UNIVERSITY OF CALIFORNIA, SAN DIEGO

Semi-invariant Forms

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by

Anne V. Shepler

Committee in charge:

Professor Peter Doyle, Chair
Professor Kate Okikiolu
Professor Douglas Smith
Professor Nolan Wallach
Professor Gabriela Wienhausen

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The dissertation of Anne V. Shepler is approved, and it is acceptable in quality and form for publication on microfilm:

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To Olav

TABLE OF CONTENTS

LIST OF TABLES

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VITA

1972	Born, Washington D.C.
1994	B.S. *magna cum laude*, Valparaiso University, Indiana
1994–1999	Teaching Assistant, Department of Mathematics, University of California, San Diego
1999	Ph.D., University of California, San Diego

PUBLICATIONS

1 *Semi-invariants of finite reflection groups*, to appear. Journal of Algebra.

2 *Logarithmic forms and anti-invariant forms of reflection groups*, with H. Terao, to appear. Singularities and Arrangements, Sapporo-Tokyo 1998, Advanced Studies in Pure Mathematics, North-Holland.

ABSTRACT OF THE DISSERTATION

Semi-invariant Forms

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Anne V. Shepler

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Let G be a finite group of complex $n \times n$ unitary matrices generated by reflections acting on $\mathbb{C}^n$. Let R be the ring of invariant polynomials, and χ be a multiplicative character of G. Consider the R-module of χ-invariant differential forms and the R-module of χ-invariants in the exterior algebra of derivations. We define a natural multiplication on these modules using ideas from arrangements of hyperplanes. We show that this multiplication gives each module the structure of an exterior algebra. We also define a multiarrangement associated to χ, and formulate the relationship between χ-invariants and logarithmic forms. We introduce a new method of computing basic derivations and the generating χ-invariants and give explicit constructions for the exceptional irreducible reflection groups.

Chapter 1

Introduction

1.1 Layperson's

If you hold two mirrors together at a right angle and then place a key in front of the mirrors, you will see four keys — one real, three images. As you angle the mirrors closer together, you see more and more keys. Mathematicians like to place infinitely tall and wide mirrors in a space and examine how imagines bounce from mirror to mirror and end up in new locations. They fix some plane to act as a mirror, and then reflect vectors about that plane. This is accomplished on paper with a well-chosen matrix: when you apply the matrix to a vector, you get a new vector which is the image upon reflecting through the given plane. We are not really interested in the physics of reflecting light or mirrors. Rather, we would like to know how rearranging a space with reflections affects properties like length, volume, orientation (right changes to left!), etc. The distance between two points doesn't change after a series of reflections. What other functions on the space are unaffected? Mathematical objects are called *invariant* when they are unaffected, and called *semi-invariant* when they are almost unaffected — they change by a constant. Semi-invariants are the subject of this thesis.

1.2 Mathematician's

The present inquiry on semi-invariants arose from some questions about dynamical systems. In 1989, P. Doyle and C. McMullen [5] solved the fifth degree polynomial using a highly symmetrical dynamical system which preserved the Galois group A_5. In 1997, S. Crass and P. Doyle [4] tackled the sixth degree polynomial by again finding a dynamical system with special symmetry — this time A_6 symmetry. Each dynamical system was formed by iterating a map that was equivariant under the projective action of a reflection group. Such maps correspond naturally to semi-invariant differential forms. Because almost nothing was known about these forms, constructing the necessary dynamical systems was a difficult step in both cases.

This thesis introduces a general theory of semi-invariants. Specifically, we show that for any finite unitary reflection group G and multiplicative character χ of G, the module of χ-invariant differential forms has a natural multiplication which turns the module into an *exterior algebra*. This exterior algebra structure allows us to understand completely the forms that give rise to highly symmetrical dynamical systems, and gives us tools to compute these forms explicitly. We extend these results to vector fields (or *derivations*), observe some applications to logarithmic forms, and show new techniques for computing semi-invariants. We also give constructions for the exceptional irreducible reflection groups.

The theory presented here builds on work by R. Stanley, who characterized the module of χ-invariant polynomials in 1977 [17]. It also builds on more recent work by P. Orlik, K. Saito, L. Solomon, H. Terao and others on invariant derivations and the theory of hyperplane arrangements (see [9], Chapter 6).

Chapter 2

Central Question

Before we can ask the central question motivating this thesis, we need to establish some terminology. We define reflection groups, differential forms, derivation forms, some particular group actions, and semi-invariance. We then pose the questions that will drive our discussions.

2.1 Reflection Groups

Begin with a complex vector space $V := \mathbb{C}^n$. A unitary $n \times n$ matrix is a *reflection* if its fixed point set is a hyperplane of V, i.e., an $(n-1)$-dimensional space. A reflection matrix is characterized by the fact that $n-1$ of its eigenvalues are 1 (corresponding to the fixed hyperplane) and the remaining eigenvalue is a non-trivial root of unity. If the non-trivial eigenvalue is a k-th root of unity, then we say that the reflection is k-fold. A *reflection group* is a finite group of matrices generated by reflections. They are often called finite reflection groups, finite pseudo-reflection groups, or u.g.g.r. (unitary groups generated by reflections).

Why must a reflection be a unitary matrix? Any finite group of complex $n \times n$ matrices is congruent to a group of unitary matrices. A unitary space comes equipped with a hermitian form, which allows us to talk about *perpendicular* objects. We wish a reflection matrix to behave as a mirror: a real reflection sends a vector perpendicular to a fixed hyperplane to its negative. A complex reflection multiplies the perpendicular vector by a root of unity.

Reflection groups are called *real reflection groups*, or *Coxeter groups*, when their matrices are real. Coxeter groups are generated by orthogonal reflections with determinant -1. The symmetry group of a regular complex polytope, called a *Shephard group*, is also a reflection group. In 1954, G. Shephard and J. Todd [11] published a paper that proved many fundamental properties of reflection groups: *Finite unitary reflection groups*. They extended results about real reflection groups to general (complex) reflection groups, collected information about the groups, and proved important new properties. Every reflection group is either irreducible or the direct product of irreducible components, each of which is itself a reflection group. Shephard and Todd list the thirty-seven irreducible reflection groups. Three of the groups on the list are infinite families. The remaining thirty-four are in dimensions 2 through 8.

2.2 Forms

Now some notation: Let $S := \mathbb{C}[x_1, \ldots, x_n]$ be the ring of polynomials on V and $F := \mathbb{C}(x_1, \ldots, x_n)$ be the field of rational functions. Denote the module of differential p-forms on V by

$$\begin{aligned} \Omega^p &:= \bigoplus_{1 \le i_1 < \ldots < i_p \le n} S \; dx_{i_1} \wedge \cdots \wedge dx_{i_p} \\ &\simeq S \otimes \bigwedge\nolimits^p V^*. \end{aligned}$$

Let $\partial x_i := \dfrac{\partial}{\partial x_i}$ and let

$$\begin{aligned} \Upsilon^p &:= \bigoplus_{1 \le j_1 < \ldots < j_p \le n} S \; \partial x_{j_1} \wedge \cdots \wedge \partial x_{j_p} \\ &\simeq S \otimes \bigwedge\nolimits^p V. \end{aligned}$$

We agree that $\Omega^0 = \Upsilon^0 = S$. Let

$$\Omega := \bigoplus_{p \ge 0} \Omega^p$$

and

$$\Upsilon := \bigoplus_{p \ge 0} \Upsilon^p.$$

Although the elements of Υ could be viewed as differential forms on V^*, we reserve the term *differential form* for elements of Ω. Note that the set of vector fields on V is

$$\Upsilon^1 = \left\{ \sum_i k_i \frac{\partial}{\partial x_i} \; : \; k_i \in S \right\}.$$

We identify Υ^1 with the set of *derivations* on V, $\mathbb{C}$-linear maps $\theta : S \to S$ satisfying

$$\theta(kh) = k\theta(h) + \theta(k)h$$

for all k, $h \in S$. We call elements of Υ^p *derivation p-forms*, or just *derivation forms.*

It is convenient to define $\mathcal{I}^p$ as the set of ordered multi-indices of $\{1, ..., n\}$ of length p:

$$\mathcal{I}^p := \{I = \{I_1, ..., I_p\} : 1 \leq I_1 < \ldots < I_p \leq n\}.$$

Note that $\mathcal{I}^0$ is the empty set. For a multi-index I, let I^c denote the (ordered) complementary index. Let $dx_I := dx_{I_1} \wedge \cdots \wedge dx_{I_p}$ and $\partial x_J := \partial x_{J_1} \wedge \cdots \wedge \partial x_{J_p}$. Then we can write any element of Ω^p as $\sum_{I \in \mathcal{I}^p} h_I \, dx_I$ and any element of Υ^p as $\sum_{J \in \mathcal{I}^p} k_J \, \partial x_J$. Denote the volume form on V by $vol := dx_1 \wedge \cdots \wedge dx_n$. If μ and ν are in Ω or Υ, we write $\mu \doteq \nu$ if $\mu = c\,\nu$ for some c in $\mathbb{C}^*$.

It is helpful to define a matrix of coefficients for forms. For $\beta_1, \ldots, \beta_n$ in Υ^1 (or in Ω^1), write each β_j as

$$\beta_j = \sum_{i=1}^{n} \beta_j^{(i)} \, \partial x_i \qquad \left(\text{or} \;\; \beta_j = \sum_{i=1}^{n} \beta_j^{(i)} \, dx_i \right)$$

and let $M(\beta_1, \ldots, \beta_n)$ be the matrix $\{\beta_j^{(i)}\}$, so that $\beta_1 \wedge \cdots \wedge \beta_n = \det M \; \partial x_1 \wedge \ldots \wedge \partial x_n$ (or $\det M \; vol$).

2.3 Group Actions

Let G be a reflection group. The group G not only acts on the space V, but also on S, Ω^p, and Υ^p. The action on S is defined by $gf := f \circ g^{-1}$ for $f \in S$, $g \in G$. The correspondence between x_i and $\partial x_i = \frac{\partial}{\partial x_i}$ extends anti-linearly to degree one polynomials by

$$\partial \left(\sum_{i=1}^{n} c_i x_i \right) := \sum_{i=1}^{n} \bar{c}_i \, \partial x_i$$

where $c_i \in \mathbb{C}$. We define an action on dx_i and ∂x_j by

$$gdx_i := d(gx_i) \quad \text{and} \quad g\partial x_j := \partial(gx_j).$$

For a multi-index $I = \{I_1, \dots, I_p\}$, define

$$gdx_I := gdx_{I_1} \wedge \cdots \wedge gdx_{I_p}$$

and

$$g\partial x_I := g\partial x_{I_1} \wedge \cdots \wedge g\partial x_{I_p}.$$

Extend the action to Ω^p and Υ^p by

$$g \sum_{I \in \mathcal{I}^p} h_I \; dx_I := \sum_{I \in \mathcal{I}^p} gh_I \; gdx_I$$

and

$$g \sum_{J \in \mathcal{I}^p} k_J \; \partial x_J := \sum_{J \in \mathcal{I}^p} gk_J \; g\partial x_J.$$

2.4 Remarks

Let W be any finite dimensional vector space over $\mathbb{C}$. The dual space W^* is anti-linearly isomorphic to W by $* : w \mapsto \bar{w}^t$ (conjugate transpose). Whenever a linear transformation L acts on W by a matrix l, there is an induced action L^* on W^* given by $L^*(u)w = u(Lw)$. If we use $*$ to define a dual basis, then the transformation L^* is given by the inverse transpose of the matrix l. Hence, one defines the matrix $l^* = (l^t)^{-1}$. Since our group is unitary, $g^* = \bar{g}$, and the group action we have defined on ∂x_i is the traditional one. In other words, the group respects the identification of V and V^* under the $*$ map. We have given the action in terms of the anti-linear map ∂ in order to exploit this fact. We will extend ∂ to a map taking polynomials to differential operators in Chapter 9.

2.5 Semi-invariants

Let χ be a multiplicative character of G, i.e. a homomorphism from G to $\mathbb{C}$. For each g in G, $\chi(g)$ is a root of unity since g has finite order. We say that β in Ω or

Υ is *χ-invariant* if β satisfies

$$g\beta = \chi(g)\,\beta$$

for all $g \in G$. Without reference to a particular χ, we say that β is *semi-invariant.*

Define the G-modules of semi-invariants

$$\begin{aligned} (\Upsilon^p)^\chi &:= \{\theta \in \Upsilon^p \ : \ g\theta = \chi(g)\ \theta \ \text{ for all } g \in G\}\,, \\ (\Omega^p)^\chi &:= \{\omega \in \Omega^p \ : \ g\omega = \chi(g)\ \omega \ \text{for all } g \in G\}\,, \end{aligned}$$

and

$$\begin{aligned} \Upsilon^\chi &:= \bigoplus_{0 \le p} (\Upsilon^p)^\chi, \\ \Omega^\chi &:= \bigoplus_{0 \le p} (\Omega^p)^\chi. \end{aligned}$$

Denote the trivial character of G by χ_o, i.e. $\chi_o(g) = 1$ for each $g \in G$. Elements of Ω^{χ_o} and Υ^{χ_o} are called *invariant.* We will usually use a more traditional notation for the modules of invariants:

$$\begin{aligned} (\Omega^p)^G &:= (\Omega^p)^{\chi_o}, \\ (\Upsilon^p)^G &:= (\Upsilon^p)^{\chi_o}, \\ \Omega^G &:= \Omega^{\chi_o}, \\ \Upsilon^G &:= \Upsilon^{\chi_o}. \end{aligned}$$

2.6 The Question

Our discussions are motivated by a central question: What is the set of semi-invariant forms? How can semi-invariants be computed? Specifically, does the set exhibit any striking algebraic structure?

Chapter 3

Invariant Theory

The best known semi-invariants are those that are invariant: forms β that satisfy $g\beta = \beta$ for all g in G. A rich theory of invariants for reflection groups has developed around a powerful theorem by G. Shephard, J. Todd, and C. Chevalley [1] (V.5.3, Theorem 3) describing the set of invariant polynomials. We recall this theorem and a theorem by L. Solomon [14] describing the set of invariant forms. We also obtain an immediate application to semi-invariants from these results.

3.1 Invariants

Let R be the set of invariant polynomials. The celebrated theorem about invariant polynomials is

Theorem 1. *There exist n homogeneous polynomials, $f_1, \ldots, f_n$, with $R = \mathbb{C}[f_1, \ldots, f_n]$. The degrees of the f_i are uniquely determined.*

We call the polynomials in the above theorem *basic invariants*, and call R the *ring of invariants*. Basic invariants have been constructed for all 37 of the irreducible reflection groups.

Notice that $(\Upsilon^p)^\chi$ and $(\Omega^p)^\chi$ are modules over R. In 1963, L. Solomon [14] showed that the R-module of invariant differential forms, Ω^G, has the beautiful structure of an exterior algebra:

Theorem 2. *The module Ω^G is generated over R as an exterior algebra by the df_i, i.e., $(\Omega^p)^G$ is generated over R by the forms $df_{i_1} \wedge \cdots \wedge df_{i_p}$, where $1 \leq i_1, \ldots, i_p \leq n$.*

3.2 Application to Semi-invariants

Given any G-module N, we can define $N^G := \{n \in N : g(n) = n \ \forall g \in G\}$. We state a well-known proposition about G-modules; for proof, see Lemma 6.45 of [9].

Proposition 3. *If M is a G-module of dimension r over $\mathbb{C}$, then the R-module $(S \otimes M)^G$ is free of rank r (over R).*

Corollary 4. *The R-modules $(\Omega^p)^\chi$ and $(\Upsilon^p)^\chi$ are both free of rank $\binom{n}{p}$.*

Proof: We follow an idea from Solomon [15]. Let Z be a one-dimensional G-module with generator z defined by $gz = \chi^{-1}(g)\, z$ for all $g \in G$. Let $M = \bigwedge^p V^* \otimes Z$, and let G act on M by $g(a \otimes b) = ga \otimes gb$. Then M is a G-module of dimension $\binom{n}{p}$. By Proposition 3, the R-module $(S \otimes M)^G$ is free of rank $\binom{n}{p}$.

The map $\omega \mapsto \omega \otimes z$ is a natural R-module isomorphism between the module of χ-invariant forms in $\Omega^p \simeq S \otimes \bigwedge^p V^*$ and the module of invariants in $(S \otimes M)$, i.e., $(\Omega^p)^\chi \simeq (S \otimes M)^G$. Hence $(\Omega^p)^\chi$ has dimension $\binom{n}{p}$. Similarly, $(\Upsilon^p)^\chi$ has dimension $\binom{n}{p}$. □

The idea behind these two results is essentially this: since $\bigwedge^p V^*$ is generated by $\binom{n}{p}$ elements, $(\Omega^p)^\chi$ is also generated by $\binom{n}{p}$ elements. But the dimension of $\bigwedge^p V^*$ is $\binom{n}{p}$ because it is generated by wedging p of the $x_i \in V^*$ together at a time — it's an exterior algebra. The last result may provoke a suspicion: could there exist some analogue of the x_i in Ω^χ? Could there exist forms that are combined together p at a time to generate $(\Omega^p)^\chi$?

Chapter 4

Arrangements of Hyperplanes

Reflection groups are often studied using results from arrangements of hyperplanes. One important result completely describes semi-invariant polynomials. In this chapter, we explain the connection between reflection groups and hyperplane arrangements, and give the fundamental result on semi-invariant polynomials. We will often follow notation from the wonderful text *Arrangements of Hyperplanes* [9], which should be consulted as a general reference.

Consider a n-dimensional vector space W over a field $\mathbb{K}$. A *hyperplane* in W is a $(n-1)$-dimensional affine subspace of W. A *hyperplane arrangement* is a finite set of hyperplanes. For each hyperplane H in a hyperplane arrangement $\mathcal{A}$, let α_H be a linear polynomial on W whose kernel is H. We call

$$Q(\mathcal{A}) = \prod_{H \in \mathcal{A}} \alpha_H$$

the *defining polynomial* of $\mathcal{A}$. The polynomial $Q(\mathcal{A})$ is uniquely defined up to a nonzero scalar multiple.

4.1 Reflection Arrangements

We now consider the arrangement defined by our reflection group G. Each reflection in our group fixes a hyperplane in $\mathbb{C}^n$. Fix $\mathcal{A}$ as the collection of all such hyperplanes. Notice that the group G permutes the hyperplanes in $\mathcal{A}$. For each $H \in \mathcal{A}$, define $\alpha_H \in S$ by $\ker(\alpha_H) = H$. Then the polynomial

$$Q := Q(\mathcal{A}) = \prod_{H \in \mathcal{A}} \alpha_H$$

defines the hyperplane arrangement $\mathcal{A}$.

4.2 Stanley Polynomials

By Corollary 4, there exists a single polynomial that generates the entire set of χ-invariant polynomials over R (the dimension of $(\Omega^0)^\chi$ is $\binom{n}{0}$). R. Stanley showed that this fundamental polynomial can be written in terms of the linear forms α_H that define the hyperplanes in $\mathcal{A}$. We raise each linear form α_H to some power determined by the character χ. Specifically, fix some $H \in \mathcal{A}$, and let G_H be the set of all elements in G that fix H pointwise. There exists a one-dimensional subspace L_H stable under G_H with $V = H \oplus L_H$ by Maschke's theorem. The subgroup G_H is cyclic. Let s_H be a generator of G_H and let $o(s_H)$ be the order of s_H. Note that $\det(s_H)$ is a primitive $o(s_H)$-th root of unity. Define $a_H(\chi)$ as the least integer satisfying $0 \le a_H(\chi) < o(s_H)$ and $\chi(s_H) = \det(s_H)^{-a_H(\chi)}$. Let

$$Q_\chi := \prod_{H \in \mathcal{A}} \alpha_H^{a_H(\chi)}.$$

The polynomial Q_χ is uniquely determined up to a nonzero scalar multiple by the group G. We call Q_χ the *Stanley polynomial* for χ because R. Stanley [17] proved that every χ-invariant polynomial is a product of Q_χ and an invariant polynomial:

Proposition 5 (Stanley).

$$(\Omega^0)^\chi = R\, Q_\chi.$$

Since *vol* is $(\det^{-1})$-invariant, the following proposition follows immediately.

Proposition 6 (Stanley).

$$(\Omega^n)^\chi = R\, Q_{\chi\cdot\det}\, vol.$$

Recall that χ_{o} is the trivial character, and note that $Q_{\chi_{\text{o}}} = 1$. An important one-dimensional character is the $\det^{-1}$ character defined by $\chi(g) := \det^{-1}(g)$. Note that when G is a Coxeter group, the $\det^{-1}$ character and the det character are identical since the reflections all have determinant -1. The polynomial Q defining the reflection arrangement is $(\det^{-1})$-invariant, and in fact,

$$Q_{\det^{-1}} = \prod_{H \in \mathcal{A}} \alpha_H = Q.$$

R. Steinberg [18] proved

Proposition 7. *Given basic invariants $f_1, \ldots, f_n$ for G,*

$$Q_{\det} = \prod_{H \in \mathcal{A}} \alpha_H^{o(s_H)-1}$$

is the determinant of the Jacobian matrix $\left\{ \frac{\partial}{\partial x_i} f_j \right\}$ up to a nonzero scalar multiple.

Chapter 5

Basic Derivations

The $\det^{-1}$ character of our reflection group plays a primary role. It is often the *only* nontrivial multiplicative character, and as we saw in Chapter 4, the corresponding hyperplane arrangement is defined by $Q_{\det^{-1}}$. Forms that are ($\det^{-1}$)-invariant have been studied in the context of Coxeter groups (see [16], for example), where they are called *anti-invariant*. We will also call them *anti-invariant*, although of course the group doesn't always multiply these forms by -1 (the det and $\det^{-1}$ characters often differ in complex groups). We will prove the main result on semi-invariants in Chapter 6 by exploiting the existence of some very special 1-forms that are ($\det^{-1}$)-invariant. We construct these forms now using a result about invariant derivations called Saito's Criterion.

5.1 Saito's Criterion

Let $\mathcal{A}'$ be an arbitrary arrangement in $\mathbb{C}^n$ defined by Q'. Define the *module of* $\mathcal{A}'$*-derivations* by

$$D(Q') := \{\theta \in \Upsilon^1 : \ \theta(Q') \in Q'S\}.$$

If $D(Q')$ is a free module over S, then $\mathcal{A}'$ is called a *free arrangement.* The following criterion was shown by K. Saito [10] in 1981:

Theorem 8 (Saito's Criterion). *Derivations* $\phi_1, \ldots, \phi_n$ *generate* $D(\mathcal{A}')$ *over* S *if and only if* $\det M(\phi_1, \ldots, \phi_n) \doteq Q'$.

H. Terao [19] showed a special relationship between invariant derivations and $\mathcal{A}$-derivations for reflection arrangements in 1987:

Theorem 9.

$$D(\mathcal{A}) \simeq S \otimes_R (\Upsilon^1)^G.$$

Corollary 4 states that $(\Upsilon^1)^G$ is free of rank n over R. Hence, $D(\mathcal{A})$ is free over S by the above theorem, and the reflection arrangement $\mathcal{A}$ is a free arrangement. Saito's Criterion has the following corollary (see [9], Corollary 6.61).

Corollary 10. *There exist n homogeneous invariant derivations $\phi_1, \ldots, \phi_n$ that generate $(\Upsilon^1)^G$ over R and whose coefficient matrix $M(\phi_1, \ldots, \phi_n)$ has determinant $Q = Q_{\det^{-1}}$.*

We call the ϕ_i in the last theorem *basic derivations.* For Coxeter Groups, one may choose the ϕ_i from the exterior derivatives of the basic invariants: $\phi_i = \sum_j \frac{\partial f_i}{\partial x_j} \partial x_j$.

5.2 Correspondence between Semi-invariants

We explain how to dualize an invariant derivation into an anti-invariant differential form: For $I \in \mathcal{I}^p$, define $\sigma(I)$ $(= \pm 1)$ by $dx_I \wedge dx_{I^c} = \sigma(I)\, vol$. Let $\Psi : \Upsilon^p \longrightarrow \Omega^{n-p}$ be the unique S-linear map satisfying $k\, \partial x_I \mapsto k\, \sigma(I)\, dx_{I^c}$ for $k \in S$ and $I \in \mathcal{I}^p$. The map Ψ is an module-isomorphism. In fact, Ψ takes χ-invariants to $(\chi \cdot \det^{-1})$-invariants:

Proposition 11. *The map Ψ induces a R-module isomorphism:*

$$(\Upsilon^p)^\chi \;\simeq\; (\Omega^{n-p})^{\chi \cdot \det^{-1}}.$$

Proof: For any matrix A, let A^i_j denote the entry in the i^{th} row and the j^{th} column, and let $A_{K,I}$ denote the submatrix of A consisting of rows $K_1, \ldots, K_p$ and columns $I_1, \ldots, I_p$, for $K, I \in \mathcal{I}^p$. Jacobi's theorem on the minors of the adjugate (1834) gives us a formula for the minors of A in terms of the complementary minors of $A^* = (A^t)^{-1}$:

$$\det(A^*_{K,I}) \;=\; \det(A^{-1})\;\; \sigma(K)\,\sigma(I)\;\; \det(A_{K^c,I^c}).$$

Suppose that $A = g^{-1} \in G$. Then as $\bar{A} = A^*$,

$$\begin{aligned}
\Psi(g\partial x_K) &= \Psi\left(\sum_i \bar{A}_i^{K_1}\,\partial x_i \wedge \cdots \wedge \sum_i \bar{A}_i^{K_p}\,\partial x_i\right) \\
&= \Psi\left(\sum_{I\in\mathcal{I}^p} \det(A^*_{K,I})\ \partial x_I\right) \\
&= \sum_{I\in\mathcal{I}^p} \det(A^*_{K,I})\ \sigma(I)\ dx_{I^c} \\
&= \sum_{I\in\mathcal{I}^p} \det(A^{-1})\ \sigma(K)\,\sigma(I)\,\sigma(I)\ \det(A_{K^c,I^c})\ dx_{I^c} \\
&= \sum_{I\in\mathcal{I}^p} \det(A^{-1})\ \sigma(K)\ \det(A_{K^c,I^c})\ dx_{I^c} \\
&= \det(A^{-1})\ \sigma(K)\ \left(\sum_i A_i^{K^c_1}\,dx_i \wedge \cdots \wedge \sum_i A_i^{K^c_{n-p}}\,dx_i\right) \\
&= \det(g)\ \sigma(K)\ g(dx_{K^c}) \\
&= \det(g)\ g\,(\sigma(K)\ dx_{K^c}) \\
&= \det(g)\ g(\Psi(\partial x_K)).
\end{aligned}$$

Thus, $g(\Psi(\theta)) = (\det g)^{-1}\,\Psi(g\theta) = (\det g)^{-1}\,\chi(g)\,\Psi(g)$ for any θ in Υ^χ. □

Although we showed the proposition directly from Jacobi's theorem on the minors of the adjugate, it is really just an application of some familiar facts about $\bigwedge^p V$ and $\bigwedge^p V^*$. The modules Υ^p and Ω^p are natural S-duals of each other. The map Φ is better known as the Hodge-Star operator: we can define $\Psi(\theta)$ as the form ω that satisfies $\omega(\alpha) = (\theta \wedge \alpha)(\partial x_1 \wedge \cdots \wedge \partial x_n)^{-1} \in \mathbb{C}$ for each α in Υ^{n-p}. The proposition then follows from the fact that $\omega(\alpha) = (g\omega)(g\alpha)$ and $g(\partial x_1 \wedge \cdots \wedge \partial x_n) = \det g\ \partial x_1 \wedge \cdots \wedge \partial x_n$.

5.3 Basic Anti-invariant Forms

We now construct differential 1-forms from the basic derivations using the map Ψ. We are particularly eager to know the implication of Saito's Criterion for our constructed 1-forms, and find the coefficient matrix explicitly.

Recall that we named our basic derivations ϕ_i and $\det M(\phi_1, \ldots, \phi_n) = Q$. Define $\mu_i := (-1)^{n-i}\,\Psi(\phi_1 \wedge \cdots \wedge \hat{\phi}_i \wedge \cdots \wedge \phi_n)$. Since each ϕ_i is invariant, $\phi_1 \wedge \cdots \wedge \hat{\phi}_i \wedge \cdots \wedge \phi_n$

is also invariant, and hence $\mu_i \in (\Omega^1)^{\det^{-1}}$. Wedging corresponds to taking subdeterminants and $\sigma(\{1,\dots,\hat{j},\dots,n\}) = n-j$. Hence, the ij^{th} entry of $M(\mu_1,\dots,\mu_n)$ is just $(-1)^{(n-i)+(n-j)} = (-1)^{i+j}$ times the subdeterminant of $M(\phi_1,\dots,\phi_n)$ obtained by deleting the i^{th} row and the j^{th} column. Thus, $M(\mu_1,\dots,\mu_n) = QM^*(\phi_1,\dots,\phi_n)$, and

$$\begin{aligned}
\det M(\mu_1,\dots,\mu_n) &= Q^n \ \det M^{-1}(\phi_1,\dots,\phi_n) \\
&= Q^n \ Q^{-1} \\
&= Q^{n-1}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\mu_1 \wedge \cdots \wedge \mu_n &= \det M(\mu_1,\dots,\mu_n) \ \ vol \\
&= Q^{n-1} \ vol.
\end{aligned}$$

We call the μ_i *basic anti-invariant forms.* We will see in the next chapter that these forms deserve the name.

Chapter 6

Exterior Algebra

In this chapter, we prove that semi-invariants enjoy a special freedom. We show that the modules Ω^χ and Υ^χ are free over R on exactly n generators. This result will completely explain the structure of the set of semi-invariants. We restrict our attention to Ω^χ in this chapter and give similar results for Υ^χ in the next. Throughout this chapter, we will use the word *form* to mean *differential form*. Most of the results here will appear in [12].

6.1 χ-wedging

The first step in understanding the structure of Ω^χ is to define a multiplication. Observe that Ω^χ is not closed under the exterior product! We will use the next lemma to show that Q_χ divides the exterior product of any two χ-invariant forms, which will allow us to define a multiplication. The lemma unwinds the definitions of Q_χ and the group action in a helpful coordinate system. Recall that

$$Q_\chi = \prod_{H \in \mathcal{A}} \alpha_H^{a_H(\chi)}$$

and that s_H is a reflection in G of maximal order that fixes H pointwise.

Lemma 12. *Suppose that μ is a χ-invariant p-form. Fix a hyperplane $H \in \mathcal{A}$ and let $a = a_H(\chi)$. Choose coordinates in which $x_1 = \alpha_H$ and s_H is diagonal, and write*

$$\mu = \sum_{I \in \mathcal{I}^p} \mu_I \; dx_{I_1} \wedge \cdots \wedge dx_{I_p}$$

in these coordinates. Then x_1^{a-1} divides μ_I whenever $I_1 = 1$ and x_1^a divides μ_I whenever $I_1 \neq 1$, for each $I = \{I_1, \ldots, I_p\} \in \mathcal{I}^p$.

Proof: Let $s = s_H$. In the given coordinates,

$$s = \begin{pmatrix} \rho & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

where ρ is the determinant of s. The group element s acts by $s^{-1}dx_i = d(x_i \circ s)$. Hence $s^{-1}dx_i = dx_i$ whenever $i \neq 1$, and $s^{-1}dx_1 = \rho \; dx_1$.

Let $I = \{I_1, I_2, ..., I_p\} \in \mathcal{I}^p$. If $I_1 = 1$, then

$$\begin{aligned} s^{-1}(\mu_I \; dx_{I_1} \wedge \cdots \wedge dx_{I_p}) &= (s^{-1}\mu_I) \;\; s^{-1}dx_1 \wedge \cdots \wedge s^{-1}dx_{I_p} \\ &= (\mu_I \circ s) \;\; \rho \quad dx_1 \wedge \cdots \wedge dx_{I_p}. \end{aligned}$$

If $I_1 \neq 1$, then

$$\begin{aligned} s^{-1}(\mu_I \; dx_{I_1} \wedge \cdots \wedge dx_{I_p}) &= (s^{-1}\mu_I) \; s^{-1}dx_{I_1} \wedge \cdots \wedge s^{-1}dx_{I_p} \\ &= (\mu_I \circ s) \qquad dx_{I_1} \wedge \cdots \wedge dx_{I_p}. \end{aligned}$$

But μ is χ-invariant, so $\rho^a \; \mu = \det(s)^a \; \mu = \chi^{-1}(s) \; \mu = s^{-1}\mu$. Thus if $I_1 = 1$, then $\rho^a \; \mu_I = \rho \; \mu_I \circ s$ and $\rho^{a-1} \; \mu_I = \mu_I \circ s$. Hence x_1^{a-1} divides μ_I whenever $I_1 = 1$. Similarly, if $I_1 \neq 1$, $\rho^a \; \mu_I = \mu_I \circ s$ and x_1^a divides μ_I.

□

Lemma 13. *Q_χ divides the exterior product of any two χ-invariant differential forms.*

Proof: Let μ be a χ-invariant p-form and ω be a χ-invariant q-form. We show that each $\alpha_H^{a_H(\chi)}$ divides $\mu \wedge \omega$. Fix $H \in \mathcal{A}$ and let $s = s_H$ and $a = a_H(\chi)$. Assume that $a \neq 0$.

Choose the coordinates from Lemma 12 in which $\alpha_H = x_1$. Write

$$\begin{aligned}\mu &= \sum_{I \in \mathcal{I}^p} \mu_I \; dx_{I_1} \wedge \cdots \wedge dx_{I_p}, \\ \omega &= \sum_{J \in \mathcal{I}^q} \omega_J \; dx_{J_1} \wedge \cdots \wedge dx_{J_q}, \qquad \text{and} \\ \mu \wedge \omega &= \sum_{K \in \mathcal{I}^{p+q}} \gamma_K \; dx_{K_1} \wedge \cdots \wedge dx_{K_{p+q}}\end{aligned}$$

in these coordinates. Then Lemma 12 implies that x_1^a divides μ_I whenever $I_1 \neq 1$ and x_1^a divides ω_J whenever $J_1 \neq 1$.

For $I \in \mathcal{I}^p$ and $J \in \mathcal{I}^q$, the polynomial $\mu_I \, \omega_J$ is divisible by x_1^a given that not both I_1 and J_1 are 1. Each γ_K is either zero or a sum of terms of the form $\pm \mu_I \, \omega_J$ where the multi-indices I and J are disjoint. Hence, x_1^a divides each γ_K and thus $\mu \wedge \omega$. So $\mu \wedge \omega$ is divisible by $\alpha_H^a = \alpha_H^{a_H(\chi)}$. Since H was arbitrary, Q_χ divides $\mu \wedge \omega$.

□

Lemma 13 prompts us to define the following multiplication in Ω^χ: For differential forms μ and ω, define the *χ-wedge* of μ and ω as

$$\mu \curlywedge \omega := \frac{\mu \wedge \omega}{Q_\chi}.$$

If μ and ω are χ-invariant forms, $\mu \curlywedge \omega$ is again χ-invariant, and thus, Lemma 13 implies

Corollary 14. *The R-module Ω^χ is closed under χ-wedging.*

6.2 Criterion

When Solomon showed that Ω^G is generated over R by the df_i as an exterior algebra (Theorem 2), he used the fact that the df_i wedge to $Q \, vol$. We follow ideas in Solomon's original proof to give a condition for n 1-forms to generate Ω^χ.

Proposition 15 (Criterion). *Let $\omega_1, \ldots, \omega_n$ be χ-invariant 1-forms. The forms $\omega_{I_1} \curlywedge \cdots \curlywedge \omega_{I_p}$, for $I \in \mathcal{I}^p$ and $p \geq 0$, generate Ω^χ over R if and only if*

$$\omega_1 \curlywedge \cdots \curlywedge \omega_n \doteq Q_{\chi \cdot \det} \, vol.$$

Proof: If the forms $\omega_{I_1} \curlywedge \cdots \curlywedge \omega_{I_p}$ generate Ω^χ, then they certainly generate $(\Omega^n)^\chi$ and thus $\omega_1 \curlywedge \cdots \curlywedge \omega_n \doteq Q_{\chi\cdot\det}\, vol$ (recall Proposition 6).

Now assume that $\omega_1 \curlywedge \cdots \curlywedge \omega_n \doteq Q_{\chi\cdot\det}\, vol$. The p-forms $\omega_{I_1} \curlywedge \cdots \curlywedge \omega_{I_p}, I \in \mathcal{I}^p$, are χ-invariant by Corollary 14.

We first argue that the forms $\omega_{I_1} \curlywedge \cdots \curlywedge \omega_{I_p}$ generate Ω^χ over the field $F = \mathbb{C}(x_1, \ldots, x_n)$. Since $\omega_1 \curlywedge \cdots \curlywedge \omega_n \neq 0$, $\omega_1 \wedge \cdots \wedge \omega_n \neq 0$, and the forms $\omega_{I_1} \wedge \cdots \wedge \omega_{I_p}$, $I \in \mathcal{I}^p$, are linearly independent over F. If not, there exist rational functions r_I with

$$0 = \sum_{I \in \mathcal{I}^p} r_I\, \omega_{I_1} \wedge \cdots \wedge \omega_{I_p}.$$

Fix $J \in \mathcal{I}^p$ and $J^c \in \mathcal{I}^{n-p}$. Then

$$\begin{aligned} 0 &= \left(\sum_{I \in \mathcal{I}^p} r_I\, \omega_{I_1} \wedge \cdots \wedge \omega_{I_p} \right) \wedge \omega_{J_1^c} \wedge \cdots \wedge \omega_{J_{n-p}^c} \\ &= \pm\, r_J\, \omega_1 \wedge \cdots \wedge \omega_n, \end{aligned}$$

and r_J must be zero. Thus the forms $\omega_{I_1} \curlywedge \cdots \curlywedge \omega_{I_p}, I \in \mathcal{I}^p$, are also linearly independent over F, and thus span the $\binom{n}{p}$-dimensional space

$$\Omega^p(V) := \bigoplus_{I \in \mathcal{I}^p} F\ dx_{I_1} \wedge \cdots \wedge dx_{I_p}.$$

Choose an arbitrary χ-invariant p-form μ. Then there exist rational functions $t_I \in F$ with

$$\mu = \sum_{I \in \mathcal{I}^p} t_I\ \omega_{I_1} \curlywedge \cdots \curlywedge \omega_{I_p}.$$

Fix $J \in \mathcal{I}^p$ and its complementary index J^c. We will show that $t_J \in R$.

By Corollary 14, the n-form $(\omega_{J_1^c} \curlywedge \cdots \curlywedge \omega_{J_{n-p}^c}) \curlywedge \mu$ is χ-invariant. Thus by Proposition 6, there exists a polynomial $f \in R$ with

$$\left(\omega_{J_1^c} \curlywedge \cdots \curlywedge \omega_{J_{n-p}^c}\right) \curlywedge \mu = f\, Q_{\chi\cdot\det}\, vol.$$

On the other hand,

$$
\begin{aligned}
&\left(\omega_{J_1^c} \curlywedge \cdots \curlywedge \omega_{J_{n-p}^c}\right) \curlywedge \mu \\
&\qquad = \left(\omega_{J_1^c} \curlywedge \cdots \curlywedge \omega_{J_{n-p}^c}\right) \curlywedge \sum_{I \in \mathcal{I}^p} t_I \; \omega_{I_1} \curlywedge \cdots \curlywedge \omega_{I_p} \\
&\qquad = \left(Q_\chi^{1-n}\right) \left(\omega_{J_1^c} \wedge \cdots \wedge \omega_{J_{n-p}^c}\right) \wedge \sum_{I \in \mathcal{I}^p} t_I \; \omega_{I_1} \wedge \cdots \wedge \omega_{I_p} \\
&\qquad = \left(Q_\chi^{1-n}\right) \; \pm t_J \, \omega_1 \wedge \cdots \wedge \omega_n \\
&\qquad = \; \pm t_J \, \omega_1 \curlywedge \cdots \curlywedge \omega_n \\
&\qquad \doteq \; \pm t_J \; Q_{\chi \cdot \det} \; vol.
\end{aligned}
$$

Thus $f \, Q_{\chi \cdot \det} \doteq t_J \, Q_{\chi \cdot \det}$, and hence, $t_J \in R$. Since J was arbitrary, μ is in the R-span of $\{\omega_{I_1} \curlywedge \cdots \curlywedge \omega_{I_p}, \; I_p \in \mathcal{I}^p\}$.

□

6.3 Criterion Satisfied

Now that we have a criterion for a set of 1-forms to generate Ω^χ, we wonder if there exist any forms that satisfy the criterion. In the case $\chi = \det^{-1}$, we have already constructed such forms. In Chapter 5, we built $(\det^{-1})$-invariant 1-forms, $\mu_1, \ldots, \mu_n$, whose exterior product is $Q^{n-1}\, vol$. We called the μ_i *basic anti-invariant forms* because they satisfy the criterion in Proposition 15:

$$
\begin{aligned}
\mu_1 \curlywedge \cdots \curlywedge \mu_n &= Q^{1-n} \; \mu_1 \wedge \cdots \wedge \mu_n \\
&= Q^{1-n} \; Q^{n-1} \;\; vol \\
&= 1 \;\; vol \\
&= Q_{\det^{-1} \cdot \det} \;\; vol.
\end{aligned}
$$

The case $\chi = \det^{-1}$ is not just an example. We will use the basic anti-invariant forms to satisfy the criterion in Proposition 15 for arbitrary χ.

We first write $Q_{\chi\cdot\det}$ in terms of Q_χ. Recall that

$$Q_\chi = \prod_{H\in\mathcal{A}} \alpha_H^{a_H(\chi)}.$$

Fix $H \in \mathcal{A}$ with $a_H(\chi) \neq 0$. The exponent $a_H(\chi \cdot \det)$ is the least nonnegative integer satisfying

$$\begin{aligned}\det(s_H)^{-a_H(\chi\cdot\det)} &= (\chi \cdot \det)(s_H)\\ &= \chi(s_H)\ \det(s_H)\\ &= \det(s_H)^{-a_H(\chi)}\ \det(s_H)\\ &= \det(s_H)^{-(a_H(\chi)-1)}.\end{aligned}$$

Hence $a_H(\chi \cdot \det) = a_H(\chi) - 1$. Now fix $H \in \mathcal{A}$ with $a_H(\chi) = 0$. Then

$$\begin{aligned}\det(s_H)^{-a_H(\chi\cdot\det)} &= (\chi \cdot \det)(s_H)\\ &= \chi(s_H)\ \det(s_H)\\ &= \det(s_H)\\ &= \det(s_H)^{-(o(s_H)-1)}.\end{aligned}$$

Hence, $a_H(\chi \cdot \det) = o(s_H) - 1$. Then

$$\begin{aligned}Q_{\chi\cdot\det} &= \prod_{H\in\mathcal{A}} \alpha_H^{a_H(\chi\cdot\det)}\\ &= \prod_{\substack{H\in\mathcal{A}\\ \chi(s_H)\neq 1}} \alpha_H^{a_H(\chi)-1} \prod_{\substack{H\in\mathcal{A}\\ \chi(s_H)=1}} \alpha_H^{o(s_H)-1},\end{aligned}$$

a fact we use to prove the next theorem.

Theorem 16. *If* $\omega_1, \ldots, \omega_n$ *generate* $(\Omega^1)^\chi$ *over* R*, then*

$$\omega_1 \curlywedge \cdots \curlywedge \omega_n \doteq Q_{\chi\cdot\det}\, vol.$$

Proof: By Corollary 14, $\omega_1 \curlywedge \cdots \curlywedge \omega_n$ is a χ-invariant n-form. Then by Proposition 6, there exists a polynomial $f \in R$ with

$$\omega_1 \curlywedge \cdots \curlywedge \omega_n = f\, Q_{\chi\cdot\det}\, vol.$$

We will show that f is a nonzero constant.

First, we show that f is nonzero. Suppose to the contrary that $\omega_1 \curlywedge \cdots \curlywedge \omega_n = 0$. Let M be the coefficient matrix $M(\omega_1, \ldots, \omega_n)$, i.e. $\omega_1 \wedge \cdots \wedge \omega_n = \det M \ vol$. Then $\det M = 0$, and one row of M is a linear combination of the other rows over $F = \mathbb{C}(x_1, \ldots, x_n)$. Multiplying by a least common multiple yields a relation over S:

$$\sum_{i=1}^{n} s_i \, \omega_i = 0.$$

After applying a group element g, multiplying by $\chi^{-1}(g)$, and then averaging over G, we get a relation over R:

$$\begin{aligned} 0 &= \sum_{g \in G} \sum_{i=1}^{n} \chi^{-1}(g) \ g s_i \ g\omega_i \\ &= \sum_{i=1}^{n} \sum_{g \in G} \chi^{-1}(g) \ g s_i \ \chi(g) \ \omega_i \\ &= \sum_{i=1}^{n} \left(\sum_{g \in G} g s_i \right) \ \omega_i . \end{aligned}$$

This contradicts the fact that $(\Omega^1)^\chi$ is free over R with basis $\omega_1, \ldots, \omega_n$. Thus, f is nonzero.

We show that f is constant by showing that f divides two polynomials that are relatively prime. We find these two polynomials by writing two particular sets of χ-invariant 1-forms in terms of the generators $\omega_1, \ldots, \omega_n$. Notice that $\det M = f \ Q_{\chi \cdot \det} \ (Q_\chi)^{n-1}$, since

$$f \, Q_{\chi \cdot \det} \, vol = \omega_1 \curlywedge \cdots \curlywedge \omega_n = (Q_\chi)^{1-n} \, \omega_1 \wedge \cdots \wedge \omega_n = (Q_\chi)^{1-n} \ \det M \, vol.$$

The first set of 1-forms is $Q_\chi df_1, \ldots, Q_\chi df_n$. Since each df_i is invariant, each $Q_\chi \, df_i$ is χ-invariant and hence a combination of $\omega_1, \ldots, \omega_n$ over R. There exists some matrix of coefficients, N, with entries in S, such that

$$\begin{aligned} Q_\chi df_1 \wedge \cdots \wedge Q_\chi df_n &= \det M \ \det N \, vol \\ &= f \, Q_{\chi \cdot \det} \, (Q_\chi)^{n-1} \ \det N \, vol. \end{aligned}$$

But

$$Q_\chi df_1 \wedge \cdots \wedge Q_\chi df_n = (Q_\chi)^n \, Q_{\det} \, vol$$

by Proposition 7. Hence,

$$f \, Q_{\chi\cdot\det} \, \det N = Q_\chi \, Q_{\det}$$

and since $\det N \in S$, f divides $Q_\chi \, Q_{\det} \, (Q_{\chi\cdot\det})^{-1}$.

The second set of 1-forms is $Q_{\chi\cdot\det}\,\mu_1, \ldots, Q_{\chi\cdot\det}\,\mu_n$, where the μ_i are basic anti-invariant forms. Each $Q_{\chi\cdot\det}\,\mu_i$ is χ-invariant (since each μ_i is $(\det^{-1})$-invariant), and thus an R-combination of $\omega_1, \ldots, \omega_n$. There exists a matrix of coefficients, N', with coefficients in S, such that

$$\begin{aligned} Q_{\chi\cdot\det}\,\mu_1 \wedge \cdots \wedge Q_{\chi\cdot\det}\,\mu_n &= \det M \;\det N' \, vol \\ &= f \, Q_{\chi\cdot\det} \, (Q_\chi)^{n-1} \, \det N' \, vol. \end{aligned}$$

But,

$$Q_{\chi\cdot\det}\,\mu_1 \wedge \cdots \wedge Q_{\chi\cdot\det}\,\mu_n = (Q_{\chi\cdot\det})^n \, (Q_{\det^{-1}})^{n-1} \, vol.$$

Hence,

$$f \, Q_{\chi\cdot\det} \, (Q_\chi)^{n-1} \, \det N' = (Q_{\chi\cdot\det})^n \, (Q_{\det^{-1}})^{n-1}.$$

Then since $\det N' \in S$, the polynomial

$$(Q_{\chi\cdot\det})^n \, (Q_{\det^{-1}})^{n-1} \, (Q_{\chi\cdot\det})^{-1} \, (Q_\chi)^{1-n} = (Q_{\chi\cdot\det} Q_{\det^{-1}})^{n-1} \, (Q_\chi)^{1-n}$$

is divisible by f. We show that the two polynomials

$$Q_\chi \, Q_{\det} \, (Q_{\chi\cdot\det})^{-1} \;\text{ and }\; (Q_{\chi\cdot\det} Q_{\det^{-1}})^{n-1} \, (Q_\chi)^{1-n}$$

have no common factors by writing them both in terms of the α_H. We expand factors:

$$Q_\chi = \prod_{\substack{H\in\mathcal{A}\\ \chi(s_H)\neq 1}} \alpha_H^{a_H(\chi)},$$

$$Q_{\det} = \prod_{\substack{H\in\mathcal{A}\\ \chi(s_H)\neq 1}} \alpha_H^{o(s_H)-1} \prod_{\substack{H\in\mathcal{A}\\ \chi(s_H)=1}} \alpha_H^{o(s_H)-1},$$

$$Q_{\chi\cdot\det} = \prod_{\substack{H\in\mathcal{A}\\ \chi(s_H)\neq 1}} \alpha_H^{a_H(\chi)-1} \prod_{\substack{H\in\mathcal{A}\\ \chi(s_H)=1}} \alpha_H^{o(s_H)-1},$$

$$Q_{\det^{-1}} = \prod_{\substack{H\in\mathcal{A}\\ \chi(s_H)\neq 1}} \alpha_H \prod_{\substack{H\in\mathcal{A}\\ \chi(s_H)=1}} \alpha_H .$$

The first polynomial, $Q_\chi\, Q_{\det}\, (Q_{\chi\cdot\det})^{-1}$, simplifies to

$$\prod_{\substack{H\in\mathcal{A}\\ \chi(s_H)\neq 1}} \alpha_H^{o(s_H)} .$$

The factor $(Q_{\chi\cdot\det}\, Q_{\det^{-1}})^{n-1}$ in the second polynomial simplifies to

$$\left(\prod_{\substack{H\in\mathcal{A}\\ \chi(s_H)\neq 1}} \alpha_H^{a_H(\chi)} \prod_{\substack{H\in\mathcal{A}\\ \chi(s_H)=1}} \alpha_H^{o(s_H)} \right)^{n-1},$$

and hence the second polynomial, $(Q_{\chi\cdot\det}\, Q_{\det^{-1}})^{n-1}\, (Q_\chi)^{1-n}$, is just

$$\left(\prod_{\substack{H\in\mathcal{A}\\ \chi(s_H)=1}} \alpha_H^{o(s_H)} \right)^{n-1}.$$

Since f divides both polynomials, f must be constant. Thus, $\omega_1, \ldots, \omega_n$ satisfy the criterion of Proposition 15.

□

Corollary 17. *There exist n χ-invariant 1-forms that freely generate Ω^χ over R via χ-wedging. Thus Ω^χ is isomorphic to an exterior algebra:*

$$(\Omega^p)^\chi = Q_\chi^{1-p} \bigwedge_R^p (\Omega^1)^\chi .$$

Proof: The generators of $(\Omega^1)^\chi$ satisfy Proposition 15 by Theorem 16, and hence generate Ω^χ. We saw in the proof of Theorem 16 that the forms $\omega_{I_1} \curlywedge \cdots \curlywedge \omega_{I_p}$ are independent over F and hence also independent over R. Thus, Ω^χ is freely generated. □

We call the generators from Corollary 17 the *basic* χ*-invariant forms*, or for brevity, *basic* χ*-forms*.

Chapter 7

Results Also Hold for Υ^χ

We showed in the last chapter that Ω^χ has the structure of an exterior algebra, and as such is freely generated by n 1-forms. We expect analogous results for Υ^χ since χ-invariant derivation forms correspond to $(\chi \cdot \det^{-1})$-invariant differential forms (Proposition 11). We define the χ-*wedge* of θ and β in Υ^χ as

$$\frac{\theta \wedge \beta}{Q_\chi}.$$

The proofs of the following results are completely analogous to those in the last chapter, except for two slight changes due to the different group action. We therefore omit the proofs.

The first slight change occurs in the next lemma: notice that whenever $I_1 = 1$, x_1^{a+1} divides θ_I instead of x_1^{a-1} as in the version for Ω^χ.

Lemma 18. *Suppose that θ is a χ-invariant derivation p-form. Fix a hyperplane $H \in \mathcal{A}$ and let $a = a_H(\chi)$. Choose coordinates in which $x_1 = \alpha_H$ and s_H is diagonal, and write*

$$\mu = \sum_{I \in \mathcal{I}^p} \theta_I \; \partial x_{I_1} \wedge \cdots \wedge \partial x_{I_p}$$

in these coordinates. Then x_1^{a+1} divides θ_I whenever $I_1 = 1$ and x_1^a divides θ_I whenever $I_1 \neq 1$, for $I = \{I_1, \ldots, I_p\} \in \mathcal{I}^p$.

Lemma 19. *If θ and β are in Υ^χ, then Q_χ divides $\theta \wedge \beta$.*

Proposition 20. *The R-module Υ^χ is closed under χ-wedging.*

The other slight change occurs below in the criterion for n derivations to generate Υ^χ via χ-wedging: they must χ-wedge to $Q_{\chi\cdot\det^{-1}}\,\partial x_1 \wedge \cdots \wedge \partial x_n$ instead of $Q_{\chi\cdot\det}\,dx_1 \wedge \cdots \wedge dx_n$. The change from $Q_{\chi\cdot\det}$ to $Q_{\chi\cdot\det^{-1}}$ occurs since $dx_1 \wedge \ldots \wedge dx_n$ is $(\det^{-1})$-invariant while $\partial x_1 \wedge \cdots \wedge \partial x_n$ is det-invariant.

Corollary 21. *Forms $\theta_1, \ldots, \theta_n \in (\Upsilon^1)^\chi$ satisfy*

$$\theta_1 \curlywedge \cdots \curlywedge \theta_n \doteq Q_{\chi\cdot\det^{-1}}\,\partial x_1 \wedge \cdots \wedge \partial x_n$$

if and only if they generate Υ^χ over R via χ-wedging.

Corollary 22. *There exist n χ-invariant derivation 1-forms that freely generate Υ^χ over R via χ-wedging. Thus Υ^χ is isomorphic to an exterior algebra:*

$$(\Upsilon^p)^\chi \;=\; Q_\chi^{1-p}\,\bigwedge_R^p (\Upsilon^1)^\chi\,.$$

Chapter 8

Logarithmic Forms

We now discuss a few applications of the previous ideas to logarithmic forms. Some of these applications will appear in [12]. We have previously only considered differential forms with polynomial coefficients, but now consider differential forms with rational functions as coefficients. The S-module of *logarithmic p-forms with poles along* $\mathcal{A}$ (see also [9], p. 124) is defined as

$$\Omega^p(\mathcal{A}) := \left\{ \frac{\omega}{Q_{\det^{-1}}} : \omega \in \Omega^p \text{ and } \omega \wedge d\alpha_H \in \alpha_H \, \Omega^{p+1} \text{ for all } H \in \mathcal{A} \right\}.$$

G. Ziegler [20] extends this definition to *multiarrangements of hyperplanes*, hyperplane arrangements in which each hyperplane is given a positive integer multiplicity. We apply his definitions to our context of reflection groups and semi-invariants: Let $\mathcal{A}_\chi$ be the multiarrangement consisting of each hyperplane $H \in \mathcal{A}$ counted with multiplicity $\alpha_H(\chi)$, i.e., the multiarrangement defined by Q_χ. We define (as in [20]) the module of *logarithmic p-forms* of $\mathcal{A}_\chi$:

$$\Omega^p(\mathcal{A}_\chi) := \left\{ \frac{\omega}{Q_\chi} : \omega \in \Omega^p \text{ and } \omega \wedge d\alpha_H \in \alpha_H^{a_H(\chi)} \, \Omega^{p+1} \text{ for all } H \in \mathcal{A} \right\}.$$

Let

$$\Omega(\mathcal{A}_\chi) := \bigoplus_{p \geq 0} \Omega^p(\mathcal{A}_\chi).$$

We first observe that any χ-invariant form can be turned into a logarithmic form of $\mathcal{A}_\chi$ by merely dividing by Q_χ, and then show that $\Omega(\mathcal{A}_\chi)$ is closed under wedging:

Proposition 23.

$$Q_\chi^{-1}\,\Omega^\chi \subset \Omega(\mathcal{A}_\chi).$$

Proof: Choose ω in $(\Omega^p)^\chi$ and fix $H \in \mathcal{A}$. Choose coordinates from Lemma 12 such that $x_1 = \alpha_H$, $\omega = \sum_{I \in \mathcal{I}^p} \omega_I \, dx_{I_1} \wedge \cdots \wedge dx_{I_p}$, and $x_1^{a_H(\chi)}$ divides ω_I whenever $1 \notin I$. Then $d\alpha_H = dx_1$, and

$$\omega \wedge d\alpha_H = \omega \wedge dx_1 = \sum_{I, 1 \notin I} \omega_I \wedge dx_1.$$

Thus $\omega \wedge d\alpha_H$ is divisible by $x_1^{a_H(\chi)}$ and $\omega \wedge d\alpha_H \in \alpha_H^{a_H(\chi)}\,\Omega^{p+1}$. As H was arbitrary, $\frac{\omega}{Q_\chi} \in \Omega(\mathcal{A}_\chi)$.

□

Proposition 24. $\Omega(\mathcal{A}_\chi)$ *is closed under the exterior product.*

Proof: Let ω/Q_χ and μ/Q_χ be in $\Omega(\mathcal{A}_\chi)$. Fix H in $\mathcal{A}$ and let $a_H(\chi) = a$. Choose coordinates such that $x_1 = \alpha_H$, and write $\omega = \sum_{I \in \mathcal{I}^p} \omega_I \, dx_{I_1} \wedge \cdots \wedge dx_{I_p}$ and $\mu = \sum_{J \in \mathcal{I}^q} \mu_J \, dx_{J_1} \wedge \cdots \wedge dx_{J_q}$ in these coordinates. Since $\omega \wedge dx_1 = \omega \wedge d\alpha_H \in \alpha_H^a \Omega = x_1^a \Omega$, ω_I is divisible by x_1^a as long as $1 \notin I$. Similarly, μ_J is divisible by x_1^a whenever $1 \notin J$. As in the proof of Lemma 13, it follows that Q_χ divides $\omega \wedge \mu$. Whenever $1 \notin I$ and $1 \notin J$, x_1^{2a} divides $\omega_I \mu_J$, and thus

$$\frac{\omega \wedge \mu}{Q_\chi} \wedge dx_1$$

is also divisible by x_1^a. Hence α_H^a divides $(1/Q_\chi)\,\omega \wedge \mu \wedge d\alpha_H$, and as H was arbitrary, $(\omega/Q_\chi) \wedge (\mu/Q_\chi)$ is in $\Omega(\mathcal{A}_\chi)$.

□

A strong relationship holds between χ-invariants and logarithmic forms when $\chi = \det^{-1}$. Recall that $\mathcal{A} = \mathcal{A}_{\det^{-1}}$ and $Q = Q_{\det^{-1}}$. We recall a few facts from *Arrangements of Hyperplanes*: The S-modules $D(\mathcal{A})$ and $\Omega(\mathcal{A})$ are dual by an interior product ([9], Theorem 4.75). This duality and Saito's Criterion imply two propositions ([9], Proposition 4.80, Proposition 4.81):

Proposition 25. *Rational forms* $\eta_1, \ldots, \eta_n$ *freely generate* $\Omega^1(\mathcal{A})$ *over* S *if and only if* $\eta_1 \wedge \cdots \wedge \eta_n \doteq Q^{-1}$ *vol.*

Proposition 26. *If rational forms* $\eta_1, \dots, \eta_n$ *freely generate* $\Omega^1(\mathcal{A})$ *over* S*, then the rational p-forms* $\eta_{I_1} \wedge \cdots \wedge \eta_{I_p}$*,* $I \in \mathcal{I}^p$*, freely generate* $\Omega^p(\mathcal{A})$ *over* S*.*

Combining this with our previous results, we realize that the 1-forms that generate $\Omega^{\det^{-1}}$ over R (the basic anti-invariant forms) also generate $\Omega(\mathcal{A})$ over S:

Corollary 27.

$$\Omega^p(\mathcal{A}) = \frac{1}{Q}\ S \otimes_R (\Omega^p)^{\det^{-1}}.$$

Proof: If $\mu_1, \dots, \mu_n$ are basic anti-invariant forms, then $\mu_1 \wedge \cdots \wedge \mu_n = Q^{n-1}\, vol$. Hence by Propositions 25 and 26, $\Omega^p(\mathcal{A})$ is freely generated by $Q^{-1}\mu_{I_1} \wedge \cdots \wedge Q^{-1}\mu_{I_p}$, $I \in \mathcal{I}^p$. □

The above corollary is shown with a different argument in [13].

Chapter 9

Constructing Generators

Corollary 17 guarantees that all χ-invariant forms can be computed from the basic χ-forms. But how does one find the basic χ-forms? We construct basic χ-forms using a new method called *knocking*. In Chapter 10, we will use knocking to give explicit constructions of the basic χ-forms for the irreducible reflection groups. Essentially, we switch dx_i (or x_i) and $\frac{\partial}{\partial x_i}$ everywhere possible in a form, bar the complex scalars, and let the resulting operator act on polynomials. We use this idea in two ways: We first construct basic derivations from basic invariants, and then construct basic χ-forms from basic derivations and Stanley polynomials. Although basic derivations for the irreducible reflection groups have been computed in [6] and [8], this method seems easier, and the construction of basic χ-forms is new.

9.1 Operators

In Chapter 2, we defined $\partial\left(\sum_{i=1}^n c_i x_i\right) := \sum_{i=1}^n \bar{c_i} \frac{\partial}{\partial x_i}$, where $c_i \in \mathbb{C}$. For each $f = f(x_1, \ldots, x_n) \in S$, we obtain a differential operator $f(\partial x_1, \ldots, \partial x_n)$ by replacing each x_i by $\frac{\partial}{\partial x_i}$ in f and replacing multiplication of x_i by composition of partial derivatives. Further extend ∂ to polynomials of arbitrary degree: For each $f \in S$, define

$$\partial f := \bar{f}(\partial x_1, \ldots, \partial x_n),$$

where $\bar{f}$ is the polynomial obtained by replacing each complex coefficient in f with its complex conjugate.

How does the group act on the set of ∂f? In Chapter 2, we defined the group action on Ω and Υ to commute with d and ∂. Define a group action on the set of operators ∂f $(f \in S)$ by $g\,\partial f := \partial(g\,f)$.

Lemma 28. *For any polynomials f, $P \in S$ and $g \in G$, $(g\,\partial f)P = g(\partial f(g^{-1}P))$. Hence, $g(\partial f P) = \partial(g\,f)(g\,P)$.*

Proof: We use the chain rule and the fact that g is unitary. Since ∂ is anti-linear, it suffices to consider a monomial $f = \alpha\, x_1^{r_1} \dots x_n^{r_n}$ where $\alpha \in \mathbb{C}$. Denote the matrix g by $\{a_j^i\}$ and the matrix g^{-1} by $\{b_j^i\}$. Then $a_j^i = \bar{b}_i^j$ since $g^{-1} = \bar{g}^t$ (g is unitary). For ease with notation, we implicitly sum over any subscript of x other than $1, 2, \dots, n$ in the following calculation:

$$
\begin{aligned}
&g\left(\partial f(g^{-1}P)\right)\\
&\quad= g\left(\bar{\alpha}\ \frac{\partial}{\partial x_1^{r_1} \dots \partial x_n^{r_n}}(P \circ g)\right)\\
&\quad= g\left(\bar{\alpha}\ a_1^{i_1} \dots a_1^{i_{r_1}} \dots\ a_n^{j_1} \dots a_n^{j_{r_n}}\ \frac{\partial P}{\partial x_{i_1} \dots \partial x_{i_{r_1}} \dots\ \partial x_{j_1} \dots \partial x_{j_{r_n}}} \circ g\right)\\
&\quad= \bar{\alpha}\ a_1^{i_1} \dots a_1^{i_{r_1}} \dots\ a_n^{j_1} \dots a_n^{j_{r_n}}\ \frac{\partial P}{\partial x_{i_1} \dots \partial x_{i_{r_1}} \dots\ \partial x_{j_1} \dots \partial x_{j_{r_n}}}\\
&\quad= \bar{\alpha}\ \bar{b}_{i_1}^1 \dots \bar{b}_{i_{r_1}}^1 \dots\ \bar{b}_{j_1}^n \dots \bar{b}_{j_{r_n}}^n\ \frac{\partial P}{\partial x_{i_1} \dots \partial x_{i_{r_1}} \dots\ \partial x_{j_1} \dots \partial x_{j_{r_n}}}\\
&\quad= \partial\left(b_{i_1}^1 \dots b_{i_{r_1}}^1 \dots\ b_{j_1}^n \dots b_{j_{r_n}}^n\ \alpha\ x^{i_1} \dots x^{i_{r_1}} \dots\ x^{j_1} \dots x^{j_{r_n}}\right) P\\
&\quad= \partial(g\,f)P\\
&\quad= g(\partial f P).
\end{aligned}
$$

Hence also, $g(\partial f P) = \partial(g\,f)(g\,P)$. □

9.2 Basic χ-Forms

We now introduce a construction of differential 1-forms from derivations and polynomials that behaves well with respect to semi-invariance. The idea is to replace each ∂x_i with dx_i and each x_i with ∂x_i, bar the complex scalars, and then apply the resulting operator to a polynomial. We will eventually use this construction to combine invariant derivations and Stanley polynomials. For a derivation $\tau = \sum_{i=1}^{n} t^i \ \partial x_i$ and polynomial P, define the differential 1-form

$$\tau \circledcirc P \quad := \quad \sum_{i=1}^{n} \partial t^i P \ dx_i,$$

read τ *knock* P. We call $\sum_{i=1}^{n} \ \partial t^i \ dx_i$ a *lowering operator*, as it lowers degree.

Proposition 29. *Let τ be a ψ-invariant derivation, and P be a χ-invariant polynomial. Then $\tau \circledcirc P$ is a $(\chi \cdot \psi^{-1})$-invariant differential 1-form.*

Proof: Again, let $g = \{a^i_j\}$ and $g^{-1} = \{b^i_j\}$, and note that $a^i_j = \bar{b}^j_i$. Since τ is ψ-invariant and ∂ is anti-linear,

$$\begin{aligned} \psi(g) \ \tau &= g \, \tau \\ &= \sum_i g \, t^i \ \partial(x_i \circ g^{-1}) \\ &= \sum_i g \, t^i \ \partial\left(\sum_j b^i_j \ x_j \right) \\ &= \sum_i g \, t^i \ \sum_j \bar{b}^i_j \ \partial x_j \\ &= \sum_i g \, t^i \ \sum_j a^j_i \ \partial x_j \end{aligned}$$

and

$$\psi(g) \ t^j = \sum_i a^j_i \ g \, t^i. \qquad (\spadesuit)$$

Then by Lemma 28 and the anti-linearity of ∂,

$$\begin{aligned} g(\tau \circledcirc P) &= \sum_i g(\partial t^i P)\ g\,dx_i \\ &= \sum_i \partial(g\,t^i)(g\,P)\ g\,dx_i \\ &= \chi(g) \sum_{i,j} \partial(g\,t^i)P\ b^i_j\ dx_j \\ &= \chi(g) \sum_{i,j} \partial(\bar{b}^i_j\ g\,t^i)P\ dx_j \\ &= \chi(g) \sum_{i,j} \partial(a^j_i\ g\,t^i)P\ dx_j. \end{aligned}$$

But by Equation ♠ above,

$$\begin{aligned} \chi(g) \sum_{i,j} \partial(a^j_i\ g\,t^i)P\ dx_j &= \chi(g) \sum_j \partial(\psi(g)\ t^j)P\ dx_j \\ &= \chi(g)\ \bar{\psi}(g) \sum_j \partial t^j P\ dx_j \\ &= \chi(g)\ \psi^{-1}(g)\quad \tau \circledcirc P. \end{aligned}$$

□

We use the next corollary to construct basic χ-forms.

Corollary 30. *Let θ be a basic derivation. Then $\theta \circledcirc Q_\chi$ is a χ-invariant differential 1-form.*

9.3 Basic Derivations

The last theorem gave a method to construct a semi-invariant 1-form from a derivation and a polynomial. Now let's do the reverse and construct a semi-invariant derivation from a differential 1-form and a polynomial. We replace each dx_i with ∂x_i and each x_i with ∂x_i, bar the complex coefficients, and let the operator act on a polynomial: For a 1-form $\nu = \sum_{i=1}^n v^i dx_i$ and a polynomial P, define the derivation

$$\nu \ominus P \quad := \quad \sum_{i=1}^n \partial v^i P\ \partial x_i,$$

read ν *co-knock* P. Again, we call $\sum_{i=1}^n\ \partial v^i\ \partial x_i$ a lowering operator.

Proposition 31. *Let ν be a ψ-invariant differential 1-form, and P be a χ-invariant polynomial. Then $\nu \ominus P$ is a $(\chi \cdot \psi^{-1})$-invariant derivation.*

Proof: Again, let $g = \{a^i_j\}$ and $g^{-1} = \{b^i_j\}$ and note that $a^i_j = \bar{b}^j_i$. Since ν is ψ-invariant and ∂ is anti-linear,

$$\begin{aligned} \psi(g)\ \nu &= g\,\nu \\ &= \sum_i g\,v^i \left(\sum_j b^i_j\ dx_j \right) \end{aligned}$$

and

$$\psi(g)\ v^j = \sum_i b^i_j\ g\,v^i.$$

Then by Lemma 28 and the anti-linearity of ∂,

$$\begin{aligned} g(\nu \circledcirc P) &= \sum_i g(\partial v^i P)\ g\,\partial x_i \\ &= \sum_i \partial(g\,v^i)(g\,P)\ g\,\partial(x_i \circ g^{-1}) \\ &= \chi(g)\ \sum_i \partial(g\,v^i) P\ \partial \left(\sum_j b^i_j\ x_j \right) \\ &= \chi(g)\ \sum_{i,j} \partial(g\,v^i) P\ \bar{b}^i_j\ \partial x_j \\ &= \chi(g)\ \sum_{i,j} \partial(b^i_j\ g\,v^i) P\ \partial x_j \\ &= \chi(g)\ \sum_j \partial(\psi(g)\ v^j) P\ \partial x_j \\ &= \chi(g)\ \bar{\psi}(g)\ \sum_j \partial v^j P\ \partial x_j \\ &= \chi(g)\ \psi^{-1}(g)\quad \tau \ominus P. \end{aligned}$$

□

We use the next corollary to construct basic derivations.

Corollary 32. *Let f_i and f_j be basic invariants. Then $df_i \ominus f_j$ is an invariant derivation.*

9.4 Remarks

The fact that knocking preserves semi-invariance is not surprising. Recall the $*$ map from Chapter 2, $* : W \to W^* = *(W)$, defined on any finite-dimensional complex vector space W. This map respects tensor products and also symmetric and exterior tensor powers: $(W \otimes W)^* \simeq W^* \otimes W^*$, $S(W \otimes W)^* \simeq S(W^* \otimes W^*)$, $(W \wedge W)^* \simeq W^* \wedge W^*$. Hence it respects the symmetric and exterior tensor powers on V, V^*, V^{**} and V^{***}. The $*$ map also obviously respects the isomorphisms $e_i \leftrightarrow \partial x_i$ ($V \simeq V^{**}$) and $x_i \leftrightarrow dx_i$ ($V^* \simeq V^{***}$) (where e_i is a basis vector of V dual to x_i).

Knocking can be viewed as an application of the $*$ map: pass the $*$ map through symmetric and exterior powers to create an operator. But $g \in G$ also preserves symmetric and exterior powers (by definition of the group actions), and can be "passed through" to individual $v \in V$ and $v^* \in V^*$. We remarked in Chapter 2 that G respects the correspondence $V \leftrightarrow V^*$ given by $*$. Hence, we expect the group to preserve knocking. Since the $*$ map is anti-linear, $\psi^{-1}(g)$ pulls out of a ψ-invariant lowering operator when g acts.

Chapter 10

In Practice

Now that we have a developed a theory of semi-invariants, how to we apply the theory in practice? Chapters 6 and 7 showed that one can build all semi-invariants from a few fundamental forms, and Chapter 9 gave a new method (knocking) for constructing these fundamental forms. We show here how these ideas are used to compute semi-invariants explicitly. Specifically, we show how to find all the semi-invariants for the exceptional irreducible reflection groups. We make some remarks about the irreducible reflection groups and display a reduced form of the Hilbert Series of χ-invariants for each of the exceptional groups. We give explicit constructions of the basic χ-forms and basic derivations for the two-dimensional exceptional groups using the constructions from Chapter 9. We indicate the basic χ-forms for the higher dimensional exceptional groups as well.

10.1 Irreducible Reflection Groups

We remarked in Chapter 1 that every reflection group is either irreducible or the direct product of irreducible components, each of which is itself a reflection group. We thus focus on semi-invariants of the irreducible reflection groups. The irreducible reflection groups consist of three infinite families and thirty-four *exceptional* groups. Shephard and Todd gave each group a serial number: the infinite families are numbered 1 through 3 and the exceptional groups are numbered 4 through 37. The exceptional groups range in dimension from 2 to 8 and are labeled G_4 through G_{37}.

Groups G_4 through G_{22} are two-dimensional and fall into three categories. For any reflection group, we can mod out by the set of scalar matrices to obtain a projective group. Each projective group corresponding to a two-dimensional exceptional group is either the tetrahedral group (size 12), octahedral group (size 24), or icosahedral group (size 60). With an abuse of terminology, we also call the corresponding reflection group tetrahedral, octahedral, or icosahedral. We focus here on the two-dimensional exceptional groups for several reasons. *Most* of the exceptional groups are two-dimensional. These groups lend themselves well to presentation — in fact, *all* of the semi-invariant forms (including basic invariants) for *all* nineteen of the two-dimensional exceptional groups can be computed from just three polynomials! Most of the two-dimensional groups have many multiplicative characters and thus a wealth of semi-invariants. The two-dimensional group G_{19} has thirty multiplicative characters, more than any other exceptional group!

In contrast, the higher dimensional exceptional groups, G_{23} through G_{37}, are lacking in multiplicative characters. The trivial and the $\det^{-1}$ characters are usually the only multiplicative characters. Only four groups have more: G_{25}, G_{26}, G_{28}, and G_{32}. We showed in Chapter 5 how to construct basic anti-invariants from basic derivations, and basic derivations have been computed for the exceptional groups in [6] and [8]. Hence, the basic χ-forms for almost all of these groups are easy to find. The basic χ-forms for the remaining four groups, G_{25}, G_{26}, G_{28}, and G_{32}, can also be computed in a straightforward way — we indicate how in Section 10.3.

Unlike the two-dimensional groups, the projective groups corresponding to groups G_{23} through G_{37} are unique with one exception: both G_{25} and G_{26} are lifts of the Hessian group of order 216. The group G_{24} is the lift of Klein's group of order 168. Group G_{27} is called the *Valentineer group* and its projective group is isomorphic to the alternating group A_6 (see S. Crass's work ([4], [2], and [3]) on this group and the highly symmetrical dynamical systems built from its semi-invariants). Groups G_{35}, G_{36}, and G_{37} are better known as the Coxeter groups E_6, E_7, and E_8.

Recall that we labeled basic invariants f_i and basic derivations ϕ_i. The degrees of the df_i are called the *exponents* of the reflection group, and the degrees of the ϕ_i are called the *coexponents* of the group. When a group is a Shephard or Coxeter group, the degree of df_i (exponent) plus the degree of ϕ_{n-i} (coexponent) is always the degree of f_n. Any group that displays this numerology is called a *duality* group. Most of the exceptional groups are duality groups, and one may ask "why do duality groups exhibit such numerology, and why *don't* other reflection groups?" This question in part motivated this thesis, and certainly motivated the constructions in the last chapter and calculations presented here. There are seven nonduality groups in dimension 2, $G_7, G_{11}, G_{12}, G_{13}, G_{15}, G_{19}$ and G_{22}, and only one higher dimensional nonduality group, G_{31}. In the next sections, we explore the influence of duality on constructing basic χ-forms.

We invite the reader to peruse Table 10.5, which indicates the Hilbert series of χ-invariant differential forms for all of the irreducible reflection groups (excepting the infinite families). We have multiplied each series by $(1-x^{d_1})\dots(1-x^{d_n})$, where d_i is the degree of f_i, to obtain a polynomial called the *reduced Hilbert series* that ignores the contribution of the basic invariants. Corollary 17 implies that the quotient is indeed a polynomial, and that this polynomial factors. The coefficient of $x^i y^j$ is the dimension over R of the space of j-forms of homogeneous polynomial degree i. The symbol ▶ indicates a nonduality group. Nolan Wallach suggested factoring the polynomials with negative exponents, which prompted the idea to knock Stanley polynomials and basic derivations together. The Hilbert series were computed from character tables using a version of Molien's theorem and the software GAP and Mathematica.

10.2 Two-dimensional Groups

We now restrict our attention to the two-dimensional exceptional groups. We compute basic derivations for the duality groups with a simple formula: $\phi_i = df_{n-i} \ominus f_n$. The numerology of a duality group suggests this formula, but perhaps the formula in some sense also sheds light on the numerology. Each basic χ-form of a duality group is either $df_i\, Q_\chi$ or $\phi_i \circledcirc Q_\chi$. With nonduality groups, these constructions may give zero for some i. We then substitute $f_n f_i$ (for some i) for f_n. (This is also the case with G_{31}, the nonduality group in dimension 4.)

For each group G_4 through G_{22}, we list the serial number, size, degrees of basic invariants (exponents $+1$), degrees of basic derivations (coexponents), and degrees of the Stanley polynomials in Table 10.1. We indicate each nonduality group with the symbol ▶ (as before) throughout our tables. The reader should compare results for duality groups and nonduality groups.

Klein [7] explored the invariant theory for the projective tetrahedral, octahedral, and icosahedral groups in detail. There are three important invariant polynomials for each projective group, f, h, and t. Up to a scalar, the polynomial h is the Hessian of f and the polynomial t is the Jacobian of f and h. Table 10.2 gives f, h, and t for the tetrahedral, octahedral, and icosahedral groups explicitly. Any invariant of the corresponding reflection group can be written as a product of the f, h, and t. In Table 10.3, we give the basic invariants in terms of f, h, and t and give the basic derivations in terms of co-knocking f_n with df_i. We write each Stanley polynomial as a product of the f, h, and t in Table 10.4. The Stanley polynomials were found by examining the effect of group generators using Mathematica. Table 10.4 gives the basic χ-forms in terms of knocking Stanley polynomials with basic derivations (we omit the trivial character). Using these tables, the reader can compute any semi-invariant form for a two-dimensional exceptional group from the appropriate polynomial f alone!

10.3 Higher Dimensional Groups and Tables

The basic χ-forms for the rest of the exceptional groups follow the same patterns. We have computed the basic derivations and basic χ-forms for the groups G_{25}, G_{26}, G_{28}, and G_{32}. The other higher dimensional groups carry only the trivial and $\det^{-1}$ multiplicative characters. Hence, their semi-invariants can be constructed from the basic derivations computed in [6] and [8]. We have also computed basic derivations and basic χ-forms for the nonduality group G_{31}. We indicate the results here, but forego the explicit calculations.

As with the two-dimensional groups, basic derivations for the duality groups can be computed with the simple formula: $\phi_i = df_{n-i} \ominus f_n$. Again, each basic χ-form of a duality group is either $df_i\, Q_\chi$ or $\phi_i \circledcirc Q_\chi$. The reduced Hilbert series in Table 10.5 indicates how to choose $df_i\, Q_\chi$ or $\phi_i \circledcirc Q_\chi$. Each factor $(1 + x^d y)$ in the reduced series corresponds to $df_i\, Q_\chi$, where d is the degree of df_i (exponent). Each factor $(1 + x^{-e} y)$ corresponds to $\phi_i \circledcirc Q_\chi$, where e is the degree of ϕ_i (coexponent). Again, with our nonduality group G_{31}, the construction $\phi_i = df_{n-i} \ominus f_n$ gives zero for one i. We then substitute f_n^2 for f_n.

The following tables can be used to construct the semi-invariant forms of all the exceptional groups.

Table 10.1: Degrees

Tetrahedral Groups

	Group	Size	Exponents +1 deg f_i	Coexponents deg ϕ_i	Stanley Polynomials deg Q_i
	4	24	$4, 6$	$1, 3$	$8, 4$
	5	72	$6, 12$	$1, 7$	$12, 12, 4, 16, 4, 8, 8, 8$
	6	48	$4, 12$	$1, 9$	$6, 10, 14, 4, 8$
▶	7	144	$12, 12$	$1, 13$	$6, 18, 18, 12, 12, 22, 10, 10, 16, 4, 4, 14, 14, 14, 8, 8, 8$

Octohedral Groups

	Group	Size	Exponents +1 deg f_i	Coexponents deg ϕ_i	Stanley Polynomials deg Q_i
	8	96	$8, 12$	$1, 5$	$12, 6, 18$
	9	192	$8, 24$	$1, 17$	$12, 12, 24, 30, 6, 18, 18$
	10	288	$12, 24$	$1, 13$	$12, 18, 6, 26, 14, 20, 8, 34, 22, 28, 16$
▶	11	576	$24, 24$	$1, 25$	$12, 12, 24, 30, 6, 18, 18, 38, 14, 26, 26, 20, 20, 32, 8, 46, 22, 34, 34, 28, 28, 40, 16$
▶	12	48	$6, 8$	$1, 11$	12
▶	13	96	$8, 12$	$1, 17$	$18, 6, 12$
	14	144	$6, 24$	$1, 19$	$12, 20, 8, 28, 16$
▶	15	288	$12, 24$	$1, 25$	$6, 18, 12, 22, 34, 28, 16, 14, 26, 20, 8$

Icosahedral Groups

	Group	Size	Exponents +1 deg f_i	Coexponents deg ϕ_i	Stanley Polynomials deg Q_i
	16	600	$20, 30$	$1, 11$	$36, 12, 48, 24$
	17	1200	$20, 60$	$1, 41$	$30, 78, 66, 54, 42, 48, 36, 24, 12$
	18	1800	$30, 60$	$1, 31$	$24, 48, 12, 36, 20, 44, 68, 32, 56, 40, 64, 88, 52, 76$
▶	19	3600	$60, 60$	$1, 61$	$30, 78, 66, 54, 42, 48, 36, 24, 12, 50, 98, 86, 74, 62, 20, 68, 56, 44, 32, 70, 118, 106, 94, 82, 40, 88, 76, 64, 52$
	20	360	$12, 30$	$1, 19$	$40, 20$
	21	720	$12, 60$	$1, 49$	$30, 70, 40, 50, 20$
▶	22	240	$12, 20$	$1, 29$	30

Table 10.2: Klein's Invariants

Tetrahedral Groups	
f	$x_1^4 + 2i\sqrt{3}\,x_1^2 x_2^2 + x_2^4$
h	$x_1^4 - 2i\sqrt{3}\,x_1^2 x_2^2 + x_2^4$
t	$x_1 x_2 (x_1^4 - x_2^4)$

Octohedral Groups	
f	$x_1 x_2 (x_1^4 - x_2^4)$
h	$x_1^8 + 14 x_1^4 x_2^4 + x_2^8$
t	$x_1^{12} - 33 x_1^8 x_2^4 - 33 x_1^4 x_2^8 + x_2^{12}$

Icosahedral Groups	
f	$x_1 x_2 (x_1^{10} + 11 x_1^5 x_2^5 - x_2^{10})$
h	$x_1^{20} - 228 x_1^{15} x_2^5 + 494 x_2^{10} x_2^{10} + 228 x_1^5 x_2^{15} + x_2^{20}$
t	$x_1^{30} + 522 x_1^{25} x_2^5 - 10005 x_1^{20} x_2^{10} - 10005 x_1^{10} x_2^{20} - 522 x_1^5 x_2^{25} + x_2^{30}$

Table 10.3: Basic Invariants and Derivations

Tetrahedral Groups

	Group	Size	Exponents +1 deg f_i	Basic Inv. f_i	Coexponents deg ϕ_i	Basic Der. ϕ_i
	4	24	$4, 6$	f t	$1, 3$	$df_2 \ominus f_2$ $df_1 \ominus f_2$
	5	72	$6, 12$	t fff	$1, 7$	$df_2 \ominus f_2$ $df_1 \ominus f_2$
	6	48	$4, 12$	tt f	$1, 9$	$df_2 \ominus f_2$ $df_1 \ominus f_2$
▶	7	144	$12, 12$	fff tt	$1, 13$	$df_2 \ominus f_2$ $df_1 \ominus f_2^2$

Octohedral Groups

	Group	Size	Exponents +1 deg f_i	Basic Inv. f_i	Coexponents deg ϕ_i	Basic Der. ϕ_i
	8	96	$8, 12$	h t	$1, 5$	$df_2 \ominus f_2$ $df_1 \ominus f_2$
	9	192	$8, 24$	h tt	$1, 17$	$df_2 \ominus f_2$ $df_1 \ominus f_2$
	10	288	$12, 24$	t hhh	$1, 13$	$df_2 \ominus f_2$ $df_1 \ominus f_2$
▶	11	576	$24, 24$	hhh tt	$1, 25$	$df_2 \ominus f_2$ $df_1 \ominus f_2^2$
▶	12	48	$6, 8$	f h	$1, 11$	$df_2 \ominus f_2$ $df_1 \ominus f_2^2$
▶	13	96	$8, 12$	h ff	$1, 17$	$df_2 \ominus f_2$ $df_1 \ominus f_2^2$
	14	144	$6, 24$	f tt	$1, 19$	$df_2 \ominus f_2$ $df_1 \ominus f_2$
▶	15	288	$12, 24$	ff tt	$1, 25$	$df_2 \ominus f_2$ $df_2 \ominus f_2^2$

Icosahedral Groups

	Group	Size	Exponents +1 deg f_i	Basic Inv. f_i	Coexponents deg ϕ_i	Basic Der. ϕ_i
	16	600	$20, 30$	h t	$1, 11$	$df_2 \ominus f_2$ $df_1 \ominus f_2$
	17	1200	$20, 60$	h tt	$1, 41$	$df_2 \ominus f_2$ $df_1 \ominus f_2$
	18	1800	$30, 60$	t hhh	$1, 31$	$df_2 \ominus f_2$ $df_1 \ominus f_2$
▶	19	3600	$60, 60$	hhh tt	$1, 61$	$df_1 \ominus f_2$ $df_1 \ominus f_2^2$
	20	360	$12, 30$	f t	$1, 19$	$df_2 \ominus f_2$ $df_1 \ominus f_2$
	21	720	$12, 60$	f tt	$1, 49$	$df_2 \ominus f_2$ $df_1 \ominus f_2$
▶	22	240	$12, 20$	f h	$1, 29$	$df_2 \ominus f_2$ $df_1 \ominus f_2^2$

Table 10.4: Basic χ-forms

Tetrahedral Groups						
Group	Character	Stanley Poly. Q_χ	Degree	Basic χ-forms		Degrees
$\mathbf{G_4}$:	χ_1	hh	8	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc Q_\chi$	$7, 5$
	χ_2	h	4	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc Q_\chi$	$3, 1$
$\mathbf{G_5}$:	χ_1	fhh	12	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc Q_\chi$	$11, 5$
	χ_2	ffh	12	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc Q_\chi$	$11, 5$
	χ_3	f	4	$\phi_1 \circledcirc Q_\chi$	$df_1 Q_\chi$	$3, 9$
	χ_4	$ffhh$	16	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc Q_\chi$	$15, 9$
	χ_5	h	4	$\phi_1 \circledcirc Q_\chi$	$df_1 Q_\chi$	$3, 9$
	χ_6	hh	8	$\phi_1 \circledcirc Q_\chi$	$df_1 Q_\chi$	$7, 13$
	χ_7	ff	8	$\phi_1 \circledcirc Q_\chi$	$df_1 Q_\chi$	$7, 13$
	χ_8	fh	8	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc Q_\chi$	$7, 1$
$\mathbf{G_6}$:	χ_1	t	6	$\phi_1 \circledcirc Q_\chi$	$df_1 Q_\chi$	$5, 9$
	χ_2	ht	10	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc Q_\chi$	$9, 1$
	χ_3	hht	14	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc Q_\chi$	$13, 5$
	χ_4	h	4	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc Q_\chi$	$3, 7$
	χ_5	hh	8	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc Q_\chi$	$7, 11$
▶ $\mathbf{G_7}$:	χ_1	t	6	$\phi_1 \circledcirc Q_\chi$	$df_1 Q_\chi$	$5, 17$
	χ_2	$ffht$	18	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc Q_\chi$	$17, 5$
	χ_3	$fhht$	18	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc Q_\chi$	$17, 5$
	χ_4	ffh	12	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc f_2 Q_\chi$	$11, 11$
	χ_5	fhh	12	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc f_2 Q_\chi$	$11, 11$
	χ_6	$ffhht$	22	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc Q_\chi$	$21, 9$
	χ_7	ft	10	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc f_2 Q_\chi$	$9, 9$
	χ_8	ht	10	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc f_2 Q_\chi$	$9, 9$
	χ_9	$ffhh$	16	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc f_2 Q_\chi$	$15, 15$
	χ_{10}	f	4	$\phi_1 \circledcirc Q_\chi$	$df_2 Q_\chi$	$3, 15$
	χ_{11}	h	4	$\phi_1 \circledcirc Q_\chi$	$df_2 Q_\chi$	$3, 15$
	χ_{12}	fht	14	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc Q_\chi$	$13, 1$
	χ_{13}	hht	14	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc f_2 Q_\chi$	$13, 13$
	χ_{14}	fft	14	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc f_2 Q_\chi$	$13, 13$
	χ_{15}	fh	8	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc f_2 Q_\chi$	$7, 7$
	χ_{16}	ff	8	$\phi_1 \circledcirc Q_\chi$	$df_2 Q_\chi$	$7, 19$
	χ_{17}	hh	8	$\phi_1 \circledcirc Q_\chi$	$df_2 Q_\chi$	$7, 19$

Table 10.4: Continued

Octohedral Groups						
Group	Character	Stanley Poly. Q_χ	Degree	Basic χ-forms		Degrees
$\mathbf{G_8}$:	χ_1	ff	12	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc Q_\chi$	$11, 7$
	χ_2	f	6	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc Q_\chi$	$5, 1$
	χ_3	fff	8	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc Q_\chi$	$17, 13$
$\mathbf{G_9}$:	χ_1	ff	12	$\phi_1 \circledcirc Q_\chi$	$df_1 Q_\chi$	$11, 19$
	χ_2	t	12	$\phi_1 \circledcirc Q_\chi$	$df_1 Q_\chi$	$11, 19$
	χ_3	fft	24	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc Q_\chi$	$23, 7$
	χ_4	$ffft$	30	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc Q_\chi$	$29, 13$
	χ_5	f	6	$\phi_1 \circledcirc Q_\chi$	$df_1 Q_\chi$	$5, 13$
	χ_6	fff	18	$\phi_1 \circledcirc Q_\chi$	$df_1 Q_\chi$	$17, 25$
	χ_7	ft	18	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc Q_\chi$	$17, 1$
$\mathbf{G_{10}}$:	χ_1	ff	12	$\phi_1 \circledcirc Q_\chi$	$df_1 Q_\chi$	$11, 23$
	χ_2	fff	18	$\phi_1 \circledcirc Q_\chi$	$df_1 Q_\chi$	$17, 29$
	χ_3	f	6	$\phi_1 \circledcirc Q_\chi$	$df_1 Q_\chi$	$5, 17$
	χ_4	$fffh$	26	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc Q_\chi$	$25, 13$
	χ_5	fh	14	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc Q_\chi$	$13, 1$
	χ_6	ffh	20	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc Q_\chi$	$19, 7$
	χ_7	h	8	$\phi_1 \circledcirc Q_\chi$	$df_1 Q_\chi$	$7, 19$
	χ_8	$fffhh$	34	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc Q_\chi$	$33, 21$
	χ_9	fhh	22	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc Q_\chi$	$21, 9$
	χ_{10}	$ffhh$	28	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc Q_\chi$	$27, 15$
	χ_{11}	hh	16	$\phi_1 \circledcirc Q_\chi$	$df_1 Q_\chi$	$15, 27$

Table 10.4: Continued

Octohedral Groups (cont.)						
Group	Character	Stanley Poly. Q_χ	Degree	Basic χ-forms		Degrees
▶ $\mathbf{G_{11}}$:	χ_1	t	12	$\phi_1 \circledcirc Q_\chi$	$df_1 Q_\chi$	$11, 35$
	χ_2	ff	12	$\phi_1 \circledcirc Q_\chi$	$df_2 Q_\chi$	$11, 35$
	χ_3	fft	24	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc f_2 Q_\chi$	$23, 23$
	χ_4	$ffft$	30	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc f_2 Q_\chi$	$29, 29$
	χ_5	f	6	$\phi_1 \circledcirc Q_\chi$	$df_2 Q_\chi$	$5, 29$
	χ_6	fff	18	$\phi_1 \circledcirc Q_\chi$	$df_2 Q_\chi$	$17, 41$
	χ_7	ft	18	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc f_2 Q_\chi$	$17, 17$
	χ_8	$fffht$	38	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc Q_\chi$	$37, 13$
	χ_9	fh	14	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc f_2 Q_\chi$	$13, 13$
	χ_{10}	$fffh$	26	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc f_2 Q_\chi$	$25, 25$
	χ_{11}	fht	26	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc Q_\chi$	$25, 1$
	χ_{12}	ffh	20	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc f_2 Q_\chi$	$19, 19$
	χ_{13}	ht	20	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc f_2 Q_\chi$	$19, 19$
	χ_{14}	$ffht$	32	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc Q_\chi$	$31, 7$
	χ_{15}	h	8	$\phi_1 \circledcirc Q_\chi$	$df_2 Q_\chi$	$7, 31$
	χ_{16}	$fffhht$	46	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc Q_\chi$	$45, 21$
	χ_{17}	fhh	22	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc f_2 Q_\chi$	$21, 21$
	χ_{18}	$fffhh$	34	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc f_2 Q_\chi$	$33, 33$
	χ_{19}	$fhht$	34	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc Q_\chi$	$33, 9$
	χ_{20}	hht	28	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc f_2 Q_\chi$	$27, 27$
	χ_{21}	$ffhh$	28	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc f_2 Q_\chi$	$27, 27$
	χ_{22}	$ffhht$	40	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc Q_\chi$	$39, 15$
	χ_{23}	hh	16	$\phi_1 \circledcirc Q_\chi$	$df_2 Q_\chi$	$15, 39$
▶ $\mathbf{G_{12}}$:	χ_1	t	12	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc Q_\chi$	$11, 1$
▶ $\mathbf{G_{13}}$:	χ_1	ft	18	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc Q_\chi$	$17, 1$
	χ_2	f	6	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc Q_\chi$	$5, 13$
	χ_3	t	12	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc Q_\chi$	$11, 7$
$\mathbf{G_{14}}$:	χ_1	t	12	$\phi_1 \circledcirc Q_\chi$	$df_1 Q_\chi$	$11, 17$
	χ_2	ht	20	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc Q_\chi$	$19, 1$
	χ_3	h	8	$\phi_1 \circledcirc Q_\chi$	$df_1 Q_\chi$	$7, 13$
	χ_4	hht	28	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc Q_\chi$	$27, 9$
	χ_5	hh	16	$\phi_1 \circledcirc Q_\chi$	$df_1 Q_\chi$	$15, 21$

Table 10.4: Continued

Octohedral Groups (cont.)						
Group	Character	Stanley Poly. Q_χ	Degree	Basic χ-forms		Degrees
▶ $\mathbf{G_{15}}$:	χ_1	f	6	$\phi_1 \circledcirc Q_\chi$	$df_2 Q_\chi$	$5, 29$
	χ_2	ft	18	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc f_2 Q_\chi$	$17, 17$
	χ_3	t	12	$\phi_1 \circledcirc Q_\chi$	$df_1 Q_\chi$	$11, 23$
	χ_4	hhf	22	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc f_2 Q_\chi$	$21, 21$
	χ_5	$fhht$	34	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc Q_\chi$	$33, 9$
	χ_6	hht	28	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc f_1 Q_\chi$	$27, 15$
	χ_7	hh	16	$\phi_1 \circledcirc Q_\chi$	$df_1 Q_\chi$	$15, 27$
	χ_8	fh	14	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc f_2 Q_\chi$	$13, 13$
	χ_9	fht	26	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc f_1 Q_\chi$	$25, 1$
	χ_{10}	ht	20	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc Q_\chi$	$19, 7$
	χ_{11}	h	8	$\phi_1 \circledcirc Q_\chi$	$df_1 Q_\chi$	$7, 19$

Table 10.4: Continued

Icosahedral Groups						
Group	Character	Stanley Poly. Q_χ	Degree	Basic χ-forms		Degrees
$\mathbf{G_{16}}$:	χ_1	fff	36	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc Q_\chi$	$35, 25$
	χ_2	f	12	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc Q_\chi$	$11, 1$
	χ_3	$ffff$	48	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc Q_\chi$	$37, 47$
	χ_4	ff	24	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc Q_\chi$	$23, 13$
$\mathbf{G_{17}}$:	χ_1	t	30	$\phi_1 \circledcirc Q_\chi$	$df_1 Q_\chi$	$29, 49$
	χ_2	$fffft$	78	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc Q_\chi$	$77, 37$
	χ_3	$ffft$	66	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc Q_\chi$	$65, 25$
	χ_4	fft	54	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc Q_\chi$	$53, 13$
	χ_5	ft	42	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc Q_\chi$	$41, 1$
	χ_6	$ffff$	48	$\phi_1 \circledcirc Q_\chi$	$df_1 Q_\chi$	$47, 67$
	χ_7	fff	36	$\phi_1 \circledcirc Q_\chi$	$df_1 Q_\chi$	$35, 55$
	χ_8	ff	24	$\phi_1 \circledcirc Q_\chi$	$df_1 Q_\chi$	$23, 43$
	χ_9	f	12	$\phi_1 \circledcirc Q_\chi$	$df_1 Q_\chi$	$11, 31$
$\mathbf{G_{18}}$:	χ_1	ff	24	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc Q_\chi$	$23, 53$
	χ_2	$ffff$	48	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc Q_\chi$	$47, 77$
	χ_3	f	12	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc Q_\chi$	$11, 41$
	χ_4	fff	36	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc Q_\chi$	$35, 65$
	χ_5	h	20	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc Q_\chi$	$19, 49$
	χ_6	ffh	44	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc Q_\chi$	$43, 13$
	χ_7	$ffffh$	68	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc Q_\chi$	$67, 37$
	χ_8	fh	32	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc Q_\chi$	$31, 1$
	χ_9	$fffh$	56	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc Q_\chi$	$55, 25$
	χ_{10}	hh	40	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc Q_\chi$	$39, 69$
	χ_{11}	$ffhh$	64	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc Q_\chi$	$63, 33$
	χ_{12}	$ffffhh$	88	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc Q_\chi$	$87, 57$
	χ_{13}	fhh	52	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc Q_\chi$	$51, 21$
	χ_{14}	$fffhh$	76	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc Q_\chi$	$75, 45$

Table 10.4: Continued

Icosahedral Groups (cont.)						
Group	Character	Stanley Poly. Q_χ	Degree	Basic χ-forms		Degrees
▶ $\mathbf{G_{19}}$:	χ_1	t	30	$\phi_1 \circledcirc Q_\chi$	$df_1 Q_\chi$	$29, 89$
	χ_2	$fffft$	78	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc f_2 Q_\chi$	$77, 77$
	χ_3	$ffft$	66	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc f_2 Q_\chi$	$65, 65$
	χ_4	fft	54	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc f_2 Q_\chi$	$53, 53$
	χ_5	ft	42	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc f_2 Q_\chi$	$41, 41$
	χ_6	$ffff$	48	$\phi_1 \circledcirc Q_\chi$	$df_2 Q_\chi$	$47, 107$
	χ_7	fff	36	$\phi_1 \circledcirc Q_\chi$	$df_2 Q_\chi$	$35, 95$
	χ_8	ff	24	$\phi_1 \circledcirc Q_\chi$	$df_2 Q_\chi$	$23, 83$
	χ_9	f	12	$\phi_1 \circledcirc Q_\chi$	$df_2 Q_\chi$	$11, 71$
	χ_{10}	ht	50	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc f_2 Q_\chi$	$49, 49$
	χ_{11}	$ffffht$	98	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc Q_\chi$	$97, 37$
	χ_{12}	$fffht$	86	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc Q_\chi$	$85, 25$
	χ_{13}	$ffht$	74	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc Q_\chi$	$73, 13$
	χ_{14}	fht	62	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc Q_\chi$	$61, 1$
	χ_{15}	h	20	$\phi_1 \circledcirc Q_\chi$	$df_2 Q_\chi$	$19, 79$
	χ_{16}	$ffffh$	68	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc f_2 Q_\chi$	$67, 67$
	χ_{17}	$fffh$	56	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc f_2 Q_\chi$	$55, 55$
	χ_{18}	ffh	44	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc f_2 Q_\chi$	$43, 43$
	χ_{19}	fh	32	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc f_2 Q_\chi$	$31, 31$
	χ_{20}	hht	70	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc f_2 Q_\chi$	$69, 69$
	χ_{21}	$ffffhht$	118	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc Q_\chi$	$117, 57$
	χ_{22}	$fffhht$	106	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc Q_\chi$	$105, 45$
	χ_{23}	$ffhht$	94	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc Q_\chi$	$93, 33$
	χ_{24}	$fhht$	82	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc Q_\chi$	$81, 21$
	χ_{25}	hh	40	$\phi_1 \circledcirc Q_\chi$	$df_2 Q_\chi$	$39, 99$
	χ_{26}	$ffffhh$	88	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc f_2 Q_\chi$	$87, 87$
	χ_{27}	$fffhh$	76	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc f_2 Q_\chi$	$75, 75$
	χ_{28}	$ffhh$	64	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc f_2 Q_\chi$	$63, 63$
	χ_{29}	fhh	52	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc f_2 Q_\chi$	$51, 51$

Table 10.4: Continued

Icosahedral Groups (cont.)						
Group	Character	Stanley Poly. Q_χ	Degree	Basic χ-forms		Degrees
$\mathbf{G_{20}}$:	χ_1	hh	40	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc Q_\chi$	$39, 21$
	χ_2	h	20	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc Q_\chi$	$19, 1$
$\mathbf{G_{21}}$:	χ_1	t	30	$\phi_1 \circledcirc Q_\chi$	$df_1 Q_\chi$	$29, 41$
	χ_2	hht	70	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc Q_\chi$	$69, 21$
	χ_3	hh	40	$\phi_1 \circledcirc Q_\chi$	$df_1 Q_\chi$	$39, 51$
	χ_4	ht	50	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc Q_\chi$	$49, 1$
	χ_5	h	20	$\phi_1 \circledcirc Q_\chi$	$df_1 Q_\chi$	$19, 31$
▶ $\mathbf{G_{22}}$:	χ_1	t	30	$\phi_1 \circledcirc Q_\chi$	$\phi_2 \circledcirc Q_\chi$	$29, 1$

Table 10.5: **Reduced Hilbert Series**

$\mathbf{G_4}$	χ_0	$(1+x^3y)(1+x^5y)$
	χ_1	$x^8(1+x^{-1}y)(1+x^{-3}y)$
	χ_2	$x^4(1+x^{-1}y)(1+x^{-3}y)$
$\mathbf{G_5}$	χ_0	$(1+x^5y)(1+x^{11}y)$
	χ_1	$x^{12}(1+x^{-1}y)(1+x^{-7}y)$
	χ_2	$x^{12}(1+x^{-1}y)(1+x^{-7}y)$
	χ_3	$x^4(1+x^{-1}y)(1+x^5y)$
	χ_4	$x^{16}(1+x^{-1}y)(1+x^{-7}y)$
	χ_5	$x^4(1+x^{-1}y)(1+x^5y)$
	χ_6	$x^8(1+x^{-1}y)(1+x^5y)$
	χ_7	$x^8(1+x^{-1}y)(1+x^5y)$
	χ_8	$x^8(1+x^{-1}y)(1+x^{-7}y)$
$\mathbf{G_6}$	χ_0	$(1+x^3y)(1+x^{11}y)$
	χ_1	$x^6(1+x^{-1}y)(1+x^3y)$
	χ_2	$x^{10}(1+x^{-1}y)(1+x^{-9}y)$
	χ_3	$x^{14}(1+x^{-1}y)(1+x^{-9}y)$
	χ_4	$x^4(1+x^{-1}y)(1+x^3y)$
	χ_5	$x^8(1+x^{-1}y)(1+x^3y)$
▶ $\mathbf{G_7}$	χ_0	$(1+x^{11}y)(1+x^{11}y)$
	χ_1	$x^6(1+x^{-1}y)(1+x^{11}y)$
	χ_2	$x^{18}(1+x^{-1}y)(1+x^{-13}y)$
	χ_3	$x^{18}(1+x^{-1}y)(1+x^{-13}y)$
	χ_4	$x^{12}(1+x^{-1}y)(1+x^{-1}y)$
	χ_5	$x^{12}(1+x^{-1}y)(1+x^{-1}y)$
	χ_6	$x^{22}(1+x^{-1}y)(1+x^{-13}y)$
	χ_7	$x^{10}(1+x^{-1}y)(1+x^{-1}y)$
	χ_8	$x^{10}(1+x^{-1}y)(1+x^{-1}y)$
	χ_9	$x^{16}(1+x^{-1}y)(1+x^{-1}y)$
	χ_{10}	$x^4(1+x^{-1}y)(1+x^{11}y)$
	χ_{11}	$x^4(1+x^{-1}y)(1+x^{11}y)$
	χ_{12}	$x^{14}(1+x^{-1}y)(1+x^{-13}y)$
	χ_{13}	$x^{14}(1+x^{-1}y)(1+x^{-1}y)$
	χ_{14}	$x^{14}(1+x^{-1}y)(1+x^{-1}y)$
	χ_{15}	$x^8(1+x^{-1}y)(1+x^{-1}y)$
	χ_{16}	$x^8(1+x^{-1}y)(1+x^{11}y)$
	χ_{17}	$x^8(1+x^{-1}y)(1+x^{11}y)$

Table 10.5: Continued

$\mathbf{G_8}$	χ_0	$(1+x^7y)(1+x^{11}y)$
	χ_1	$x^{12}(1+x^{-1}y)(1+x^{-5}y)$
	χ_2	$x^6(1+x^{-1}y)(1+x^{-5}y)$
	χ_3	$x^{18}(1+x^{-1}y)(1+x^{-5}y)$
$\mathbf{G_9}$	χ_0	$(1+x^7y)(1+x^{23}y)$
	χ_1	$x^{12}(1+x^{-1}y)(1+x^7y)$
	χ_2	$x^{12}(1+x^{-1}y)(1+x^7y)$
	χ_3	$x^{24}(1+x^{-1}y)(1+x^{-17}y)$
	χ_4	$x^{30}(1+x^{-1}y)(1+x^{-17}y)$
	χ_5	$x^6(1+x^{-1}y)(1+x^7y)$
	χ_6	$x^{18}(1+x^{-1}y)(1+x^7y)$
	χ_7	$x^{18}(1+x^{-1}y)(1+x^{-17}y)$
$\mathbf{G_{10}}$	χ_0	$(1+x^{11}y)(1+x^{23}y)$
	χ_1	$x^{12}(1+x^{-1}y)(1+x^{11}y)$
	χ_2	$x^{18}(1+x^{-1}y)(1+x^{11}y)$
	χ_3	$x^6(1+x^{-1}y)(1+x^{11}y)$
	χ_4	$x^{26}(1+x^{-1}y)(1+x^{-13}y)$
	χ_5	$x^{14}(1+x^{-1}y)(1+x^{-13}y)$
	χ_6	$x^{20}(1+x^{-1}y)(1+x^{-13}y)$
	χ_7	$x^8(1+x^{-1}y)(1+x^{11}y)$
	χ_8	$x^{34}(1+x^{-1}y)(1+x^{-13}y)$
	χ_9	$x^{22}(1+x^{-1}y)(1+x^{-13}y)$
	χ_{10}	$x^{28}(1+x^{-1}y)(1+x^{-13}y)$
	χ_{11}	$x^{16}(1+x^{-1}y)(1+x^{11}y)$

Table 10.5: Continued

▶ $\mathbf{G_{11}}$	χ_0	$(1+x^{23}y)(1+x^{23}y)$
	χ_1	$x^{12}(1+x^{-1}y)(1+x^{23}y)$
	χ_2	$x^{12}(1+x^{-1}y)(1+x^{23}y)$
	χ_3	$x^{24}(1+x^{-1}y)(1+x^{-1}y)$
	χ_4	$x^{30}(1+x^{-1}y)(1+x^{-1}y)$
	χ_5	$x^{6}(1+x^{-1}y)(1+x^{23}y)$
	χ_6	$x^{18}(1+x^{-1}y)(1+x^{23}y)$
	χ_7	$x^{18}(1+x^{-1}y)(1+x^{-1}y)$
	χ_8	$x^{38}(1+x^{-1}y)(1+x^{-25}y)$
	χ_9	$x^{14}(1+x^{-1}y)(1+x^{-1}y)$
	χ_{10}	$x^{26}(1+x^{-1}y)(1+x^{-1}y)$
	χ_{11}	$x^{26}(1+x^{-1}y)(1+x^{-25}y)$
	χ_{12}	$x^{20}(1+x^{-1}y)(1+x^{-1}y)$
	χ_{13}	$x^{20}(1+x^{-1}y)(1+x^{-1}y)$
	χ_{14}	$x^{32}(1+x^{-1}y)(1+x^{-25}y)$
	χ_{15}	$x^{8}(1+x^{-1}y)(1+x^{23}y)$
	χ_{16}	$x^{46}(1+x^{-1}y)(1+x^{-25}y)$
	χ_{17}	$x^{22}(1+x^{-1}y)(1+x^{-1}y)$
	χ_{18}	$x^{34}(1+x^{-1}y)(1+x^{-1}y)$
	χ_{19}	$x^{34}(1+x^{-1}y)(1+x^{-25}y)$
	χ_{20}	$x^{28}(1+x^{-1}y)(1+x^{-1}y)$
	χ_{21}	$x^{28}(1+x^{-1}y)(1+x^{-1}y)$
	χ_{22}	$x^{40}(1+x^{-1}y)(1+x^{-25}y)$
	χ_{23}	$x^{16}(1+x^{-1}y)(1+x^{23}y)$
▶ $\mathbf{G_{12}}$	χ_0	$(1+x^{5}y)(1+x^{7}y)$
	χ_1	$x^{12}(1+x^{-1}y)(1+x^{-11}y)$
▶ $\mathbf{G_{13}}$	χ_0	$(1+x^{7}y)(1+x^{11}y)$
	χ_1	$x^{18}(1+x^{-1}y)(1+x^{-17}y)$
	χ_2	$x^{6}(1+x^{-1}y)(1+x^{7}y)$
	χ_3	$x^{12}(1+x^{-1}y)(1+x^{-5}y)$
$\mathbf{G_{14}}$	χ_0	$(1+x^{5}y)(1+x^{23}y)$
	χ_1	$x^{12}(1+x^{-1}y)(1+x^{5}y)$
	χ_2	$x^{20}(1+x^{-1}y)(1+x^{-19}y)$
	χ_3	$x^{8}(1+x^{-1}y)(1+x^{5}y)$
	χ_4	$x^{28}(1+x^{-1}y)(1+x^{-19}y)$
	χ_5	$x^{16}(1+x^{-1}y)(1+x^{5}y)$

Table 10.5: Continued

▶ $\mathbf{G_{15}}$	χ_0	$(1+x^{11}y)(1+x^{23}y)$
	χ_1	$x^{6}(1+x^{-1}y)(1+x^{23}y)$
	χ_2	$x^{18}(1+x^{-1}y)(1+x^{-1}y)$
	χ_3	$x^{12}(1+x^{-1}y)(1+x^{11}y)$
	χ_4	$x^{22}(1+x^{-1}y)(1+x^{-1}y)$
	χ_5	$x^{34}(1+x^{-1}y)(1+x^{-25}y)$
	χ_6	$x^{28}(1+x^{-1}y)(1+x^{-13}y)$
	χ_7	$x^{16}(1+x^{-1}y)(1+x^{11}y)$
	χ_8	$x^{14}(1+x^{-1}y)(1+x^{-1}y)$
	χ_9	$x^{26}(1+x^{-1}y)(1+x^{-25}y)$
	χ_{10}	$x^{20}(1+x^{-1}y)(1+x^{-13}y)$
	χ_{11}	$x^{8}(1+x^{-1}y)(1+x^{11}y)$
$\mathbf{G_{16}}$	χ_0	$(1+x^{19}y)(1+x^{29}y)$
	χ_1	$x^{36}(1+x^{-1}y)(1+x^{-11}y)$
	χ_2	$x^{12}(1+x^{-1}y)(1+x^{-11}y)$
	χ_3	$x^{48}(1+x^{-1}y)(1+x^{-11}y)$
	χ_4	$x^{24}(1+x^{-1}y)(1+x^{-11}y)$
$\mathbf{G_{17}}$	χ_0	$(1+x^{19}y)(1+x^{59}y)$
	χ_1	$x^{30}(1+x^{-1}y)(1+x^{19}y)$
	χ_2	$x^{78}(1+x^{-1}y)(1+x^{-41}y)$
	χ_3	$x^{66}(1+x^{-1}y)(1+x^{-41}y)$
	χ_4	$x^{54}(1+x^{-1}y)(1+x^{-41}y)$
	χ_5	$x^{42}(1+x^{-1}y)(1+x^{-41}y)$
	χ_6	$x^{48}(1+x^{-1}y)(1+x^{19}y)$
	χ_7	$x^{36}(1+x^{-1}y)(1+x^{19}y)$
	χ_8	$x^{24}(1+x^{-1}y)(1+x^{19}y)$
	χ_9	$x^{12}(1+x^{-1}y)(1+x^{19}y)$

Table 10.5: Continued

$\mathbf{G_{18}}$	χ_0	$(1+x^{29}y)(1+x^{59}y)$
	χ_1	$x^{24}(1+x^{-1}y)(1+x^{29}y)$
	χ_2	$x^{48}(1+x^{-1}y)(1+x^{29}y)$
	χ_3	$x^{12}(1+x^{-1}y)(1+x^{29}y)$
	χ_4	$x^{36}(1+x^{-1}y)(1+x^{29}y)$
	χ_5	$x^{20}(1+x^{-1}y)(1+x^{29}y)$
	χ_6	$x^{44}(1+x^{-1}y)(1+x^{-31}y)$
	χ_7	$x^{68}(1+x^{-1}y)(1+x^{-31}y)$
	χ_8	$x^{32}(1+x^{-1}y)(1+x^{-31}y)$
	χ_9	$x^{56}(1+x^{-1}y)(1+x^{-31}y)$
	χ_{10}	$x^{40}(1+x^{-1}y)(1+x^{29}y)$
	χ_{11}	$x^{64}(1+x^{-1}y)(1+x^{-31}y)$
	χ_{12}	$x^{88}(1+x^{-1}y)(1+x^{-31}y)$
	χ_{13}	$x^{52}(1+x^{-1}y)(1+x^{-31}y)$
	χ_{14}	$x^{76}(1+x^{-1}y)(1+x^{-31}y)$
► $\mathbf{G_{19}}$	χ_0	$(1+x^{59}y)(1+x^{59}y)$
	χ_1	$x^{30}(1+x^{-1}y)(1+x^{59}y)$
	χ_2	$x^{78}(1+x^{-1}y)(1+x^{-1}y)$
	χ_3	$x^{66}(1+x^{-1}y)(1+x^{-1}y)$
	χ_4	$x^{54}(1+x^{-1}y)(1+x^{-1}y)$
	χ_5	$x^{42}(1+x^{-1}y)(1+x^{-1}y)$
	χ_6	$x^{48}(1+x^{-1}y)(1+x^{59}y)$
	χ_7	$x^{36}(1+x^{-1}y)(1+x^{59}y)$
	χ_8	$x^{24}(1+x^{-1}y)(1+x^{59}y)$
	χ_9	$x^{12}(1+x^{-1}y)(1+x^{59}y)$
	χ_{10}	$x^{50}(1+x^{-1}y)(1+x^{-1}y)$
	χ_{11}	$x^{98}(1+x^{-1}y)(1+x^{-61}y)$
	χ_{12}	$x^{86}(1+x^{-1}y)(1+x^{-61}y)$
	χ_{13}	$x^{74}(1+x^{-1}y)(1+x^{-61}y)$
	χ_{14}	$x^{62}(1+x^{-1}y)(1+x^{-61}y)$
	χ_{15}	$x^{20}(1+x^{-1}y)(1+x^{59}y)$
	χ_{16}	$x^{68}(1+x^{-1}y)(1+x^{-1}y)$
	χ_{17}	$x^{56}(1+x^{-1}y)(1+x^{-1}y)$
	χ_{18}	$x^{44}(1+x^{-1}y)(1+x^{-1}y)$
	χ_{19}	$x^{32}(1+x^{-1}y)(1+x^{-1}y)$
	χ_{20}	$x^{70}(1+x^{-1}y)(1+x^{-1}y)$
	χ_{21}	$x^{118}(1+x^{-1}y)(1+x^{-61}y)$
	χ_{22}	$x^{106}(1+x^{-1}y)(1+x^{-61}y)$
	χ_{23}	$x^{94}(1+x^{-1}y)(1+x^{-61}y)$
	χ_{24}	$x^{82}(1+x^{-1}y)(1+x^{-61}y)$
	χ_{25}	$x^{40}(1+x^{-1}y)(1+x^{59}y)$
	χ_{26}	$x^{88}(1+x^{-1}y)(1+x^{-1}y)$
	χ_{27}	$x^{76}(1+x^{-1}y)(1+x^{-1}y)$
	χ_{28}	$x^{64}(1+x^{-1}y)(1+x^{-1}y)$
	χ_{29}	$x^{52}(1+x^{-1}y)(1+x^{-1}y)$

Table 10.5: Continued

$\mathbf{G_{20}}$	χ_0	$(1+x^{11}y)(1+x^{29}y)$
	χ_1	$x^{40}(1+x^{-1}y)(1+x^{-19}y)$
	χ_2	$x^{20}(1+x^{-1}y)(1+x^{-19}y)$
$\mathbf{G_{21}}$	χ_0	$(1+x^{11}y)(1+x^{59}y)$
	χ_1	$x^{30}(1+x^{-1}y)(1+x^{11}y)$
	χ_2	$x^{70}(1+x^{-1}y)(1+x^{-49}y)$
	χ_3	$x^{40}(1+x^{-1}y)(1+x^{11}y)$
	χ_4	$x^{50}(1+x^{-1}y)(1+x^{-49}y)$
	χ_5	$x^{20}(1+x^{-1}y)(1+x^{11}y)$
▶ $\mathbf{G_{22}}$	χ_0	$(1+x^{11}y)(1+x^{19}y)$
	χ_1	$x^{30}(1+x^{-1}y)(1+x^{-29}y)$
$\mathbf{G_{23}}$	χ_0	$(1+xy)(1+x^{5}y)(1+x^{9}y)$
	χ_1	$x^{15}(1+x^{-1}y)(1+x^{-5}y)(1+x^{-9}y)$
$\mathbf{G_{24}}$	χ_0	$(1+x^{3}y)(1+x^{5}y)(1+x^{13}y)$
	χ_1	$x^{21}(1+x^{-1}y)(1+x^{-9}y)(1+x^{-11}y)$
$\mathbf{G_{25}}$	χ_0	$(1+x^{5}y)(1+x^{8}y)(1+x^{11}y)$
	χ_1	$x^{12}(1+x^{-1}y)(1+x^{-4}y)(1+x^{-7}y)$
	χ_2	$x^{24}(1+x^{-1}y)(1+x^{-4}y)(1+x^{-7}y)$
$\mathbf{G_{26}}$	χ_0	$(1+x^{5}y)(1+x^{11}y)(1+x^{17}y)$
	$\chi_{\det^{-1}}$	$x^{21}(1+x^{-1}y)(1+x^{-7}y)(1+x^{-13}y)$
	$\chi_{\det}$	$x^{33}(1+x^{-1}y)(1+x^{-7}y)(1+x^{-13}y)$
	$\chi_{\det^4}$	$x^{24}(1+x^{-1}y)(1+x^{-7}y)(1+x^{5}y)$
	$\chi_{\det^2}$	$x^{12}(1+x^{-1}y)(1+x^{-7}y)(1+x^{5}y)$
	$\chi_{\det^3}$	$x^{9}(1+x^{-1}y)(1+x^{5}y)(1+x^{11}y)$
$\mathbf{G_{27}}$	χ_0	$(1+x^{5}y)(1+x^{11}y)(1+x^{29}y)$
	χ_1	$x^{45}(1+x^{-1}y)(1+x^{-19}y)(1+x^{-25}y)$
$\mathbf{G_{28}}$	χ_0	$(1+xy)(1+x^{5}y)(1+x^{7}y)(1+x^{11}y)$
	χ_1	$x^{24}(1+x^{-1}y)(1+x^{-5}y)(1+x^{-7}y)(1+x^{-11}y)$
	χ_2	$x^{12}(1+x^{-1}y)(1+x^{-5}y)(1+x^{1}y)(1+x^{5}y)$
	χ_3	$x^{12}(1+x^{-1}y)(1+x^{-5}y)(1+x^{1}y)(1+x^{5}y)$

Table 10.5: Continued

$\mathbf{G_{29}}$	χ_0	$(1+x^3y)(1+x^7y)(1+x^{11}y)(1+x^{19}y)$
	χ_1	$x^{40}(1+x^{-1}y)(1+x^{-9}y)(1+x^{-13}y)(1+x^{-17}y)$
$\mathbf{G_{30}}$	χ_0	$(1+xy)(1+x^{11}y)(1+x^{19}y)(1+x^{29}y)$
	χ_1	$x^{60}(1+x^{-1}y)(1+x^{-11}y)(1+x^{-19}y)(1+x^{-29}y)$
▶ $\mathbf{G_{31}}$	χ_0	$(1+x^7y)(1+x^{11}y)(1+x^{19}y)(1+x^{23}y)$
	χ_1	$x^{60}(1+x^{-1}y)(1+x^{-13}y)(1+x^{-17}y)(1+x^{-29}y)$
$\mathbf{G_{32}}$	χ_0	$(1+x^{11}y)(1+x^{17}y)(1+x^{23}y)(1+x^{29}y)$
	χ_1	$x^{80}(1+x^{-1}y)(1+x^{-7}y)(1+x^{-13}y)(1+x^{-19}y)$
	χ_2	$x^{40}(1+x^{-1}y)(1+x^{-7}y)(1+x^{-13}y)(1+x^{-19}y)$
$\mathbf{G_{33}}$	χ_0	$(1+x^3y)(1+x^5y)(1+x^9y)(1+x^{11}y)(1+x^{17}y)$
	χ_1	$x^{45}(1+x^{-1})(1+x^{-7}y)(1+x^{-9}y)(1+x^{-13}y)(1+x^{-15}y)$
$\mathbf{G_{34}}$	χ_0	$(1+x^5y)(1+x^{11}y)(1+x^{17}y)(1+x^{23}y)(1+x^{29}y)(1+x^{41}y)$
	χ_1	$x^{126}(1+x^{-1}y)(1+x^{-13}y)(1+x^{-19}y)(1+x^{-25}y)(1+x^{-31}y)(1+x^{-37}y)$
$\mathbf{G_{35}}$	χ_0	$(1+xy)(1+x^4y)(1+x^5y)(1+x^7y)(1+x^8y)(1+x^{11}y)$
	χ_1	$x^{36}(1+x^{-1}y)(1+x^{-4}y)(1+x^{-5}y)(1+x^{-7}y)(1+x^{-8}y)(1+x^{-11}y)$
$\mathbf{G_{36}}$		
	χ_0	$(1+xy)(1+x^5y)(1+x^7y)(1+x^9y)(1+x^{11}y)(1+x^{13}y)(1+x^{17}y)$
	χ_1	$x^{63}(1+x^{-1}y)(1+x^{-5}y)(1+x^{-7}y)(1+x^{-9}y)(1+x^{-11}y)(1+x^{-13}y)(1+x^{-17}y)$
$\mathbf{G_{37}}$		
	χ_0	$(1+xy)(1+x^7y)(1+x^{11}y)(1+x^{13}y)(1+x^{17}y)(1+x^{19}y)(1+x^{23}y)(1+x^{29}y)$
	χ_1	$x^{120}(1+x^{-1}y)(1+x^{-7}y)(1+x^{-11}y)(1+x^{-13}y)(1+x^{-17}y)(1+x^{-19}y)(1+x^{-23}y)(1+x^{-29}y)$

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