QUAL PREP SESSION 1

Problem 1

Suppose X is a compact metric space, let $\{x_n\}$ be a sequence in X such that every convergent subsequence converges to $y \in X$ (the same y). Show that $x_n \to y$.

Problem 2

Let $\varphi(x) := \frac{1}{x \log(x+1)}$, defined on $[1, \infty)$. Further suppose we have the sequence $f_n := c_n \chi_{E_n}$ for some sequence of real numbers $c_n \geq 0$ and measurable sets $E_n \subset [1, \infty)$, such that $f_n \leq \varphi$ on $[1, \infty)$ and such that $f_n \to 0$ almost everywhere.

Note: φ is not in $L^1([1, \infty))$.

a) Show that for any fixed N such that $1 \leq N < \infty$

$$\int_1^N f_n(x)dx \to 0.$$

b) Now show

$$\int_1^\infty f_n(x)dx \to 0.$$

Problem 3

a) Let μ and ν be two finite signed measures. Show that there *exists* a *unique* largest measure η such that $\eta \leq \mu$ and $\eta \leq \nu$, (i.e., if η' is also smaller than μ and ν, then $\eta' \leq \eta$.

b) Suppose μ, ν are positive measures. Prove that μ and ν are mutually singular if and only if $\eta = 0$.
