Research Statement

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In the study of dynamical systems, one seeks to answer questions about the trajectory of a point in some mathematical space that evolves according to a prescribed set of rules. For example, consider a ball rolling without slipping on a billiard table in the shape of a polygon. The ball will continue rolling in a straight line at a constant speed until it hits one of the sides of the table, at which point it will bounce off in such a way that the angle of incidence equals the angle of reflection. If we assume that no energy is lost, then the ball will continue bouncing around this table forever, tracing out some potentially complicated path, or *orbit*. Some interesting questions we might ask about this system include: Are there any orbits that are dense in the whole space? Are there any that are periodic? Are there any that have fractal closure? What does a typical orbit look like? Although the "rules" for this dynamical system seem quite simple, they lead to surprisingly complex behavior, and many basic facts remain unknown. For example, it is still not known whether every triangular billiard table admits a periodic orbit.

Classically, dynamical systems have been used to model natural phenomena, such as the motion of planets or the growth and dispersal of populations. These models generally take the form of either a discrete-time dynamical system, characterized by a $\mathbb{Z}$-group (or $\mathbb{Z}^+$-semigroup) action on the underlying space, or a continuous dynamical system, characterized by a $\mathbb{R}$-group (or $\mathbb{R}^+$-semigroup) action. It turns out that many of the methods used to study such systems can be generalized to study the actions of other groups with a sufficient amount of structure, and that doing so can yield valuable insight into more theoretical areas of mathematics, such as number theory and geometry. This point of view has been particularly fruitful in the field of homogeneous dynamics, which studies the actions of subgroups of a Lie group G on the quotient of G by a discrete subgroup. Examples of dynamical systems of this form include linear flows on tori and geodesic and horocycle flows on the modular surface.

Many recent breakthroughs in a variety of areas are the result of reformulating old problems into the language of homogeneous dynamics. Foundational works such as Margulis's proof of the Oppenheim conjecture [32] and Ratner's classification of unipotent orbits and unipotent-invariant measures [44, 45] have led the way for others to seek rigidity phenomena in different contexts with an abundance of useful applications. Such results include Lindenstrauss's proof of arithmetic quantum unique ergodicity [27], Einsiedler-Katok-Lindenstrauss's partial result toward Littlewood's conjecture [12], Benoist-Quint's rigidity theorems [1], Eskin-Mozes-Shah and Gorodnik-Oh's counting results for integer and rational points on homogeneous varieties [17, 20], and Venkatesh's work on the subconvexity problem for L-functions [51].

Adapting the techniques of homogeneous dynamics to analogous settings has also proven quite powerful. For example, the groundbreaking work of McMullen [34], Eskin-Mirzakhani [15], and Eskin-Mirzakhani-Mohammadi [16] classifying the invariant measures and orbit closures for the $\mathrm{SL}_2(\mathbb{R})$-action on moduli spaces of abelian differentials was inspired largely by results and techniques from homogeneous dynamics, such as Ratner's classification theorems for unipotent flows. This work has numerous applications in geometry and physics, including in the study of billiard trajectories described above.

I am interested in exploring the rich and surprising connections of homogeneous dynamics to other areas of math, particularly number theory. In my research, I study effective equidistribution of horospherical flows and use these effective results to obtain information about sparse subsets of these flows. In the next section, I will define these terms and give a brief history of related work, and in Section 2 I describe in more detail my past work. Finally, in Section 3 I discuss my current projects, and in Section 4 I propose some directions for future investigation.

1 Effective Equidistribution and Horospherical Flows

Equidistribution results play an important role in dynamical systems and their applications. A subset of an orbit is said to equidistribute with respect to a given probability measure if it spends the expected amount of time in different regions of the space, i.e., if the proportion of the subset landing within any measurable set converges to the measure of that set. Often in applications to number theory it is important that an equidistribution result be effective—that is, that the rate of convergence is known. Effective results can be used to derive explicit bounds for number theoretic questions, such as quantitative solutions to the Oppenheim conjecture [21,28], or to obtain information about the distribution of certain "sparse" subsets of the orbit, such as prime times or polynomial sequences (e.g. [22, 46, 51]).

Despite their usefulness for applications in number theory and beyond, effective results for many homogeneous systems remain elusive. For example, effective versions of Ratner's theorems would have far-reaching consequences and are highly sought after; however the problem in its complete generality appears to be very challenging (although some progress has been made, see e.g. [13, 14, 28, 48]). Nonetheless, there are two contexts for which we have strong effective results—that of nilflows, established by Green-Tao in [22], and that of horospherical flows, which have a long and rich history.

Geodesic and horocycle flows on $\mathrm{T}^1(\mathbb{H}^2) \cong \mathrm{PSL}_2(\mathbb{R})$ and $\mathrm{SL}_2(\mathbb{R})$ (or quotients of these) are classical objects. They are defined, respectively, by actions of the subgroups

$$\left\{ a_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} \right\}_{t\in\mathbb{R}} \qquad \left\{ u_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \right\}_{t\in\mathbb{R}}$$

via multiplication. Observe that $\lim_{t\to\infty} a_t^{-1} u_s a_t = e$ for any $s \in \mathbb{R}$, and moreover that the only elements for which this holds are of the form u_s for some such s. We thus expand this notion in the following way: A subgroup U of a Lie group G is said to be *horospherical* if there exists $g \in G$ such that

$$U = \{u \in G \,|\, g^{-n} u g^n \text{ as } n \to \infty\}.$$

In work originating with Hedlund [23] and Furstenberg [18] and extended by Veech [50] and Ellis-Perrizo [11], horospherical flows were shown to be minimal and uniquely ergodic on compact quotients of suitable Lie groups. For quotients by a non-uniform lattice, Margulis proved that orbits of unipotent (hence horospherical) flows cannot diverge to infinity [31], which was later refined in Dani's nondivergence theorem [7]. Dani also showed that horospherical flows on noncompact homogeneous spaces have nice (finite volume, homogeneous submanifold) orbit closures and that every ergodic probability measure invariant under such a flow is the natural Lebesgue measure on some such orbit closure [8]. Many of these results have since been effectivized, in particular with a polynomial error rate (see, e.g., [5, 24, 30, 46, 47, 51]).

Questions about the distribution of sparse sequences have long been of interest in dynamical systems, especially regarding primes. In [2], Bourgain showed that ergodic averages over primes converge almost-everywhere, but this unfortunately says nothing about any particular orbit. In contrast, Sarnak's Möbius disjointness conjecture seeks to formalize the heuristic that primes are essentially randomly distributed by positing that the Möbius function

$$\mu(n) = \begin{cases} 0 & \text{if } n \text{ is not squarefree} \\ (-1)^k & \text{if } n \text{ is the product of } k \text{ distinct primes} \end{cases}$$

is asymptotically uncorrelated with *any* sequence of the form $f(T^n x)$ for $x \in X$, a compact metric space, $f \in C(X)$, and $T : X \to X$ a surjection of zero topological entropy. The conjecture has been established in a variety of settings [3,6,10,22,29], including for unipotent flows [4,43] but these results are not effective and therefore do not provide information about the distribution of primes in the space. Although the noncompact setting lies outside the scope of the Möbius disjointness conjecture, there is reason to believe that even in this case a sort of randomness heuristic holds for horospherical flows—namely, that prime times are equidistributed in the orbit closure containing them (see, e.g., Question 16 of [19]). In this direction, Sarnak and Ubis have shown that primes are dense in a set of positive measure for equidistributing horocycle orbits on $\mathrm{SL}_2(\mathbb{R})/\mathrm{SL}_2(\mathbb{Z})$ and that almost-primes are dense in the whole space [46].

In my research, I have extended work of Venkatesh and Sarnak-Ubis to study the distribution of sparse subsets of horospherical orbits consisting of almost-prime entries, i.e. integer entries having fewer than a fixed number of prime factors. These results are significant in that they lend further support to the conjecture that prime times in horospherical flows are generically dense and possibly equidistributed.

2 Almost-Primes in Horospherical Flows

In [33], I prove the density of almost-prime times (i.e. integer times having fewer than a fixed number of prime factors) in abelian horospherical flows on compact quotients of $\mathrm{SL}_n(\mathbb{R})$, as well as on the space of lattices $\mathrm{SL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{Z})$, where in this case the number of prime factors allowed in the almost-primes depends on a Diophantine property of the basepoint.

To be more precise, let $X = \mathrm{SL}_n(\mathbb{R})/\Gamma$ where $\Gamma < \mathrm{SL}_n(\mathbb{R})$ is a lattice (i.e. a discrete subgroup of finite covolume). Define $a_t = \exp(tY)$ where $Y \in \mathfrak{sl}_n(\mathbb{R})$ is a diagonal matrix with only two eigenvalues, the first m of which are positive and the rest of which are negative. Observe that

$$U = \left\{ \left(\begin{array}{c|c} I_m & * \\ \hline 0 & I_{n-m} \end{array} \right) \right\}$$

is the horospherical subgroup corresponding to a_t for $t > 0$. Note that $U \cong \mathbb{R}^d$ for $d = m(n-m)$ under identification with the upper-right block. Let $u(\mathbf{t})$ denoted the element of U identified with $\mathbf{t} \in \mathbb{R}^d$. Any abelian horospherical subgroup of $\mathrm{SL}_n(\mathbb{R})$ is conjugate to a subgroup of this form.

Let $\mathcal{A}_\ell(x)$ be the sparse subset of the orbit of $x \in X$ defined by

$$\mathcal{A}_\ell(x) = \{u(k_1, k_2, \cdots, k_d)x \mid k_i \in \mathbb{Z} \text{ has fewer than } \ell \text{ prime factors}\}.$$

Then we have the following theorem.

Theorem.

(i) *If Γ is cocompact, then there exists a constant $\ell = \ell(n, d, \Gamma)$ such that for any $x \in X$, the set $\mathcal{A}_\ell(x)$ is dense in X.*

(ii) *If $\Gamma = \mathrm{SL}_n(\mathbb{Z})$ and $x = g\Gamma \in X$ satisfies the property that there exists $\delta > 0$ and $T_i \to \infty$ such that*

$$\inf_{\substack{w \in \Lambda^j(\mathbb{Z}^n)\setminus\{0\} \\ j=1,\cdots,n-1}} \sup_{\mathbf{t}\in[0,T_i]^d} \|u(\mathbf{t})gw\| > T_i^\delta$$

then there exists $\ell = \ell(n, d, \delta)$ such that $\mathcal{A}_\ell(x)$ is dense in X.

This theorem says that for a compact quotient, there is a uniform ℓ such that the orbit of any point at ℓ-almost-prime times is dense in the whole space, whereas for $\mathrm{SL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{Z})$ the value of ℓ depends on a property of the basepoint. This property can be thought of as a Diophantine condition that measures how far the basepoint is from the orbit of an intermediate subgroup $U < L < \mathrm{SL}_n(\mathbb{R})$ which meets Γ in a lattice. I prove the theorem in three main steps.

In the first step, I use the "thickening" method developed by Margulis in his thesis [30], which leverages the known exponential rate of mixing for the flow $\{a_t\}_{t\in\mathbb{R}}$ to obtain a polynomial rate of equidistribution for an arbitrary horospherical subgroup U (not necessarily abelian). This method employs the idea that a long piece of the horospherical orbit can be viewed as a much shorter piece of horospherical orbit that has been expanded under conjugation by a_t. Furthermore, we can approximate the value of a smooth function along a piece of horospherical orbit by its value in a neighborhood of the orbit, and when we conjugate by a_t this will remain a good approximation because (by definition of the horospherical subgroup) expansion only occurs in the direction of the flow. We may then consider the effect of the flow a_t on the "moving basepoint" which is our original basepoint moved by both the horospherical flow and a_t^{-1}. In the case of $\Gamma = \mathrm{SL}_n(\mathbb{Z})$, I use a quantitative nondivergence result of Kleinbock-Margulis [25] to deal with difficulties that arise due to the space being noncompact, and this is where I require a Diophantine property of the basepoint.

In the second step, I use a method developed by Venkatesh [51] to obtain an effective equidistribution result for arithmetic sequences of times in abelian flows. The method relies on a result that is similar in spirit to the van der Corput lemma (see Lemma 3.1 in [51]). In this method, we approximate a test function by its ergodic average over a small number of translates in the flow direction. So long as the number of translates is small in comparison to the length of the piece of orbit we are considering, then the approximation will be good. We can then show that the average of this new function along arithmetic progressions can be approximated by its average over a neighborhood of each point inside the horospherical subgroup (this is where we require the abelian assumption). We then use effective equidistribution for the continuous flow together with bounds on the decay of matrix coefficients for the unitary representations of $\mathrm{SL}_n(\mathbb{R})$ to show that the different translates of the function become essentially uncorrelated, which yields the result.

Finally, in the third step I use an upper- and lower-bound sieve to obtain the stated result for almost-primes from the effective equidistribution of arithmetic sequences. I use a combinatorial sieve as in [40], which requires averages along arithmetic sequences of step size K (squarefree) to be approximated by a multiplicative function in K satisfying several arithmetic properties, and such that the sum of errors over K can be sufficiently bounded.

To do this, I use a generalization of Pillai's arithmetical function and apply my effective equidistribution result for arithmetic sequences to establish the necessary error bounds. The lower bound obtained from the sieve implies that any set with positive Haar measure (hence any open set) will contain a point of the orbit with almost-prime entries.

The work I did for this project lends itself to a variety of interesting generalizations, some of which I am making progress on now, and some of which I plan to work on in the future. In the rest of this document, I will describe some of this ongoing work.

3 Improvements and Generalizations

Recently, my collaborator M. Luethi and I have shown that Venkatesh's method can be applied along a single (central) direction in such a way that in the other flow directions we may select any uniformly discrete set satisfying a mild asymptotic condition, which in particular is satisfied for primes. For example, if $U < \mathrm{SL}_3(\mathbb{R})$ is the Heisenberg group,

$$U = \left\{ u(t_1, t_2, t_3) = \begin{pmatrix} 1 & t_1 & t_3 \\ 0 & 1 & t_2 \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, t_1, t_2, t_3 \in \mathbb{R} \right\}$$

then the center of U is given by $\{u(0,0,t_3)\}_{t_3 \in \mathbb{R}}$, and we can show that there is an integer ℓ such that the sparse set

$$\mathcal{A}_\ell(x) = \left\{ u(k_1, k_2, k_3)x \,\middle|\, \begin{array}{c} k_1, k_2 \text{ prime} \\ k_3 \text{ has fewer than } \ell \text{ prime factors} \end{array} \right\}.$$

is dense in $X = \mathrm{SL}_3(\mathbb{R})/\Gamma$ for any $x \in X$ (for Γ cocompact) or for x satisfying a Diophantine condition (for Γ non-uniform). Moreover, we could replace the primality condition in the t_1 and t_2 directions with many other sparse subsets of number-theoretic interest, as we require only that there is a minimum distance between points and that there are not too few points as time goes to infinity. Notice that we no longer require our horospherical subgroup to be abelian, as we can always choose a direction in the center of U with which to carry out our analysis. We are currently in the process of writing up these results for the more general setting of a connected, semisimple Lie group quotiented by a lattice, and we plan to present the results in a forthcoming paper.

More ambitiously, we are working to remove dependence on the basepoint for the number of prime factors allowed in the almost-prime times when the lattice is non-uniform. In both [33] and the work described above, dependence on the basepoint results from the fact that the number of prime factors depends on the rate of equidistribution for the continuous flow, which itself varies depending on the basepoint. In the noncompact setting, it is not possible to obtain a uniform rate of equidistribution for the full flow, since the rate becomes arbitrarily bad for points near a proper orbit closure. Nonetheless, Sarnak-Ubis are able to show in [46] that almost-prime times of a fixed order are dense in $\mathrm{SL}_2(\mathbb{R})/\mathrm{SL}_2(\mathbb{Z})$ for any nonperiodic horocycle orbit, independent of basepoint. In doing this, they develop estimates that are more precise than ours, but which are somewhat particular to the case of $\mathrm{SL}_2(\mathbb{R})/\mathrm{SL}_2(\mathbb{Z})$. Despite this, we believe that the core ideas they use can be transported to a more general setting. Namely, the "linearization" technique of Dani-Margulis [9] can be used to show that if a basepoint equidistributes too slowly, then it must spend a significant proportion of time near a proper orbit closure which is itself a lower-dimensional homogeneous space, such that the almost-prime times in the orbit we are considering are quantitatively close to the almost-prime times of a basepoint within this proper subspace. We hope to use an induction-type argument to show that the almost-prime times in this nearby orbit are dense in the proper subspace, and then use known results regarding the equidistribution of such proper subspaces within the whole space to complete the proof.

4 Directions for Future Work

Many of the methods used in my past and ongoing work are not specific to almost-primes, and I expect that they can be employed to study a wide variety of interesting sequences. For example, Venkatesh uses effective equidistribution for arithmetic sequences in the horocycle flow on compact quotients of $\mathrm{SL}_2(\mathbb{R})$ to obtain equidistribution for sequences comprised of integer points raised to a small power [51]. He does this using a Taylor series approximation, which suggests that the approach could be replicated for other sequences that can be similarly approximated. I would like to investigate whether this is possible in a multiparameter horospherical flow on spaces that are not necessarily compact, such as $\mathrm{SL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{Z})$. There are also generalizations of the sieve methods that could be used to show that certain irreducible polynomials of the entries have no small prime factors. Such a result would yield a statement about integer points in horospherical flows which are not necessarily almost-prime, but which have other interesting arithmetic properties. More generally, I would like to axiomatize the types of sets these methods work for, i.e. provide explicit conditions that guarantee density for a broad class of discrete subsets of orbits.

I also want to investigate effective equidistribution and sparse subsets of horospherical orbits in similar settings for which such questions have been less thoroughly explored. A key example of this would be the quotient of a semisimple Lie group by a discrete, geometrically finite, Zariski dense subgroup of infinite covolume, i.e. a thin group. Such spaces can be found throughout geometry and number theory and are an exciting area of current research. For example, the curvatures of circles in an Apollonian circle packing are described by the orbit of a thin group in the orthogonal group preserving a particular quadratic form, and any geometrically finite, infinite volume hyperbolic 3-manifold can be represented as the quotient of $\mathrm{Isom}^+(\mathbb{H}^3)$ by a thin Kleinian group.

For such spaces, we may still define natural geometric flows such as geodesic and horospherical flows, but many basic dynamical tools break down in the infinite volume setting. In fact, in this setting neither the geodesic nor horospherical flows will even be recurrent (not to mention ergodic) with respect to the Haar measure, as most trajectories will disappear toward infinity. Nonetheless, it is possible to define a subset of the space and certain special measures supported on this subset with respect to which some of the previous dynamical methods can be applied (see, e.g., recent works of Kontorovich-Oh [26], Oh-Shah [41], Mohammadi-Oh [38, 39], and McMullen-Mohammadi-Oh [35–37]).

For example, let $G = \mathrm{SO}(2,1) \cong \mathrm{Isom}(\mathbb{H}^2)$, let $\Gamma < G$ be a discrete, finitely generated, Zariski-dense subgroup of infinite covolume, and let $A = \{a_t\}$ and $U = \{u_t\}$ be the usual geodesic and horocycle flows. Define the convex core to be the quotient by Γ of the smallest convex set in G containing all geodesics connecting points in the limit set corresponding to Γ. Using Patterson-Sullivan conformal densities developed in [42] and [49], one can define the Bowen-Margulis-Sullivan measure m^{BMS} and the Burger-Roblin measure m^{BR} on G/Γ. For the above setting, m^{BMS} is a finite measure with support in the convex core of Γ that is invariant and mixing for the A-action. On the other hand, m^{BR} is an infinite measure which is the only locally-finite U-invariant ergodic measure on G/Γ that is not supported on a closed U-orbit. We may now ask questions of the form: For any U-orbit that is recurrent to the convex core of G/Γ, will prime times return to the convex core infinitely often? Will they be dense in the convex core? I hope to use the above framework in order to explore whether it is possible to adapt my past work to this and similar settings.

Finally, I would like to expand my knowledge in numerous areas of geometry, number theory, and dynamics in order to better understand the many interactions of these fields and the tools and techniques that could be used to shed new light on difficult problems using a dynamical approach. I plan to read and learn more about a variety of topics, including symbolic dynamics, interval exchange transformations, Teichmüller theory, billiards and flat spaces, random walks on groups, smooth dynamics, complex dynamics, and other topics.

My interest in dynamical systems and their applications throughout mathematics is wide ranging. In the future, I hope to use this dynamical perspective to generate valuable insights and uncover novel connections between diverse subjects.

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