LIE GROUPS: Notes and Exercises

1. Let $\{v_i\}$ be a collection of weight vectors such that ω_i is the weight of v_i with $\omega_i \neq \omega_j$ for $i \neq j$. Show that $\{v_i\}$ is linearly independent. *Hint* : Assume we have a linear combination $\sum_{i=1}^r \alpha_i v_i = 0$ with some $\alpha_i \neq 0$. After re-indexing and changing r, if necessary, we can assume all $\alpha_i \neq 0$ for $1 \leq i \leq r$. Obviously $r = 1$ is not possible. Show that if $r > 1$, we could find a new nontrivial linear combination $\sum \beta_i v_i = 0$ with fewer nonzero β_i, by using the same idea as in the proof of the Lemma discussed in 2 (but at least one nonzero β_i).

2. *Remark* : We had shown in class that if the $\mathbf{sl}_3$ module V has a basis of weight vectors, then so has any submodule $W \subset V$. The way we had set up the proof, we had already implicitly assumed that weight vectors belonging to different weights are linearly independent, as stated in 1.

3. Show that $\mathbf{g}$ solvable is equivalent to $(X, [Y, Z]) = 0$ for all $X, Y, Z \in \mathbf{g}$, where $(X, Y) = Tr(ad_X ad_Y)$ is the Killing form on $\mathbf{g}$. *Hint* : Use Cartan's criterium (Lemma 2 in the lecture) saying that $(X, Y) = 0$ for all $X, Y \in \mathbf{g}$ implies $\mathbf{g}$ solvable. Also observe that $[\mathbf{g}, \mathbf{g}]$ solvable implies $\mathbf{g}$ solvable (why?).

4. Let $A \in \mathrm{End}(V)$ be diagonalizable with eigenvalues λ_i such that $\lambda_i \neq \lambda_j$ for $i \neq j$. Let

$$Q_i(x) \;=\; \prod_{j \neq i} \frac{x - \lambda_j}{\lambda_i - \lambda_j}.$$

Verify that $P_i = Q_i(A)$ is the projection onto the eigenspace for λ_i. *Hint* : Calculate $P_i(A)v$ for an eigenvector with eigenvalue λ_k.

5. (a) Let $\alpha \in \mathbf{R}^n$ and let $\beta = w(\alpha)$ for some $w \in Gl(n, \mathbf{R})$. Show that $s_\beta = w s_\alpha w^{-1}$, where

$$s_\gamma(v) \;=\; \frac{2(v, \gamma)}{(\gamma, \gamma)}\gamma$$

for $\gamma, v \in \mathbf{R}^n$ and (,) the usual dot product for $\mathbf{R}^n$.
(b) Let β be a positive root. Show that there exists a positive root β' and a simple root α such that $\beta = \beta' + \alpha$. *Hint* : It was shown in class that for $\beta > 0$ there exists a simple root α such that $(\beta, \alpha) > 0$.
(c) Let β be a positive root. Show that there exists a *simple* root α and $w \in W$ such that $\beta = w(\alpha)$. *Hint* : Proceed by induction on the length $\ell(\beta)$, using part (b).
(d) Show that for any root β there exists $w \in W$ and a simple root α such that $\beta = w(\alpha)$.
(d) Recall that W was generated by the reflections s_α, α a root. Show that W is already generated by the simple reflections $s_i = s_{\alpha_i}$, where $\Delta = \{\alpha_i\}$.

6. Let ε_i, $1 \leq i \leq 8$ be a basis for $\mathbf{R}^8$. Let $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$, $1 \leq i < 8$.
(a) Find $\alpha_8 \in \mathbf{R}^8$ such that the set of vectors $\Delta = \{\alpha_i,\ 1 \leq i \leq 8\}$ generates the Coxeter graph E_8 (this is the graph A_7 with an additional vertex connected with α_5 by a single edge).
(b) Calculate the orbit Φ of Δ under the group W generated by the reflections s_i defined by

$$s_i(\mathbf{x}) = \mathbf{x} - (\mathbf{x}, \alpha_i)\alpha_i, \quad \mathbf{x} \in \mathbf{R}^8.$$

(*Answer* : $\Phi = \{\pm\varepsilon_i \pm \varepsilon_j,\ i \neq j\} \ \cup\ \{\alpha = \sum_{i=1}^8 \pm\frac{1}{2}\varepsilon_i\}$, where we have an odd number of minus signs for the summands). Problem 5 can be useful for showing that we do indeed get all the indicated vectors in the orbit.
(c) Determine positive and negative roots with respect to Δ.
(d) Calculate the dimension of the Lie algebra corresponding to the graph E_8 (Don't forget the Cartan subalgebra!).