LIE GROUPS: Homework Solutions

Assignment 2

1. (b) The kernel of the map
$$\Phi : (t, a) \in \mathbf{R} \times SU(n) \mapsto e^{it} a \in U(n)$$
is equal to $\{(2k\pi/n, e^{-2k\pi i/n} I),\ k \in \mathbf{Z}\}$.

2. (a) Unfortunately, the problem was set up incorrectly in the book. We have to choose the partition
$$\frac{s_0}{2}, \frac{s_1}{2}, \ ..., \ \frac{s_M}{2} = \frac{1}{2}, \frac{1}{2} + \frac{t_1}{2}, \ ..., \ \frac{1}{2} + \frac{t_m}{2} = 1.$$
The corresponding path $C(t)$ is then given by $C(t) = B(2t)$ for $t \leq \frac{1}{2}$, and $C(t) = A(2t-1)B$ for $t \geq \frac{1}{2}$. It then follows that $C(\frac{1}{2} + \frac{t_j}{2})C(\frac{1}{2} + \frac{t_{j-1}}{2})^{-1} = A(t_j)A(t_{j-1})^{-1}$, from which the claim is easy to prove.

3. In part (a), you need to point out that the Lie algebra homomorphism from $\mathbf{sl}_n$ to $\mathbf{g}$ extends to a homomorphism of the complexifications as $\mathbf{g} \subset M_m(\mathbf{C})$ for some $m \in \mathbf{N}$. This defines a homomorphism from $SL(n, \mathbf{C})$ into $Gl(m, \mathbf{C})$ as $SL(n, \mathbf{C})$ is simply connected. We get the desired homomorphism by restricting to $SL(n, \mathbf{R})$.
For (b), let $\hat{\Phi} : \hat{S}l(n, \mathbf{R}) \to Gl(m, \mathbf{C})$ be a representation. This induces a Lie algebra map $\psi : \mathbf{sl}_n \to M_m(\mathbf{C})$. By part (a), this map induces a homomorphism $\Phi : Sl(n, \mathbf{R}) \to Gl(m, \mathbf{C})$. Φ can be constructed along paths in $Sl(n, \mathbf{C})$, as done in class. For Φ to be well-defined, we have to obtain $\binom{n+d-1}{d-1}$ the identity $I \subset Gl(m, \mathbf{C})$ for any closed path from $I \subset Sl(n, \mathbf{R})$ to itself, i.e. it has to factor over $\pi_1(Sl(n, \mathbf{R}))$.

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1. As sent out in my email, there was a misprint. The correct dimension is $\binom{n+d-1}{n-1}$.

2. Let f_i be the partial derivative with respect to the i-th coordinate. If $g \in SO(3)$, we have $g^{-1} = g^t$. So if $w = g^{-1}x$, $w, x \in \mathbf{C}^3$, we have for the j-th coordinate of w
$$w_j = \sum_{i=1}^{3} g_{ij} x_i.$$
Let f_j be the partial derivative of f with respect to its j-th coordinate. Then it follows from the chain rule that
$$\frac{\partial(g.f)}{\partial x_i}(x) = \sum_{j=1}^{3} f_j(g^{-1}x) g_{ij}.$$
Applying this again to $\frac{\partial(g.f)}{\partial x_i}$, we obtain
$$\frac{\partial^2(g.f)}{\partial x_i^2}) = \sum_{j,l=1}^{3} f_{lj}(g^{-1}x) g_{ij} g_{il}.$$
It follows that
$$\Delta(g.f)(x) \ = \ \sum_{j,l=1}^{3} (\sum_{i=1}^{3} g_{ij} g_{il}) f_{lj}(g^{-1}x) \ = \ \sum_{j=1}^{3} f_{jj}(g^{-1}x) \ = \ g.\Delta f(x),$$

where we used $\sum_{i=1}^3 \; g_{ij}g_{il} \;=\; \delta_{jl}$ for $g \in SO(3)$.

4. (a) We know that we have a surjective map $SU(2) \to SO(3)$ which implies an isomorphism of Lie algebras. Moreover, we have seen that the Lie algebras of $SU(2)$ and $Sl(2, \mathbf{R})$ have isomorphic complexifications. Hence it suffices to show that the kernel of Δ in $P_d(\mathbf{C}^3)$ is an irreducible $\mathbf{sl}_2^{\mathbf{C}}$ module, where the action on $\mathbf{C}^3$ is given via its irreducible 3-dimensional representation.

   Step 1: Let $H \in \mathbf{sl}_2$ be, as usual, the diagonal matrix with diagonal entries ± 1. Then $exp(iH)$ acts via eigenvalues $e^{\pm 2i}$ and 1 in the 3-dimensional representation of $SU(2)$. It follows that

   $$exp(iH).Y_3^d = e^{2di} Y_3^d,$$

   where $Y_3(v) = v_3$ for $v \in \mathbf{C}^3$. Hence H has an eigenvalue $2d$ in $P_d(\mathbf{C}^3)$. ($Warning$: Observe that the action of $SU(2)$ on $\mathbf{C}^3$ uses a different basis than the action of $SO(3)$, i.e. the image of $exp(iH)$ in $SO(3)$ is NOT a diagonal matrix. It follows from one of our previous homeworks that orthogonal matrices usually can not be diagonalized over $\mathbf{R}$. Hence this does not contradict contradict the fact that $\Delta X_3^d \neq 0$ for $d > 1$.).

   Step 2: We now prove that this eigenvector of H in $P_d(\mathbf{C}^3)$ with eigenvalue $2d$ must be in the kernel of Δ. For this, it suffices to prove by induction on d that the eigenvalues of H on $P_d(\mathbf{C}^3)$ are given by $\pm 2d, \pm(2d-2), ...$ This is trivially true for $d = 0$ and $d = 1$. For the induction step $d-2$ to d, it follows from Exercise 3 and the induction assumption for $P_{d-2}(\mathbf{C}^3)$ that the only possible eigenvalues greater than $2d - 2$ must be in the kernel of Δ. In particular, the eigenvalue $2d$ must occur in the kernel. But then the kernel must have at least dimension $2d + 1$, by our classification of irreducible representations of $\mathbf{sl}_2$. But by Exercise 3, the kernel has dimension $2d + 1$, which forces it to be irreducible. Hence we get the following decomposition of $P_d(\mathbf{C}^3)$ into irreducible $SO(3)$-modules:

   $$P_d(\mathbf{C}^3) \;\cong\; V_{2d} \oplus V_{2d-4} \oplus V_{2d-8} \; ....$$

   E.g. $P_4(\mathbf{C}^3) \;\cong V_8 \oplus V_4 \oplus V_0$, where dim $V_m = m + 1$.