernels in the definition of $\mu_{\mathbb{G}'}$. Thus, the density of $\mu_{\mathbb{G}'}$ will take the form

$$\rho_{\tilde{t}_i}(c_i\gamma)\delta,$$

where γ is a word in edge variables other than c_i and where δ is a product of heat kernels evaluated on edge variables other than c_i. Thus, if t_i' denotes the area of F_i', we have

$$\begin{aligned}\frac{\partial}{\partial t_i'}\int f'\ d\mu_{\mathbb{G}'} &= \frac{1}{2}\int f'\ \Delta_{c_i}[\rho_{\tilde{t}_i}(c_i\gamma)]\delta\ d\text{Haar}\\ &= \frac{1}{2}\int(\Delta_{c_i}f')\ d\mu_{\mathbb{G}'}\\ &= 0,\end{aligned}$$

since f' is, by construction, independent of c_i.

We next consider the possibility that for some $i \neq j$, two bounded faces F_i and F_j in the original graph coincide. In that case, the face $F_i = F_j$ is divided into three faces in the new graph, F_i', F_j', and one other face G. Thus,

$$t_i = t_i' + t_j' + s,$$

where s is the area of G, which means that varying t_i' has the same effect as varying t_i. It follows from this observation and the consistency of the Yang–Mills measure that

$$\frac{\partial}{\partial t_i'} \int f' \, d\mu_{\mathbb{G}'} = \frac{\partial}{\partial t_i} \int f \, d\mu_{\mathbb{G}}. \tag{44}$$

If three or more bounded faces in the original graph coincide, a very similar argument shows that (44) still holds.

Meanwhile, using the geometric interpretation of $\nabla^{a_1} \cdot \nabla^{a_2}$ in Figure 10, we can verify that

$$(\nabla^{a_1} \cdot \nabla^{a_2} f)' = \nabla^{\tilde{a}_1} \cdot \nabla^{\tilde{a}_2} f'. \tag{45}$$

Finally, using the geometric interpretation of extended gauge invariance in Figure 11, we can verify that if f has extended gauge invariance, so does f'. Since, also, $\mathbb{G}'$ is generic at v, the Makeenko–Migdal equation at v will holds for f'. Then, using (44), (45), and the consistency of the Yang–Mills measure, we obtain the Makeenko–Migdal equation for f.

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