LIE GROUPS: 3rd Assignment

1. (*Optional*) Let $\mathcal{P}_d(\mathbf{C}^n)$ be the space of all homogeneous polynomials of degree d in n variables. Show that it has dimension $\binom{n+d-1}{d-1}$. *Hint*: Show that

$$\mathcal{P}_d(\mathbf{C}^n) \;=\; \mathcal{P}_d(\mathbf{C}^{n-1}) \oplus z_n \mathcal{P}_{d-1}(\mathbf{C}^n),$$

and use induction on n and d.

2. (*Optional*) Let $\Delta = \Delta_3$ be the Laplacian in three variables, i.e.

$$\Delta\ f = \frac{\partial^2 f}{\partial z_1^2} + \frac{\partial^2 f}{\partial z_2^2} + \frac{\partial^2 f}{\partial z_3^2}.$$

Show that Δ commutes with the $SO(3)$ action on $\mathcal{P}_d(\mathbf{C}^3)$, i.e. show that

$$\Delta(g.f) = g.(\Delta\ f) \quad \text{for all } g \in SO(3).$$

Observe that for $g \in SO(3)$ we have $g^{-1} = g^t$, where g^t is the transposed matrix.

3. (a) Show that $\Delta : \mathcal{P}_{d+2}(\mathbf{C}^3) \to \mathcal{P}_d(\mathbf{C}^3)$ is surjective for all $d \in \mathbf{N}$. *Hint*: You may show that $z_1^{e_1} z_2^{e_2} z_3^{e_3}$, $e_1 + e_2 + e_3 = d$, is in the image of Δ by induction on alphabetical order of (e_1, e_2, e_3).
(b) Prove that we have an isomorphism as $SO(3)$ modules

$$\mathcal{P}_{d+2}(\mathbf{C}^3) \;\cong\; \ker\,\Delta \;\oplus\; \mathcal{P}_d(\mathbf{C}^3).$$

You may use that for any $SO(3)$ subrepresentation $W \subset V$ there exists an $SO(3)$ representation $W' \subset V$ such that $V \cong W \oplus W'$ as $SO(3)$ representations.

4. (a) Prove that $\ker\,\Delta$ as in the last problem is an irreducible $SO(3)$ representation. (*Hint*: Prove that it is an irreducible sl_2 representation, with the action of sl_2 on $\mathbf{C}^3$ given by its unique irrreducible 3-dimensional representation).
(b) Determine the decomposition of $\mathcal{P}_d(\mathbf{C}^3)$ into a direct sum of irreducible $SO(3)$ representations.