LIE GROUPS: 1st Assignment

1. (a) Let $X = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Show that $exp(tX) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$.
(b) Show that any $g \in SO(2)$ can be written as $exp(tX)$ for some $t \in \mathbf{R}$. This implies that the map exp from the Lie algebra $\mathbf{so_2}$ to the Lie group $SO(2)$ is surjective.
(c) Show that $O(2)$ has the same Lie algebra as $SO(2)$. (Hint: consider the map $t \in \mathbf{R} \mapsto det(exp(tX))$ for any X in the Lie algebra of $O(2)$.
(d) Find a Lie group H and two homomorphisms $\Phi_i : O(2) \to H$, $i = 1, 2$ which have the same Lie algebra map but are not the same.

2. *(Orthogonal diagonalization)* It can be easily seen that the eigenvalues of $exp(tX)$ in the previous problem are equal to $e^{\pm it}$. So in general we can *not* diagonalize orthogonal matrices over the real numbers. We will show here that we can still conjugate any orthogonal matrix to a matrix with a diagonal block with eigenvalues ± 1 and 2×2 diagonal blocks as in Problem 1.
(a) Let λ be an eigenvalue of the orthogonal matrix $g \in O(n)$. Show that $|\lambda| = 1$.
(b) Let v be an eigenvector of g with eigenvalue $\lambda \notin \mathbf{R}$. Then we can write $v = v_1 + iv_2$, with $v_1, v_2 \in \mathbf{R}^n$. Show that
$$(v, v) = 0 \quad \text{and} \quad (v_1, v_1) = (v_2, v_2).$$
(c) Assume $\lambda = e^{it}$, $\lambda \notin \mathbf{R}$. Show that the action of g on the span of v_1 and v_2 is given by the matrix $exp(tX)$ as in the first problem.
(d) Show that there exists an orthonormal basis $\{u_1, u_2, \ ... \ u_n\}$ of $\mathbf{R}^n$ consisting of eigenvectors of g with eigenvalues ± 1 or of pairs of vectors v_1 and v_2 as in (c) belonging to the eigenvalues $e^{\pm it}$.

3. Show that $SO(n)$ is contractible for all $n \in \mathbf{N}$. (*Hint* : Use the last problem to show that for given $g \in SO(n)$ there exists an orthogonal matrix u such that $g = udu^{-1}$, where d only consists of 2×2 diagonal blocks as in Problem 1 or diagonal entries equal to ± 1).

4. Show that the Lie algebra of $U(n)$ is given by all matrices X such that $X^* = -X$. What is the Lie algebra of $SU(n)$? What are the dimensions of these Lie algebras?

5. Do Problem 10 and 12 on page 60 of the course book.