C^∞ STRUCTURE

Here we give some details about how to prove that a matrix Lie group G is a Lie group, i.e. it is a C^∞ manifold for which the group operations are C^∞ maps. Let us first recall the following basic lemma:

Lemma Let $f(z) = \sum_k a_k z^k$ be a power series with radius of convergence $R > 0$, and let $X = (x_{ij})$ be an $n \times n$ matrix. Then $f(X) = \sum_k a_k X^k$ and all of its partial derivatives, of any order, are well-defined for all matrices X with $\|X\| < R$. In particular, $X \mapsto f(X)$ is a C^∞ function.

Proof. (Sketch) Using $\|AB\| \leq \|A\|\|B\|$ and $\|A + B\| \leq \|A\| + \|B\|$, it is easy to check that $f(X)$ is given by a convergent series for $\|X\| < R$. Using the product rule and the fact that $\|E_{ij}\| = 1$ for the matrix unit $E_{ij} = \frac{\partial}{\partial x_{ij}} X$, one also easily checks that $\|\frac{\partial}{\partial x_{ij}} X^k\| \leq k\|X\|^{k-1}$.

Let now $G \subset Gl(n)$ be a matrix Lie group with Lie algebra $\mathfrak{g} \subset M_n$, the set of all (complex or real) $n \times n$ matrices. We also assume $X_1, X_2, \ ... \ X_m$ to be a basis for $\mathfrak{g}$ which defines an isomorphism between $\mathfrak{g}$ and $\mathbf{R}^m$ via

$$(x_1, x_2 \ ... \ x_m) \ \mapsto \ \sum x_i X_i.$$

Let U be a neighborhood of $1 \in G$ as in the last section, and let $U_g = gU$. Let $V \subset \mathbf{R}^m$ be the image of $\log(U)$ under this isomorphism. We then get homeomorphisms

$$\phi_g = \log \circ \ell_{g^{-1}} : U_g \to V,$$

where $\ell_h : g \mapsto hg$ is left multiplication by $h \in G$.

Proposition If G is a matrix Lie group, it is a Lie group. This means it is a group and a manifold with respect to the charts (U_g, ϕ_g) such that multiplication and inverse operations are C^∞ maps.

Proof. Fix $g_0, h_0 \in G$. Then the maps $\phi_{g_0}^{-1}$ and $\phi_{h_0}^{-1}$ from a neighborhood $V \subset \mathfrak{g}$ of 0 are given by

$$\phi_{g_0}^{-1} : (x_1, \ ..., x_m) \ \mapsto \ g_0 \exp(\sum \ x_i X_i), \quad \phi_{h_0}^{-1} : (y_1, \ ..., y_m) \ \mapsto \ h_0 \exp(\sum \ y_i X_i).$$

Both of these maps are C^∞ maps from $\mathbf{R}^m$ into the $n \times n$ matrices. We now consider the map

$$(x_1, \ ... \ x_m, y_1, \ ... \ y_m) \ \mapsto \ \phi_{g_0 h_0}(g_0 \exp(\sum \ x_i X_i) h_0 \exp(\sum \ y_i X_i)), \qquad (*)$$

where we assume $(x_1, \ ... \ x_m, y_1, \ ... \ y_m)$ close enough to 0 so that the right hand side is well-defined; this means that we restrict to smaller neighborhoods $\tilde{U}_{g_0} \subset U_{g_0}$ of g_0 and $\tilde{U}_{h_0} \subset U_{h_0}$ of h_0 such that $\tilde{U}_{g_0} \tilde{U}_{h_0} \subset U_{g_0 h_0}$. As matrix multiplication and the exponential maps are C^∞ maps, we see that the argument of $\phi_{g_0 h_0}$ in $(*)$ is a matrix whose entries are C^∞ functions in the x and y coordinates. By our assumptions on the coordinates, this matrix is in the domain of $\phi_{g_0h_0}$, i.e. after multiplying it with $(g_0h_0)^{-1}$ we get a matrix Z for which $\log(Z)$ converges. In particular, the entries of the matrix $\log(Z)$ are again C^∞ functions in the x and y coordinates, and $\log(Z)$ is in $\mathfrak{g}$. This means that

$$\log(Z) = \sum_{i=1}^{m} f_i(x_1,\ ...\ x_m, y_1,\ ...\ y_m)X_i,$$

where the f_i are C^∞ functions. Hence the coordinate function $\phi_{g_0h_0} \circ m \circ (\phi_{g_0}^{-1}, \phi_{h_0}^{-1})$ for the multiplication

$$m:\ (g,h) \subset \tilde{U}_{g_0} \times \tilde{U}_{h_0} \to U_{g_0h_0}$$

is given by the function

$$(x_1,\ ...\ x_m, y_1,\ ...\ y_m)\ \mapsto\ (f_i(x_1,\ ...\ x_m, y_1,\ ...\ y_m))_i.$$

This shows that multiplication is a C^∞ function.

We also need to show that our chart maps are consistent. Let $g_0, h_0 \in G$ such that $U_{g_0} \cap U_{h_0}$ is not empty. Then we have

$$(x_1,\ ...\ x_m) \in \phi_{g_0}^{-1}(U_{g_0} \cap U_{h_0}) \mapsto g_0 \exp(\sum_i x_iX_i) \mapsto \log(h_0^{-1}g_0 \exp(\sum_i x_iX_i)).$$

is a composition of C^∞ functions from an open set in $\mathbf{R}^m$ into $\mathfrak{g} \subset M_n$. Using the isomorphism of $\mathfrak{g}$ with $\mathbf{R}^m$ as above, we see that the coordinate functions are given by C^∞ functions as before. This proves that the map

$$(x_1,\ ...\ x_m) \in \phi_{g_0}^{-1}(U_{g_0} \cap U_{h_0})\ \mapsto \phi_{h_0}(\phi_{g_0}^{-1}(x_1,\ ...\ x_m)) \in \phi_{h_0}^{-1}(U_{g_0} \cap U_{h_0})$$

is a C^∞ map.