MATH 251 LIE GROUPS

EXPONENTIAL MAPS AND C^∞ MANIFOLDS

The following remarks and exercises are meant to review some facts about C^∞ manifolds and about exponential maps. They are particularly recommended to those who would like to get a better feel about how to work with manifolds. If you are a little stuck, please ask.

1. The following exercise is supposed to give you an idea how to produce matrix C^∞ functions via power series. Let $f(z) = \sum a_n z^n$ be a power series which converges for $|z| < R$, and let $|f|(z) = \sum |a_n| z^n$ (which also converges for $|z| < R$). Moreover, let $X = (x_{ij})$ be an $n \times n$ matrix, and let E_{ij} be the matrix unit, i.e. a matrix all of whose entries are zero except in the i-th row and j-th column, where it is 1.
(a) Calculate for fixed i, j, r, s the partial derivatives $\frac{\partial}{\partial x_{ij}} X^m$ and $\frac{\partial^2}{\partial x_{ij} \partial x_{rs}} X^m$.
(b) Show that $\| \frac{\partial^r}{\partial x_{i_1 j_1} \partial x_{i_2 j_2} \ldots \partial x_{i_r j_r}} X^m \| \leq m(m-1)...(m-r+1) \|X\|^{m-r}$.
(c) Show that $\frac{\partial^r}{\partial x_{i_1 j_1} \partial x_{i_2 j_2} \ldots \partial x_{i_r j_r}} f(X) = \sum_{m=r}^{\infty} \frac{\partial^r}{\partial x_{i_1 j_1} \partial x_{i_2 j_2} \ldots \partial x_{i_r j_r}} X^m$,
where the right hand side converges absolutely for $\|X\| < R$.

2. (a) Let $X = \begin{bmatrix} x & y \\ z & -x \end{bmatrix}$, and let $\lambda = \sqrt{x^2 + yz} = \sqrt{-\det(X)}$. Show that $\exp(X) = \cosh(\lambda) I + \lambda^{-1} \sinh(\lambda) X$; rewrite this in terms of real functions if $x^2 + yz < 0$. Observe that this gives a description of a neighborhood of 1 in $SL(2, \mathbf{R})$ in terms of the coordinates (x, y, z). (Why is there no problem for $\sqrt{x^2 + yz}$ at $x^2 + yz = 0$ as far as differentiablilty goes?)
(b) Show that the matrix $= \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$ is in $SL(2, \mathbf{R})$, but not in the image of its Lie algebra under the exponential map.
(c) Show that for every invertible matrix A there exists a matrix X such that $\exp(X) = A$. Why does this not contradict (b)?
(d) Show that whenever $\|X\| < \ln 2$, then $\exp(X)$ is in the domain of the matrix logarithm, i.e. $\| \exp X - I \| < 1$.

3. Here are some indications how to explicitly check the C^∞ structure of $SL(2, \mathbf{R})$. It follows from Problem 1 that we have the map

$$(x, y, z) \mapsto \exp X = (a_{ij}(x, y, z))_{1 \leq i,j \leq 2},$$

where the a_{ij} are C^∞ functions, and the map is invertible if we restrict exp to a sufficiently small neighborhood of 0 (see Problem 2(d)).
(a) Show that the matrix coefficients of $(x, y, z) \mapsto \exp(X)^{-1}$ are C^∞ maps.
(b) Show that the matrix coefficients of

$$\Phi : (x_1, y_1, z_1, x_2, y_2, z_2) \mapsto \exp(X_1) \exp(X_2)$$

are C^∞ maps.
(c) Show that for X_1, X_2 in sufficiently small neighborhoods U_1, U_2 of 0 in the Lie algebra of SL_2 the map $\log \circ \Phi$ is a C^∞ map (this is a special case of what is mentioned in Problem 4).
(d) Explain what remains to be done to prove that $SL(2, \mathbf{R})$ is a Lie group.

4. By definition, a map f from an open set $V \subset \mathbf{R}^m$ to a d-dimensional Lie group (or any C^∞-manifold) is a C^∞ map if the maps $\phi_\lambda \circ f : V \to \mathbf{R}^d$ are $\mathbf{C}^\infty$ maps for all coordinate maps $\phi_\lambda : U_\lambda \to \mathbf{R}^d$. If G is a matrix Lie group, one can show that this is equivalent to the fact that all matrix coefficients of f are C^∞ functions. Check this for the example of $G = SL(2, \mathbf{R})$, where you can assume that the image of f is contained in the domain of log, i.e. in the set $\{A, \|A - I\| < 1\}$.

5. Let $H \subset G$ be matrix Lie groups. Show that the Lie algebra $\mathfrak{h}$ of H is a sub Lie algebra of the Lie algebra $\mathfrak{g}$ of G, i.e. it is a linear subspace $\mathfrak{h} \subset \mathfrak{g}$ such that $[X, y] \in \mathfrak{h}$ if $X, Y \in \mathfrak{h}$

6. *Remarks*. For practical calculations one usually does not use the exponential maps for charts. Many examples are given via algebraic equations (such as determinant = 1, or $A^T A = I$ (orthogonality), $U^* U = I$(unitarity) etc). The existence of charts then follows from the implicit function theorem (or, in simpler cases, by just solving for sufficiently many matrix entries using the defining equations).

Answer: 1(a) $\sum_{p=1}^m X^{p-1} E_{ij} X^{m-p}$ and the sum of all possible expressions obtained by replacing two of the factors in X^m by E_{ij} and E_{rs} (check there are $m(m-1)$ such expressions).