The proof of the following theorem can be found in the lecture notes for which a link is given on the course web site. See Proposition 13.1, Lemma 13.2 and Proposition 13.3 in these notes:

Theorem 5.1 Any connected abelian Lie group G is isomorphic to $(\mathbf{R}/\mathbf{Z})^m \times \mathbf{R}^k$ for non-negative integers k and m. In particular, if G is compact, it is isomorphic to $(\mathbf{R}/\mathbf{Z})^m$ for some m.

Definition A compact connected abelian Lie group T is called a torus. By the previous theorem, $T \cong (\mathbf{R}/\mathbf{Z})^m$ for some m.

If T is a torus as in the definition, its Lie algebra $\mathfrak{t}$ can be identified with $\mathbf{R}^m$, and its exponential map is given by

$$exp : \mathbf{t} = (t_1, t_2, \ ... \ t_m) \in \mathbf{R}^m \ \mapsto \mathbf{z} = (z_1, z_2, \ ... \ z_m), \tag{$*$}$$

where $z_j = e^{2\pi i t_j}$. Observe that this is just a way how to explicitly describe the quotient map $\mathbf{R}^m \to (\mathbf{R}/\mathbf{Z})^m$, where $\mathbf{R}/Z$ is identified with the circle $S^1 = \{z \in \mathbf{C}, |z| = 1\}$.

Theorem 5.2 Any 1-dimensional representation $\Phi : T = (\mathbf{R}/\mathbf{Z})^m \to \mathbf{C}$ is given in the form

$$\mathbf{z} = (z_j)_j \in (S^1)^m \ \mapsto \ \mathbf{z}^\lambda = z_1^{\lambda_1} z_2^{\lambda_2} \ ... \ z_m^{\lambda_m},$$

where the numbers $\lambda_1, \lambda_2, \ ...$ are integers.

Proof. Let $\mathfrak{t} = \mathbf{R}^m$ be the Lie algebra of T. Then the Lie algebra map $\phi : \mathbf{R}^m \to \mathbf{C}$ is linear, hence of the form

$$(t_1, t_2, \ ... \ t_m) \mapsto \sum_j \tilde{\lambda}_j z_j,$$

for some scalars $\tilde{\lambda}_j$. Hence

$$\Phi(\mathbf{z}) = \Phi(exp(t_1, t_2, \ ... \ t_m)) = \exp(\phi(t_1, \ ... \ t_m)) = exp(\sum \tilde{\lambda}_j t_j). \tag{1}$$

Now observe that if we take for $\mathbf{t}$ one of the standard basis vectors $\mathbf{e}_k$, i.e. $t_j = \delta_{kj}$, then $\exp \mathbf{t} = 1$ by $(*)$. This also forces $\Phi(\exp \mathbf{t}) = e^{\tilde{\lambda}_k} = 1$, from which we conclude that $\tilde{\lambda}_j = 2\pi i \lambda_j$ for some integer λ_j. Hence we get from (1)

$$\Phi(\mathbf{z}) = \prod_j e^{2\pi i \lambda_j t_j} = \prod_j z_j^{\lambda_j},$$

by definition of z_j, below $(*)$.