MATH 251 LIE GROUPS

EXPONENTIAL MAPS FOR MATRIX GROUPS

We review some of the key theorems relating the structures of a Lie group and its corresponding Lie algebra. Recall that for a matrix Lie group $G \subset Gl(n)$ (real or complex invertible $n \times n$ matrices), we can view the Lie algebra $\mathfrak{g}$ as a subspace of $n \times n$ matrices. In particular, we can define the exponential function on $\mathfrak{g}$ via its power series.

Theorem 1 Let G be a matrix Lie group with Lie algebra $\mathfrak{g}$. Then the differential $d\ exp : \mathfrak{g} \to \mathfrak{g}$ at an arbitrary $X \in \mathfrak{g}$ is given by

$$d\exp(X) = \exp(X) \ \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!}\ ad_X^n \ = \ \exp(X)\ f(ad_X),$$

where $f(ad_X)$ is defined via the power series of the function $f(z) = \frac{1-e^{-z}}{z}$.

Outline of proof: This was done in class via a brute-force power series calculation as follows: By definition of differential, $d\ exp(X)$ is the linear map from $\mathfrak{g}$ to $exp(X)\mathfrak{g}$, the tangent space at $exp(X)$ which can be calculated as

$$d\exp(X)Y = \frac{d}{dt}_{|t=0} \exp(X+tY) \ = \ \sum_{m=0}^{\infty} \frac{1}{m!} \frac{d}{dt}_{|t=0} (X+tY)^m \ = \ \sum_{m=0}^{\infty} \sum_{i=0}^{m-1} \frac{1}{m!} X^i Y X^{m-1-i}.$$

Expanding $exp(-X) = exp(X)^{-1}$ as a power series, and doing some resummations of the resulting power series in the non-commuting variables X and Y, one can show that

$$\exp(-X)\ d\exp(X)Y \ = \ \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \sum_{s=0}^{n} (\sum_{k=0}^{s} (-1)^k \binom{n+1}{k}) X^s Y X^{n-s} \ =$$

$$= \ \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \sum_{s=0}^{n} (-1)^s \binom{n}{s}) X^s Y X^{n-s} \ = \ \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!}\ ad_X^n(Y).$$

It can now be checked easily that this is equal to $f(ad_X)Y$.

We have seen before that if $f(z)$ and $h(z)$ are given by power series with radius of convergence $\geq R$, with $f(z)h(z) = k(z)$ and if X is a matrix with $\|X\| < R$, then also $f(X)h(X) = k(X)$. In particular, if we have

$$h(z) = \frac{z}{1-e^{-z}} = \sum_{k}^{\infty} \frac{(-1)^{k+1}}{k+1}\ (e^{-z} - 1)^k, \qquad (*)$$

we have $h(ad_X)f(ad_X) = id_{\mathfrak{g}}$ for $\|ad_X\|$ sufficiently small, i.e. $h(ad_X)$ is the inverse of $f(ad_X)$.

Corollary Let $Z(t) = \log(\exp X \exp tY)$. Then

$$Z'(t) \;=\; h(ad_{Z(t)})Y \;=\; \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{(k+1)!} \;(\sum_{\ell,m} (-1)^{\ell+m} \frac{1}{\ell!\; m!}\; t^{\ell} ad_Y^{\ell} ad_X^{m})^k \; Y,$$

where the summation goes over all $\ell, m \geq 0$ for which $\ell + m > 0$.

Proof. On the one hand we have

$$\frac{d}{dt}(\exp Z(t)) = \frac{d}{dt}(\exp X \exp tY) = \exp X \exp(tY)\; Y.$$

On the other hand, using the chain rule, we get

$$\frac{d}{dt}(\exp Z(t)) = d\exp(Z(t))\; Z'(t) \;=\; f(ad_{Z(t)})Z'(t).$$

Setting these two expressions for $\frac{d}{dt}(\exp Z(t))$ equal and solving for $Z'(t)$ by multiplying both sides by $h(ad_{Z(t)})$, we get the first equality. For the second equality, we use the power series expansion for $h(z)$ in terms of $(e^{-z} - 1)$, see $(*)$. We first observe that

$$\exp(ad_{\log(\exp X \exp tY)}) \;=\; \exp ad_X \;\; \exp ad_{tY};$$

indeed, this follows from $\exp(ad_Z) = Ad_{\exp Z}$ for $Z \in \mathfrak{g}$ and the homomorphism property of Ad, i.e. $Ad_{gh} = Ad_g Ad_h$. It then follows that

$$h(ad_{Z(t)})Y = \sum_{k}^{\infty} \frac{(-1)^{k+1}}{k+1} \;(\exp(-ad_{\log(\exp X \exp \; tY)}) - 1)^k \; Y \;=$$

$$= \sum_{k}^{\infty} \frac{(-1)^{k+1}}{k+1} \;(\exp ad_{-tY} \;\; \exp ad_{-X} - 1)^k \; Y.$$

The second equality now follows by expanding the last expression in terms of power series in ad_X and $t\; ad_Y$.

Theorem We can express $\log(\exp X \exp Y)$ in terms of a series consisting of brackets in X and Y which converges for X and Y sufficiently close to 0. In particular, the product $\exp X \exp Y$ is determined by the Lie algebra structure of $\mathfrak{g}$.

Proof. It follows from $Z(0) = X$, the previous corollary and the fundamental theorem of calculus that

$$\log(\exp X \exp Y) \;=\; Z(0) + \int_0^1 Z'(t)\; dt \;=$$

$$=\; X + \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{(k+1)!} \;(\sum_{\ell,m} (-1)^{\ell+m} \frac{1}{(\ell+1)!\; m!}\; ad_Y^{\ell} ad_X^{m})^k \; Y,$$

where the summation over ℓ and m is as in the corollary. By definition of ad_X and ad_Y, this describes a rather complicated series of brackets in X and Y.

Remark It is possible to calculate explicit expressions for the brackets in the power series of the last theorem via certain recursive formulas. This is the context of what is generally known as Campbell-Baker-Hausdorff theorem. However, for our purposes, it is not necessary to know these explicit expressions.

Theorem (*Local group homomorphism*) Let G and H be (matrix) Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, and let $\phi : \mathfrak{g} \to \mathfrak{h}$ be a Lie algebra homomorphism. Then there exists an open neighborhood U of $1 \in G$ and a map $\Phi : G \to H$ such that

$$\Phi(u_1 u_2) = \Phi(u_1)\Phi(u_2), \quad \text{for } u_1, u_2 \in U,$$

and $\Phi(\exp X) = \exp(\phi(X))$ for $X \in \log(U) \subset \mathfrak{g}$. We call Φ a local group homomorphism associated to ϕ.

Proof. As the group multiplication is continuous, we can find an open neighborhood U of $1 \in G$ such that $u_1 u_2$ is in the domain of log whenever $u_1, u_2 \in U$. Let $X_i = \log u_i$ for $i = 1, 2$. We define

$$\Phi(u) = \exp(\phi(\log u)) \quad \text{for } u \in U^2 = \{u_1 u_2,\ u_i \in U,\ i = 1, 2\}.$$

Then we have, for $u_1, u_2 \in U$,

$$\begin{aligned}\Phi(u_1 u_2) &= \Phi(\exp(\log(\exp X_1 \exp X_2))) = \exp(\phi(\log(\exp X_1 \exp X_2))) = \\ &= \exp(\phi(\text{series of brackets in } X_1 \text{ and } X_2)) = \\ &= \exp(\text{same series of brackets in } \phi(X_1) \text{ and } \phi(X_2)) =\end{aligned}$$

where we used the series for $\log(\exp X_1 \exp X_2)$ from the last theorem and the fact that ϕ is a Lie algebra homomorphism,

$$= \exp(\log(\exp \phi(X_1) \exp \phi(X_2))) = \exp \phi(X_1) \exp \phi(X_2)) = \Phi(u_1)\Phi(u_2).$$

Theorem (*Global group homomorphism*) We use the same notations as in the last theorem. If G is a *simply connected* group, then there exists a unique group homomorphism $\Phi : G \to H$ such that $\Phi(\exp X) = \exp(\phi(X))$ for all X in an open neighborhood of $1 \in G$.

Proof. The idea of the proof is to express a given element $g \in G$ as a product of $u_i \in U$, $1 \le i \le k$, where $u_i = g_{i-1}^{-1} g_i$ and the g_i's are group elements on a path from $1 = g_0$ to $g = g_k$ chosen sufficiently close to each other that the u_i's are in U as in the previous theorem. Using the local homomorphism property of the last theorem, one can show that

$$\Phi(g) = \Phi(u_1)\Phi(u_2) \ldots \Phi(u_k)$$

does not depend on the choice of path or choice of elements g_i on the path, and that Φ is indeed a group homomorphism. See your notes for details.