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q-Enumeration of Classical Combinatorial Structures

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by

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The dissertation of David P. Little is approved, and it is acceptable in quality and form for publication on microfilm:

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To Mom and Dad

Beep beep!

—A. F. Ward, Jr.

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ABSTRACT OF THE DISSERTATION

q-Enumeration of Classical Combinatorial Structures

by

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Doctor of Philosophy in Mathematics

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The primary focus of this dissertation is to establish generating functions for specific instances of two classical combinatorial structures, namely permutations and partitions. We start by giving two simple correspondences between Dyck paths and permutations that avoid certain patterns. These correspondences immediately yield generating functions in the shape of continued fractions. Next, we construct permutations with exactly one occurrence of a certain pattern by again manipulating Dyck paths. These constructions immediately give way to a variety of generating functions in the form of Chebyshev polynomials of the second kind. Next, we examine an involution on Dyck paths that yields numerous results regarding pattern avoidances. Lastly, we turn our attention to partitions, and examine Franklin's involution on partitions with distinct parts.

Chapter 1

Pattern avoidances and Dyck paths

In this chapter we establish the basic relationships between lattice paths and permutations that avoid a given pattern. These correspondences will provide a foundation for subsequent chapters. We also show how these constructions relate to previously known results.

1.1 Introduction

Given permutations $\alpha \in S_k$ and $\sigma \in S_n$ for $n \geq k$, we say that σ contains the pattern α if there exists indices $i_1 < i_2 < \cdots < i_k$ such that $\sigma_{i_r} < \sigma_{i_s}$ if and only if $\alpha_r < \alpha_s$ for all $1 \leq r, s \leq k$. For example, the permutation $(3, 2, 4, 1)$ has 2 occurrences of the pattern $(2, 3, 1)$, namely $(2, 4, 1)$ and $(3, 4, 1)$, 1 occurrence of the pattern $(3, 2, 1)$, and 1 occurrence of the pattern $(2, 1, 3)$.

In the event that σ does not contain the pattern α, we say that σ is α-avoiding or more simply σ avoids α. The set consisting of all α-avoiding permutations on n is denoted by $S_n(\alpha)$ and its cardinality is denoted by $s_n(\alpha)$. Given several permutations $\alpha_1, \ldots, \alpha_l$, $S_n(\alpha_1, \ldots, \alpha_l)$ denotes $\cap_{j=1}^{l} S_n(\alpha_j)$. Notice that the example given above is an element of $S_4(123, 132, 312)$. In subsequent chapters, we will examine particular cases when a permutation has a given number of occurrences of a pattern. Consequently, we will also adopt the convention that $S_n^k(\alpha)$ denotes the subset of S_n that consists of permutations with exactly k occurrences of the pattern α.

In this chapter, we will primarily deal with patterns of length 3. Given any $\alpha \in S_3$, it is well known [5] that $s_n(\alpha)$ is given by the n^{th} Catalan number, $\binom{2n}{n}/(n+1)$. It is also well known that this is the number of Dyck paths from $(0,0)$ to (n,n). In order to establish our results, we will introduce many correspondences between permutations, σ, and paths, π. We use the convention that σ_π is the permutation that corresponds to the path π_σ, and vice versa. While this has the possibilty of introducing some ambiguity, each correspondence will be identified by a set of avoided patterns, and any uncertainty can be erased by noting the relavant patterns avoided by σ_π. We will also use these symbols interchangably, meaning that to say π_σ is in $S_n(\alpha)$ is the same as saying σ_π is in $S_n(\alpha)$.

1.2 Parking functions and Dyck paths

In order to establish our results in latter chapters, it will be necessary to construct some basic correspondences between $S_n(\alpha)$ and the set of Dyck paths. These correspondences will all stem from proving the following fundamental fact.

Theorem 1.1 *$s_n(\alpha) = \frac{1}{n+1}\binom{2n}{n}$ for all $\alpha \in S_3$.*

In order to prove the above theorem, we recall the definition of a *parking function.* Consider the situation of parking n cars on a one-way street and each car has a preferred spot in which to park. When the i^{th} car arrives at its preferred spot, park the car in that spot if possible, otherwise park in the next available spot. We call any function, f, from $\{1, 2, \ldots, n\}$ to itself, a *preference function* and interpret it to mean that the i^{th} car prefers to park in the $f(i)^{th}$ spot. Of course there is no guarantee that for any preference function, each car will be able to park, much less park in its preferred spot. For instance, if there are two cars that prefer to park in the last spot, the second car that drives to the last spot will not be able to park. If each car is able to park, then f is referred to as a *parking function.* See [6] for more details regarding parking functions.

Let f_i be the number of cars that want to park in the first i spots. If $f_i < i$ for some i, then there will be $n \Leftrightarrow f_i$ cars that want to park in the last $n \Leftrightarrow i$ spots, and they will be unable to do so since $n \Leftrightarrow f_i > n \Leftrightarrow i$. Thus f is not a parking function. On the other hand, if f is not a parking function, then let I be the first car that is unable to park and let $f(I) = J$. Since car I is unable to park, cars 1 through $I \Leftrightarrow 1$ have taken spots J through n and possibly others. Let K be the last available spot when car I attempts to park. There are at least $n \Leftrightarrow K + 1$ cars that prefer to park in spots $K + 1$ through n, the $n \Leftrightarrow K$ cars that actually parked there plus car I. Therefore f_K is at most $K \Leftrightarrow 1$.

This shows that $f_i \geq i$ for all i is a necessary and sufficient condition for f to be a parking function. Using this as the defining property of a parking function, we can associate to f a path from $(0, 0)$ to (n, n) using only the vectors $(0, 1)$ and $(1, 0)$, which we will refer to as steps NORTH and EAST, respectively. To do this, we specify that the n steps EAST go from $(i \Leftrightarrow 1, f_i)$ to (i, f_i) for each i. The condition that $f_i \geq i$ for all i simply means that our path must remain weakly above the line $y = x$. These paths corresponding to parking functions are called *Dyck paths*[1] and the set of all such paths for a fixed n is denoted by $\mathcal{D}_n$. For example, let f be given by:

$$\begin{array}{llll} f(1) = 3 & f(2) = 6 & f(3) = 1 & f(4) = 1 \\ f(5) = 7 & f(6) = 3 & f(7) = 1 & f(8) = 6 \end{array}$$

and therefore we have that $f_1 = f_2 = 3$, $f_3 = f_4 = f_5 = 5$, $f_6 = 7$, and $f_7 = f_8 = 8$. We have incorporated both the parking function and Dyck path into a single diagram by labeling each step NORTH from $(I \Leftrightarrow 1, J \Leftrightarrow 1)$ to $(I \Leftrightarrow 1, J)$ with the label of a car that prefers to park in spot I. Figure 1.1 illustrates this process with the sample parking function given above.

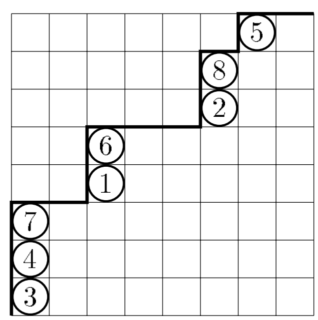


Figure 1.1: Parking functions and Dyck paths

1.3 Three basic involutions

Before we prove Theorem 1.1, it will be useful to point out three very natural operations acting on a permutation $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$. First, the *complement* of σ, denoted by σ^c, refers to the permutation $(n+1 \Leftrightarrow \sigma_1, n+1 \Leftrightarrow \sigma_2, \ldots, n+1 \Leftrightarrow \sigma_n)$. Second, the

[1] In most texts, the set of Dyck paths refers to the paths from $(0, 0)$ to $(0, 2n)$ that stay weakly above the x-axis while using only NORTHEAST and SOUTHEAST steps.

reverse of σ, denoted by σ^r, refers to the permutation $(\sigma_n, \sigma_{n-1}, \ldots, \sigma_1)$. And last, the *inverse* of σ, denoted by σ^i, is the usual functional inverse of σ. With these definitions, it is clear that we must have the following lemma.

Lemma 1.2 *Let $\alpha \in S_k$ for some $k \leq n$. The following are equivalent:*

1. $\sigma \in S_n(\alpha)$
2. $\sigma^c \in S_n(\alpha^c)$
3. $\sigma^r \in S_n(\alpha^r)$
4. $\sigma^i \in S_n(\alpha^i)$

Proof. Assume that σ is not in $S_n(\alpha)$ and let $(\sigma_{i_1}, \sigma_{i_2}, \ldots, \sigma_{i_k})$ be an instance of the pattern α in σ. It is clear that $(n+1 \Leftrightarrow \sigma_{i_1}, n+1 \Leftrightarrow \sigma_{i_2}, \ldots, n+1 \Leftrightarrow \sigma_{i_k})$, $(\sigma_{n+1-i_k}, \sigma_{n+1-i_{k-1}}, \ldots, \sigma_{n+1-i_1})$, and $(i_{\alpha^i_1}, i_{\alpha^i_2}, \ldots, i_{\alpha^i_k})$ will be instances of the patterns α^c, α^r, and α^i in σ^c, σ^r, and σ^i, respectively. This shows that 2, 3, and 4 each imply 1. But since each operation is an involution, we can simply replace σ by σ^c (resp. σ^r, σ^i) and α by α^c (resp. α^r, α^i) and use the same argument as above to show that 1 implies 2 (resp. 3,4). $\square$

Using Lemma 1.2, we immediately have that $s_n(132) = s_n(312) = s_n(231) = s_n(213)$ and $s_n(321) = s_n(123)$. The connections between each set is diagramed in Figure 1.2. This allows us to break up the proof of Theorem 1.1 into two parts. First, we will prove that $s_n(132) = \frac{1}{n+1}\binom{2n}{n}$ and second, $s_n(321) = \frac{1}{n+1}\binom{2n}{n}$. It is important to note here that we have singled out the patterns (132) and (321) to simplify the notation as much as possible. It is not our intention to suggest that these patterns are more relevant than the others. In most presentations, (132)-avoiding and (123)-avoiding permutations

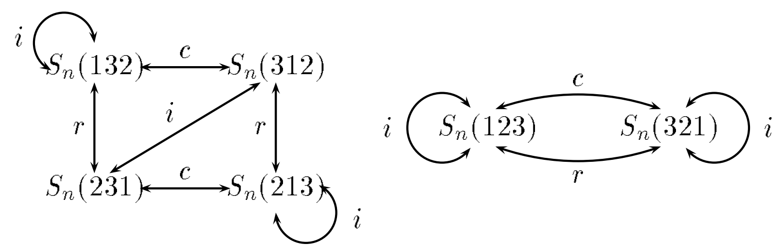


Figure 1.2: Diagram of three involutions

are studied. Also, $S_n(231)$ is of particular interest since it is the set of all stack-sortable permutations [6, 15].

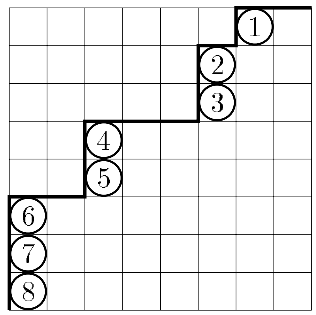
We should also point out that we can use Lemma 1.2 to establish correspondences between sets that avoid any number of patterns. In other words, once we have enumerated the set $S_n(\alpha_1, \ldots, \alpha_l)$, we would then have equivalent results for $S_n(\alpha_1^c, \ldots, \alpha_l^c)$, $S_n(\alpha_1^r, \ldots, \alpha_l^r)$, and $S_n(\alpha_1^i, \ldots, \alpha_l^i)$. Instead of repeatedly referring to Lemma 1.2, we will refrain from listing each of these equivalent results after a new identity. Consequently, we will not include any result that follows immediately from Lemma 1.2, unless a different approach would prove more fruitful.

1.4 Constructing (132)-avoiding permutations

As mentioned above, we will begin our proof of Theorem 1.1 by showing that $s_n(132) = \frac{1}{n+1}\binom{2n}{n}$. To do so, we will construct a bijection between $S_n(132)$ and $\mathcal{D}_n$.

Given $\pi \in \mathcal{D}_n$, form the parking function, f, that corresponds to π by numbering the vertical portions of the path from top to bottom. Now simply park the cars and let σ_i be the car that parks in the i^{th} spot.

Let us first establish that this process results in an element of $S_n(132)$. Assume



$$\longleftrightarrow \quad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 7 & 4 & 5 & 8 & 2 & 1 & 3 \end{pmatrix}$$

Figure 1.3: Example of bijection between $\mathcal{D}_n$ and $S_n(132)$

that $(\sigma_I, \sigma_J, \sigma_K)$ is a (132) pattern in the resulting permutation σ. Since car σ_K is parked after car σ_I, it was unable to park in its preferred spot. This implies that when car σ_K attempted to park, spots $f(K)$ through $K \Leftrightarrow 1$ (which includes spot I) were already taken. Therefore when car σ_J tries to park, there are no available spots between cars σ_I and σ_K, whichs contradicts our assumption that $(\sigma_I, \sigma_J, \sigma_K)$ was a (132) pattern. Therefore $\sigma \in S_n(132)$.

Now, let us establish that this is an injection. Given two different paths, π_1 and π_2, find the first place where the paths differ when traversing the path from (n, n) to $(0, 0)$ and assume that this occurs after I steps SOUTH. Furthermore, assume that the next step in π_1 is to the WEST and that the next step in π_2 is to the SOUTH. When we park cars according to π_1, car $I + 1$ will park in its preferred spot, $f_1(I + 1)$. When we park cars according to π_2, car $I+1$ will park weakly to the right of its preferred spot, $f_2(I + 1)$. But since $f_1(I + 1) < f_2(I + 1)$, the resulting permutations will be distinct.

Given $\sigma \in S_n(132)$, we can construct π_σ as a path going from (n, n) to $(0, 0)$. We start by pointing out that car 1 will always park in its preferred spot since it is the first car to park. Let π_σ be the path that starts at (n, n) and proceeds to the point $(\sigma^{-1}(1) \Leftrightarrow 1, n \Leftrightarrow 1)$ by taking exactly $n + 1 \Leftrightarrow \sigma^{-1}(1)$ steps WEST followed by a single step SOUTH. This not only insures that $f(1) = \sigma^{-1}(1)$, but it also guarantees that the final portion of π_σ will be weakly above the line $y = x$. The remaining portion of π_σ is constructed by systematically determining where cars 2 through n prefer to park.

To determine $f(i)$, look at car i's position relative to car $i \Leftrightarrow 1$. If car i resides in a spot prior to car $i \Leftrightarrow 1$, then car i has parked in its preferred spot. Append to π_σ exactly $f(i \Leftrightarrow 1) \Leftrightarrow f(i)$ steps WEST followed by a single step SOUTH. Thus π_σ now goes through the point $(f(i) \Leftrightarrow 1, n \Leftrightarrow i)$ where $f(i) = \sigma^{-1}(i)$. Notice that if π_σ was above the line $y = x$, it will remain so after this operation since $f(i \Leftrightarrow 1) \Leftrightarrow f(i) \geq 1$.

If car i resides in a spot after car $i \Leftrightarrow 1$, then there can be no empty spots between car $i \Leftrightarrow 1$ and car i once car i has parked. Otherwise, this spot will be eventually filled by car j, with $j > i$, and $(i \Leftrightarrow 1, j, i)$ would form a (132) pattern. In other words, we have shown that car i parked in the first available spot following car $i \Leftrightarrow 1$. We therefore say that car i wanted to park in the same spot that car $i \Leftrightarrow 1$ wanted to park and we append to π_σ a single step SOUTH. Now π_σ goes through the point $(f(i) \Leftrightarrow 1, n \Leftrightarrow i)$ where $f(i) = f(i \Leftrightarrow 1)$. Again notice that this operation cannot produce a π_σ that crosses the line $y = x$. If it did then $f(i) > n \Leftrightarrow i + 1$, which says that at least i cars want to park in the last $i \Leftrightarrow 1$ spots and this cannot happen.

Using this process, ultimately the path π_σ will reach the point $(f(n) \Leftrightarrow 1, 0)$ only after all n cars have parked. Notice that the car that parked in the first spot must have preferred to park there, otherwise it would have driven past it. Thus each subsequent car must park after the first spot, which means that its preferred spot was the first spot. In particular, $f(n) = 1$, and thus $\pi_\sigma \in \mathcal{D}_n$. Therefore we have established that this correspondence is in fact a bijection.

This of course provides us with

Lemma 1.3 $s_n(132) = \frac{1}{n+1}\binom{2n}{n}$,

which was the first step in proving Theorem 1.1.

We should point out that Christian Krattenthaler [7] independently discovered this construction described in terms of (123)-avoiding permutations, however, he neglects to describe the process of converting a Dyck path into a (132)-avoiding permutation.

1.5 An alternate construction

In the above construction, if a car did not park in its preferred spot, then it parked in the next available location. The fact that there was always a next location was taken care of by the assumption that our path remained weakly above the diagonal. Unfortunately, this does not provide us with adequate information regarding where this "next location" will occur. In this section we present an alternate construction for elements of $S_n(132)$ that will make this point more clear.

To construct an element of $S_n(132)$, simply take a path $\pi \in \mathcal{D}_n$ and label the vertical segments from top to bottom as before. Here, it will be easier to think of our path π going from $(0,0)$ to $(2n,0)$, as is the usual custom. So we are really labeling the NORTHEAST steps from right to left. Now imagine that there is a horizontal string connecting each NORTHEAST step to the first SOUTHEAST step that occurs to its immediate left. For example, the path in Figure 1.3 would be labelled as follows:

Now simply slide each label on the NORTHEAST edges along the string until it reaches the corresponding SOUTHEAST edge, as is done in Figure 1.5. Reading the numbers from left to right yields a permutation in $S_n(132)$. Note that this is exactly the same as the permutation in Figure 1.3.

It is easy to see that this forms a permutation in $S_n(132)$ according to the same

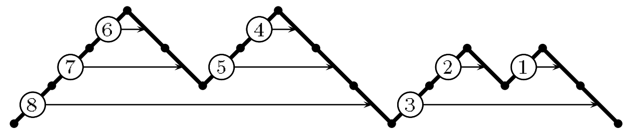


Figure 1.4: Alternate labelling of a path

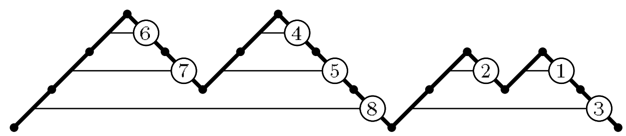


Figure 1.5: Alternate construction of $S_n(132)$

recipe as provided in the previous section. Car 1 will always park in its preferred spot. If car i parks in its preferred spot, it will do so prior to the spot in which car $i \Leftrightarrow 1$ parked. If car i does not park in its preferred spot, it will park in the next available location following car $i \Leftrightarrow 1$. However, in this light, it is more clear as to where this next location will be. For instance, in Figure 1.4, it is clear that car 8 will park in the very next spot following car 5. And we can determine this without having to park cars 6 and 7.

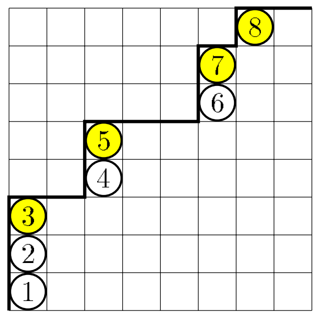
Also in this setting, it is clear what the effect of our three involutions is on this construction. For instance, the complement of σ, which is a (312)-avoiding permutation, is formed by labelling the NORTHEAST steps from left to right instead of right to left. The reverse of σ, which is a (231)-avoiding permutation, is formed by simply reading off the permutation from right to left instead of left to right. The inverse of σ, which is a (132)-avoiding permutation, is formed by labelling the SOUTHEAST steps from left to right, sliding the labels to the left and then reading the permuation from right to left. In other words, σ^i is the (132)-avoiding permutation that corresponds to the path formed by reflecting π across the line $x = n$.

We can also combine these procedures to form other types of permutations. If we slide the labels from right to left, we see that $\sigma^{ci} \in S_n(231)$ is formed by labelling the SOUTHEAST steps from left to right and $\sigma^{cic} \in S_n(213)$ is formed by labelling the SOUTHEAST steps from right to left. In fact it is exactly this interpretation of σ^{ci} that gives West [15] his correspondence between bracketing sequences and $S_n(231)$, the set of stack-sortable permutations.

1.6 Constructing (321)-avoiding permutations

To complete our proof of Theorem 1.1, it remains to construct a correspondence between $S_n(321)$ and $\mathcal{D}_n$. To do so, we will return to the setting of parking functions. Imagine that we are parking n cars and that some cars have very important passengers, meaning that these cars *must* park in their preferred spot. But since important passengers have important things to do, they are typically late, and therefore we identify *vip cars* as the last car that wants to park in a given spot. These cars appear shaded in Figure 1.6. To construct our bijection, pick an element of $\mathcal{D}_n$ and number the vertical segments of the path from bottom to top. Now park the cars by first taking care of the vip cars and then parking each of the remaining cars in the first available spot.

Let us first establish that this process results in a (321)-avoiding permutation. Assume that $(\sigma_I, \sigma_J, \sigma_K)$ is a (321) pattern in the resulting permutation σ. If car σ_I is a vip car, then cars σ_J and σ_K are regular cars since vip cars are parked in increasing order. But regular cars are also parked in increasing order and thus $\sigma_J < \sigma_K$, which contradicts our assumption. If car σ_I is a regular car, then cars σ_J and σ_K are vip cars. Thus $\sigma_J < \sigma_K$, which again contradicts our assumption. Therefore $\sigma \in S_n(321)$.



$$\longleftrightarrow \quad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 1 & 5 & 2 & 4 & 7 & 8 & 6 \end{pmatrix}$$

Figure 1.6: Example of bijection between $\mathcal{D}_n$ and $S_n(321)$

Now let us establish that this is an injection. It is clear that a path is uniquely determined by the set of vip cars in conjunction with their preferred spots. This is because once we know where the vip cars prefer to park, the remaining cars prefer to park in the same spot as the vip car with the smallest label that is bigger than theirs. Therefore, given two distinct paths, the set of vip cars and their preferred spots corresponding to these paths must be different. Since the vip cars must park in their preferred spots, the permutations corresponding to these paths will be distinct.

Given $\sigma \in S_n(321)$, we can construct the path that corresponds to it by simply determining which cars are vip cars. We say that σ_i is a *salient point* if for all $j < i$, $\sigma_j < \sigma_i$. We identify car σ_i to be a vip car if and only if σ_i is a salient point. Notice that the remaining cars must be parked in increasing order. If not, then there are two cars α and β such that $\alpha > \beta$ and car α is parked before car β. Since car α was not a salient point, there exists at least one car γ such that (γ, α, β) is a (321) pattern, and this contradicts our assumption.

Now that we have identified the vip cars, we can easily reconstruct the path. It remains to show that the path stays weakly above the line $y = x$. Clearly the path will begin and end weakly above the line $y = x$ since σ_1 and n will always be classified as vip cars. Therefore, if the path crosses the line $y = x$ once, it must cross again. Assume that our path crosses the line $y = x$ from below at the point (J, J). This implies that σ_{J+1} is a salient point and that J is not. On the other hand, the J spots prior to σ_{J+1} will eventually be filled with cars 1 through J, which implies that J is a salient point. Therefore the path cannot cross the line $y = x$. This of course implies that $\pi_\sigma \in \mathcal{D}_n$ and thus our correspondence is a bijection. This in turn establishes

Lemma 1.4 $s_n(321) = \frac{1}{n+1}\binom{2n}{n}$,

which completes the proof of Theorem 1.1.

We should again point out that Christian Krattenthaler [7] independently discovered this construction described in terms of (123)-avoiding permutations.

Notice that to any Dyck path, we can now associate several permutations. In particular, we can associate a (132)⇔avoiding permutation and a (321)⇔avoiding permutation to a specific Dyck path rather easily. This yields a very natural bijection between $S_n(132)$ and $S_n(321)$ by associating $\sigma_1 \in S_n(132)$ with $\sigma_2 \in S_n(321)$ if and only if σ_1 and σ_2 correspond to the same element of $\mathcal{D}_n$. Using Lemma 1.2, we can convert σ_2 into an element of $S_n(123)$ in two ways, each one representing a different bijection between $S_n(132)$ and $S_n(123)$.

In view of Figure 1.2, once a correspondence between $S_n(132)$ and $S_n(123)$ has been made, we can create several other bijections. Of all the bijections that we have encountered, the only difference has been in making use of Lemma 1.2. In the remaining sections of this chapter we will examine the relationship between these correspondences and these other well known bijections.

1.7 Simion-Schmidt algorithm

Rodica Simion and Frank Schmidt [12] establish a bijection between the sets $S_n(123)$ and $S_n(132)$. In addition to being one of the first, their bijection has the added benefit that elements of $S_n(123, 132)$ are left unchanged. The bijection is broken down into the following two algorithms.

Algorithm A

Input:	$\sigma = (\sigma_1, \ldots, \sigma_n) \in S_n(123)$
Output:	$\tau = (\tau_1, \ldots, \tau_n) \in S_n(132)$
Step 1:	$i = 1$
Step 2:	$\tau_1 = \sigma_1$, $x = \sigma_1$
Step 3:	$i = i + 1$ if $i > n$ then exit.
Step 4:	If $x > \sigma_i$ then $\tau_i = \sigma_i$ and $x = \sigma_i$ else $\tau_i = min\{k \vert x < k \leq n, k \neq \tau_j \text{ for all } j < i\}$. Repeat step 3

Algorithm B

Input:	$\tau = (\tau_1, \ldots, \tau_n) \in S_n(132)$
Output:	$\sigma = (\sigma_1, \ldots, \sigma_n) \in S_n(123)$
Step 1:	$i = 1$
Step 2:	$\sigma_1 = \tau_1$, $x = \tau_1$
Step 3:	$i = i + 1$ if $i > n$ then exit.
Step 4:	If $x > \tau_i$ then $\sigma_i = \tau_i$ and $x = \tau_i$ else $\sigma_i = max\{k \vert k \leq n, k \neq \sigma_j \text{ for all } j < i\}$. Repeat step 3

We will now demonstrate how our algorithm relates to that given by Simion and Schmidt. Let τ be an arbitrary (132)-avoiding permutation of length n. If $\sigma \in S_n(123)$ is the result of applying Algorithm B to τ and $\alpha \in S_n(321)$ is the result of converting τ into $\pi \in \mathcal{D}_n$ and then interpreting π as an element of $S_n(321)$, then we aim to show that

$$\sigma = \alpha^c. \tag{1.1}$$

The act of complementing α in terms of π means that the vertical segments of

π are labeled from top to bottom, vip cars are identified as being the earliest car that prefers to park in a given spot and that non-vip cars are parked in reverse order. In our original treatment, we choose to deal with $S_n(321)$ instead of $S_n(123)$ so as to be able to maintain the order in which cars are parked. Nevertheless, using this method of converting π into $\beta = \alpha^c$, it is easy to establish 1.1.

First, notice that in both our methods of converting π into $\tau \in S_n(132)$ and $\beta \in S_n(123)$, the first car that wants to park in spot 1 will necessarily do so, and since in both methods we label the vertical segments from top to bottom, this will be the same car. In other words, $\beta_1 = \tau_1$. Let us call this common value x, so as to compare with Algorithm B.

Now for all $i \geq 2$, if $\tau_i < x$ then τ_i must have parked in its preferred spot. This would identify it as being a vip car and therefore $\beta_i = \tau_i$ and we reset x to be τ_i. If $\tau_i > x$, then τ_i did not park in its preferred spot and β_i would be filled by the highest labelled non-vip car remaining, which is equivalent to setting β_i to be $max\{k|k \leq n, k \neq \beta_j$ for all $j < i\}$.

For example, the permutation $\tau = (67324158) \in S_8(132)$ would correspond to the element of $\mathcal{D}_8$ in Figure 1.7. This path also corresponds to $\alpha = (31672845) \in S_8(321)$ and therefore $\beta = (68327154) \in S_8(123)$.

Simion and Schmidt also pointed out that for any value $1 \leq x \leq n$,

$$|\{\sigma \in S_n(123)|\sigma_1 = x\}| = |\{\sigma \in S_n(132)|\sigma_1 = x\}|. \tag{1.2}$$

This can be easily seen using our construction for elements of $S_n(132)$ and $S_n(321)$. Fix a path $\pi \in \mathcal{D}_n$ that starts with exactly s steps north. The permutation $\alpha \in S_n(132)$ that corresponds to π will have the property that $\alpha_1 = n \Leftrightarrow s + 1$, since the vertical segments were labeled from top to bottom and the first car that wants to park in a particular spot

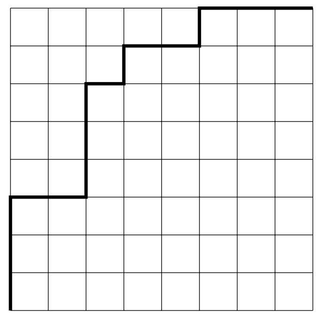


Figure 1.7: Path corresponding to $(67324158) \in S_8(132)$ and $(68327154) \in S_8(123)$

will do so. The permutation $\beta \in S_n(321)$ that corresponds to π will have the property that $\beta_1 = s$ and therefore $\beta^c \in S_n(123)$ will have the property that $\beta_1^c = n + 1 \Leftrightarrow s$. Therefore, the common value in 1.2 is the number of elements of $\mathcal{D}_n$ that start with exactly $n + 1 \Leftrightarrow x$ steps north, which is given by

$$\frac{n + 1 \Leftrightarrow x}{n} \binom{n + x \Leftrightarrow 2}{n \Leftrightarrow 1}.$$

1.8 West's bijection

In [15], Julian West uses bracketing sequences to enumerate (231) avoiding permutations, the so-called stack-sortable permutations, see [6]. Using a natural bijection between bracketing sequences and Dyck paths, we can easily establish the relationship between. The bracketing sequences, $\mathcal{B}_n$, is the set of all sequences consisting of n left parentheses, '(', and n right parentheses, ')', so that at no point does the number of ')' exceed the number of '(' when reading from left to right. The natural bijection between $\mathcal{B}_n$ and $\mathcal{D}_n$ is to replace each '(' with a step NORTH and each ')' with a step EAST.

The path shown in Figure 1.1 would translate into the following bracketing sequence.

$$((())(()))(()()).$$

To convert the above sequence into an element of $S_n(231)$, simply label each ')' from left to right and label each '(' with the same label of the ')' that closes it off. The above sequence would be labelled as follows

$$\underset{5}{(}\underset{2}{(}\underset{1}{(}\overset{1}{)}\overset{2}{)}\underset{4}{(}\underset{3}{(}\overset{3}{)}\overset{4}{)}\overset{5}{)}\underset{8}{(}\underset{6}{(}\overset{6}{)}\underset{7}{(}\overset{7}{)}\overset{8}{)}.$$

The corresponding (231)-avoiding permutation would be $(5, 2, 1, 4, 3, 8, 6, 7)$. Notice that this matches exactly our interpretation of σ^{ci} as given in Section 1.5.

To complete the bijection between $S_n(132)$ and $S_n(123)$, West uses rooted trees to enumerate elements of $S_n(123)$.

Chapter 2

Generating functions

Using the techniques established in the previous chapter, we can identify generating functions that count the number of particular patterns in permutations that avoid (132) or (321).

2.1 Introduction

To any Dyck path, π, we associate a word, $w(\pi)$, in the alphabet

$$\{a_0, a_1, \ldots, c_0, c_1, \ldots\}.$$

To this end, label each step NORTH from height i to $i+1$ with an a_i only if it is followed by another step NORTH, otherwise label it with c_i. Read off the labels from bottom to top to form $w(\pi)$. For example, the path in Figure 1.1 is associated with the word $a_0a_1c_2a_1c_2a_0c_1c_1$. We also define a shift operator S that acts on $w(\pi)$ by replacing each a_i by a_{i+1} and each c_i by c_{i+1}. For example, $S[a_0a_1c_2a_1c_2a_0c_1c_1] = a_1a_2c_3a_2c_3a_1c_2c_2$.

Let $C(z; a, c)$ be the generating function for words associated to Dyck paths defined by

$$C(z;a,c) = \sum_{n \geq 0} \sum_{\pi \in \mathcal{D}_n} w(\pi) z^n.$$

We can further decompose the words associated with Dyck paths based on whether the first letter of $w(\pi)$ is an a_0 or a c_0. This enables us to write down the following recursive formula for $C(z;a,c)$,

$$C(z;a,c) = 1 + za_0(S[C(z;a,c)] \Leftrightarrow 1)C(z;a,c) + zc_0 C(z;a,c).$$

Solving for $C(z;a,c)$ yields the following continued fraction expansion

$$C(z;a,b) = \cfrac{1}{1 + (a_0 \Leftrightarrow c_0)z \Leftrightarrow \cfrac{a_0 z}{1 + (a_1 \Leftrightarrow c_1)z \Leftrightarrow \cfrac{a_1 z}{1 + (a_2 \Leftrightarrow c_2)z \Leftrightarrow \cdots}}}. \tag{2.1}$$

In particular, if we let $C(z)$ be $C(z;a,c)$ with $a_i = c_i = 1$ for all $i \geq 0$, then

$$C(z) = 1 + zC(z)^2, \tag{2.2}$$

and therefore we can solve for $C(z)$ explicitly

$$C(z) = \frac{1 \Leftrightarrow \sqrt{1 \Leftrightarrow 4z}}{2z}, \tag{2.3}$$

which is the well-known generating function for the Catalan sequence. By making specific replacements for a_i and c_i, we yield various generating functions for specific pattern avoiding permutations.

2.2 A generating function for (132)-avoiding permutations

Let $\sigma \in S_n(132)$ and let $\pi_\sigma \in \mathcal{D}_n$ be the path corresponding to σ. Recall that a step NORTH originating from the point $(x, x+i)$ means that car $n \Leftrightarrow x \Leftrightarrow i$ prefers to park in spot $x+1$. This implies that after car $n \Leftrightarrow x \Leftrightarrow i$ parks, spots 1 through x will all be available. Since each of the remaining $x+i$ cars prefers to park in one of the first x spots, exactly i of them will be forced to park in the first available spot after car $n \Leftrightarrow x \Leftrightarrow i$. Therefore, for any $k \geq 1$, there are exactly $\binom{i}{k-1}$ patterns of the form $(123\ldots k)$ in σ that use $n \Leftrightarrow x \Leftrightarrow i$ as the "1". In other words, if we define the sequence $\{s_{n,r}^k\}$ by the following,

$$\sum_{r\geq 0} s_{n,r}^k q^r = \sum_{\pi \in \mathcal{D}_n} w(\pi)\Big|_{a_i, c_i \to q^{\binom{i}{k-1}}} \tag{2.4}$$

then $s_{n,r}^k$ will denote the number of elements of $S_n(132)$ that have exactly r occurrences of the pattern $(123\ldots k)$. More generally, if we replace a_i and c_i by $q^{\binom{i}{k-1}}$ in (2.1), we have

Theorem 2.1 *The generating function for the sequence* $\{s_{n,r}^k\}_{n,r\geq 0}$ *is given by:*

$$\sum_{n,r\geq 0} s_{n,r}^k z^n q^r = \cfrac{1}{1 \Leftrightarrow \cfrac{zq_0}{1 \Leftrightarrow \cfrac{zq_1}{1 \Leftrightarrow \cfrac{zq_2}{1 \Leftrightarrow \cdots}}}}$$

where $q_i = q^{\binom{i}{k-1}}$.

Note that this was originally proven by Robertson, Wilf, and Zeilberger [11] in the case when $k=3$ and in full generality by Mansour and Vainshtein [9], however this method of proof is new.

Corollary 2.2 $s_n(132, 123\ldots k)$ = *the number of Dyck paths from* $(0,0)$ *to* (n,n) *that are bounded above by the line* $y = x + k \Leftrightarrow 1$.

Proof. Let $\pi \in \mathcal{D}_n$ and let σ_π be the (132)-avoiding permutation corresponding to π. From the proof of Theorem 2.1, we know that if $w(\pi)$ contains an a_{k-1} or a c_{k-1}, then σ_π will contain a $(12\ldots k)$ pattern. Therefore π is bounded above by $y = x + k \Leftrightarrow 1$. □

Corollary 2.3 *For all $n \geq 1$, $s_n(132, 123) = 2^{n-1}$*

Proof. Using Corollary 2.2, a path $\pi \in \mathcal{D}_n$ which corresponds to an element of $S_n(132, 123)$ must remain weakly below the line $y = x + 2$. Since π necessarily starts with a step NORTH, each of the remaining $n \Leftrightarrow 1$ steps NORTH can be paired off with a step EAST that immediately precedes or immediately follows it. In other words, the generating function for such paths is given by

$$a_0(a_0 + c_1)^{n-1}$$

since $a_0 = c_0$. Therefore there are 2^{n-1} such paths. □

Corollary 2.4 *For all $n \geq 2$, $|S_n(132) \cap S_n^1(123)| = (n \Leftrightarrow 2)2^{n-3}$*

Proof. Making use of the replacements given in (2.4), it is clear that if σ_π has exactly one (123) pattern, then $w(\pi)$ must contain exactly one c_2, and in fact this c_2 must be immediately preceded by an a_1. We can construct such a path as follows. Starting with a path $\pi \in \mathcal{D}_{n-2}$ such that σ_π is an element of $S_{n-2}(132, 123)$, simply insert the word a_1c_2 into $w(\pi)$ after any one of its $n \Leftrightarrow 2$ letters. Applying Corollary 2.3 completes the proof. □

Another way to look at the above argument is to think of the c_2 as "gluing" together a pair of nonempty paths $(\pi_1, \pi_2) \in \mathcal{D}_\alpha \times \mathcal{D}_\beta$, where $\alpha + \beta = n \Leftrightarrow 1$. The process of "gluing" two paths together is formally defined by first replacing the last step of π_1

with a step NORTH, then replacing the first step of π_2 with a step EAST and finally connecting these two paths together with a step NORTH followed by a step EAST. This process is illustrated in Figure 2.1.

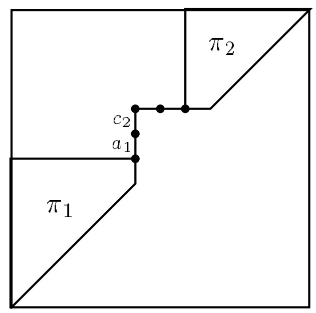


Figure 2.1: "Gluing" together two Dyck paths

Corollary 2.5 *For all $n \geq 3$, $|S_n(132) \cap S_n^2(123)| = n(n \Leftrightarrow 3)2^{n-6}$*

Proof. Again making use of the replacements made in (2.4), it is clear that for σ_π to have exactly two (123) patterns, $w(\pi)$ must contain exactly two c_2's. We can construct such a path as follows. Take a triple of paths, (π_1, π_2, π_3), from $\mathcal{D}_\alpha \times \mathcal{D}_\beta \times \mathcal{D}_\gamma$ where $\alpha + \beta + \gamma = n \Leftrightarrow 2$ and only β is allowed to be zero. Make sure that each path corresponds to a permutation that avoids both (132) and (123). Form π by gluing the three paths together. When $\beta = 0$, the two c_2's in $w(\pi)$ are adjacent. This means that the two (123) patterns will have the same "2" and "3". In this case there are exactly $2^{\alpha-1}2^{\gamma-1}$ pairs of paths for each of the $n \Leftrightarrow 3$ solutions to $\alpha + \gamma = n \Leftrightarrow 2$. If $\beta > 0$, then the two c_2's are not adjacent and the two (123) patterns will either be disjoint or share the "3". In this case, there are exactly $2^{\alpha-1}2^{\beta-1}2^{\gamma-1}$ triples of paths for each of the $\binom{n \Leftrightarrow 3}{2}$ solutions to $\alpha + \beta + \gamma = n \Leftrightarrow 2$. Therefore we have that the total number of paths is given by

$$(n \Leftrightarrow 3)2^{n-4} + \binom{n \Leftrightarrow 3}{2} 2^{n-5} = n(n \Leftrightarrow 3)2^{n-6}.$$

$\square$

Corollary 2.6 *For all* $n \geq 4$, $|S_n(132) \cap S_n^3(123)| = \frac{(n-4)(n-2)(n+3)}{3} 2^{n-8}$

Proof. Following the method of proofs established in Corollaries 2.4 and 2.5, we see that in order to construct elements of $S_n(132) \cap S_n^3(123)$, we will need 4-tuples of paths $(\pi_1, \pi_2, \pi_3, \pi_4)$, from $\mathcal{D}_\alpha \times \mathcal{D}_\beta \times \mathcal{D}_\gamma \times \mathcal{D}_\delta$ where $\alpha + \beta + \gamma + \delta = n \Leftrightarrow 3$ and only β and γ can be zero.

If both β and γ are zero, then there are exactly $2^{\alpha-1}2^{\delta-1}$ pairs of paths for each of the $n \Leftrightarrow 4$ solutions to $\alpha + \delta = n \Leftrightarrow 3$. If exactly one of β and γ is zero (assume $\gamma = 0$) then there are exactly $2^{\alpha-1}2^{\beta-1}2^{\delta-1}$ triples of paths for each of the $\binom{n-4}{2}$ solutions to $\alpha + \beta + \delta = n \Leftrightarrow 3$. And finally, if neither β nor γ is zero, then there are $2^{\alpha-1}2^{\beta-1}2^{\gamma-1}2^{\delta-1}$ 4-tuples of paths for each of the $\binom{n-4}{3}$ solutions to $\alpha + \beta + \gamma + \delta = n \Leftrightarrow 3$. Therefore we have that the total number of paths is given by

$$(n \Leftrightarrow 4)2^{n-5} + 2\binom{n \Leftrightarrow 4}{2}2^{n-6} + \binom{n \Leftrightarrow 4}{3}2^{n-7} = \frac{(n \Leftrightarrow 4)(n \Leftrightarrow 2)(n+3)}{3}2^{n-8}.$$

$\square$

If we were to continue to look at specific values of $|S_n(132) \cap S_n^r(123)|$ for $r \geq 4$, we would have to start taking into account the additional possibly of c_3 and a_2 appearing in $w(\pi)$. In the next section, we will have a different specialization for a_i starting with a_2, so for the time being, we will stop here.

2.3 A generating function for (321)-avoiding permutations

Let $\sigma \in S_n(321)$ and let $\pi_\sigma \in \mathcal{D}_n$ be the path corresponding to σ. Recall that a step NORTH originating from the point $(x, x+i)$ means that car $x+i+1$ prefers to park in spot $x+1$. If car $x+i+1$ is a vip car, or in other words, the step NORTH is immediately followed by a step EAST, then car $x+i+1$ will necessarily park in spot $x+1$. It also means that at most x of the cars 1 through $x+i$ are vip cars and thus exactly i non-vip cars will park in the first available spot after car $x+i+1$. Therefore, for any $k \geq 1$, there are exactly $\binom{i}{k-1}$ patterns of the form $(k12\ldots k \Leftrightarrow 1)$ in σ that use $x+i+1$ as the "k". If car $x+i+1$ is not a vip car, no such pattern can be formed. In other words, if we define the sequence $\{t^k_{n,r}\}$ by the following,

$$\sum_{r\geq 0} t^k_{n,r} q^r = \sum_{\pi\in S_n(321)} w(\pi)|_{a_i\to 1, c_i\to q^{\binom{i}{k-1}}}$$

then $t^k_{n,r}$ will denote the number of elements of $S_n(321)$ that have exactly r occurrences of the pattern $(k12\ldots k \Leftrightarrow 1)$. More generally, if we replace a_i by 1 and c_i by $q^{\binom{i}{k-1}}$ in 2.1, we have

Theorem 2.7 *The generating function of the sequence $\{t^k_{n,r}\}_{n,r\geq 0}$ is given by:*

$$\sum_{n,r\geq 0} t^k_{n,r} z^n q^r = \cfrac{1}{1 \Leftrightarrow zq_0 \Leftrightarrow \cfrac{z}{1 \Leftrightarrow zq_1 \Leftrightarrow \cfrac{z}{1 \Leftrightarrow zq_2 \Leftrightarrow \cfrac{z}{1 \Leftrightarrow zq_3 \Leftrightarrow \cdots}}}}$$

where $q_i = q^{\binom{i}{k-1}} \Leftrightarrow 1$.

Note that this was originally proven by Krattenthaler [7].

Each of the next three corollaries is a consequence of Corollaries 2.3-2.5 and the fact that a_0 and a_1 have the same weight in the construction of (132)-avoiding permutations as they do in the construction of (321)-avoiding permutations.

Corollary 2.8 $|S_n(321, k12\ldots k-1)|$ = *the number of Dyck paths from* $(0,0)$ *to* (n,n) *that are bounded above by the line* $y = x + k - 1$.

Corollary 2.9 *For all* $n \geq 1$, $|S_n(321, 312)| = 2^{n-1}$

Corollary 2.10 *For all* $n \geq 2$, $|S_n(321) \cap S_n^1(312)| = (n-2)2^{n-3}$

Corollary 2.11 *For all* $n \geq 3$, $|S_n(321) \cap S_n^2(312)| = n(n-3)2^{n-6}$

For this last corollary, we may use Corollary 2.6 as a starting point, but notice that it is entirely possible for a path to consist only of the letters a_0, a_1, c_1, a_2 and c_3. In particular, a path having exactly one c_3 appearing in it and no c_2's would correspond to an element of $S_n(321) \cap S_n^3(312)$. Mimicking the proof of Corollary 2.4, we see that there are exactly $(n-3)2^{n-4}$ such paths. Therefore,

Corollary 2.12 *For all* $n \geq 4$, $|S_n(321) \cap S_n^3(312)| = \frac{2^{n-8}}{3}(n^3 - 3n^2 + 38n - 120)$.

Chapter 3

Permutations with one occurrence of the pattern (132)

In this chapter we offer constructions for generating a variety of permutations that have exactly one occurrence of the pattern (132).

3.1 Constructing permutations with one (132) pattern

Miklós Bóna [2] established that

Theorem 3.1 $s_n^1(132) = \binom{2n-3}{n-3}$.

Bóna's proof relies heavily on the following recursion

$$B_n = \sum_{I=4}^{n} B_{I-1}C_{n-I} + \sum_{I=1}^{n-3} C_{I-1}B_{n-I} + \sum_{I=2}^{n-1} C_{I-2}C_{n-I} \tag{3.1}$$

with initial conditions $B_0 = B_1 = B_2 = 0$, where for convenience we have set $B_n = s_n^1(132)$ and $C_n = \binom{2n}{n}/(n+1)$. This recursion comes from the fact that we can partition the set $S_n^1(132)$ into three subsets. If we let I be the number such that $\sigma_I = n$, then the first subset consists of permutations such that the (132) pattern occurs *before* the I^{th} position, the second subset consists of permutations such that the (132) pattern occurs *after* the I^{th} position and the last subset consists of permutations such that n acts as the "3" in the (132) pattern. In our case, we would like to prove the above theorem by directly constructing the relevant permutations. We begin with the following lemma.

Lemma 3.2 *Let $\sigma \in S_n^1(132)$. Then the only (132) pattern that occurs in σ must be of the form*

$$(\sigma_I, \sigma_{I+1}, \sigma_I + 1).$$

for some $1 \leq I \leq n - 2$ and $1 \leq \sigma_I < n - I$

Proof. Let $\sigma \in S_n^1(132)$ and let $\{\sigma_I, \sigma_J, \sigma_K\}$ be the only (132) pattern that appears in σ for some $I < J < K$. If $J > I + 1$ and $\sigma_{I+1} < \sigma_K$ then $(\sigma_{I+1}, \sigma_J, \sigma_K)$ is another (132) pattern. If $J > I + 1$ and $\sigma_{I+1} > \sigma_K$ then $(\sigma_I, \sigma_{I+1}, \sigma_K)$ is another (132) pattern. Therefore we must have $J = I + 1$. If $\sigma_K > \sigma_I + 1$ then $\sigma_i = \sigma_I + 1$ for some $i \neq K$. If $i < I$ then $(\sigma_i, \sigma_{I+1}, \sigma_K)$ is another (132) pattern. If $i > I$ then $(\sigma_I, \sigma_{I+1}, \sigma_i)$ is another (132) pattern. Therefore we conclude that $\sigma_K = \sigma_I + 1$. Finally, since each of the $\sigma_I - 1$ numbers less σ_I must appear after spot $I + 1$, we conclude that $\sigma_I < n - I$. $\square$

A consequence of the above lemma is that we can construct all elements of $S_n^1(132)$ for a fixed I and σ_I in the following manner. Let $\pi \in \mathcal{D}_n$ be a path that has the following properties:

1. The I^{th} step EAST is preceded by at least two steps NORTH and followed by at least one step EAST,

2. The two steps NORTH that precede the I^{th} step EAST cannot start on the line $y = x$.

For instance, the path illustrated in Figure 3.1 has these properties for $I = 2$, 4, and 6. To form the corresponding element of $S_n^1(132)$, first form the (132)-avoiding permutation, α, corresponding to π by labelling the vertical segments from top to bottom and then parking the cars accordingly. Property 1 means that we must have $\alpha_{I+1} = 1+\alpha_I$. Property 2 insures that there is at least one value $k > I+1$ such that $\alpha_k > \alpha_I + 1$. Let K be the smallest such value of k. Now switch the cars in spot $I+1$ with the car in spot K. Let σ be the permutation that sends i to the car that now resides in spot i.

Clearly σ has at least one (132) pattern, namely $(\sigma_I, \sigma_{I+1}, \sigma_I + 1)$. Notice also that for all i and j such that $\sigma_i, \sigma_j < \sigma_I$ where $I+1 < i < K < j$, we must also have $\sigma_i > \sigma_j$. This insures that the process of switching the cars parked in spots $I+1$ and K cannot introduce any other (132) patterns, and thus $\sigma \in S_n^1(132)$.

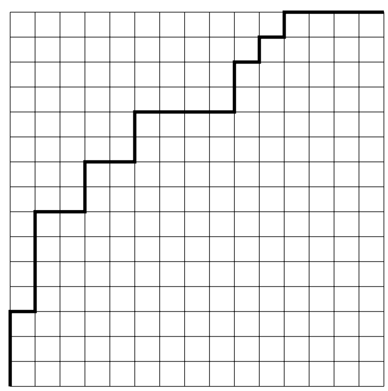


Figure 3.1: $\pi \in \mathcal{D}_{15}$ corresponding to an element of $S_{15}^1(132)$

For $I = 2$, the permutation corresponding to the path in Figure 3.1 is given by

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ 13 & \underline{9} & \underline{11} & 7 & 8 & 5 & 6 & \underline{10} & 12 & 3 & 2 & 1 & 4 & 14 & 15 \end{pmatrix},$$

for $I = 4$, it is given by

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ 13 & 9 & 10 & \underline{7} & \underline{11} & 5 & 6 & \underline{8} & 12 & 3 & 2 & 1 & 4 & 14 & 15 \end{pmatrix},$$

and for $I = 6$, it is given by

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ 13 & 9 & 10 & 7 & 8 & \underline{5} & \underline{11} & \underline{6} & 12 & 3 & 2 & 1 & 4 & 14 & 15 \end{pmatrix}.$$

In each case, the only (132) pattern is underlined.

Given our construction, we can decompose π into π_1, the portion of π that occurs prior to the two steps NORTH followed by the I^{th} step EAST, and π_2, the portion of π that occurs after the $(I+1)^{st}$ step EAST. This decomposition is illustrated in Figure 3.2. For $I = 6$. the path in Figure 3.1 is decomposed in Figure 3.3.

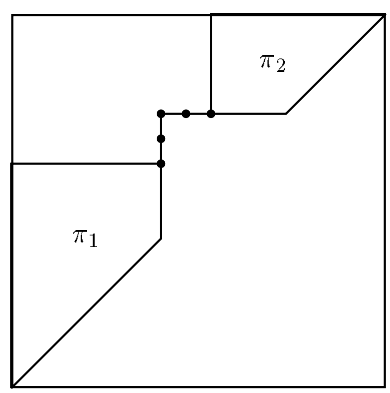


Figure 3.2: Decomposition of π

Since the number of paths from $(0, 0)$ to (a, b) for $b \geq a$ that stay weakly above the line $y = x$ is given by

$$\frac{b + 1 \Leftrightarrow a}{b + 1}\binom{a+b}{a},$$

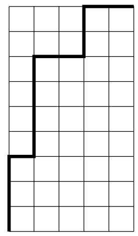
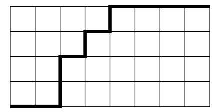
$\pi_1 =$  $\pi_2 =$ 

Figure 3.3: Sample decomposition of π

we see that the total number of relevant paths is

$$\sum_{I=1}^{n-2}\sum_{\sigma_I=1}^{n-I-1}\frac{(n-I-\sigma_I+1)^2}{(n-I)(n-\sigma_I)}\binom{n-I+\sigma_I-2}{n-I-1}\binom{n-\sigma_I+I-2}{n-\sigma_I-1}. \tag{3.2}$$

Making the change of variable $d = I + \sigma_I$, (3.2) becomes

$$\sum_{d=2}^{n-1}\sum_{\sigma_I=1}^{d-1}\frac{(n-d+1)^2}{(n-d+\sigma_I)(n-\sigma_I)}\binom{n-d+2\sigma_I-2}{n-d+\sigma_I-1}\binom{n-2\sigma_I+d-2}{n-\sigma_I-1},$$

which can be transformed into the following summation

$$\sum_{d=2}^{n-1}\frac{n-d+1}{n-1}\binom{n+d-4}{d-2}\sum_{s=0}^{d-2}\frac{(2-d)_s(1-n)_s(\frac{n-d}{2}+1)_s(\frac{n-d+1}{2})_s}{(1)_s(n-d+2)_s(\frac{-n-d}{2}+2)_s(\frac{-n-d+1}{2}+2)_s} \tag{3.3}$$

where $(a)_n = a(a+1)\cdots(a+n-1)$. Now the inner summation is in the form of a well-poised hypergeometric series. Using Clausen's ${}_4F_3$ formula, (3.3) becomes

$$\sum_{d=2}^{n-1}\frac{n-d+1}{n-1}\binom{2n-2}{d-2} = \binom{2n-3}{n-3}. \tag{3.4}$$

which follows from the binomial theorem.

Our next task is to enumerate certain subsets of $S_n^1(132)$. To this end, we will need the following

Lemma 3.3 *Let $\sigma \in S_n^1(132)$ and let $I < I+1 < K$ be the indices of the lone (132) pattern in σ. For all $k > K$ such that $\sigma_k > \sigma_{I+1}$,*

$$(\sigma_I, \sigma_{I+1}, \sigma_k) \quad \text{and} \quad (\sigma_I, \sigma_I+1, \sigma_k)$$

are (123) patterns in σ. Additionally, for all pairs $j, k > K$, $j < k$ such that $\sigma_j, \sigma_k >$ σ_{I+1},

$$(\sigma_I, \sigma_j, \sigma_k) \quad \text{and} \quad (\sigma_{I+1}, \sigma_j, \sigma_k) \quad \text{and} \quad (\sigma_I + 1, \sigma_j, \sigma_k)$$

are (123) patterns in σ.

3.2 Constructing elements of $S_n^1(132) \cap S_n(123)$

Using Corollary 2.10 and Lemma 1.2, we know that the number of (123)-avoiding permutations that have exactly one occurrence of the pattern (132) is $(n \Leftrightarrow 2)2^{n-3}$. Recall that this is also the number of (132)-avoiding permutations that have exactly one occurrence of the pattern (123), see Corollary 2.4. In light of Lemma 3.3, a path π that corresponds to an element of $S_n^1(132) \cap S_n(123)$ can be decomposed into (132,123)-avoiding paths π_1 and π_2 as illustrated in Figure 2.1. In other words, to construct an element of $S_n^1(132) \cap S_n(123)$, simply take an element of $S_n(132) \cap S_n^1(123)$ and swap the "2" and the "3" of the (123) pattern. For example, the path illustrated in Figure 3.4 corresponds to the following element of $S_n(132) \cap S_n^1(123)$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 11 & 12 & 9 & 8 & \underline{6} & \underline{7} & 5 & 4 & 3 & \underline{10} & 2 & 1 \end{pmatrix},$$

where the lone (123) pattern has been underlined. Swapping the "2" and the "3" yields

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 11 & 12 & 9 & 8 & \underline{6} & \underline{10} & 5 & 4 & 3 & \underline{7} & 2 & 1 \end{pmatrix},$$

which is an element of $S_n^1(132) \cap S_n(123)$. Notice that since all of the numbers between the "2" and the "3" must be less than the "2" and are in decreasing order, this process of swapping will not introduce any more (132) patterns and it will not create any new (123) patterns.

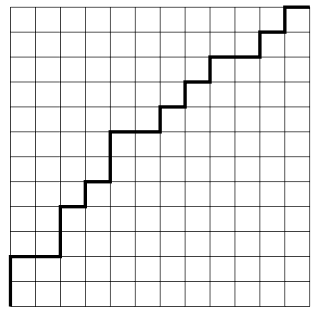


Figure 3.4: Constructing an element of $S^1_{12}(132) \cap S_{12}(123)$

Theorem 3.4 *The number of permutations of length $n \geq 4$ that have exactly one occurrence of the pattern (132) and avoid the pattern (123) is given by*

$$(n - 2)2^{n-3}.$$

.

3.3 Constructing elements of $S^1_n(132) \cap S^1_n(123)$

To construct an element of $S^1_n(132) \cap S^1_n(123)$, we start with an element of $S_n(132) \cap S^2_n(123)$ and convert one of the (123) patterns into a (132) pattern, while leaving the other (123) pattern unchanged. This can only be accomplished if the (123) patterns are disjoint or if they only share the "3". The path in Figure 3.5 illustrates such a permutation. From the proof of Corollary 2.5, the number of such permutations is

$$\binom{n - 3}{2} 2^{n-5}.$$

Since there are two choices for which (123) pattern is to be converted into a (132) pattern, we have

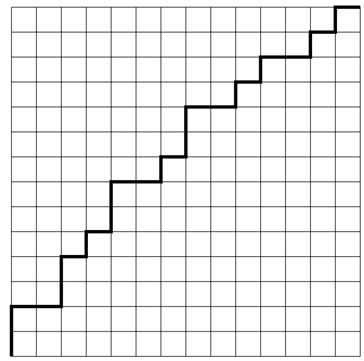


Figure 3.5: Constructing an element of $S^1_{14}(132) \cap S^1_{14}(123)$

Theorem 3.5 *The number of permutations of length $n \geq 4$ that have exactly one occurrence of the pattern (132) and exactly one occurrence of the pattern (123) is given by*

$$(n \Leftrightarrow 3)(n \Leftrightarrow 4)2^{n-5}.$$

Alternatively, we could start with a path π in $S^1_{n-2}(132) \cap S_{n-2}(123)$ and insert two steps NORTH followed by two steps EAST at any one of the $n \Leftrightarrow 3$ places where π touches the line $y = x + 1$. Using Theorem 3.4, this again results in $(n \Leftrightarrow 3)(n \Leftrightarrow 4)2^{n-5}$ permutations. For example, the path in Figure 3.5 would be interpretted as

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\ 13 & 14 & 11 & 10 & \underline{8} & \underline{12} & 7 & \overline{5} & \overline{6} & 4 & 3 & \underline{9} & 2 & 1 \end{pmatrix},$$

if we converted the first (123) pattern into a (132) pattern,or as

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\ 13 & 14 & 11 & 10 & \overline{8} & \overline{9} & 7 & \underline{5} & \overline{\underline{12}} & 4 & 3 & \underline{6} & 2 & 1 \end{pmatrix},$$

if we converted the second (123) pattern into a (132) pattern. In both permutations, the (132) pattern is underlined and the (123) pattern is barred.

Chapter 4

Permutations with one occurrence of the pattern (321)

In this chapter we offer constructions for generating a variety of permutations that have exactly one occurrence of the pattern (321).

4.1 Constructing permutations with one (321) pattern

John Noonan [10] established that

Theorem 4.1 $s_n^1(321) = \frac{3}{n}\binom{2n}{n+3}$.

Noonan offers a proof that looks at these permutations in a more general setting and shows that they must satisfy a certain recursion. From our point of view, it will prove more beneficial to have a constructive proof of the above theorem.

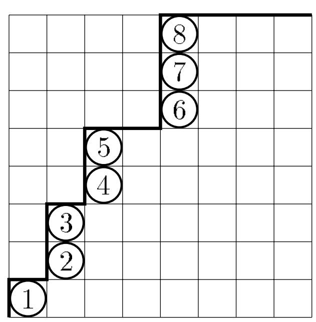
Proof. We begin by assuming that the values $\{\sigma_I, \sigma_J, \sigma_K\}$ form the lone (321) pattern in $\sigma \in S_n$ for some $I < J < K$. Thus if $i < J$ and $i \neq I$, then $\sigma_i < \sigma_J$, otherwise $\{\sigma_i, \sigma_J, \sigma_K\}$ would form another (321) pattern. Similarly, if $i > J$ and $i \neq K$, then

$\sigma_i > \sigma_J$. Therefore σ maps $\{1, 2, \ldots, J\}$ bijectively on to the set $\{1, 2, \ldots, \sigma_K \Leftrightarrow 1, \sigma_K + 1, \ldots, \sigma_J, \sigma_I\}$, and we must conclude that $\sigma_J = J$.

We can now construct all permutations $\sigma \in S_n^1(321)$ such that $\sigma_J = J$ acts as the "2" in the only (321) pattern in σ. Let $\pi_1 \in \mathcal{D}_J$ and $\pi_2 \in \mathcal{D}_{n-J+1}$. First, park the cars $\{1, 2, \ldots, J\}$ in spots 1 through J according to π_1, avoiding a (321) pattern. Let I be the position in which car J parks and let σ_K be the car that parks in spot J. Notice that we must exclude the case when car J parks in spot J since $\sigma_K < J$. This implies that π_1 ends with at least two steps EAST because car J will always park in its preferred spot. Therefore there are exactly

$$C_J \Leftrightarrow C_{J-1}$$

choices for π_1, where $C_n = \binom{2n}{n}/(n+1)$. This part of the process is illustrated in Figure 4.1.



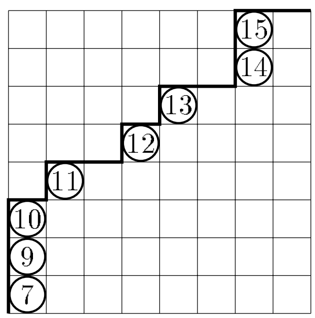
$$\longleftrightarrow \quad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 3 & 5 & 2 & 8 & 4 & 6 & 7 \end{pmatrix}$$

Figure 4.1: Constructing elements of $S_n^1(321)$: Step 1

Now park cars $\{\sigma_K, J+1, J+2, \ldots, n\}$ in spots J through n according to π_2, avoiding a (321) pattern. Note that car σ_K is in fact being parked again. Let K be the position in which car σ_K now resides and let σ_I be the car that is now located in spot J. Notice that we must exclude the case when car σ_K parks in spot J since $K > J$. This implies that π_2 starts with at least two steps NORTH because car σ_K always prefers to park in spot J, but can only do so if it is a vip car. Therefore there are exactly

$$C_{n-J+1} \Leftrightarrow C_{n-J}$$

choices for π_2. This part of the process is illustrated in Figure 4.2.



$$\longleftrightarrow \quad \begin{pmatrix} 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ 10 & 11 & 7 & 12 & 13 & 9 & 15 & 14 \end{pmatrix}$$

Figure 4.2: Constructing elements of $S_n^1(321)$: Step 2

Finally, switch the two cars J and σ_I, which are in spots I and J respectively, and define σ to be the function that sends i to the car that now resides in spot i. Clearly σ has at least one (321) pattern created by $(\sigma_I, \sigma_J, \sigma_K)$, but a careful inspection of the above construction yields that this is the only (321) pattern. All of the cars that are eventually parked in spots 1 through J are labelled with a number less than or equal to J, except for the car that is in spot I. But car σ_I is conveniently placed in the spot that is filled with car J so as to not introduce any (321) patterns amoung spots 1 through J. Additionaly, σ_I's location relative to cars $J+1$ through n does not change. Similarly, when car σ_K is re-parked, it's location relative to cars 1 through J does not change. Therefore, the only way to have a (321) pattern is for the "3" to occur before spot J and the "1" to occur after spot J. J must act as the "2", otherwise we would have had a (321) pattern due to π_1 or π_2. Since σ_I is the only number prior to J that is bigger than J, it must act as the "3". Likewise, σ_K is the only number that occurs after J that is smaller than J, and thus acts as the "1". Therefore σ is in fact an element of $S_n^1(321)$.

The element in $S_{15}^1(321)$ given by Figures 4.1 and 4.2 is

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ 1 & 3 & 5 & 2 & \underline{10} & 4 & 6 & \underline{8} & 11 & \underline{7} & 12 & 13 & 9 & 15 & 14 \end{pmatrix},$$

where the numbers that make up the only (321) pattern are underlined.

Given an arbitrary element of $S_n^1(321)$, we can easily reverse each of the above steps. Therefore the total number of permutations with exactly one (321) pattern is given by

$$\sum_{J=2}^{n-1}(C_J \Leftrightarrow C_{J-1})(C_{n-J+1} \Leftrightarrow C_{n-J}). \tag{4.1}$$

If we let $F(x) = \sum_{i\geq 1}(C_i \Leftrightarrow C_{i-1})x^i$, then 4.1 is none other than the coefficient of x^{n+1} in $F(x)^2$. Notice that $F(x) = (1 \Leftrightarrow x)C(x) \Leftrightarrow 1$ and using 2.2, we have that

$$\begin{aligned} F(x)^2 &= (1 \Leftrightarrow x)^2C(x)^2 \Leftrightarrow 2(1 \Leftrightarrow x)C(x) + 1 \\ &= (1 \Leftrightarrow 2x + x^2)\frac{C(x) \Leftrightarrow 1}{x} \Leftrightarrow 2(1 \Leftrightarrow x)C(x) + 1 \\ &= (\frac{1}{x} \Leftrightarrow 4 + 3x)C(x) \Leftrightarrow (\frac{1}{x} \Leftrightarrow 3 + x). \end{aligned}$$

Taking the coefficient of x^{n+1} for $n \geq 2$ on both sides reveals that 4.1 is

$$C_{n+2} \Leftrightarrow 4C_{n+1} + 3C_n = \frac{3}{n}\binom{2n}{n+3}.$$

$\square$

In the above proof, it was shown that each element of $S_n^1(321)$ corresponds to a pair of Dyck paths $(\pi_1, \pi_2) \in \mathcal{D}_{n_1} \times \mathcal{D}_{n_2}$, where $n_1 + n_2 = n + 1$. Since π_1 must end with at least two consecutive steps EAST and π_2 must start with at least two consecutive steps NORTH, we can decompose these paths still further. Let π_{11} represent the initial portion of π_1 continuing until π_1 touches the line $y = x$ for the last time, not including the last step EAST in π_1. Of the remaining part of π_1, let π_{12} be the portion that starts on the line $y = x + 1$ continuing until π_1 touches the line $y = x + 1$ for the last time, not including the next to last step EAST in π_1. Let π_{13} be the remaining portion of π_1, starting on the line $y = x + 2$ and continuing until π_1 crosses the line $y = x + 2$. Let π_{21} represent the portion of π_2 that starts immediately after the two initial steps NORTH and continues until it crosses the line $y = x + 2$ for the first time. Let π_{22} represent the remaining portion of π_2 that starts on the line $y = x + 1$ and continues until it crosses the line $y = x$. And finally, let π_{23} represent the remaining portion of π_2 that starts on the line $y = x$ and continues until the end of π_2. This process is illustrated in Figure 4.3. The decomposition of π_1 and π_2 as given in Figures 4.1 and 4.2 is shown in Figure 4.4.

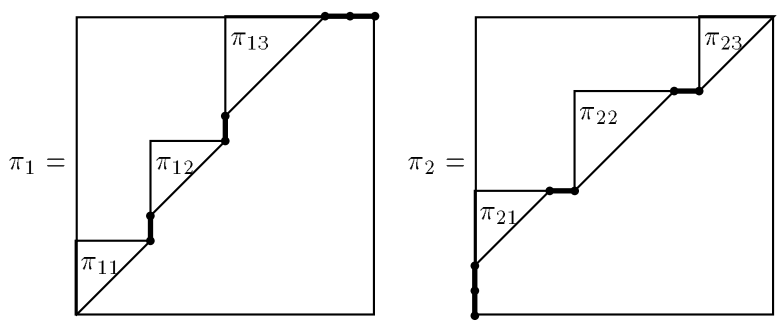


Figure 4.3: Decomposition of π

This leads us immediately to the following.

Corollary 4.2

$$\sum_{n\geq 3} \frac{3}{n}\binom{2n}{n+3}x^n = \frac{(1 - \sqrt{1 - 4x})^6}{64x^3}$$

Proof. Since each element of $S_n^1(321)$ can be decomposed into a 6-tuple of paths, $(\pi_{11}, \pi_{12}, \ldots, \pi_{23}) \in \mathcal{D}_{n_1} \times \mathcal{D}_{n_2} \times \cdots \times \mathcal{D}_{n_6}$ where $n_1 + n_2 + \cdots + n_6 = n - 3$, we see that

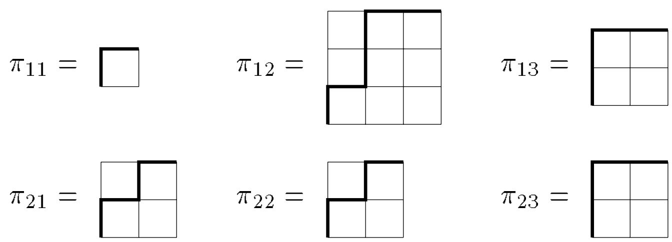


Figure 4.4: Sample decomposition of π_1 and π_2

the generating function for $s_n^1(321)$ is given by $x^3 C(x)^6$. Applying (2.3) completes the proof. $\square$

Remarkably, this says that an ordered set of 6 permutations, each *avoiding* the pattern (321), corresponds to a single permutation that has exactly *one* occurrence of the pattern (321).

Our next task is to enumerate certain subsets of $S_n^1(321)$, but in order to do so, we will need the aid of the following

Lemma 4.3 *Let $\sigma \in S_n^1(321)$ and let $I < J < K$ be the indices of the lone (321) pattern in σ. For all i such that $I < i < J$,*

$$(\sigma_I, \sigma_i, J) \quad \text{and} \quad (\sigma_I, \sigma_i, \sigma_K)$$

are (312) patterns in σ. Similarly, for all j such $J < j < \sigma_I$,

$$(\sigma_I, J, j) \quad \text{and} \quad (\sigma_I, \sigma_K, j)$$

are (312) patterns in σ.

Proof. Note that for all i such that $I < i < J$, we must have $\sigma_i < \sigma_K$, otherwise $(\sigma_I, \sigma_i, \sigma_K)$ would form a (321) pattern. Therefore (σ_I, σ_i, J) and $(\sigma_I, \sigma_i, \sigma_K)$ are (312) patterns. Also note that the set of cars $\{\sigma_K, J+1, \ldots, \sigma_I\}$ all prefer to park in spot J, but only σ_I is allowed to park there. The other cars are parked in increasing order. Therefore (σ_I, J, j) and (σ_I, σ_K, j) are (312) patterns for each j such that $J < j < \sigma_I$. $\square$

4.2 Constructing elements of $S_n^1(321) \cap S_n(312)$

Using Corollary 2.4 and Lemma 1.2 we know that the number of (312)-avoiding permutations that have exactly one occurrence of the pattern (321) is $(n \Leftrightarrow 2)2^{n-3}$. Recall that this is the same as the number of elements of $S_n(321) \cap S_n^1(312)$, which was shown using elements of $\mathcal{D}_n$. In the above construction of $\sigma \in S_n^1(321)$, π_σ was an element of $\mathcal{D}_{n+1}$. This suggests that there is a simple embedding of $\mathcal{D}_n$ onto $\mathcal{D}_{n+1}$ that explains why these sets are equinumerous.

Using Lemma 4.3, we must have $I = J \Leftrightarrow 1$ and $\sigma_I = J + 1$ if $\sigma \in S_n^1(321) \cap S_n(312)$. In other words, both π_{13} and π_{21} are empty, where $\pi = \pi_\sigma$ and π is decomposed as in Figure 4.3. Additionally, π_1 and π_2 must correspond to (312)-avoiding permutations. Furthermore, since π_{12} and π_{22} start on the line $y = x + 1$, $w(\pi_{12})$ and $w(\pi_{22})$ can only be powers of c_0. Therefore

$$w(\pi) = w(\pi_{11}) a_0 c_1^i c_1 a_0 c_1 c_1^j w(\pi_{23})$$

for some $i, j \geq 0$. If $\bar{\pi}$ corresponds to an element of $S_n(321) \cap S_n^1(312)$, then

$$w(\bar{\pi}) = w(\bar{\pi}_{11}) a_0 c_1^i a_1 c_2 c_1^j w(\bar{\pi}_{23})$$

for some $i, j \geq 0$. In other words, we can switch between the two sets by simply replacing $c_1 a_0 c_1$ in $w(\pi)$ with $a_1 c_2$, and vice versa. Graphically, this amounts to Figure 4.5, where the relevant portions of the path have been shaded.

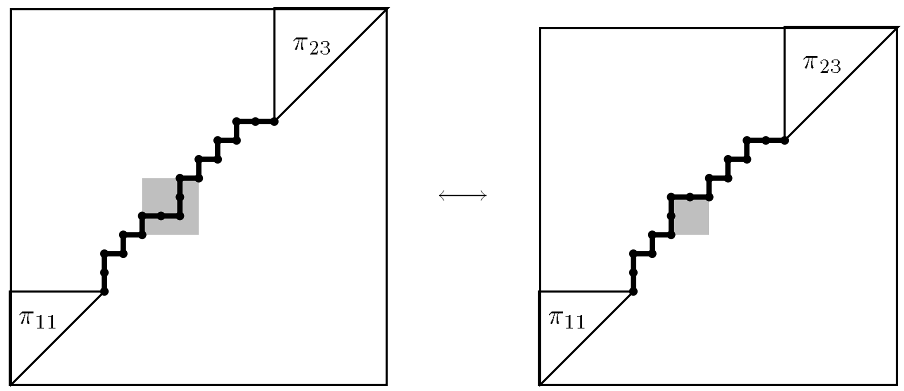


Figure 4.5: A bijection between $S_n^1(321) \cap S_n(312)$ and $S_n(321) \cap S_n^1(312)$

This correspondence establishes

Theorem 4.4 *The number of permutations of length $n \geq 4$ that have exactly one occurrence of the pattern (321) and avoid the pattern (312) is given by*

$$(n \Leftrightarrow 2)2^{n-3}.$$

4.3 Constructing elements of $S_n^1(321) \cap S_n^1(312)$

Using Lemma 4.3, we must have $I = J \Leftrightarrow 1$ and $\sigma_I = J + 1$ if $\sigma \in S_n^1(321) \cap S_n^1(312)$. In other words, both π_{13} and π_{21} are empty. Notice that so far, this is exactly the same as the situation that occurred when we were constructing elements of $S_n^1(321) \cap S_n(312)$. In fact, we can use that construction to help us here. In particular, $w(\pi)$ will have a single occurrence of the letter c_2, because the (312) pattern that occurs must have come from π_1 or π_2, which are (321) avoiding permutations. If we can remove the c_2, we should be able to produce a (312)-avoiding permutation with exactly one decreasing sequence of length three.

Alternatively, starting with $\alpha \in S^1_{n-2}(321) \cap S_{n-2}(312)$, we can construct $\sigma \in S^1_n(321) \cap S^1_n(312)$. The idea is to choose a vertical segment of π_α and replace it by a series of three vertical segments, thereby converting it from an element of $\mathcal{D}_{n-1}$ to $\mathcal{D}_{n+1}$. If we let $\pi = \pi_\alpha$, then

$$w(\pi) = w(\pi_{11})a_0 S[w(\pi_{12})]c_1 a_0 c_1 S[w(\pi_{22})]w(\pi_{23}).$$

Excluding the $c_1 a_0$ pair immediately following $S[w(\pi_{12})]$, we can accomplish this using the following substitutions:

$$a_0 \rightarrow a_0 a_1 c_2 \quad c_0 \rightarrow a_0 a_1 c_2 \quad c_1 \rightarrow c_1 a_1 c_2. \tag{4.2}$$

Graphically, this can be represented as in Figure 4.6. An example of this substitution is given in Figure 4.7.

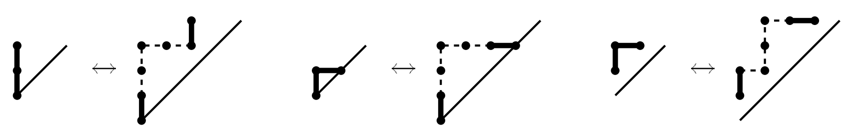


Figure 4.6: Graphical representation of replacements in 4.2

The reason we can not apply these substitutions to the $c_1 a_0$ pair immediately following $S[w(\pi_{12})]$ is that the resulting path would not correspond to an element of $S^1_n(321)$. Therefore we can simply choose any of the other $n \Leftrightarrow 3$ vertical segments in π to make this replacement, and the resulting path will correspond to an element of $S^1_n(321) \cap S^1_n(312)$. Conversely, given an element of $S^1_n(321) \cap S^1_n(312)$, we can easily identify which replacement was made and find the corresponding element of $S^1_{n-2}(321) \cap S_{n-2}(312)$. Thus we have established

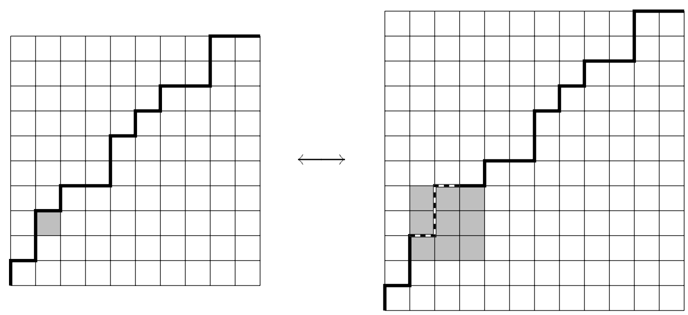


Figure 4.7: Correspondence between $S^1_{n-2}(321) \cap S_{n-2}(312)$ and $S^1_n(321) \cap S^1_n(312)$

Theorem 4.5 *The number of permutations of length $n \geq 4$ that have exactly one occurrence of the pattern (321) and exactly one occurrence of the pattern (312) is given by*

$$(n \Leftrightarrow 3)(n \Leftrightarrow 4)2^{n-5}.$$

Notice that this a simple consequence of Lemma 1.2 and Theorem 3.5, however, the proofs of Theorems 3.5 and 4.5 are significantly different. In this presentation, we used paths in $\mathcal{D}_{n+1}$ to construct a single element of $S^1_n(321) \cap S^1_n(312)$, while in the proof of Theorem 3.5, we used paths in $\mathcal{D}_n$ to construct two elements of $S^1_n(132) \cap S^1_n(123)$.

4.4 Constructing elements of $S^1_n(321) \cap S^2_n(312)$

Let $\sigma \in S^1_n(321) \cap S^2_n(312)$ and let $\pi = \pi_\sigma$. Using Lemma 4.3, we must have $I = J \Leftrightarrow 2$ or $I = J \Leftrightarrow 1$ if we want exactly two occurrences of the pattern (312).

Case: $I = J \Leftrightarrow 2$

In this case, $(\sigma_I, \sigma_{J-1}, J)$ and $(\sigma_I, \sigma_{J-1}, \sigma_K)$ will constitute our only allowed (312) patterns. Again using Lemma 4.3, we must have $\sigma_I = J+1$, or in other words, π_{21} is empty. Additionally, if $\sigma_K < J \Leftrightarrow 1$, then there exists an $i < J \Leftrightarrow 2$ such that $\sigma_i = J \Leftrightarrow 1$, and thus

$(\sigma_i, \sigma_{J-1}, \sigma_K)$ would be a third (312) pattern. Therefore $\sigma_K = J - 1$, or in other words, $w(\pi_{13}) = c_0$. Lastly, since π_{12} and π_{22} start on the line $y = x + 1$, the words associated with these paths can be nothing more than powers of c_0, since the two existing (312) patterns can already be attributed to π_{13}. But this is exactly like the construction for elements of $S_n^1(321) \cap S_n(312)$, except that π_{13} is not empty. Therefore, if we simply remove π_{13}, we will have an element of $S_{n-1}^1(321) \cap S_{n-1}(312)$. Since this process is easily reversed, we have that there are $(n - 3)2^{n-4}$ elements of $S_n^1(321) \cap S_n^2(312)$ with $I = J - 2$.

Case: $I = J - 1$

In this situation, π_{13} must be empty. Using Lemma 4.3, additionally, we must have $\sigma_I = J + 2$ or $\sigma_I = J + 1$.

Subcase: $\sigma_I = J + 2$

Here our only allowed (312) patterns are $(J+2, J, J+1)$ and $(J+2, \sigma_K, J+1)$. If $K > J+1$ then $(\sigma_{J+1}, \sigma_K, J + 1)$ will be another (312) pattern. Therefore, $K = J + 1$, or in other words, $w(\pi_{21}) = c_0$. Since we already have our two (312) patterns, π_{11}, π_{12}, π_{22} and π_{23} must correspond to (312)-avoiding permutations. Again, this is exactly like constructing elements of $S_n^1(321) \cap S_n(312)$, except that π_{21} is not empty. If we simply remove π_{21}, then we would obtain a path corresponding to an element of $S_{n-1}^1(321) \cap S_{n-1}(312)$. Since this process can be reversed, these two sets must have the same cardinality, namely $(n - 3)2^{n-4}$.

Subcase: $\sigma_I = J + 1$

This means that π_{21} is also empty. In this situation neither σ_I nor σ_J can be involved in a (312) pattern. This means that the (312) patterns must come from either π_1 or π_2, but not from the interaction between the two. It also means that we can think of π_1 (resp. π_2) as corresponding to an element of S_{J-1} (S_{n-J}), by simply removing the last (first) c_1 that occurs in $w(\pi_1)$ ($w(\pi_2)$).

If both (312) patterns come from π_1, then applying Corollaries 2.9 and 2.11, we have

$$(J-1)(J-4)2^{J-7}2^{n-J-1}$$

such paths. Summing over all possible J yields

$$2^{n-8}\sum_{J=5}^{n-1}(J-1)(J-4) = 2^{n-8}\frac{n(n-4)(n-5)}{3}.$$

The same argument applies to the case when both (312) patterns come from π_2.

If both π_1 and π_2 contain exactly one (312) pattern, then applying Corollary 2.10, we have

$$(J-3)2^{J-4}(n-J-2)2^{n-J-3}$$

such paths. Summing over all possible J yields

$$2^{n-7}\sum_{J=4}^{n-2}(J-3)(n-J-2) = 2^{n-7}\frac{(n-4)(n-5)(n-6)}{6}.$$

In total, we have

$$(n-3)2^{n-3} + 2^{n-7}\frac{n(n-4)(n-5)}{3} + 2^{n-7}\frac{(n-4)(n-5)(n-6}{6}$$

relevant paths, which yields

Theorem 4.6 *The number of permutations of length $n \geq 4$ that have exactly one occurrence of the pattern (321) and exactly two occurrences of the pattern (312) is given by*

$$2^{n-8}(n^3 - 11n^2 + 70n - 136).$$

4.5 Constructing elements of $S_n^1(321) \cap S_n^3(312)$

Let $\sigma \in S_n^1(321) \cap S_n^3(312)$ and let $\pi = \pi_\sigma$. Using Lemma 4.3, we must have $I = J \Leftrightarrow 2$ or $I = J \Leftrightarrow 1$ if we want exactly three occurrences of the pattern (312).

Case: $I = J \Leftrightarrow 2$

In this case $(\sigma_I, \sigma_{J-1}, J)$ and $(\sigma_I, \sigma_{J-1}, \sigma_K)$ will constitute two of our three (312) patterns. Again using Lemma 4.3, we must have $\sigma_I = J + 1$, or in other words, π_{21} is empty. If $\sigma_K < J \Leftrightarrow 2$, then there exists an $i, j < I$ such that $\sigma_i = J \Leftrightarrow 1$ and $\sigma_j = J \Leftrightarrow 2$, and thus $(\sigma_i, \sigma_{J-1}, \sigma_K)$ and $(\sigma_j, \sigma_{J-1}, \sigma_K)$ would be a third and fourth (312) pattern. Therefore $\sigma_K = J \Leftrightarrow 1$ or $\sigma_K = J \Leftrightarrow 2$.

Subcase: $\sigma_K = J \Leftrightarrow 1$

This forces $w(\pi_{13})$ to be c_0, otherwise car $J \Leftrightarrow 1$ would park in its preferred spot which would be *before* spot I. We are now left with the task of constructing one more (312) pattern. But this is the same situation as constructing elements of $S_n^1(321) \cap S_n^1(312)$, except π_{13} is not empty. If we were to simply remove π_{13}, a path corresponding to an element of $S_{n-1}^1(321) \cap S_{n-1}^1(312)$ would remain. Since this process is reversible, applying Theorem 4.5 yields that there are $(n \Leftrightarrow 4)(n \Leftrightarrow 5)2^{n-6}$ such permutations.

Subcase: $\sigma_K = J \Leftrightarrow 2$

This implies that $(J \Leftrightarrow 1, \sigma_{J-1}, J \Leftrightarrow 2)$ is our third and final (312) pattern. Additionally, we must have $\sigma_{J-3} = J \Leftrightarrow 1$, otherwise, $(J \Leftrightarrow 1, \sigma_{J-3}, J \Leftrightarrow 2)$ would be a fourth (312) pattern. This implies that π_{13} is the path in $\mathcal{D}_2$ corresponding to the word c_0^2. The remainder of the path cannot contribute anymore (312) patterns. But this is the same situation as constructing elements of $S_n^1(321) \cap S_n(312)$, except π_{13} is not empty. If we were to simply remove π_{13}, a path corresponding to an element of $S_{n-2}^1(321) \cap S_{n-2}(312)$

would remain. Since this process is reversible, applying Theorem 4.4 yields that there are $(n \Leftrightarrow 4)2^{n-5}$ such permutations.

Case: $I = J \Leftrightarrow 1$

This means that π_{13} is empty and we still need to create three (312) patterns. Using Lemma 4.3, we can only have $\sigma_I = J + 2$ or $\sigma_I = J + 1$.

Subcase: $\sigma_I = J + 2$

In this case, two of our allowed (312) patterns are $(J+2, J, J+1)$ and $(J+2, \sigma_K, J+1)$. If $K > J+2$ then $(\sigma_{J+1}, \sigma_K, J+1)$ and $(\sigma_{J+2}, \sigma_K, J+1)$ would be two more additional (312) patterns. Therefore, $K = J+1$ or $K = J+2$.

If $K = J+1$, then π_{21} is in $\mathcal{D}_1$ and we still need to construct one more (312) pattern. But this is the same situation as constructing elements of $S_n^1(321) \cap S_n^1(312)$, except π_{21} is not empty. If we were to simply remove π_{21}, a path corresponding to an element of $S_{n-1}^1(321) \cap S_{n-1}^1(312)$ would remain. Since this process is reversible, applying Theorem 4.5 yields that there are $(n \Leftrightarrow 4)(n \Leftrightarrow 5)2^{n-6}$ such permutations.

If $K = J+2$, then $(\sigma_{J+1}, \sigma_K, J+1)$ is our third and final (312) pattern. Additionally, we must have $\sigma_{J+1} = J+3$, otherwise, $(\sigma_{J+1}, \sigma_K, J+3)$ would be a fourth (312) pattern. This implies that π_{21} is the path in $\mathcal{D}_2$ corresponding to the word c_0^2. The remainder of the path cannot contribute anymore (312) patterns. But this is the same situation as constructing elements of $S_n^1(321) \cap S_n(312)$, except π_{21} is not empty. If we were to simply remove π_{21}, a path corresponding to an element of $S_{n-2}^1(321) \cap S_{n-2}(312)$ would remain. Since this process is reversible, applying Theorem 4.4 yields that there are $(n \Leftrightarrow 4)2^{n-5}$ such permutations.

At this point we would like to mention that we have so far classified

$$(n-4)(n-5)2^{n-5}+(n-4)2^{n-4}=(n-3)(n-4)2^{n-5}$$

relevant patterns. What is (not so) surprising about this is that it coincides exactly with the number of elements of $S_n^1(321)\cap S_n^1(312)$. We leave it as an exercise to the reader to construct a bijection between these two sets.

Subcase: $\sigma_I = J+1$

In this case, π_{21} is also empty and neither σ_I nor σ_J can be involved in a (312) pattern. This means that the (312) patterns must come from either π_1 or π_2, but not from the interaction between the two. It also means that we can think of π_1 and π_2 as corresponding to elements of S_{J-1} and S_{n-J}, just as we did in the previous section.

If all three (312) patterns come from π_1, then applying Corollaries 2.9 and 2.12, we have

$$\frac{2^{J-9}}{3}(J^3-6J^2+47j-162)2^{n-J-1}$$

such paths. Summing over all possible J yields

$$\frac{2^{n-10}}{3}\sum_{J=6}^{n-1}(J^3-6J^2+47j-162)=\frac{2^{n-12}}{3}(n-5)(n^3-5n^2+82n-336).$$

The same argument applies to the case when all three (312) patterns come from π_2. If π_1 contains exactly two (312) patterns and π_2 contains exactly one (312) pattern, then applying Corollaries 2.10 and 2.11, we have

$$(J-1)(J-4)2^{J-7}(n-J-2)2^{n-J-3}$$

such paths. Summing over all possible J yields

$$2^{n-10}\sum_{J=5}^{n-2}(J-1)(J-4)(n-J-2)=2^{n-12}\frac{n(n-5)(n-6)(n-7)}{3}.$$

The same argument applies to the case when π_1 contains one (312) pattern and π_2 contains two (312) patterns. In total, we have

$$(n-3)(n-4)2^{n-5}+\frac{2^{n-11}}{3}(n-5)\left[(n^3-5n^2+82n-336)+n(n-6)(n-7)\right]$$

relevant paths, which yields

Theorem 4.7 *The number of permutations of length $n \geq 4$ that have exactly one occurrence of the pattern (321) and exactly three occurrences of the pattern (312) is given by*

$$\frac{2^{n-10}}{3}(n-4)(n^3-10n^2+163n-498).$$

Chapter 5

More generating functions

Using the theory of orthogonal polynomials and its relationship to Dyck paths, we can further generalize many of the results from previous chapters.

5.1 Introduction

Let the sequence of polynomials $\{Q_n\}_{n\geq 0}$ be defined recursively by

$$Q_{n+1} = zQ_n - Q_{n-1},$$

with initial conditions $Q_0 \equiv 1$ and $Q_1 = z$. These polynomials are related to the n^{th} Chebyshev polynomial of the second kind, U_n, by the simple relation

$$Q_n(z) = U_n(z/2).$$

We state without proof some basic results from the theory of orthogonal polynomials. For further background, see [3, 14].

Theorem 5.1

$$\frac{Q_n(1/\sqrt{z})}{\sqrt{z}Q_{n+1}(1/\sqrt{z})} = \sum_{k\geq 0} c_{n,k} z^k$$

where $c_{n,k}$ is the number of elements of $\mathcal{D}_k$ that stay weakly below the line $y = x + n$.

Theorem 5.2

$$\frac{\sqrt{z}^{n-1}}{Q_{n+1}(1/\sqrt{z})} = \sum_{k \geq n} d_{n,k} z^k$$

where $d_{n,k}$ is the number of paths from $(0,0)$ to $(k \Leftrightarrow n, k)$ using only steps NORTH and EAST that stay weakly below the line $y = x + n$ and weakly above the line $y = x$.

In view of the continued fraction expansions that we saw in previous chapters, we should point out that

Theorem 5.3

$$\frac{Q_n(1/\sqrt{z})}{\sqrt{z}Q_{n+1}(1/\sqrt{z})} = \cfrac{1}{1 \Leftrightarrow \cfrac{z}{1 \Leftrightarrow \cfrac{z}{1 \Leftrightarrow \cfrac{\cdots}{\cdots \cfrac{\cdots}{1 \Leftrightarrow z}}}}}.$$

where there are a total of n z's that appear in the above continued fraction.

5.2 (132)-avoiding permutations

Each of the following corollaries follows from the above theorems and the specializations for a_i and c_i as given in Section 2.2.

Corollary 5.4 *Let $k \geq 2$. The generating function for the number of (132)-avoiding permutations of length n that also avoid the pattern $(12 \ldots k)$ is given by*

$$\frac{Q_{k-1}(1/\sqrt{z})}{\sqrt{z}Q_k(1/\sqrt{z})}.$$

Proof. This is a simple consequence of Theorem 5.1 and Corollary 2.2. □

Corollary 5.5 *Let $k \geq 2$. The generating function for the number of (132)-avoiding permutations of length n that have exactly one occurrence of the pattern $(12\dots k)$ is given by*

$$\frac{1}{Q_k^2(1/\sqrt{z})}$$

Proof. This follows immediately from Theorem 5.2 and Figure 5.1. □

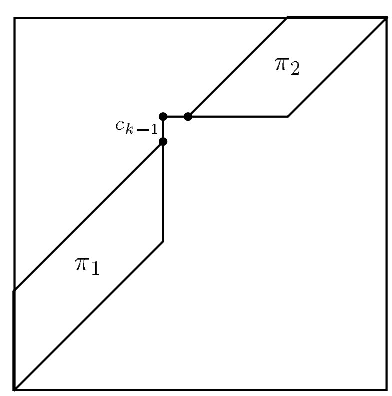


Figure 5.1: Generic element of $S_n(132) \cap S_n^1(12\dots k)$

Corollary 5.6 *Let $k \geq 2$. The generating function for the number of (132)-avoiding permutations of length n that have exactly two occurrences of the pattern $(12\dots k)$ is given by*

$$\frac{\sqrt{z}Q_{k-1}(1/\sqrt{z})}{Q_k^3(1/\sqrt{z})}$$

Proof. Using Figure 5.2 as a guide, apply Theorem 5.1 to path π_2 and apply Theorem 5.2 to paths π_1 and π_3. The generating function is thus given by

$$\frac{z^{k/2}}{Q_k(1/\sqrt{z})} \times \frac{Q_{k-1}(1/\sqrt{z})}{\sqrt{z}Q_k(1/\sqrt{z})} \times \frac{z^{k/2}}{z^{k-1}Q_k(1/\sqrt{z})},$$

where the additional powers of z come from the vertical segments separating the three paths and the fact that π_3 is turned on its side. □

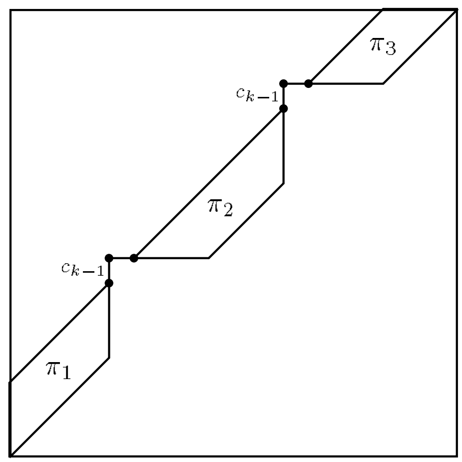


Figure 5.2: Generic element of $S_n(132) \cap S_n^2(12\ldots k)$

Using a similar method of proof, we can also obtain

Corollary 5.7 *Let $k \geq 2$. The generating function for the number of (132)-avoiding permutations of length n that have exactly three occurrences of the pattern $(12\ldots k)$ is given by*

$$\frac{zQ_{k-1}^2(1/\sqrt{z})}{Q_k^4(1/\sqrt{z})}.$$

5.3 (321)-avoiding permutations

Each of the following corollaries is an immediate consequence of the analogous corollary from the previous section, in addition to the following observation. The specializations for a_i and c_i that we used to count the number of $(12\ldots k)$ patterns in (132)-avoiding permutations were exactly the same as the ones we used to count the number of $(k12\ldots k \Leftrightarrow 1)$ patterns in (321)-avoiding permutations, for $i < k \Leftrightarrow 1$. For $i = k \Leftrightarrow 1$, the only difference was that we replaced a_{k-1} by q to count $(12\ldots k)$ patterns and we replaced a_{k-1} by 1 to count $(k12\ldots k \Leftrightarrow 1)$ patterns, but since our paths are bounded above by the line $y = x + k$ (with one exception), the words associated to these cannot use the letter a_{k-1}.

Corollary 5.8 *Let $k \geq 3$. The generating function for the number of (321)-avoiding permutations of length n that also avoid the pattern $(k12\ldots k \Leftrightarrow 1)$ is given by*

$$\frac{Q_{k-1}(1/\sqrt{z})}{\sqrt{z}Q_k(1/\sqrt{z})}.$$

Corollary 5.9 *Let $k \geq 3$. The generating function for the number of (321)-avoiding permutations of length n that have exactly one occurrence of the pattern $(k12\ldots k \Leftrightarrow 1)$ is given by*

$$\frac{1}{Q_k^2(1/\sqrt{z})}.$$

Corollary 5.10 *Let $k \geq 3$. The generating function for the number of (321)-avoiding permutations of length n that have exactly two occurrences of the pattern $(k12\ldots k \Leftrightarrow 1)$ is given by*

$$\frac{\sqrt{z}Q_{k-1}(1/\sqrt{z})}{Q_k^3(1/\sqrt{z})}.$$

Corollary 5.11 *Let $k \geq 4$. The generating function for the number of (321)-avoiding permutations of length n that have exactly three occurrences of the pattern $(k12\ldots k \Leftrightarrow 1)$ is given by*

$$\frac{zQ_{k-1}^2(1/\sqrt{z})}{Q_k^4(1/\sqrt{z})}.$$

Note that in this last corollary, we do not include the case $k = 3$. This is because it is possible for a path to cross the line $y = x + 3$ and still correspond to a permutation with exactly three (312) patterns. This subtlety was already encountered in proving Corollary 2.12.

5.4 Permutations with one (321) pattern

Following the constructions used in the previous chapter, we can also obtain generating functions for permutations with one (321) pattern.

Corollary 5.12 *Let $k \geq 2$. The generating function for the number of elements of $S_n^1(321)$ that also avoid the pattern $(k12\ldots k \Leftrightarrow 1)$ is given by*

$$\frac{Q_{k-3}^2(1/\sqrt{z})}{Q_k^2(1/\sqrt{z})}.$$

Proof. Using Theorem 5.1 (three times) and following the decomposition of π as given in Figure 4.3, we see that the above generating function is given by

$$\left(\frac{Q_{k-1}(1/\sqrt{z})}{\sqrt{z}Q_k(1/\sqrt{z})} z \frac{Q_{k-2}(1/\sqrt{z})}{\sqrt{z}Q_{k-1}(1/\sqrt{z})} z \frac{Q_{k-3}(1/\sqrt{z})}{\sqrt{z}Q_{k-2}(1/\sqrt{z})}\right)^2 \Bigg/ z.$$

We must divide by z due to the fact that permutations in $S_n^1(321)$ correspond to paths in $\mathcal{D}_{n+1}$. $\square$

Chapter 6

An involution on Dyck paths

In this chapter, we examine how an involution on paths can result in a wide array of information regarding pattern avoidances.

6.1 Introduction

For any path, π, we say that π has a *descent* at i if the i^{th} step is to the EAST and is immediately followed by a step NORTH. The set of all descents of π will be denoted by $Des(\pi)$ and its cardinality is given by $des(\pi)$. The *major index of* π, denoted $maj(\pi)$, is defined to be the sum of the elements of $Des(\pi)$. Note that while two distinct paths can have the same set of descents, the positions of these descents uniquely determine a path. In other words we can identify a path by a pair of increasing sequences, $(a_1, a_2, \dots, a_k | b_1, b_2, \dots, b_k)$, specifying that the descents in π only occur after a_i steps NORTH and b_i steps EAST for each $1 \leq i \leq k$. In short, we write $\pi = (a_\pi | b_\pi)$. A path π is a Dyck path if and only if all of its descents occur above the diagonal. Put another way,

$$\pi \in \mathcal{D}_n \Leftrightarrow \forall i \;\; a_i \geq b_i.$$

In Figure 6.1, the descents are labelled by their appropriate index, with the rows and columns labelled by the sequences a_π and b_π, respectively.

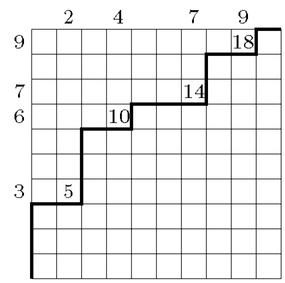


Figure 6.1: Descents of a path π

We can similarly define descent and major index for permutations, $\sigma \in S_n$. We say that σ has a *descent* at i if $\sigma_i > \sigma_{i+1}$. The set of all descents of σ will be denoted by $Des(\sigma)$ and its cardinality is given by $des(\sigma)$. The *major index of a permutation*, denoted $maj(\sigma)$, is the sum of the indices where σ has a descent. In this chapter, we are only considering the case when σ is a (132)-avoiding permutation. For example, the path given in Figure 6.1 corresponds to the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 8 & 9 & 5 & 6 & 4 & 7 & 10 & 2 & 3 & 1 \end{pmatrix},$$

which has descents occurring in positions 2, 4, 7, and 9, which is exactly b_π.

Lemma 6.1 *For all* $\pi \in \mathcal{D}_n$, $maj(\sigma_\pi) = b_1 + b_2 + \cdots + b_k$

Proof. Let $\pi = (a_\pi | b_\pi) \in \mathcal{D}_n$ and let $\sigma = \sigma_\pi$ be the corresponding element of $S_n(132)$. Using the alternate construction of σ given in Section 1.5, it is clear that $\sigma_i > \sigma_{i+1}$ if and only if $i = b_j$ for some j. $\square$

6.2 The involution

Since b_π yields information about the major index of σ_π, it is natural to ask what information does a_π provide about σ_π or is there some other path π' where $\sigma_{\pi'}$ has descent set a_π.

If $A = \{a_1, a_2, \ldots, a_k\}$ and $B = \{b_1, b_2, \ldots, b_k\}$ are subsets of $[n-1]$ such that $\pi = (A|B) \in \mathcal{D}_n$ then let $\pi' = (B^c|A^c) \in \mathcal{D}_n$, where A^c and B^c are the set complements of A and B. Clearly A^c and B^c will satisfy the inequalities $b_j^c \geq a_j^c$ so that π' is an element of $\mathcal{D}_n$. It is also clear that this construction is an involution. For example, if π is the path shown in Figure 6.1, then π' is shown in Figure 6.2.

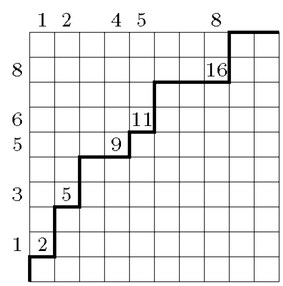


Figure 6.2: Descents of π'

Using the above construction, we can now interpret a_π as being the set complement of the descent set of $\sigma_{\pi'}$. In other words, we have the following relationship.

Lemma 6.2 *For all* $\pi \in \mathcal{D}_n$, $maj(\pi) = maj(\sigma_\pi) + \binom{n}{2} - maj(\sigma_{\pi'})$

This of course gives us an exact formula for the difference between $maj(\sigma_\pi)$ and $maj(\sigma_{\pi'})$. Amazingly, the sum of these two numbers is also restricted.

Lemma 6.3 *For all* $\pi \in \mathcal{D}_n$, $maj(\sigma_\pi) + maj(\sigma_{\pi'}) \leq \binom{n}{2}$.

Proof.

$$\begin{aligned} maj(\sigma_\pi) + maj(\sigma_{\pi'}) &= b_1 + b_2 + \cdots + b_k + b'_1 + b'_2 + \cdots + b'_l \\ &= b_1 + b_2 + \cdots + b_k + \binom{n}{2} \Leftrightarrow (a_1 + a_2 + \cdots + a_k) \\ &= \binom{n}{2} \Leftrightarrow (a_1 \Leftrightarrow b_1) \Leftrightarrow (a_2 \Leftrightarrow b_2) \Leftrightarrow \cdots \Leftrightarrow (a_k \Leftrightarrow b_k) \\ &\leq \binom{n}{2} \end{aligned}$$

□

Notice that equality is equivalent to having $a_i = b_i$ for all i. This describes exactly those paths which Haglund referred to as *balance paths*. As it turned out, balance paths played a pivotal role in finding a statistic for Garsia and Haiman's qt-Catalan sequence [4]. They also have a particular interpretation when it comes to pattern avoidances.

6.3 More pattern avoidances

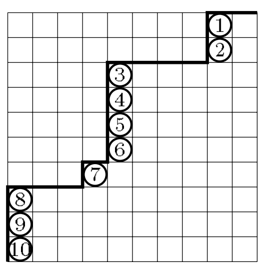
Lemma 6.4 *For all* $\pi \in \mathcal{D}_n$, $maj(\sigma_\pi) + maj(\sigma_{\pi'}) = \binom{n}{2}$ *if and only if* $\sigma_\pi \in S_n(132, 213)$.

Proof. Assume $\sigma \in S_n(132, 213)$ and let $\pi = \pi_\sigma = (a_\pi | b_\pi)$. We proceed by induction on n. If $\sigma_n = 1$ then π starts with a path $\pi_0 \in \mathcal{D}_{n-1}$ and ends with a step NORTH followed by a step EAST. If $\sigma_n = i > 1$ then $\sigma_{n-j} = i \Leftrightarrow j$ for all $1 \leq j < i$. If not, let J be the minimum j such that $\sigma_{n-j} \neq i \Leftrightarrow j$. If $\sigma_{n-J} < i \Leftrightarrow J$ then $(i \Leftrightarrow J, \sigma_{n-J}, i)$ is a (213) pattern. If $\sigma_{n-J} > i \Leftrightarrow J$ then σ_{n-J} must also be bigger than i, otherwise this would contradict our choice of J, and thus $(i \Leftrightarrow J, \sigma_{n-J}, i)$ is a (132) pattern. Therefore π starts with a path $\pi_0 \in \mathcal{D}_{n-i}$ and ends with exactly i steps NORTH followed by i steps EAST.

In either case, $a_k = b_k = n \Leftrightarrow \sigma_n$ where $k = des(\sigma)$ and $a_i = b_i$ for all $i < k$

by applying the inductive hypotesis to π_0. In other words we have established that $\pi = (a_\pi | a_\pi)$, which means that $maj(\sigma_\pi) + maj(\sigma_{\pi'}) = \binom{n}{2}$.

Assume $maj(\sigma_\pi) + maj(\sigma_{\pi'}) = \binom{n}{2}$, or equivalently $a_i = b_i$ for all i. It is clear from Figure 6.3 that σ consists of sequences of consecutive numbers in increasing order. Furthermore, these sequences are ordered in decreasing order according to their first element. Therefore if there exists two numbers $i < j$ such that $\sigma_i < \sigma_j$ then σ_i and σ_j both come from the same increasing sequence. and thus σ avoids the pattern (213). $\square$



$$\longleftrightarrow \quad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 8 & 9 & 10 & 7 & 3 & 4 & 5 & 6 & 1 & 2 \end{pmatrix}$$

Figure 6.3: An element of $S_{10}(132, 213)$

Corollary 6.5 *For all* $\pi \in \mathcal{D}_n$, $\sigma_\pi \in S_n(132, 213)$ *iff* $\sigma_{\pi'} \in S_n(132, 213)$

Proof. Use the above lemma and the fact that the map $\pi \to \pi'$ is an involution. $\square$

Corollary 6.6 *For all* $n \geq 1$, $s_n(132, 213) = 2^{n-1}$.

Proof. Using Lemma 6.4, each element of $S_n(132, 213)$ corresponds to a path $\pi = (a_\pi | a_\pi)$. Since a_π can be any subset of $[n \Leftrightarrow 1]$, there are exactly 2^{n-1} such paths. $\square$

We should point out that the above corollary does not follow from any of our previous results in conjuction with Lemma 1.2.

Corollary 6.7 *Let $\sigma \in S_n(132, 213)$ and $\pi_\sigma = (a_\pi | a_\pi)$. The number of $(12 \ldots k)$ patterns in σ is given by*

$$\sum_{i=1}^{des(\sigma)} \binom{a_i}{k}.$$

Corollary 6.8 *Let $\sigma \in S_n(132, 213)$ and $\pi_\sigma = (a_\pi | a_\pi)$. The number of $(k \ldots 21)$ patterns in σ is given by*

$$e_k(a_\pi) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} a_{i_1} a_{i_2} \cdots a_{i_k},$$

where e_k denotes the k^{th} elementary symmetric function.

Corollary 6.9 *Let F_n^k denote $s_n(132, 213, 12 \ldots k)$ for $k \geq 2$. Then F_n^k satisfies the following recursion*

$$F_n^k = F_{n-1}^k + F_{n-2}^k + \cdots + F_{n-k+1}^k, \tag{6.1}$$

with initial conditions $F_0^k = 1$, $F_i^k = 2^{i-1}$ for $1 \leq i \leq k \Leftrightarrow 2$.

Proof. Let $\sigma \in S_n(132, 213)$ and let $\pi_\sigma = (a_\pi | a_\pi)$. If σ also avoids the pattern $(12 \ldots k)$, then $a_i < k$ for each i. The terms on the right hand side of 6.1 decomposes paths based on the value of a_1. The initial conditions follow from the previous corollary and the fact that all permutations of length n avoid the pattern $(12 \ldots k)$ if $n < k$. $\Box$

This generalizes a result given in [12] for $k = 3$, in which case these numbers coincide with the Fibonacci sequence.

Lemma 6.10 *For all $\pi \in \mathcal{D}_n$, $\sigma_\pi \in S_n(132, 231)$ iff $\sigma_{\pi'} \in S_n(132, 312)$.*

Proof. Assume $\sigma \in S_n(132, 231)$. Let $\pi_\sigma = (a|b)$ and let f be the parking function that corresponds to σ. If car i parks in its preferred spot, $f(i)$, then $f(i \Leftrightarrow 1) \Leftrightarrow f(i) = 1$. This is because if $f(i \Leftrightarrow 1) \Leftrightarrow f(i) > 1$ then the car that parks in spot $f(i)+1$ will come along after cars $i \Leftrightarrow 1$ and i. In other words, $(\sigma_{f(i)}, \sigma_{f(i)+1}, \sigma_{f(i-1)})$ will be a (231) pattern. In particular, this means that

$$b = \{1, 2, \ldots, k\}$$

where $k = f(1)$. And in fact this is a sufficient condition for σ to avoid (132) and (231).

Assume $\alpha \in S_n(132, 312)$. Let $\pi_\alpha = (c|d)$ and let g be the parking function that corresponds to α. If $g(i) > 1$ then $g(i \Leftrightarrow 1) \Leftrightarrow g(i) > 0$. Otherwise $g(i) = g(i \Leftrightarrow 1)$ and $(\sigma_1, \sigma_{g(i)}, \sigma_{g(i)+1})$ will form a (312) pattern. In particular, this means that

$$c = \{l+1, l+2, \ldots, n\}$$

where $l = n + 1 \Leftrightarrow \sigma_1$. And in fact this is a sufficient condition for α to avoid (132) and (312).

To complete the proof, we simply need to point out that

$$\pi = (a|1, 2, \ldots, k) \quad \Leftrightarrow \quad \pi' = (k+1, k+2, \ldots n|a^c).$$

$\square$

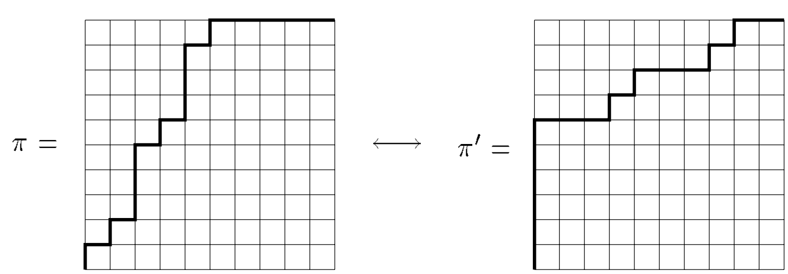


Figure 6.4: Example of $\pi \in S_{10}(132, 231)$ and $\pi' \in S_{10}(132, 312)$

This correspondence between elements of $S_n(132, 231)$ and $S_n(132, 312)$ is illustrated in Figure 6.4. It should be pointed out that this is a different bijection than the one presented in [12]. Furthermore, since there are $\binom{n-1}{k}$ choices for $a_1, a_2, \ldots, a_k$, summing over k yields

Corollary 6.11 *For all $n \geq 1$, $s_n(132, 231) = s_n(132, 312) = 2^{n-1}$*

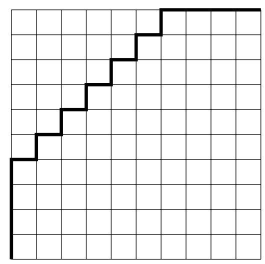


Figure 6.5: An element of $S_{10}(132, 231, 312)$

It should be pointed out that the equality of $s_n(132, 231)$ and $s_n(132, 312)$ is not a consequence of Lemma 1.2. Also notice that paths such as those shown in Figure 6.5 are the only paths that fall into both categories and thus correspond to the elements of $S_n(132, 231, 312)$. This immediately yields

Corollary 6.12 *For all $n \geq 1$, $s_n(132, 213, 312) = n$.*

6.4 A qt-Catalan sequence

Let us define a qt-analog of the Catalan sequence, $d_n(q,t)$, by

$$d_n(q,t) = \sum_{\pi \in \mathcal{D}_n} q^{maj(\sigma_\pi)} t^{maj(\sigma_{\pi'})}.$$

Clearly $d_n(q,t)$ is symmetric in q and t because the map $\pi \to \pi'$ is an involution. Remarkably, $d_n(q,t)$, has the following properties.

Lemma 6.13

$$q^{\binom{n}{2}} d_n(q, 1/q) = \frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n \end{bmatrix}_q$$

Proof. Using Lemma 6.2, we immediately have

$$q^{\binom{n}{2}} d_n(q, 1/q) = \sum_{\pi \in \mathcal{D}_n} q^{maj(\sigma_\pi) - maj(\sigma_{\pi'}) + \binom{n}{2}} = \sum_{\pi \in \mathcal{D}_n} q^{maj(\pi)},$$

with the last step being a classic result due to MacMahon [8]. □

Lemma 6.14

$$\sum_{\sigma \in S_n(132,213)} q^{maj(\sigma)} t^{maj(\sigma')} = \prod_{i=1}^{n-1} (q^i + t^i)$$

Proof. Using Lemma 6.4, we know that if $\sigma \in S_n(132, 213)$, then $\pi_\sigma = (a_\pi | a_\pi)$ and thus $\pi'_\sigma = (a_{\pi'} | a_{\pi'})$ where the sets a_π and $a_{\pi'}$ partition the set $\{1, 2, \ldots, n \Leftrightarrow 1\}$. In other words, each i between 1 and $n \Leftrightarrow 1$ either contributes to $maj(\sigma)$ by being in the set a_π or to $maj(\sigma')$ by being in the set $a_{\pi'}$. □

Chapter 7

Extending Franklin's Involution

We are dealing here with the power series expansion of the product $\prod_{n>m}(1 \Leftrightarrow q^n)$. This expansion may be readily obtained from an identity of Sylvester and the latter, in turn, may be given a relatively simple combinatorial proof. Nevertheless, the problem remains to give a combinatorial explanation for the massive cancelations which produce the final result. The case $m = 0$ is clearly explained by Franklin's proof of the Euler Pentagonal Number Theorem. Efforts to extend the same mechanism of proof to the general case $m > 0$ have led to the discovery of an extension of the Franklin involution which explains all the components of the final expansion.

7.1 Introduction

Sylvester [13, p. 281] used Durfee squares to prove the following result.

Theorem 7.1

$$\prod_{n\geq 1}(1+zq^n) = 1 + \sum_{n\geq 1} z^n q^{\frac{3n^2-n}{2}}(1+zq^{2n})(\Leftrightarrow zq)_{n-1}/(q)_n \tag{7.1}$$

where $(z)_n = (1 - z)(1 - zq)\cdots(1 - zq^{n-1})$. Multiplying the above equation by $1+z$ and then setting $z = -q^{m+1}$ for any $m \geq 0$ yields

$$\prod_{n>m}(1-q^n) = \sum_{n\geq 0}(-1)^n \begin{bmatrix} n+m \\ m \end{bmatrix}_q q^{\frac{3n^2+n}{2}+nm}(1-q^{2n+m+1}) \tag{7.2}$$

where

$$\begin{bmatrix} n+m \\ m \end{bmatrix}_q = \frac{(q)_{n+m}}{(q)_n(q)_m} \tag{7.3}$$

is the usual q-analog of the binomial coefficients. When $m = 0$, equation (7.2) is none other than Euler's Pentagonal Number Theorem,

$$\prod_{n>0}(1-q^n) \quad = \quad \sum_{n\geq 0}(-1)^n q^{\frac{3n^2+n}{2}}(1-q^{2n+1}). \tag{7.4}$$

Of course, in the process of setting $z = -q^{m+1}$, we invite a tremendous amount of cancellation to occur, none of which is explained by Sylvester's proof of (7.1), which has been included in the following section for the sake of completeness. However, Franklin's proof of (7.4) does exactly that, and in fact, offers an explanation for *every* single cancellation which occurs. It would be of historical interest to extend Franklin's ideas to explain as many of the cancellations as possible in (7.2) for any $m \geq 1$. This will be the focus of the remainder of the paper.

7.2 Sylvester's Proof of Theorem 7.1

The left-hand side of (7.1) can be thought of as the generating function for partitions λ, with k distinct parts > 0 weighted by $z^k q^{|\lambda|}$, where $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_k$. To prove Sylvester's identity, we need to show that the right-hand side of (7.1) enumerates the exact same objects.

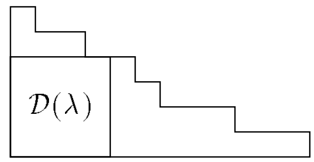


Figure 7.1: Typical partition with distinct parts, $\lambda_{n+1} < n$

We begin by noting that the Durfee square associated with λ, $\mathcal{D}(\lambda)$, is the largest square contained in the Ferrers diagram [1, p. 7] of λ. The dimension, $d(\lambda)$, of this square can be defined as the maximum i such that $\lambda_i \geq i$. Using the Durfee square to classify these partitions, we see that λ can fall into one of two distinct categories. The first category is comprised of partitions λ such that $\lambda_{n+1} < n$, where for convenience we have set $n = d(\lambda)$. A typical partition in this category would look like Figure 7.1.

Directly above $\mathcal{D}(\lambda)$ can be any partition with distinct parts $< n$. These partitions are generated by $(\Leftrightarrow zq)_{n-1}$. Directly to the right of $\mathcal{D}(\lambda)$ can be any partition with exactly n distinct parts ≥ 0. The generating function for these partitions is $z^n q^{\binom{n}{2}}/(q)_n$. Putting this all together, any partition falling into this category can be accounted for in the following term

$$z^n q^{n^2+\binom{n}{2}}(\Leftrightarrow zq)_{n-1}/(q)_n. \tag{7.5}$$

The second category is comprised of partitions λ such that $\lambda_{n+1} = n$. Note that this is the only other possibility since λ_{n+1} cannot be $\geq n+1$ by the definition of $d(\lambda)$. In this case, λ must be of the form illustrated in Figure 7.2.

Directly above $\mathcal{D}(\lambda)$ can be any partition with distinct parts $\leq n$ and largest part equal to n. Directly to the right of $\mathcal{D}(\lambda)$ can be any partition with exactly n distinct

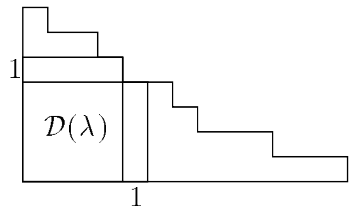


Figure 7.2: Typical partition with distinct parts, $\lambda_{n+1} = n$

parts > 0. The following term accounts for any partition falling into this category.

$$z^{n+1}q^{n^2+\binom{n}{2}+2n}(\Leftrightarrow zq)_{n-1}/(q)_n. \tag{7.6}$$

Combining (7.5) and (7.6), we get the summand in the right-hand side of (7.1), and summing over all values of $n \geq 1$ completes the proof.

7.3 Extending Franklin's Bijection

Franklin's proof [1, p. 10] of Euler's Pentagonal Number Theorem begins by defining two sets of cells contained in the Ferrers diagram associated with a fixed partition. For our purposes we will need to extend these definitions as well as further classify the cells involved.

Fix $m \geq 0$ and λ, a partition with n distinct parts $> m$. Define a *stair* to be a cell in the Ferrers diagram associated with λ at the end of a row or the top of one of the $\lambda_n \Leftrightarrow m \Leftrightarrow 1$ left-most columns. Of the remaining cells, define a *landing* to be any cell that does not have another cell above it. The *m-landing staircase* is the sequence of neighboring stairs and landings, starting with the stair at the end of the first row, with exactly m landings, using as many stairs occurring at the end of a row as possible. Let $\mathcal{S}_m(\lambda)$ refer to the cells in the m-landing staircase, with $s_m(\lambda)$ defined to be $|\mathcal{S}_m(\lambda)|$,

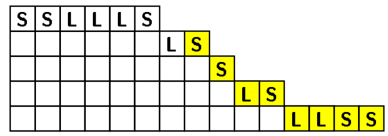


Figure 7.3: Example of a 3-landing staircase

and let $\mathcal{T}(\lambda)$ refer to the cells in the top row of λ, with $t(\lambda)$ defined to be $|\mathcal{T}(\lambda)| = \lambda_n$. Lastly, we define the weight of λ, $w(\lambda)$, to be $(\Leftrightarrow 1)^n q^{|\lambda|}$.

For example, let $m = 3$ and $\lambda = (14, 11, 9, 8, 6)$, then the Ferrers diagram would be labelled as in Figure 7.3, with stairs and landings denoted by S's and L's, respectively and cells belonging to $\mathcal{S}_3(\lambda)$ shaded.

By definition, an m-landing staircase must have exactly m landings and can have anywhere from 1 to n stairs. Since it will be an extremely useful fact for proving 7.2, we shall restate this in the following lemma.

Lemma 7.2 *Let λ be a partition with n distinct parts $> m$. Then the following inequalities must hold.*

$$m + 1 \leq s_m(\lambda) \leq m + n \tag{7.7}$$

Armed with these definitions and the above lemma, we are now in a position to prove the following

Lemma 7.3

$$\prod_{n>1} (1 \Leftrightarrow q^n) = \sum_{n \geq 0} (\Leftrightarrow 1)^n q^{\frac{3n^2+n}{2}} (1 + q + q^2 + \cdots + q^{2n}). \tag{7.8}$$

Although its validity can be readily checked by dividing both sides of (7.4) by $(1 \Leftrightarrow q)$, it will prove more insightful to obtain this identity through a combinatorial means which can be easily extended to prove (7.2).

Proof of Lemma 7.3

Notice that the left-hand side of (7.8) can be written in the form

$$\sum_{n\geq 0}\sum_{\lambda=(\lambda_1>\cdots>\lambda_n)} w(\lambda). \tag{7.9}$$

We will proceed by defining a bijection, I, that pairs off a partition, λ, with $I(\lambda)$, in such a way that $w(I(\lambda)) = \Leftrightarrow w(\lambda)$ whenever $\lambda \neq I(\lambda)$. This will allow us to reduce the inner summation of (7.9) to a finite sum that accounts only for the fixed points of I. The idea is to use 1-landing staircases in a manner similar to the way Franklin used staircases (i.e. 0-landing staircases) to prove (7.4) . The basic principle of the involution is this,

1. If $t(\lambda) \leq s_1(\lambda)$, move $\mathcal{T}(\lambda)$, if possible, to the outside of $\mathcal{S}_1(\lambda)$ so that $s_1(I(\lambda)) = t(\lambda)$ and

2. If $t(\lambda) > s_1(\lambda)$, move $\mathcal{S}_1(\lambda)$, if possible, to the empty row above $\mathcal{T}(\lambda)$.

The best way to see what is meant by "if possible", is to break up the definition of I into two cases. Case 1 is when $s_1(\lambda) < 1 + n$, which means that $\mathcal{S}_1(\lambda)$ *cannot* reach the top row of λ, and thus it will always be possible to move either $\mathcal{T}(\lambda)$ or $\mathcal{S}_1(\lambda)$. In the event that $t(\lambda) \leq s_1(\lambda)$, move the landing in $\mathcal{T}(\lambda)$ so that it is directly above the landing in the first $t(\lambda) \Leftrightarrow 2$ rows. If there is no landing in these rows, then place the landing at the end of the first row. Now move the stairs in $\mathcal{T}(\lambda)$ by placing one at the end of the first $t(\lambda) \Leftrightarrow 1$ rows. Moving $\mathcal{T}(\lambda)$ in this manner will guarantee that $s_1(I(\lambda)) = t(\lambda)$, as required. This procedure is illustrated in Figure 7.4.

In the event that $t(\lambda) > s_1(\lambda)$, move $\mathcal{S}_1(\lambda)$ to the top row, as in Figure 7.5. Notice that this operation will not result in a partition with a part < 2, since $t(I(\lambda)) = s_1(\lambda) \geq 2$, by Lemma 7.2.

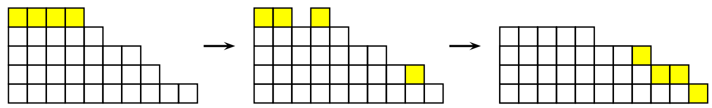


Figure 7.4: Case 1: $t(\lambda) \leq s_1(\lambda)$

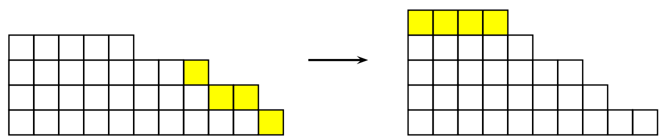


Figure 7.5: Case 1: $t(\lambda) > s_1(\lambda)$

Case 2 of the involution is when $s_1(\lambda) = 1 + n$. In this case, $\mathcal{S}_1(\lambda)$ *must* reach the top row of λ, and thus it might not be possible to move either $\mathcal{T}(\lambda)$ or $\mathcal{S}_1(\lambda)$. In other words, $\mathcal{S}_1(\lambda)$ shares at least one cell with $\mathcal{T}(\lambda)$ and possibly two, if the landing in $\mathcal{S}_1(\lambda)$ occurs in the last row of λ. For this reason, we'll denote the row of λ in which the landing occurs by $r(\lambda)$. For Case 2a, we will assume that $r(\lambda) < n$. If $t(\lambda) \leq s_1(\lambda) \Leftrightarrow 1$, move $\mathcal{T}(\lambda)$ in a similar manner to Figure 7.4 and if $t(\lambda) \Leftrightarrow 1 > s_1(\lambda)$, move $\mathcal{S}_1(\lambda)$ in a similar manner to Figure 7.5.

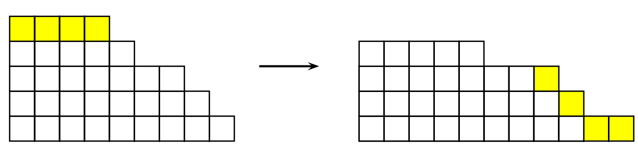


Figure 7.6: Case 2a: $t(\lambda) \leq s_1(\lambda) \Leftrightarrow 1$

For Case 2b, we will assume that $r(\lambda) = n$. If $t(\lambda) \leq s_1(\lambda) \Leftrightarrow 1$, then the involution is performed just as in Figures 7.4 and 7.6.

And finally, if $t(\lambda) \Leftrightarrow 2 > s_1(\lambda)$, then the involution is similar to that in Figures

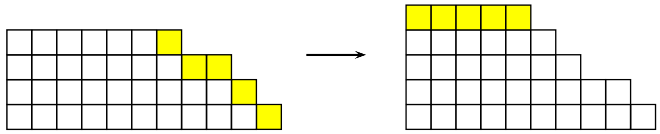


Figure 7.7: Case 2a: $t(\lambda) \Leftrightarrow 1 > s_1(\lambda)$

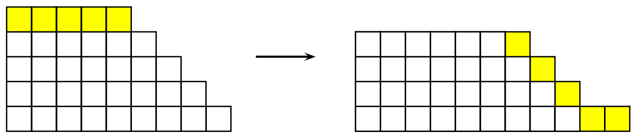


Figure 7.8: Case 2b: $t(\lambda) \leq s_1(\lambda) \Leftrightarrow 1$

7.5 and 7.7.

In the event that λ does not fit into one of the above categories, simply define $I(\lambda) = \lambda$. For example, moving $\mathcal{T}(\lambda)$ could shorten $\mathcal{S}_1(\lambda)$ to the point that $\mathcal{T}(\lambda)$ is too big to move, as in Figure 7.10a. Similarly, moving $\mathcal{S}_1(\lambda)$ could shorten $\mathcal{T}(\lambda)$ to the point where $\mathcal{S}_1(\lambda)$ is also too big, as in Figure 7.10b. Table 7.1 summarizes the fixed points of I.

We can now replace the inner summation in (7.9) with

$$\sum_{\lambda = I(\lambda)} w(\lambda) = (\Leftrightarrow 1)^n q^{\frac{3n^2+n}{2}}(1 + q + q^2 + \cdots + q^{2n}).$$

□

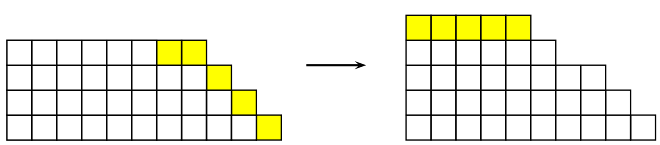


Figure 7.9: Case 2b: $t(\lambda) \Leftrightarrow 2 > s_1(\lambda)$

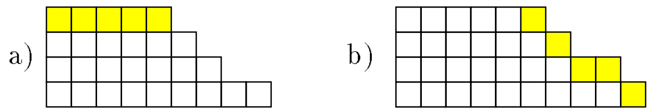


Figure 7.10: Sample fixed points of I

Table 7.1: Fixed points of I for $m = 1$

$s_1(\lambda)$	$t(\lambda)$	$r(\lambda)$	$\lvert\lambda\rvert$
$n+1$	$n+1$	$\{1, 2, \ldots, n-1\}$	$n^2 + \binom{n+1}{2} + r(\lambda)$
$n+1$	$n+2$	$\{1, 2, \ldots, n-1\}$	$n^2 + \binom{n+1}{2} + n + r(\lambda)$
$n+1$	$n+1$	n	$n^2 + \binom{n+1}{2}$
$n+1$	$n+2$	n	$n^2 + \binom{n+1}{2} + n$
$n+1$	$n+3$	n	$n^2 + \binom{n+1}{2} + 2n$

We are now in possession of a mechanism that can be easily generalized to prove formula (7.2). However, we must first formalize the definition of our involution for a fixed $m \geq 1$. Having done that, a simple observation regarding m-landing staircases will provide the key to determining a necessary and sufficient characteristic of fixed points.

Proof of equation (7.2)

Let λ be a partition with n distinct parts $> m$. Let $\tau(\lambda)$ be the result of moving $\mathcal{T}(\lambda)$ to the outside of $\mathcal{S}_m(\lambda)$. This is accomplished by placing a landing from $\mathcal{T}(\lambda)$ on top of each landing in the $t(\lambda) - m - 1$ bottommost rows of $\mathcal{S}_m(\lambda)$. Any landings still remaining in $\mathcal{T}(\lambda)$ should be placed at the end of the first row. Next, place the stairs from $\mathcal{T}(\lambda)$ at the ends of the $t(\lambda) - m$ bottommost rows. This process will insure that $s_m(\tau(\lambda)) = t(\lambda)$, which is necessary in order to reverse the process. Let $\sigma(\lambda)$ be the result of moving $\mathcal{S}_m(\lambda)$ to the empty row above $\mathcal{T}(\lambda)$. Notice that we cannot apply τ and σ to just any partition λ with parts $> m$, so to make up for this, we define I as follows.

$$I(\lambda) = \begin{cases} \tau(\lambda) & \text{if } t(\lambda) \leq s_m(\lambda) \quad \& \quad t(\lambda) < m+n, \\ \sigma(\lambda) & \text{if } t(\lambda) \Leftrightarrow |\mathcal{T}(\lambda) \cap \mathcal{S}_m(\lambda)| > s_m(\lambda), \\ \lambda & \text{otherwise.} \end{cases}$$

I is an involution since τ and σ are inverses of each other and if $\mu = \tau(\lambda)$, then

$$t(\mu) \Leftrightarrow |\mathcal{T}(\mu) \cap \mathcal{S}_m(\mu)| = \lambda_{n-1} > \lambda_n = t(\lambda) = s_m(\mu)$$

and if $\mu = \sigma(\lambda)$, then

$$t(\mu) = s_m(\lambda) \leq s_m(\mu) \quad \& \quad t(\mu) = s_m(\lambda) \leq m+n.$$

Notice that if λ is a fixed point, then $t(\lambda) \geq m+n$ and $s_m(\lambda) = m+n$. This means that the partition $\lambda^* = (2n \Leftrightarrow 1 + m, 2n \Leftrightarrow 2 + m, \ldots, n+m)$ is the smallest fixed point of I with exactly n parts. The weight of λ^* is given by

$$w(\lambda^*) = (\Leftrightarrow 1)^n q^{|\lambda^*|} = (\Leftrightarrow 1)^n q^{\frac{3n^2-n}{2}+nm}. \tag{7.10}$$

Unfortunately, it is not enough for $t(\lambda) \geq m+n$ and $s_m(\lambda) = m+n$. In order to come up with a necessary and sufficient condition for λ to be a fixed point, we need the following observation.

If $s_m(\lambda) = m+n$ then $\mathcal{S}_m(\lambda)$ will start and finish
at opposite corners of an $n \times m+n$ rectangle.

Of course this is none other than a simple fact regarding taxicab distances, but using this observation, we can prove the following crucial lemma.

Lemma 7.4 *Let* $\lambda = (\mu_1 + 2n \Leftrightarrow 1 + m, \mu_2 + 2n \Leftrightarrow 2 + m, \ldots, \mu_n + n + m)$ *where* $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n \geq 0$. *Then* λ *is a fixed point if and only if*

$$\mu_1 \leq m \qquad \text{or} \qquad \mu_1 = m+1 \;\; \& \;\; \mu_n \geq 1.$$

Proof

Let us start by assuming that λ is a fixed point. In particular, this means that $s_m(\lambda) = m + n$ and that $\mathcal{S}_m(\lambda)$ cannot be moved, or symbolically,

$$t(\lambda) \Leftrightarrow |\mathcal{T}(\lambda) \cap \mathcal{S}_m(\lambda)| \leq m + n. \tag{7.11}$$

Notice that the observation we made above allows us to compute the left-hand side of (7.11) exactly.

$$t(\lambda) \Leftrightarrow |\mathcal{T}(\lambda) \cap \mathcal{S}_m(\lambda)| = \mu_1 + n \Leftrightarrow 1 \tag{7.12}$$

Therefore, $\mu_1 \leq m + 1$. If $\mu_1 \leq m$, then we are done. If $\mu_1 = m + 1$, then using the observation again, the left-most cell of $\mathcal{S}_m(\lambda)$ occurs in the top row of μ, and thus we must also have that $\mu_n \geq 1$.

Now we need to show that this condition is sufficient. If $\mu_1 \leq m$, then one of the stairs in $\mathcal{S}_m(\lambda^*)$ will be used as a landing in $\mathcal{S}_m(\lambda)$. This insures that $s_m(\lambda) = m+n$. It also allows us to use equation (7.12) again to see that

$$t(\lambda) \Leftrightarrow |\mathcal{T}(\lambda) \cap \mathcal{S}_m(\lambda)| = \mu_1 + n \Leftrightarrow 1 \leq m + n \Leftrightarrow 1,$$

which means that $I(\lambda) = \lambda$.

In the event that $\mu_1 = m + 1$ and $\mu_n \geq 1$, one of the cells in the first column of μ will be used as a landing, insuring that $s_m(\lambda) = m + n$. Again we see that

$$t(\lambda) \Leftrightarrow |\mathcal{T}(\lambda) \cap \mathcal{S}_m(\lambda)| = \mu_1 + n \Leftrightarrow 1 = m + n,$$

which means that $I(\lambda) = \lambda$ in this case as well. □

Using this lemma, we see that any partition μ that fits in an $n \times m$ box will lead to a fixed point, as will any partition $\tilde{\mu}$ that fits in an $n \times m+1$ box with $\tilde{\mu}_1 = m+1$

and $\tilde{\mu}_n \geq 1$. Therefore, the weights of all fixed points with exactly n parts are accounted for in

$$w(\lambda^*)\left(\begin{bmatrix} n+m \\ m \end{bmatrix}_q + q^{n+m}\begin{bmatrix} n+m \Leftrightarrow 1 \\ m \end{bmatrix}_q\right). \tag{7.13}$$

Summing (7.13) over all values of $n \geq 0$, we see that

$$\prod_{n>m}(1 \Leftrightarrow q^n) = \sum_{n\geq 0}(\Leftrightarrow 1)^n q^{\frac{3n^2-n}{2}+nm}\begin{bmatrix} n+m \Leftrightarrow 1 \\ m \Leftrightarrow 1 \end{bmatrix}_q \frac{1 \Leftrightarrow q^{2n+m}}{1 \Leftrightarrow q^m}. \tag{7.14}$$

Multiplying both sides of equation (7.14) by $(1 \Leftrightarrow q^m)$ and making a change of variable $m \to m+1$ yields (7.2). □

One property of Franklin's bijection is that it accounts for all of the cancellation occurring in the left-hand side of equation (7.4). Unfortunately, this is not always the case for I. In fact, as soon as $m = 3$ there is some unexplained cancellation. For example, the two partitions $(14, 13, 12, 11)$ and $(12, 11, 10, 9, 8)$ are both partitions of 50 and both are fixed points of I. On the other hand, there are 31,571,191 partitions of 250 with parts > 10. Of those 31,571,191 partitions, 3,537 are fixed points of I. Of those 3,537 fixed points, just 47 have a positive sign associated with them, and can therefore be cancelled out.

The text of this chapter, in part or in full, is a reprint of the material as it appears in Volume 42 of *Séminaire Lotharingien de Combinatorie* (The Andrews Festschrift). The dissertation author was the primary researcher and author.

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