UNIVERSITY OF CALIFORNIA, SAN DIEGO

Jacobi Forms over Number Fields

A dissertation submitted in partial satisfaction of the
requirements for the degree
Doctor of Philosophy

in

Mathematics

by

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1999

The dissertation of Howard Skogman is approved, and it is acceptable in quality and form for publication on microfilm:

Chair

University of California, San Diego

1999

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ABSTRACT OF THE DISSERTATION

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Doctor of Philosophy in Mathematics

University of California San Diego, 1999

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We define Jacobi Forms over an algebraic number field K and construct examples by first embedding the group and the space into the symplectic group and the symplectic upper half space respectively. We then create symplectic modular forms and create Jacobi forms by taking the appropriate Fourier coefficients. We also prove some relations of these Jacobi forms over certain fields to other types of modular forms.

Chapter 1

Introduction

The functions considered in this paper are generalizations of the classical Jacobi forms. Classical Jacobi forms are functions satisfying two transformation properties

$$\phi\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) = (c\tau+d)^k e^{2\pi i m \frac{cz^2}{c\tau+d}} \phi(\tau, z) \tag{1.1}$$

and

$$\phi(\tau, z+\lambda\tau+\mu) = e^{-2\pi i m(\lambda^2\tau+2\lambda z)}\phi(\tau, z) \tag{1.2}$$

for all matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $Sl_2(\mathbb{Z})$, and vectors $[\lambda, \mu]$ in $\mathbb{Z}^2$. The original examples of such forms were created by Jacobi in [11]. These examples were types of theta functions of quadratic forms. For example, given a positive definite $l \times l$ matrix Q with rational integer entries and even diagonal entries, and given a fixed vector $\vec{b}$ in $\mathbb{Z}^l$, define the function

$$\theta_{Q,b}(\tau, z) = \sum_{\vec{r}\in\mathbb{Z}^l} e^{\pi i\, {}^t\vec{r}Q\vec{r}\tau + 2\, {}^t\vec{r}Q\vec{b}z}.$$

These theta functions have the special transformation properties described above, however there was no investigation into general functions satisfying the transformation properties presented above until [6].

It is clear that these functions are generalizations of modular forms which are functions satisfying

$$f\left(\frac{a\tau+b}{c\tau+d}\right)=(c\tau+d)^k f(\tau),\quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sl_2(\mathbb{Z})$$

because setting the variable $z=0$ and considering the first transformation formula 1.1 yields the modular form transformation formula. The most interesting feature of Jacobi forms is that although they are only slightly more complicated than classical modular forms, they provide information about much more complicated types of modular forms such as vector valued and symplectic modular forms. There are a number of isomorphisms between spaces of Jacobi forms and these other types of modular forms and therefore understanding the Jacobi forms gives information about the other spaces of modular forms. Many of these connections are presented in [6] but others are presented in [12],[13],[18], [19].

There have also been a large number of investigations into more general types of Jacobi forms. Some of these articles focus on replacing the variable z above with a general matrix which has produced more connections to symplectic modular forms and vector valued modular forms see for example [15], [22],[24],[25].

There are two principal constructions for most types of modular forms. One is through theta functions, and the other is through Eisenstein series. These Eisenstein series have also been created in the case of Jacobi forms and this has allowed the study of the space of Jacobi forms as an algebraic structure. These techniques have also led to more connections between Jacobi forms and other types of forms see [1],[2],[7].

Another type of generalization of Jacobi forms is functions satisfying the same sort of transformation properties with larger groups of matrices and translation vectors. This is the primary focus of this paper. There have been very few investigations into this area. Gritsenko in [9] studied functions which satisfied generalizations of the transformation formulas where the invariance properties were with respect to all $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $Sl_2(\mathbb{Z}[\sqrt{-1}])$ and $[\lambda,\mu]$ in $\mathbb{Z}[\sqrt{-1}]^2$. Krieg in [14] studied similar forms with the ring $\mathbb{Z}[\sqrt{-1}]$ replaced by the ring of integers in any imaginary quadratic field, and studied their connection to symplectic modular forms. Haverkamp in [10] studied forms where the first transformation properties are with respect to $Sl_2(\mathbb{Z})$ but the second is with respect to vectors in the ring of integers in an imaginary quadratic field.

The aim of this paper is a still further generalization where the transformation formulas are with respect to the ring of integers in any algebraic number field, (i.e. finite degree extension of the rationals $\mathbb{Q}$). So the first transformation formula is with respect to $Sl_2(\mathcal{O}_K)$ and the second is with respect to $\mathcal{O}_K^2$ where $\mathcal{O}_K$ is the ring of algebraic integers in a number field K. The construction relies on extending the connection between Jacobi forms and symplectic modular forms to algebraic number fields. The actual construction produces the "natural" generalization of Jacobi's initial examples. One interesting difference between this work and some of the above cited works [9],[10],[15], is the use of a different symplectic group. The cited works all used the Hermitian symplectic group which is all matrices M with entries in the ring of integers in the field such that ${}^t\overline{M}JM = J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$ where the bar denotes complex conjugation. The symplectic group employed here has a similar requirement without the bar on the first matrix.

Another interesting feature of the Jacobi forms constructed in this paper is the need to introduce the index vector. The m which appears in the classical transformation formulas is replaced by the inner product of a vector with itself. The vector only appears in the variables corresponding to complex conjugates of the field and it causes a slight modification of the transformation formulas. The difference is that instead of all of the factors having the inner product of the vector with itself, some appear with the inner product of the vector and its complex conjugate. This index vector does not appear in Jacobi forms of totally real fields and this allows (along with some other facts) to extend the connection to vector valued modular forms for totally real fields.

1.1 Some Notation

Throughout the paper some standard notation will be used which is listed here. $\mathbb{Q}, \mathbb{Z}, \mathbb{R}, \mathbb{R}^+, \mathbb{C}$, will be used to denote the rational numbers, the rational integers, the real numbers, the positive real numbers and the complex numbers respectively.
We will use K to denote an algebraic number field and $\mathcal{O}_K$ will be the ring of algebraic

integers in this number field. A totally positive element $\alpha \in K$ will be denoted $\alpha \gg 0$ and the notation $\alpha \geq\geq 0$ is used to denote an element that is totally positive or zero. We will denote by δ_K, δ_K^{-1} the different and the inverse different of the field K.
The $n \times n$ matrices with entries in $\mathbb{F}$ is denoted $M_n(\mathbb{F})$. Given a matrix $M \in M_n(\mathbb{F})$ the transpose of M will be denoted ${}^tM$. The $l \times l$ identity matrix will be I_l.
The imaginary part of α is denoted $Im(\alpha)$.
The n dimensional $\mathbb{Z}$-module spanned by the $\{a_i\}_{1\leq i\leq n}$ is written $[a_1, a_2, ...a_n]_{\mathbb{Z}}$.

Chapter 2

Jacobi Forms over $\mathbb{Q}$

All of the results presented in this chapter are presented in [6].

2.1 The Jacobi Group

Jacobi forms are functions satisfying transformation formulas related to a group acting on the domain of the function. In order to define Jacobi forms, the first thing is to define is the Jacobi group and then introduce the space on which this group discretely acts. The Jacobi group is $Sl_2(\mathbb{Z}) \ltimes \mathbb{Z}^2 \equiv \Gamma^J(\mathbb{Z})$ where the group $Sl_2(\mathbb{Z})$ consists of 2×2 matrices

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sl_2(\mathbb{Z}).$$

such that each entry is a rational integer $a, b, c, d \in \mathbb{Z}$ and $\det(M) = ad - bc = 1$. Note this is not a direct product but a semi-direct product where the group operation is given by: for all M, M' in $Sl_2(\mathbb{Z})$, and $[\lambda, \mu], [\alpha, \beta]$ in $\mathbb{Z}^2$

$$(M, [\lambda, \mu]) \circ (M', [\alpha, \beta]) = (MM', [\lambda, \mu]M' + [\alpha, \beta]). \tag{2.1}$$

The second entry is the product of the first vector and the second matrix, added to the second vector.

We also define, as usual, the congruence subgroup of level N, for N in $\mathbb{Z}^+$ as

$$\Gamma_0(N) := \left\{ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, M \in \Gamma \text{ and } c \equiv 0 \bmod N \right\}.$$

and we may choose to look at forms transforming under $\Gamma_0(N) \ltimes (m\mathbb{Z})^2$ where m is in $\mathbb{Z}$. The space on which this group acts is $\mathfrak{h} \times \mathbb{C}$ where $\mathfrak{h} := \{z \in \mathbb{C}, \operatorname{Im} z > 0\}$ and $\mathbb{C}$ is the field of complex numbers. The actions of $\Gamma^J(\mathbb{Z})$ on $\mathfrak{h} \times \mathbb{C}$ are for M in $Sl_2(\mathbb{Z})$,

$$M \circ (\tau, z) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \circ (\tau, z) = \left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d} \right) \tag{2.2}$$

and for $[\lambda, \mu]$ in $\mathbb{Z}^2$,

$$[\lambda, \mu] \circ (\tau, z) = (\tau, z + \lambda\tau + \mu). \tag{2.3}$$

The general element of this Jacobi group acts first by the matrix and then by the vector which insures there is a group action on this space. To have a group action means that:

$$\forall g_1, g_2 \in \Gamma^J(\mathbb{Z}),\ (\tau, z) \in \mathfrak{h} \times \mathbb{C} \quad g_1 \circ (g_2 \circ (\tau, z)) = g_1 g_2 \circ (\tau, z).$$

This group action fixes the space and can be shown to be discrete. Briefly, a discrete action is one with no sequences of distinct elements approaching a limit in the group.

2.2 Jacobi Forms

Definition. *A function $\phi(\tau, z) : \mathfrak{h} \times \mathbb{C} \longrightarrow \mathbb{C}$ that satisfies*

$$\phi\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d} \right) = (c\tau + d)^k e^{2\pi i m \frac{cz^2}{c\tau + d}} \phi(\tau, z) \tag{2.4}$$

$$\phi(\tau, z + \lambda\tau + \mu) = e^{-2\pi i m(\lambda^2 \tau + 2\lambda z)} \phi(\tau, z) \tag{2.5}$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sl_2(\mathbb{Z}), [\lambda, \mu] \in \mathbb{Z}^2$, *and is analytic in both variables is called a* **Jacobi Form** *of weight k and index m.*

The first transformation formula 2.4 is a generalization of the classical modular transformation formula, that is a function which satisfies a formula like

$$f\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \circ \tau\right) = (c\tau+d)^k f(\tau), \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sl_2(\mathbb{Z}). \tag{2.6}$$

A Jacobi form restricted to $z = 0$ is a modular form. The second transformation formula 2.5 is a type of elliptic transformation, which means it expresses an invariance under translations by the rational integer lattice spanned by $\{1, \tau\}$. Because of these formulas, a Jacobi form satisfies $\phi(\tau+1, z) = \phi(\tau, z+1) = \phi(\tau, z)$ and therefore possesses a Fourier expansion with respect to both τ, z i.e.

$$\phi(\tau, z) = \sum_{n=0}^{\infty} \sum_{r\in\mathbb{Z},\ r^2\leq 4nm} c(n,r)q^n\xi^r$$

where $q = e^{2\pi i\tau}, \xi = e^{2\pi iz}$. Note that the $r^2 \leq 4nm$ is a condition to make the function analytic at infinity, i.e. analytic as $\tau \longrightarrow i\infty$. In the definition of Jacobi forms one may also restrict to a subgroup of the Jacobi group, and one may allow a multiplier system in the first transformation formula. Specifically we may replace 2.4 by

$$\phi\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) = \chi(M)(c\tau+d)^k e^{2\pi im\frac{cz^2}{c\tau+d}}\phi(\tau, z)$$

where $\chi(M)$ is a root of unity depending only on the matrix M. We will discuss the multiplier systems more in the following sections.

2.3 Relations to other types of modular forms

Jacobi forms are related to other types of modular forms, notably symplectic and vector valued modular forms. We will only sketch some of the major results of the correspondences here.

Symplectic modular forms of genus n are functions on

$$\mathfrak{h}^{(n)} = \{Z = X + iY \in M_n(\mathbb{C}) \mid {}^tZ = Z,\ Y > 0\},$$

i.e. $n \times n$ symmetric matrices over the complex numbers with positive definite imaginary part. The group $Sp_{2n}(\mathbb{Z})$ acts discretely on this space, where

$$Sp_{2n}(\mathbb{Z}) = \left\{ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in M_{2n}(\mathbb{Z}) \mid {}^tMJM = J = \begin{pmatrix} 0_n & -I_n \\ I_n & 0_n \end{pmatrix} \right\}$$

and $0_n, I_n$ denote the $n \times n$ zero and identity matrix respectively and all of A, B, C, D are $n \times n$ matrices over the rational integers. This group may also be written as

$$Sp_{2n}(\mathbb{Z}) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in M_{2n}(\mathbb{Z}) \mid {}^tAC = {}^tCA,\, {}^tDB = {}^tBD,\, {}^tAD - {}^tCB = I_n \right\}.$$

The action of $Sp_{2n}(\mathbb{Z})$ on $\mathfrak{h}^{(n)}$ is given by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \circ Z = (AZ + B)(CZ + D)^{-1}, \qquad \forall \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_{2n}(\mathbb{Z}),\ Z \in \mathfrak{h}^{(n)}.$$

Definition. *A Symplectic modular form of weight k and genus n is a function $f(Z) : \mathfrak{h}^{(n)} \longrightarrow \mathbb{C}$ such that*

$$f(M \circ Z) = \det(CZ + D)^k f(Z), \qquad \forall\ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_{2n}(\mathbb{Z}). \tag{2.7}$$

We may allow a multiplier system in the transformation formula.

When the genus is two the variable Z may be written as

$$Z = \begin{pmatrix} \tau & z \\ z & \tau_2 \end{pmatrix}, \qquad \tau, \tau_2 \in \mathfrak{h}, \quad z \in \mathbb{C},$$

where $Im(\tau)Im(\tau_2) > Im(z)^2$. The correspondence between Jacobi forms and symplectic modular forms is given by

Theorem. *If f is a symplectic modular form of weight k and genus two, $f(Z) = f(\tau, z, \tau_2)$ has a Fourier expansion with respect to τ_2 of the form*

$$f(\tau, z, \tau_2) = \sum_{m=0}^{\infty} \phi_{k,m}(\tau, z) e^{2\pi i m \tau_2}$$

where each of the $\phi_{k,m}(\tau, z)$ are Jacobi forms of weight k and index m.

This theorem shows the close connection between Jacobi forms and symplectic modular forms. A generalization of this theorem will be used to create Jacobi forms over an algebraic number field. It is interesting to note that this correspondence is somewhat reversible, in that symplectic modular forms may be created by taking certain Jacobi forms as the coefficients in a Fourier expansion when the genus is two.

The relation to vector valued modular functions arises from a periodicity in the Fourier coefficients of a Jacobi form. To state this connection first define for each congruence class μ modulo $2m$

$$\theta_{m,\mu}(\tau,z) = \sum_{r\in\mathbb{Z},\ r\equiv\mu \bmod 2m} q^{r^2/4m}\xi^r, \quad q=e^{2\pi i\tau},\ \xi=e^{2\pi iz}$$

which are Jacobi forms of weight $\frac{1}{2}$ and index m on a subgroup of $\Gamma^J(\mathbb{Z})$. If $\phi(\tau,z)$ is a Jacobi form of weight k and index m, then there exist functions $h_\mu(\tau)$ such that

$$\phi(\tau,z) = \sum_{\mu \bmod 2m} h_\mu(\tau)\theta_{m,\mu}(\tau,z).$$

Where the $h_\mu(\tau)$ satisfy

$$h_\mu(\tau+1) = e^{2\pi i m\mu^2/4m}h_\mu(\tau) \tag{2.8}$$

$$h_\mu\left(\frac{-1}{\tau}\right) = \frac{\tau^k}{\sqrt{2\pi\tau/i}}\sum_{\nu \bmod 2m} e^{2\pi i\mu\nu/2m}h_\nu(\tau) \tag{2.9}$$

where the square root in 2.9 is given by the principal value.

Theorem. *The correspondence $\phi_{k,m}(\tau,z) \longleftrightarrow (h_\mu)_{\mu \bmod 2m}$ gives an isomorphism between the space of Jacobi forms of weight k and index m and vector valued modular forms satisfying 2.8, 2.9 and bounded as $Im(\tau)$ goes to infinity.*

By vector valued modular forms of weight $k-\frac{1}{2}$ we mean functions $\vec{h}(\tau) = (h_\mu(\tau))_{\mu \bmod 2m}$ satisfying $\vec{h}(M\circ\tau) = (c\tau+d)^{k-\frac{1}{2}}U(M)\vec{h}(\tau)$ where $U(M)$ is some $2m\times 2m$ matrix and $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $Sl_2(\mathbb{Z})$.

Both of these correspondences will be extended for certain types of number fields and then proofs of the results will be given.

2.4 Multiplier systems

The more general definitions of modular forms, Jacobi forms, and symplectic modular forms all include multiplier systems. This is especially important for those forms that have non-integer weight. The importance of the multiplier system can be seen most easily in the case of classical modular forms. We now state a definition of modular forms on $Sl_2(\mathbb{Z})$ which incorporates a multiplier system and then give some justification for its necessity.

Definition. *A* **modular form** *of weight k and multiplier system $\chi(...)$ is an analytic function $f : \mathfrak{h} \longrightarrow \mathbb{C}$ such that* $\forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sl_2(\mathbb{Z})$

$$f\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \circ \tau\right) = f\left(\frac{a\tau + b}{c\tau + d}\right) = \chi\begin{pmatrix} a & b \\ c & d \end{pmatrix}(c\tau + d)^k f(\tau),$$

where $\chi(...)$ is a root of unity (i.e. $\chi(...)^f = 1$ for some $f \in \mathbb{Z}$) depending only on the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

This may be seen as a generalization of the usual definition of modular forms which is given with the multiplier system $\chi(M) \equiv 1$ for all matrices M in $Sl_2(\mathbb{Z})$.

In order to see how these multiplier systems arise let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a matrix in $Sl_2(\mathbb{Z})$ and let $\nu(M, \tau) = (c\tau + d)$. In order to have the group $Sl_2(\mathbb{Z})$ act on the space $\mathfrak{h}$ it is necessary that for all M_1, M_2 in $Sl_2(\mathbb{Z}), \tau$ in $\mathfrak{h}$,

$$M_1 \circ (M_2 \circ \tau) = M_1 M_2 \circ \tau.$$

So in order to be a modular form of weight 1 without a multiplier the function ν must have the property that

$$\nu(M_1 M_2, \tau) = \nu(M_1, M_2 \circ \tau)\nu(M_2, \tau)$$

and it is easy to check that ν has this property. However, if f is a modular form of weight $\frac{1}{2}$ without a multiplier system then the requirement is

$$\nu(M_1 M_2, \tau)^{\frac{1}{2}} = \nu(M_1, M_2 \circ \tau)^{\frac{1}{2}}\nu(M_2, \tau)^{\frac{1}{2}}$$

which is not necessarily true due to the ambiguity with how each of the square roots are taken. All that is known

$$\nu(M_1M_2,\tau)^{\frac{1}{2}} = \pm\nu(M_1, M_2\circ\tau)^{\frac{1}{2}}\nu(M_2,\tau)^{\frac{1}{2}}.$$

A multiplier system is introduced to handle the sign trouble in this case and is defined as $\chi(...) : Sl_2(\mathbb{Z}) \longrightarrow U_l$ such that

$$\chi(M_1M_2)\nu(M_1M_2,\tau)^{\frac{1}{2}} = \chi(M_1)\nu(M_1, M_2\circ\tau)^{\frac{1}{2}}\chi(M_2)\nu(M_2,\tau)^{\frac{1}{2}}.$$

for all $M_1, M_2 \in Sl_2(\mathbb{Z})$ where U_l is the finite group of the l-th roots of unity. Note that it is not in general true that $\chi(M_1M_2) = \chi(M_1)\chi(M_2)$.

In general it is very complicated to figure out the multiplier χ explicitly in terms of the matrix. Most important to the work presented here is the multiplier system of the symplectic theta function which is a symplectic modular form of weight $\frac{1}{2}$ and satisfies

$$f\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}\circ Z\right) = f((AZ+B)(CZ+D)^{-1}) = \chi\begin{pmatrix} A & B \\ C & D \end{pmatrix} det(CZ+D)^{\frac{1}{2}} f(Z)$$

for all matrices in a certain subgroup of $Sp_{2n}(\mathbb{Z})$. The multiplier system χ in this case is an eighth root of unity depending only on the matrix and the value of the square root that is taken. This is the most important case to this work because the symplectic theta function will be used to create the new Jacobi forms and then the transformation properties of the symplectic theta function are used to derive the transformation properties of the Jacobi forms. Stark in [21] determined the explicit multiplier system for the symplectic theta function in some important special cases and this could be used to determine the exact root of unity in the Jacobi transformation properties.

Chapter 3

Number fields

In this chapter we present some notation and facts about algebraic number fields. The notation will be used throughout the rest of the paper. All of the results presented here are standard and contained in most books about algebraic number theory for example [5], [20].

3.1 The conjugates of a number field

Let K be an algebraic number field, that is $K \subseteq \mathbb{C}$, K is a finite degree extension of $\mathbb{Q}$. It is well known that K may be written as $K = \mathbb{Q}(\alpha)$ where α satisfies an irreducible nth degree polynomial whose coefficients are rational integers, and then n is the degree of the extension. Given α, every β in K may be expressed as a rational combination of powers of α, for example $\beta = \sum_{j=0}^{n-1} b_j \alpha^j$ where each b_i is in $\mathbb{Q}$. If the other roots of the polynomial satisfied by α are denoted $\alpha^{(1)}, \alpha^{(2)}, ..., \alpha^{(n)}$, assume the roots are ordered so that $\alpha = \alpha^{(1)}$, then the conjugates of the field K are defined as the fields $K^{(i)} = \mathbb{Q}(\alpha^{(i)})$. Therefore, if an element β is in K, and $\beta = \sum_{j=0}^{n-1} b_j \alpha^j$ then the conjugates of β are denoted $\beta^{(i)} = \sum_{j=0}^{n-1} b_j \alpha^{(i)j}$ in $K^{(i)}$. If $\alpha^{(i)}$ is a real number, $K^{(i)}$ is called a real conjugate of the field K and if $\alpha^{(i)}$ is in $\mathbb{C}$ but not in $\mathbb{R}$, $K^{(i)}$ is called a complex conjugate of K. Note that complex conjugates of α always come in pairs, if $\alpha^{(i)} = a + bi$ is one root of the polynomial satisfied by α then $\overline{\alpha}^{(i)} = a - bi$ is also a root. If $K^{(i)} = \mathbb{Q}(\alpha^{(i)})$ is a complex conjugate then denote the corresponding complex conjugate field $\overline{K}^{(i)} = \mathbb{Q}(\overline{\alpha}^{(i)})$.

Assume K is an nth degree extension, denoted $[K : \mathbb{Q}] = n, n = r_1 + 2r_2$ where r_1 is the number of real conjugates of K and $2r_2$ is the number of complex conjugates of K. Define $\mathcal{O}_K$ as the ring of integers in K, which is all elements of K satisfying a monic polynomial with coefficients in $\mathbb{Z}$. It is known that $\mathcal{O}_K$ is an n-dimensional $\mathbb{Z}$ module written $\mathcal{O}_K = [\omega_1, \omega_2, ...\omega_n]_{\mathbb{Z}}$, so for all λ in $\mathcal{O}_K$, there exists $l_1, l_2, ..., l_n \in \mathbb{Z}$ such that $\lambda = \sum_{j=1}^{n} l_i \omega_i$. For the rest of the paper the conjugates are labelled so that the first r_1 are the real conjugates and the rest are such that $K^{r_1+j} = \overline{K}^{r_1+r_2+j}$, for all $1 \leq j \leq r_2$.

The invertible elements of $\mathcal{O}_K$ are called the units of K and denoted $\mathfrak{U}_K$,

$$\mathfrak{U}_K = \{\varepsilon \in \mathcal{O}_K \mid \varepsilon \neq 0,\ \varepsilon^{-1} \in \mathcal{O}_K\}.$$

An element a in K is called totally positive, denoted by $a \gg 0$, if all the real conjugates of a are positive. It will also be useful to use the notation $a \geq\geq 0$ to mean that a is totally positive or zero. The trace and the norm of an element a in K are respectively given by

$$Tr_{K/\mathbb{Q}}(a) := \sum_{j=1}^{n} a^{(j)},$$

and

$$\mathcal{N}_{K/\mathbb{Q}}(a) := \mathcal{N}(a) := \prod_{j=1}^{n} a^{(j)}.$$

For any a in K both $Tr_{K/\mathbb{Q}}(a), \mathcal{N}_{k/\mathbb{Q}}(a)$ are in $\mathbb{Q}$; in particular if a is in $\mathcal{O}_K$ then both $Tr_{K/\mathbb{Q}}(a), \mathcal{N}_{k/\mathbb{Q}}(a)$ are in $\mathbb{Z}$. The **discriminant** of K denoted $\Delta_K = \det(W)^2$ where

$$W = \begin{pmatrix} \omega_1^{(1)} & \ldots & \omega_n^{(1)} \\ \vdots & & \vdots \\ \omega_1^{(n)} & \ldots & \omega_n^{(n)} \end{pmatrix}.$$

Where the ω_j were defined above as the rational integer basis elements of $\mathcal{O}_K$. The discriminant of a number field is a rational integer.

3.2 Ideals in number fields, the Different

There are certain subsets of the field K which are known as ideals. Ideals play an important role in a number field because the arithmetic of numbers is replaced by the arithmetic of ideals.

Definition. *An* **ideal** *of K is a subset $\mathfrak{a} \subseteq \mathcal{O}_K$ such that*

(1) $\forall a, b \in \mathfrak{a}, \quad a + b \in \mathfrak{a}$

(2) $\forall b \in \mathcal{O}_K, \quad b\mathfrak{a} \subseteq \mathfrak{a}$.

However, zero is not considered to be an ideal even though it does satisfy all of the hypotheses. The ideals of a number field possess an arithmetic structure in that ideals may be added or multiplied together, or divided one by the other. Define for two ideals $\mathfrak{a}, \mathfrak{b} \subseteq K$, $\mathfrak{a} + \mathfrak{b} = (\mathfrak{a} \cup \mathfrak{b})$ that is the ideal generated by the union of the two ideals, and define $\mathfrak{ab}$ as the ideal generated by all elements of the form $\alpha\beta$ for α in $\mathfrak{a}, \beta$ in $\mathfrak{b}$. In order to define division of ideals it is necessary to define the inverse of an ideal. The **inverse** of an ideal $\mathfrak{a}$ is

$$\mathfrak{a}^{-1} = \{b \in K \mid \forall a \in \mathfrak{a}, \quad ba \in \mathcal{O}_K\}.$$

It is not difficult to check that the above definition is an ideal. There are also notions of prime ideals and the unique factorization of ideals into the product of prime ideals but this will not be needed. Now define the inverse different of K to be δ_K^{-1} where

$$\delta_K^{-1} = \{a \in K \mid Tr_{K/\mathbb{Q}}(ab) \in \mathbb{Z}, \ \forall b \in \mathcal{O}_K\}.$$

The **Different** of K is δ_K, i.e. the inverse of the ideal δ_K^{-1}.

All ideals in a number field of degree n have a n-dimensional rational integer basis. So if $\mathfrak{a}$ is an ideal in K and α is in $\mathfrak{a}$ then $\alpha = \sum_{j=1}^{n} c_j a_j$ where the a_j are fixed elements of K and the c_j are in $\mathbb{Z}$. This representation is denoted $\mathfrak{a} = [a_1, a_2, ...a_n]_{\mathbb{Z}}$. The conjugate ideals are then given by $\mathfrak{a}^{(j)} = [a_1^{(j)}, a_2^{(j)}, ...a_n^{(j)}]_{\mathbb{Z}}$.

This type of basis may be used to turn a sum over elements of the ideal and the conjugates of the element, into a sum over the rational integers. For example if $\alpha \in \mathfrak{a}$

and $\alpha = \sum_{j=1}^{n} c_j a_j$ as above then

$$\begin{pmatrix} \alpha^{(1)} \\ \vdots \\ \alpha^{(n)} \end{pmatrix} = \begin{pmatrix} a_1^{(1)} & ... & a_n^{(1)} \\ \vdots & & \vdots \\ a_1^{(n)} & ... & a_n^{(n)} \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \mathfrak{A} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}.$$

The matrix representation of the ideal $\mathfrak{a}$ is invertible and the inverse of the matrix $\mathfrak{A}$ gives the basis for the ideal $\mathfrak{a}^{-1}\delta_K^{-1}$. The inverse matrix has the same structure as $\mathfrak{A}$, for example there exist $b_1, b_2, ...b_n \in K$ such that

$$\mathfrak{A}^{-1} = \begin{pmatrix} b_1^{(1)} & ... & b_n^{(1)} \\ \vdots & & \vdots \\ b_1^{(n)} & ... & b_n^{(n)} \end{pmatrix}$$

then the ideal $\mathfrak{a}^{-1}\delta_K^{-1}$ has a rational integer basis $\mathfrak{a}^{-1}\delta_K^{-1} = [b_1, b_2, ...b_n]\mathbb{Z}$.

Chapter 4

Jacobi forms over number fields

In this chapter the Jacobi group over a number field K is defined as well as the space on which it acts, and the definition of a Jacobi form over a number field is given. All of the notation from the previous chapter carries over into the rest of this paper including the definitions of $K, r_1, r_2, K^{(j)}, \mathcal{O}_K, W$, etc.

4.1 The Jacobi group

The Jacobi group of the number field K will be denoted $\Gamma^J(K)$ where

$$\Gamma^J(K) = Sl_2(\mathcal{O}_K) \ltimes \mathcal{O}_K^2.$$

The group operation is the same as in the classical case, see 2.1. This group acts on

$$\mathcal{H} = \mathfrak{h}^{r_1} \times \mathfrak{h}^{(\mathcal{Q})r_2} \times \mathbb{C}^{r_1} \times \mathcal{Q}^{r_2}$$

where $\mathcal{Q}$ is used to denote

$$\mathcal{Q} = \{x + y\kappa \mid x, y \in \mathbb{C}, \kappa^2 = -1,\ a\kappa = \kappa\overline{a}, \quad \forall a \in \mathbb{C}\}$$

which is known as the full ring of **quaternions**. It is sometimes written as

$$\mathcal{Q} = \{a + bi + cj + d\kappa \mid a, b, c, d \in \mathbb{R}, i^2 = j^2 = \kappa^2 = -1, \quad ij = \kappa, j\kappa = i, ji = -\kappa\}.$$

$\mathfrak{h}^{(\mathcal{Q})}$ is used to denote the upper half plane of quaternions

$$\mathfrak{h}^{(\mathcal{Q})} = \{x + y\kappa \in \mathcal{Q} \mid y \in \mathbb{R}^+\}$$

which may also be written as all quaternions that have no j component and positive κ component. It will be crucial in all of the following calculations to be careful about the ordering of elements of the quaternions since elements of this ring do not commute.

This space $\mathcal{H}$ where this Jacobi group acts is composed of one copy of $\mathfrak{h} \times \mathbb{C}$ for each real conjugate of the field, and one copy of $\mathfrak{h}^{(\mathcal{Q})} \times \mathcal{Q}$ for every pair of complex conjugates. Variables of this space will be listed as

$$(\tau_1, \tau_2, ..., \tau_{r_1+r_2}, z_1, ..., z_{r_1+r_2})$$

where each of the τ variables is in the appropriate upper half space and each of the z variables is in the appropriate field. Note the ordering of the conjugates is retained from chapter 4. So the first r_1 of the upper half plane τ_j (or full field z_j) variables are in $\mathfrak{h}$ (or $\mathbb{C}$) and the next r_2 of the variables are in $h^{(\mathcal{Q})}$ (or $\mathcal{Q}$). The quaternionic variables will be represented as

$$\tau_j = x_j + y_j\kappa, \quad \forall \tau_j \in \mathfrak{h}^{(\mathcal{Q})} \text{ and } z_j = u_j + v_j\kappa, \quad \forall z_j \in \mathcal{Q}.$$

The actions of $\Gamma^J(\mathcal{O}_K)$ on the space $\mathcal{H}$ are given by

$$\forall \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in Sl_2(\mathcal{O}_K), \quad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \circ (\tau_1, ...\tau_{r_1+r_2}, z_1, ...z_{r_1+r_2}) =$$

$$(\frac{\alpha^{(1)}\tau_1 + \beta^{(1)}}{\gamma^{(1)}\tau_1 + \delta^{(1)}}, ... \frac{\alpha^{(r_1)}\tau_{r_1} + \beta^{(r_1)}}{\gamma^{(r_1)}\tau_{r_1} + \delta^{(r_1)}}, (\alpha^{(r_1+1)}\tau_{r_1+1} + \beta^{(r_1+1)})(\gamma^{(r_1+1)}\tau_{r_1+1} + \delta^{(r_1+1)})^{-1}, ...$$

$$\frac{z_1}{\gamma^{(1)}\tau_1 + \delta^{(1)}}, ... \frac{z_{r_1}}{\gamma^{(r_1)}\tau_{r_1} + \delta^{(r_1)}}, (\tau_{r_1+1}\gamma^{(r_1+1)} + \delta^{(r_1+1)})^{-1} z_{r_1+1} ...), \tag{4.1}$$

$$\forall [\lambda, \mu] \in \mathcal{O}_K^2, \quad [\lambda, \mu] \circ (\tau_1, ...\tau_{r_1+r_2}, z_1, ...z_{r_1+r_2}) =$$

$$(\tau_1, ...\tau_{r_1+r_2}, z_1 + \tau_1\lambda^{(1)} + \mu^{(1)}, ..., z_{r_1} + \tau_{r_1}\lambda^{(r_1)} + \mu^{(r_1)},$$
$$z_{r_1+1} + \tau_{r_1+1}\lambda^{(r_1+1)} + \mu^{(r_1+1)}, ...). \tag{4.2}$$

As before, a general element of the Jacobi group acts first by the matrix and then by the vector. These actions are just the different conjugates of the group elements acting on the different copies of the upper half spaces and the full fields by the same actions as in the classical case.

The definition of congruence subgroups as above may be extended to K and thereby define subgroups of the Jacobi group using subgroups of $Sl_2(\mathcal{O}_K)$ and sublattices in the ring of integers. For example, extending the notation from earlier, given an ideal $\mathfrak{N} \subseteq \mathcal{O}_K$ define

$$\Gamma_0(\mathfrak{N}) = \left\{ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \ M \in Sl_2(\mathcal{O}_K) \text{ and } c \in \mathfrak{N} \right\}.$$

and so a subgroup of $\Gamma^J(\mathcal{O}_K)$ of the form $\Gamma_0(\mathfrak{N}) \ltimes \mathfrak{a}^2$ for ideals $\mathfrak{a}, \mathfrak{N} \subseteq \mathcal{O}_K$ may be used in place of $\Gamma^J(\mathcal{O}_K)$.

4.2 Jacobi forms

In order to define Jacobi forms over K it is necessary to define the transformation formulas and transformation factors for the Jacobi group. It will be useful to define an exponential of a quaternion as

$$e[...] = e^{2\pi i[...]}, \quad e[a + b\kappa] = e[a + \overline{a} + i(b + \overline{b})] = e^{2\pi i(a+\overline{a}+i[b+\overline{b}])}, \quad \forall a + b\kappa \in \mathfrak{Q}.$$

To reduce some notation in the formulas the factor $\mathcal{N}(\gamma\tau+\delta)$ is used to denote the factor that replaces the $(c\tau + d)$ in the classical transformation formula 2.4, so denote for τ in $\mathfrak{h}^{r_1} \times \mathfrak{h}^{(\mathfrak{Q})r_2}$, γ, δ in $\mathcal{O}_K$

$$\mathcal{N}(\gamma\tau + \delta) = \prod_{j=1}^{r_1}(\gamma^{(j)}\tau_j + \delta^{(j)}) \prod_{j=r_1+1}^{r_1+r_2}(|\gamma^{(j)}x_j + \delta^{(j)}|^2 + y_j^2|\gamma^{(j)}|^2) \tag{4.3}$$

where the notation $\tau_j = x_j + y_j\kappa$, for all τ_j in $\mathfrak{h}^{(\mathfrak{Q})}$ is used. The usual complex norm is denoted by $|...|$, $|a + bi|^2 = a^2 + b^2$.

In the case of number fields, especially fields with complex conjugates, it is necessary to define the **index vector** associated with the index $\mathbf{m}$. The index vector will be a certain complex vector, denoted $\vec{m}$ of length equal to twice the weight k of the form, such that ${}^t\vec{m}^{(j)}\vec{m}^{(j)} = \mathbf{m}^{(\mathbf{j})}$, for all $1 \leq j \leq n$. The explicit nature and the appearance of the index vector as well as what is meant by its conjugates will be made clear in the construction.

Definition. *A Jacobi form of weight k and index $\mathbf{m}$ and index vector $\vec{m}$ for the number field K is a function*

$$\Phi(\vec{\tau}, \vec{z}) : \mathfrak{h}^{r_1} \times \mathfrak{h}^{(\mathcal{Q})r_2} \times \mathbb{C}^{r_1} \times \mathcal{Q}^{r_2} \longrightarrow \mathbb{C}$$

satisfying

$$\Phi\left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \circ (\vec{\tau}, \vec{z})\right) = \chi\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mathcal{N}(\gamma\tau + \delta)^k \left(\prod_{j=1}^{r_1} e[\mathbf{m}^{(j)} \frac{\gamma^{(j)} z_j^2}{\gamma^{(j)}\tau_j + \delta^{(j)}}]\right)$$
$$\left(\prod_{j=r_1+1}^{r_1+r_2} e[{}^t\vec{m}^{(j)}(u_j + \overline{v_j}\kappa)(\gamma^{(j)}\tau_j + \delta^{(j)})^{-1}\gamma^{(j)}(u_j + v_j\kappa)\vec{m}^{(j)}]\right) \Phi(\vec{\tau}, \vec{z}) \quad (4.4)$$

and

$$\Phi([\lambda, \mu] \circ (\vec{\tau}, \vec{z})) = \left(\prod_{j=1}^{r_1} e[-\mathbf{m}^{(\mathbf{j})}(\lambda^{(j)2}\tau_j + 2\lambda^{(j)} z_j)]\right)$$
$$\left(\prod_{j=r_1+1}^{r_1+r_2} e[-{}^t\vec{m}^{(j)}(\lambda^{(j)}\tau_j\lambda^{(j)} + 2\lambda^{(j)} z_j)\vec{m}^{(j)}]\right) \Phi(\vec{\tau}, \vec{z}) \quad (4.5)$$

for all $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in Sl_2(\mathcal{O}_K)$, $[\lambda, \mu] \in \mathcal{O}_K^2$, *where χ is a root of unity depending only on the matrix.*

It is important to note that the forms which will be created have weight k in $\mathbb{Z}$ or k in $\mathbb{Z} + \frac{1}{2}$, however the index $\mathbf{m}$ is allowed to be any integer in the field. These transformation formulas are easily seen to be generalizations of the classical case. The transformation factors are essentially one conjugate of the factor from the classical action for each conjugate of K, where one must be careful in the ordering of the terms for the quaternionic variables. This definition may be restricted to subgroups of the Jacobi group. The root of unity χ depending on the matrix will be an eighth root of unity, i.e. $\chi(M)^8 = 1$ for all M in $Sl_2(\mathcal{O}_K)$ which may be determined using work of Stark in [21].

Note there is no multiplier system in the second transformation formula 4.5. The group action by elements of the form $[\lambda, \mu]$ along with the fact that theta functions will be used in the construction, force the multiplier system to be trivial on these Jacobi group elements. The only differences from the exact form of the classical formulas and factors results from the noncommutative nature of the quaternionic variables. It is also useful to note that the index vector is only necessary for fields with complex conjugates. So for totally real fields it is only required to specify the index and not the index vector. Also for totally real fields, it is possible to require that the Jacobi forms be analytic and in fact this will be necessary in order to prove the connection to vector valued modular forms. However, for the general algebraic number field there is no notion of an analytic function so it is left out of the definition.

Chapter 5

Construction

In this chapter a construction will be outlined that produces Jacobi forms of weight $\frac{1}{2}$, indexes of the form λ^2, and index vectors of the form (λ), for λ in $\mathcal{O}_K$. There is a method of producing arbitrary weights using quadratic forms which we will delay until chapter 7 and suggest [17] for more information. The idea of the construction is to use the relationship between Jacobi forms and symplectic modular forms. Specifically, first create a symplectic modular form which transforms over a number field, then take a Fourier expansion with respect to the appropriate variables and the Fourier coefficients will be Jacobi forms. It will be necessary to explicity show this since it is not obvious from the previous result how to generalize this property to number fields. The specific function we will create as a first example is a generalization of the classical Jacobi theta function

$$\theta_1(\tau, z) = \sum_{n \in \mathbb{Z}} e^{\pi i (n^2 \tau + 2nz)}.$$

which is a Jacobi form of weight $\frac{1}{2}$ and index 1 on a subgroup of the Jacobi group $\Gamma^J(\mathbb{Z})$.

5.1 The index vector

To introduce the index vector, it is easiest to describe how it appears. First, one creates a more general type of Jacobi theta function for any symmetric, positive definite quadratic form Q. That is, if Q is an l dimensional symmetric quadratic form over the rational integers with even diagonal entries, i.e. for a in $\mathbb{Z}^l$, $Q(a) = {}^t a Q a$ and $Q(a) > 0$, for all a in $\mathbb{Z}^l, a \neq 0$, then fix a vector b in $\mathbb{Z}^l$ and define

$$\theta_Q(\tau, z) = \sum_{a \in \mathbb{Z}^l} e^{\pi i({}^t a Q a \tau + 2\,{}^t a Q b z)}$$

which is a Jacobi form of weight $\frac{l}{2}$ and index $Q(b)$. The natural generalization of this form may be given for an l dimensional symmetric quadratic form Q_K whose entries are in $\mathcal{O}_K$. It is known that such a form decomposes as $Q_K^{(j)} = {}^t L^{[j]} L^{[j]}$, and for a fixed b in $\mathcal{O}_K^l$ the general Jacobi theta fuction of a quadratic form may be written as

$$\theta_{Q_K,b}(\vec{\tau}, \vec{z}) = \sum_{a \in \mathcal{O}_K^l} exp(\pi i(\sum_{j=1}^{r_1+r_2} {}^t a^{(j)}\,{}^t L^{[j]} \tau_j L^{[j]} a^{(j)} + 2\,{}^t a^{(j)}\,{}^t L^{[j]} z_j L^{[j]} b^{(j)})),$$

where we are using the previously defined exponential for quaternions. Such a form will be constructed and shown to be a Jacobi form over K of weight $\frac{l}{2}$ and the index ${}^t b Q_K b = {}^t b\,{}^t L L b$, except that the non complex parts of the quaternionic transformation factors will have an index of ${}^t b\,{}^t L \overline{L b}$. So if the index $\mathbf{m} = \sum_{i=1}^{l} c_i^2$ the non complex parts of the quaternionic transformation factors have an index of $\sum_{i=1}^{l} c_i \bar{c}_i$. This creates the need to specify the index vector which is $\vec{m}^{(j)} = L^{[j]} b^{(j)}$ for all of the conjugates in order to completely specify the transformation factors. Actually knowing the index vector is overkill. The transformation formulas are completely detemined by the index pair $Q(b)$ and the index for the non complex part of the quaternionic factors ${}^t b\,{}^t L \overline{L b}$. So if another index vector has the same index pair as Lb, the Jacobi form with this index vector will transform identically to the Jacobi form with the index vector Lb.

This means that instead of requiring that forms have the same index vector, the definition of Jacobi forms over K actually requires that the index vector for the form be in the same class as the index vector in the definition. Where the class of an index vector is determined by the index pair. The first construction will follow this idea in the case where $Q_K = L = (1)$ the 1×1 identity matrix. The problems with the class of the index vector will not arise until the general construction in chapter 7.

5.2 Symplectic theta function

We now introduce the function which will be used to create a symplectic modular form, namely the **symplectic theta function**

$$\Theta_{Sp}\left(Z, \begin{pmatrix} U \\ V \end{pmatrix}\right) = \sum_{m\in\mathbb{Z}^n} e^{\pi i({}^t(m+V)Z(m+V)-2\,{}^tmU-{}^tVU)} \tag{5.1}$$

where Z is in $\mathfrak{h}^{(n)}$, and U, V in $\mathbb{C}^n$ are fixed. $\Theta_{Sp}(Z)$ is a symplectic modular form of weight $\frac{1}{2}$ with a multiplier system for $\Gamma_\theta^{(n)}$ where

$$\Gamma_\theta^{(n)} = \left\{ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_{2n}(\mathbb{Z}) \mid A\,{}^tB,\ C\,{}^tD \text{ have even diagonal entries} \right\}.$$

Specifically, this means $\forall\ \ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_\theta^{(n)}$

$$\Theta_{Sp}\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix} \circ Z, M \circ \begin{pmatrix} U \\ V \end{pmatrix}\right) = \chi(M) det(CZ+D)^{\frac{1}{2}}\Theta_{Sp}\left(Z, \begin{pmatrix} U \\ V \end{pmatrix}\right) \tag{5.2}$$

where $\chi(M)$ is an eighth root of unity depending only on the matrix M and the value of the square root that is taken. This root of unity can be explicitly determined in certain cases using results of [21].

There is an alternate presentation of this theta function which changes the group under which this function transforms. Namely by replacing the πi that appears in 5.1 by a $2\pi i$ produces a function that does not transform properly under $\Gamma_\theta^{(n)}$ but under the group

$$\Gamma_0^{(n)}(4) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_{2n}(\mathbb{Z}) \mid C \text{ has entries divisible by } 4 \right\}.$$

This type of group is more common in the literature, and make calculations simpler to present however it is quite a bit smaller than $\Gamma_\theta^{(n)}$ so the simplification comes at the price of a large number of transformation formulas. In this paper the "$2\pi i$" version will be used to simplify the statements of results and to simplify the construction however the construction with the "πi" version works just as well.

5.3 Embeddings

The method of creating a symplectic theta function that transforms appropriately is to embed $\Gamma^J(\mathcal{O}_K)$ into $Sp_{4n}(\mathbb{Z})$ and the space $\mathfrak{h}^{r_1} \times \mathfrak{h}^{(\mathfrak{Q})r_2} \times \mathbb{C}^{r_1} \times \mathfrak{Q}^{r_2}$ into $\mathfrak{h}^{(2n)}$. We will introduce notation to make this easier to state. We set for $(\vec{\tau}, \vec{z})$ in $\mathfrak{h}^{r_1} \times \mathfrak{h}^{(\mathfrak{Q})r_2} \times \mathbb{C}^{r_1} \times \mathfrak{Q}^{r_2}$

$$d_1(\vec{\tau}) = \begin{pmatrix} \tau_1 & 0 & \dots & \\ 0 & \tau_2 & & \\ \vdots & & \ddots & \\ & & & \tau_{r_1} \end{pmatrix}, \quad d_1(\vec{z}) = \begin{pmatrix} z_1 & 0 & \dots & \\ 0 & z_2 & & \\ \vdots & & \ddots & \\ & & & z_{r_1} \end{pmatrix}$$

$$d_2(\vec{x}) = \begin{pmatrix} x_{r_1+1} & 0 & \dots & \\ 0 & x_{r_1+2} & & \\ \vdots & & \ddots & \\ & & & x_{r_1+r_2} \end{pmatrix}, \quad d_2(\vec{u}) = \begin{pmatrix} u_{r_1+1} & 0 & \dots & \\ 0 & u_{r_1+2} & & \\ \vdots & & \ddots & \\ & & & u_{r_1+r_2} \end{pmatrix}$$

$$d_3(\vec{x}) = \begin{pmatrix} \bar{x}_{r_1+1} & 0 & \dots & \\ 0 & \bar{x}_{r_1+2} & & \\ \vdots & & \ddots & \\ & & & \bar{x}_{r_1+r_2} \end{pmatrix}, \quad d_3(\vec{u}) = \begin{pmatrix} \bar{u}_{r_1+1} & 0 & \dots & \\ 0 & \bar{u}_{r_1+2} & & \\ \vdots & & \ddots & \\ & & & \bar{u}_{r_1+r_2} \end{pmatrix}.$$

As before, set $\tau_j = x_j + y_j\kappa, \quad z_j = u_j + v_j\kappa$ for $r_1 < j \leq r_1 + r_2$ and similarly define $d_2(\vec{y}), d_2(\vec{v}), d_3(\vec{y}), d_3(\vec{v})$. It should be noted that all of the following quaternionic embeddings are based on the same representation, i.e.

$$a + b\kappa \in \mathfrak{Q} \hookrightarrow \begin{pmatrix} a & ib \\ i\bar{b} & \bar{a} \end{pmatrix} \in M_2(\mathbb{C}). \tag{5.3}$$

Now embed the space $\mathcal{H}$ into $\mathfrak{h}^{(2n)}$ by sending $(\vec{\tau}, \vec{z})$ to

$$Z = \begin{pmatrix} d_1(\vec{\tau}) & 0 & 0 & d_1(\vec{z}) & 0 & 0 \\ 0 & d_2(\vec{x}) & id_2(\vec{y}) & 0 & d_2(\vec{u}) & id_2(\vec{v}) \\ 0 & id_3(\vec{y}) & d_3(\vec{x}) & 0 & id_3(\vec{v}) & d_3(\vec{u}) \\ d_1(\vec{z}) & 0 & 0 & d_1(\vec{\tau}') & 0 & 0 \\ 0 & d_2(\vec{u}) & id_3(\vec{v}) & 0 & d_2(\vec{x}') & id_2(\vec{y}') \\ 0 & id_2(\vec{v}) & d_3(\vec{u}) & 0 & id_3(\vec{y}') & d_3(\vec{x}') \end{pmatrix}.$$

There are an extra set of upper half plane variables $\vec{\tau}'$ (where the quaternionic variables are denoted $\tau_j' = x_j' + y_j'\kappa$), which were embedded in the lower right $n \times n$ corner of the matrix. The Fourier expansion will be formed with respect to these extra variables, after creating the symplectic modular form.

Another embedding as in chapter 4 allows a sum over rational integers instead of integers in $\mathcal{O}_K$, recall the definition from Chapter 3,

$$W = \begin{pmatrix} \omega_1^{(1)} & \dots & \omega_n^{(1)} \\ \vdots & & \vdots \\ \omega_1^{(n)} & \dots & \omega_n^{(n)} \end{pmatrix} \text{ where } \mathcal{O}_K = [\omega_1, ...\omega_n]_{\mathbb{Z}}.$$

This matrix W converts rational integers into elements of $\mathcal{O}_K$, i.e. given λ in $\mathcal{O}_K$, $\lambda = \sum_{j=1}^n a_i \omega_i$ with a_j in $\mathbb{Z}$, $1 \le j \le n$, then

$$\begin{pmatrix} \lambda^{(1)} \\ \lambda^{(2)} \\ \vdots \\ \lambda^{(n)} \end{pmatrix} = \begin{pmatrix} \omega_1^{(1)} & \dots & \omega_n^{(1)} \\ \vdots & & \vdots \\ \omega_1^{(n)} & \dots & \omega_n^{(n)} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}.$$

Define $\widehat{W} = \begin{pmatrix} W & 0 \\ 0 & W \end{pmatrix}$ and create the symplectic theta function

$$\Theta\left(Z, \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) = \sum_{m \in \mathbb{Z}^{2n}} e^{2\pi i\, {}^t m\, {}^t\widehat{W} Z \widehat{W} m} = \sum_{m \in \mathbb{Z}^{2n}} e^{2\pi i\, {}^t m \tilde{Z} m} \text{ where } \widetilde{Z} = {}^t\widehat{W} Z \widehat{W}.$$

In order for this to be a symplectic theta function it is necessary to know that $\widetilde{Z}$ is in the space $\mathfrak{h}^{(2n)}$. For this we need the following lemma.

Lemma. *Assume V and T are invertible complex matrices, then the imaginary part of $Z = {}^t\overline{T} V T$ is positive definite if $\frac{1}{2i}(V - {}^t\overline{V})$ is positive definite.*

Proof. The imaginary part of ${}^t\overline{T} V T$ is $\frac{1}{2i}[{}^t\overline{T} V T - {}^tT\overline{V T}] = \frac{1}{2i}[{}^t\overline{T} V T - {}^t\overline{T}\, {}^t\overline{V} T]$ since Z is symmetric. Simplifying we have this is ${}^t\overline{T}[\frac{1}{2i}(V - {}^t\overline{V})]T$ and the expression in the parentheses is positive definite by assumption. Multiplying by ${}^t\overline{T}$ and T does not affect the positive definite nature of the expression in parentheses. $\square$

The converse of the lemma is also true but will be unnecessary in the construction.

To see that $\widetilde{Z}$ is in $\mathfrak{h}^{(2n)}$, define

$$S = \begin{pmatrix} I_{r_1} & 0 & \dots & & & & \\ 0 & 0 & I_{r_2} & & & & \\ \vdots & I_{r_2} & 0 & & & & \\ & & & I_{r_1} & & & \\ & & & & 0 & I_{r_2} \\ & & & & I_{r_2} & 0 \end{pmatrix}$$

then SZS is the same matrix as Z with the complex conjugate pairs switched in all of the variables. Similarly $S\widehat{W}$ is the same as the matrix $\widehat{W}$ with the complex conjugate pairs of rows switched which will be written as $\overline{W}$. Note $S^2 = I_{2n}$ and write ${}^t\widehat{W}Z\widehat{W} = {}^t\widehat{W}SSZ\widehat{W} = {}^t\overline{W}SZ\widehat{W}$ and the imaginary part of SZ is (using the notation from above)

$$\begin{pmatrix} d_1(y) & & & d_1(v) & & \\ & d_3(y_1) & & & d_3(v_1) & d_3(u_2) \\ & & d_2(y_1) & & d_2(u_2) & d_2(v_1) \\ d_1(v) & & & d_1(y') & & \\ & d_2(v_1) & d_3(u_2) & & d_3(y_1') & \\ & d_2(u_2) & d_3(v_1) & & & d_2(y_1') \end{pmatrix}. \tag{5.4}$$

This matrix can be forced to be positive definite simply by picking the y' and the y_1' variables to be large enough based on the other variables in the matrix. So some conditions on the extra upper half plane variables forces the imaginary part of the matrix in 5.4 to be positive definite and therefore the matrix ${}^t\widehat{W}Z\widehat{W} = {}^t\overline{W}SZ\widehat{W}$ has positive definite imaginary part and since it is obviously symmetric it is in $\mathfrak{h}^{(2n)}$.

Expanded out $\Theta(Z) = \Theta(\vec{\tau}, \vec{z}, \vec{\tau}')$ looks like

$$\Theta(\vec{\tau}, \vec{z}, \vec{\tau}') = \sum_{\lambda_1, \lambda_2 \in \mathcal{O}_K} e[\sum_{j=1}^{r_1} \lambda_1^{(j)2} \tau_j + 2\lambda_1^{(j)} \lambda_2^{(j)} z_j] e[\sum_{j=r_1+1}^{r_1+r_2} \lambda_1^{(j)} \tau_j \lambda_1^{(j)} + 2\lambda_1^{(j)} z_j \lambda_2^{(j)}]$$
$$e[\sum_{j=1}^{r_1} \lambda_2^{(j)2} \tau_j'] e[\sum_{j=r_1+1}^{r_1+r_2} \lambda_2^{(j)} \tau_j' \lambda_2^{(j)}]. \tag{5.5}$$

This first line mimics the classical Jacobi theta functions where one thinks of the λ_2 as fixed and the quadratic form $Q = (1)$. In fact the first line is the Fourier expansion with respect to the τ_j', so the Fourier expansion with respect to $\vec{\tau}'$ may be written

$$\Theta(\vec{\tau}, \vec{z}, \vec{\tau}') = \sum_{\lambda_2 \in \mathcal{O}_K} \Phi_{\frac{1}{2},\lambda_2}(\vec{\tau}, \vec{z}) e[\sum_{j=1}^{r_1} \lambda_2^{(j)2} \tau_j'] e[\sum_{j=r_1+1}^{r_1+r_2} \lambda_2^{(j)} \tau_j' \lambda_2^{(j)}]$$

where

$$\Phi_{\frac{1}{2},\lambda_2}(\vec{\tau}, \vec{z}) = \sum_{\lambda_1 \in \mathcal{O}_K} e[\sum_{j=1}^{r_1} \lambda_1^{(j)2} \tau_j + 2\lambda_1^{(j)} \lambda_2^{(j)} z_j] e[\sum_{j=r_1+1}^{r_1+r_2} \lambda_1^{(j)} \tau_j \lambda_1^{(j)} + 2\lambda_1^{(j)} z_j \lambda_2^{(j)}].$$

To write this expansion properly we should group the coefficients corresponding to λ_2 and $-\lambda_2$ should be grouped together, however since these terms are identical we leave them separate for now.

The next step in the construction is to embed the Jacobi group into $Sp_{4n}(\mathbb{Z})$ and show that the action on the embedded variables $(\vec{\tau}, \vec{z})$ as an element of $\mathfrak{h}^{(2n)}$ is the claimed group action on the space. First define the notation $\check{d}(\alpha)$, for α in K and extend the notation from above for d_1, d_2, d_3 to elements of K

$$\check{d}(\alpha) = \begin{pmatrix} \alpha^{(1)} & 0 & \dots & 0 \\ 0 & \alpha^{(2)} & & \\ \vdots & & \ddots & \\ 0 & & & \alpha^{(n)} \end{pmatrix} = \begin{pmatrix} d_1(\alpha) & & \\ & d_2(\alpha) & \\ & & d_3(\alpha) \end{pmatrix}.$$

The embeddings of the elements of the Jacobi group are:

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in Sl_2(\mathcal{O}_K) \hookrightarrow \begin{pmatrix} \check{d}(\alpha) & 0 & \check{d}(\beta) & 0 \\ 0 & I_n & 0 & 0 \\ \check{d}(\gamma) & 0 & \check{d}(\delta) & 0 \\ 0 & 0 & 0 & I_n \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

$$[\lambda, \mu] \in \mathcal{O}_K^2 \hookrightarrow \begin{pmatrix} I_n & 0 & 0 & \check{d}(\mu) \\ \check{d}(\lambda) & I_n & \check{d}(\mu) & \check{d}(\lambda\mu) \\ 0 & 0 & I_n & \check{d}(-\lambda) \\ 0 & 0 & 0 & I_n \end{pmatrix} = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix}.$$

With these embeddings $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \circ (\vec{\tau}, \vec{z}, \vec{\tau}') \hookrightarrow$

$$\begin{pmatrix} d_1(\alpha\vec{\tau}+\beta) & 0 & 0 & d_1(\alpha\vec{z}) & 0 & 0 \\ 0 & d_2(\alpha\vec{x}+\beta) & id_2(\alpha\vec{y}) & 0 & d_2(\alpha\vec{u}) & id_2(\alpha\vec{v}) \\ 0 & id_3(\alpha\vec{y}) & d_3(\alpha\vec{x}+\beta) & 0 & id_3(\alpha\vec{v}) & d_3(\alpha\vec{u}) \\ d_1(\vec{z}) & 0 & 0 & d_1(\vec{\tau}') & 0 & 0 \\ 0 & d_2(\vec{u}) & id_3(\vec{v}) & 0 & d_2(\vec{x}') & id_2(\vec{y}') \\ 0 & id_2(\vec{v}) & d_3(\vec{u}) & 0 & id_3(\vec{y}') & d_3(\vec{x}') \end{pmatrix} \circ$$

$$\begin{pmatrix} d_1(\gamma\vec{\tau}+\delta) & 0 & 0 & d_1(\gamma\vec{z}) & 0 & 0 \\ 0 & d_2(\gamma\vec{x}+\delta) & id_2(\gamma\vec{y}) & 0 & d_2(\gamma\vec{u}) & id_2(\gamma\vec{v}) \\ 0 & id_3(\gamma\vec{y}) & d_3(\gamma\vec{x}+\beta) & 0 & id_3(\gamma\vec{v}) & d_3(\gamma\vec{u}) \\ 0 & 0 & 0 & I_n & 0 & 0 \\ 0 & 0 & 0 & 0 & I_n & 0 \\ 0 & 0 & 0 & 0 & 0 & I_n \end{pmatrix}^{-1}.$$

It may be verified that this action with these embeddings takes the variables

$$\tau_j \longrightarrow \frac{\alpha^{(j)}\tau_j + \beta^{(j)}}{\gamma^{(j)}\tau_j + \delta^{(j)}}, \quad z_j \longrightarrow \frac{z_j}{\gamma^{(j)}\tau_j + \delta^{(j)}}, \quad \forall\ 1 \le j \le r_1 \tag{5.6}$$

$$\tau_j \longrightarrow (\alpha^{(j)}\tau_j + \beta^{(j)})(\gamma^{(j)}\tau_j + \delta^{(j)})^{-1},\ z_j \longrightarrow (\tau_j\gamma^{(j)} + \delta^{(j)})^{-1}z_j, \quad \forall\ r_1 \le j \le r_1 + r_2 \tag{5.7}$$

$$\tau_j' \longrightarrow \tau_j' - \frac{\gamma^{(j)}z_j^2}{\gamma^{(j)}\tau_j + \delta^{(j)}}, \quad \forall\ 1 \le j \le r_1 \tag{5.8}$$

$$\tau_j' \longrightarrow \tau_j' - (u_j + \overline{v}_j\kappa)(\gamma^{(j)}\tau_j + \delta^{(j)})^{-1}\gamma^{(j)}(u_j + v_j\kappa), \quad \forall\ r_1 \le j \le r_1 + r_2. \tag{5.9}$$

Where the quaternionic factors are still embedded using 5.3. These are exactly the proscibed actions of the matrix $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ on the space $\mathcal{H}$, and the extra variables τ_j' are transformed by subtracting the exponential transformation factor from formula 4.4 for the appropriate conjugate to the τ_j'.

It should be noted that the action $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \circ Z$ is the same as the action

$$\begin{pmatrix} {}^tWA\,{}^tW^{-1} & {}^tWBW \\ W^{-1}C\,{}^tW^{-1} & W^{-1}DW \end{pmatrix} \circ \tilde{Z}$$

(recall $\tilde{Z} = {}^tWZW$). In order for this embedding of $\Gamma^J(\mathcal{O}_K)$ to act on the symplectic theta function it is necessary that these embedded matrices be in $Sp_{4n}(\mathbb{Z})$ or to be more precise $\Gamma_0^{(2n)}(4)$. The next step is to calculate the conditions on $\alpha, \beta, \gamma, \delta$ that will force the matrix

$$\begin{pmatrix} \check{A} & \check{B} \\ \check{C} & \check{D} \end{pmatrix} = \begin{pmatrix} {}^tWA\,{}^tW^{-1} & {}^tWBW \\ W^{-1}C\,{}^tW^{-1} & W^{-1}DW \end{pmatrix} \in \Gamma_0^{(2n)}(4).$$

Since W is a matrix of integers in the field and contains a basis for the ring of integers $\mathcal{O}_K$ the inverse matrix W^{-1} is a basis for δ_K^{-1} the inverse different of the field K. It is easy to check that each of the entries in the matrices $\check{A}, \check{B}, \check{C}, \check{D}$ is the trace of an element of K. Since W^{-1} is a basis for the inverse different, as long as α, β, δ are in $\mathcal{O}_K$ the entries of $\check{A}, \check{B}, \check{D}$ will be traces of elements of δ_K^{-1} and therefore rational integers. In order for $\check{C}$ to be integral it is required that γ in δ_K, i.e. γ must be in the different. This is because the entries of $\check{C}$ are traces of elements in $\delta_K^{-2}(\gamma)$ and by the definition of the inverse different, an element of the inverse different has integral trace. So as long as γ is in δ_K, $\check{C}$ will have entries that are traces of elements in δ_K^{-1}. In order for this matrix to be in $\Gamma_0^{(2n)}(4)$ it is sufficient to require that 4 divide γ. Therefore as long as $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_0(4\delta_K)$ the matrix $\begin{pmatrix} \check{A} & \check{B} \\ \check{C} & \check{D} \end{pmatrix} \in \Gamma_0^{(2n)}(4)$.

It is easy to verify that

$$\det(\check{C}\tilde{Z} + \check{D}) = \det(CZ + D) = \mathcal{N}(\gamma\tau + \delta).$$

Therefore matching the Fourier coefficients of the symplectic theta function and the transformed version by 5.2 and using the fact that $\Theta(Z)$ is a symplectic modular form, the coefficients satisfy

$$\Phi_{\frac{1}{2},\lambda_2}\left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \circ \vec{\tau}, \vec{z}\right) = \chi(M)\mathcal{N}(\gamma\tau + \delta)^{\frac{1}{2}} e[\sum_{j=1}^{r_1} \lambda_2^{(j)2} \frac{\gamma^{(j)} z_j^2}{\gamma^{(j)}\tau_j + \delta^{(j)}}]$$

$$e[\sum_{j=r_1+1}^{r_1+r_2} \lambda_2^{(j)}(u_j+\overline{v}_j\kappa)(\gamma^{(j)}\tau_j+\delta^{(j)})^{-1}\gamma^{(j)}(u_j+v_j\kappa)\lambda_2^{(j)}]\Phi_{\frac{1}{2},\lambda_2}(\vec{\tau},\vec{z}).$$

This agrees with the formula 4.4 given in the definition of a Jacobi form over K of weight $\frac{1}{2}$, index λ_2^2 and index vector (λ_2).

The second transformation law is proved by looking at the actions of $[\lambda,\mu]$ in $\mathcal{O}_K^2$ embedded in $Sp_{4n}(\mathbb{Z})$ and its action on the variables $(\vec{\tau},\vec{z},\vec{\tau}')$ and comparing Fourier coefficients. The embedded $[\lambda,\mu]$ matrix takes

$$\tau_j \longrightarrow \tau_j, \quad z_j \longrightarrow z_j+\tau_j\lambda^{(j)}+\mu^{(j)}, \ \forall\ 1\leq j\leq r_1+r_2$$

$$\tau_j' \longrightarrow \tau_j'+\lambda^{(j)2}\tau_j+2\lambda^{(j)}z_j+\lambda^{(j)}\mu^{(j)} \quad \forall\ 1\leq j\leq r_1$$

$$\tau_j' \longrightarrow \tau_j'+\lambda^{(j)}\tau_j\lambda^{(j)}+2\lambda^{(j)}z_j+\lambda^{(j)}\mu^{(j)}, \quad \forall\ r_1+1\leq j\leq r_1+r_2.$$

It is trivial to see that $\det(C'\widetilde{Z}+D')=1$, and therefore by the same arguement as above that by matching the Fourier coefficients on both sides of the symplectic theta function transformation formula 5.2 one extracts the transformation formulas for these Fourier coefficients. These formulas are exactly those given in 4.5 of the definition for a Jacobi form of weight $\frac{1}{2}$ index λ_2^2 and index vector (λ_2). It is also easy to check that in order for

$$\begin{pmatrix} {}^t\!WA'\,{}^t\!W^{-1} & {}^t\!WB'W \\ W^{-1}C'\,{}^t\!W^{-1} & W^{-1}D'W \end{pmatrix} \in \Gamma_0^{(2n)}(4)$$

the only requirement is that $[\lambda,\mu]$ be in $\mathcal{O}_K^2$ since $W^{-1}C'\,{}^t\!W^{-1}=0$. Therefore by this construction, one is able to produce Jacobi forms on $\Gamma_0(4\delta_K)\ltimes\mathcal{O}_K^2$ of weight $\frac{1}{2}$ and index λ_2^2 and index vector (λ_2) for all λ_2 in $\mathcal{O}_K$.

Chapter 6

Relation to vector valued modular forms

In this chapter the algebraic number field K is required to be totally real, i.e. $K \subseteq \mathbb{R}$, though K is still assumed to be an nth degree extension. This restriction is necessary because it is only for the totally real fields that one may require the forms to be analytic, (it is only in the case of real fields and complex variables that analytic functions make sense). In order for the forms to be analytic the construction in chapter 7 would need to be limited to quadratic forms all of whose conjugates are positive definite (since K is totally real this makes sense). The analyticity of the Jacobi forms over K plus the fact that the Jacobi forms are periodic in both variables produces a "nice" Fourier expansion. The periodicity may be seen from letting $\Phi(\vec{\tau}, \vec{z})$ be a Jacobi form over K, then because of the transformation formulas 4.4,4.5

$$\Phi(\vec{\tau}, \vec{z}) = \Phi(\vec{\tau} + 1, \vec{z}) = \Phi(\vec{\tau}, \vec{z} + 1)$$

where for example $\vec{\tau} + 1$ means to add 1 to each element of the vector. Therefore the Fourier expansion of Φ has the form (note $r_2 = 0$, so $n = r_1$)

$$\Phi(\vec{\tau}, \vec{z}) = \sum_{\alpha, \beta \in \delta_K^{-1}} c(\alpha, \beta) e[\sum_{j=1}^{n} \alpha^{(j)} \tau_j + \beta^{(j)} z_j] \tag{6.1}$$

where each of the $c(\alpha, \beta)$ is constant and δ_K^{-1} is the inverse different as introduced earlier. In fact, for the forms that will be constructed in chapter 7 the Fourier expansion is only over $\mathcal{O}_K \subseteq \delta_K^{-1}$ however in the general case this may not be assumed.

The "nice" property of these Fourier expansions is that the Fourier expansion is with respect to τ, z instead of just x, and u. For a general algebraic number field, we do not have enough information about the Fourier expansions of our Jacobi forms to prove many of these results. We introduce notation which will be helpful. Earlier the trace $Tr_{K/\mathbb{Q}}$ of an element of the field K was defined, now extend this definition to include the variables in the space $\mathcal{H}$. For example, let $\vec{\tau}$ be an n dimensional upper half plane variable as usual and let $\alpha \in K$ then set

$$Tr_{K/\mathbb{Q}}(\alpha\vec{\tau}) = \sum_{j=1}^{n} \alpha^{(j)}\tau_j.$$

We similarly define the trace of the n dimensional full field variable which has been written as $\vec{z}$.

6.1 Restriction in the Fourier expansion

For Jacobi forms of index m over $\mathbb{Q}$, the functions are required to be analytic and this forces an extra condition on the Fourier expansion which has the form

$$\Phi(\tau, z) = \sum_{n=0}^{\infty} \sum_{r\in\mathbb{Z},\ r^2\leq 4nm} c(n,r)q^n\xi^r, \quad q = e^{2\pi i\tau}, \quad \xi = e^{2\pi iz}.$$

The condition that $r^2 \leq 4nm$ is a condition to make the function analytic as $\tau \to i\infty$. A similar condition arises with analytic Jacobi forms over totally real number fields.

Theorem. *If $\Phi(\vec{\tau}, \vec{z})$ is an analytic Jacobi form of weight k and index m over a totally real number field K of degree n then $m \gg 0$ and*

$$\Phi(\vec{\tau}, \vec{z}) = \sum_{\alpha\in\delta_K^{-1},\ \alpha\geq\geq 0} \ \sum_{(\beta\in\mathcal{O}_K\ |\ \beta^2\leq 4\alpha m)} c(\alpha,\beta)e^{2\pi iTr_{K/\mathbb{Q}}(\alpha\tau+\beta z)}. \tag{6.2}$$

The condition on the second sum is that for all $1 \leq j \leq n$, $\beta^{(j)2} \leq 4\alpha^{(j)}m^{(j)}$.

Proof. Assume that $\Phi(\vec{\tau}, \vec{z})$ is a Jacobi form over K and has a Fourier expansion of the form 6.1. Since the Jacobi form is required to be analytic, the form restricted to $(\vec{\tau}, 0)$

is also an analytic function of $\vec{\tau}$. Therefore $\Phi(\vec{\tau},0)$ will have a Fourier expansion with respect to only totally positive elements of δ_K^{-1} and zero, because if there was an α in the Fourier expansion such that $\alpha^{(j)} < 0$ for some $1 \leq j \leq n$ then by letting $\tau_j \to i\infty$, the function would diverge and therefore not be analytic. Similarly for any $\lambda \in \mathcal{O}_K$ the function

$$\Phi(\vec{\tau},0) = f(\vec{\tau}) = e^{2\pi i Tr_{K/\mathbb{Q}}(m\lambda^2\tau)}\Phi([\lambda,0]\circ(\vec{\tau},0))$$

is analytic in $\vec{\tau}$. The Fourier expansion of $f(\vec{\tau})$ is

$$f(\vec{\tau}) = \sum_{\alpha,\beta\in\delta_K^{-1}} c(\alpha,\beta)e[Tr_{K/\mathbb{Q}}(\alpha\tau+\lambda\beta\tau+\lambda^2\tau m)].$$

In order for this function to be analytic in all of the τ_j it is necessary that

$$\alpha^{(j)}+\lambda^{(j)}\beta^{(j)}+\lambda^{(j)2}m^{(j)} > 0, \quad \forall 1\leq j\leq n,\ \lambda\in\mathcal{O}_K.$$

This is a quadratic polynomial in $\lambda^{(j)}$ so it will always be positive provided the discriminant is negative or zero, i.e. $\beta^{(j)2}-4m^{(j)}\alpha^{(j)}\leq 0$ and since $\alpha^{(j)}>0$, $m^{(j)}>0$ and this was what was claimed in the theorem. $\square$

Therefore for totally real number fields the index is always totally positive as well as the condition on the Fourier expansion $\beta^2-4\alpha m \ll 0$ or $\beta^2-4\alpha m = 0$.

6.2 Periodicity in the Fourier coefficients

Given $\Phi(\vec{\tau},\vec{z})$ is a Jacobi form over a totally real number field K, the Fourier expansion may be written as

$$\Phi(\vec{\tau},\vec{z}) = \sum_{\alpha\in\delta_K^{-1},\ \alpha\geq\geq 0}\ \sum_{(\beta\in\mathcal{O}_K\ \mid\ \beta^2\leq 4\alpha m)} c(\alpha,\beta)e^{2\pi i Tr_{K/\mathbb{Q}}(\alpha\tau+\beta z)}.$$

Since there is no need of an index vector for this case we will just specify that the index of the form is $m\in\mathcal{O}_K$.

Theorem. *Given that $\Phi(\vec{\tau},\vec{z})$ is an analytic Jacobi form of index m over a totally real number field K, Φ has a Fourier expansion of the form given above and the Fourier coeffiecients $c(\alpha,\beta)$ depend only on $\beta \bmod 2m$ and on $4\alpha m-\beta^2$.*

Proof. The second transformation formula 4.5 for Jacobi forms over K yields, for any $\lambda, \mu \in \mathcal{O}_K$

$$\Phi([\lambda,\mu]\circ(\vec{\tau},\vec{z})) = \sum_{\alpha,\beta\in\delta_K^{-1}} c(\alpha,\beta)e[Tr_{K/\mathbb{Q}}(\alpha\tau+\beta(z+\lambda\tau+\mu))] =$$
$$e[-Tr_{K/\mathbb{Q}}(m\lambda^2\tau+2m\lambda z)]\Phi(\vec{\tau},\vec{z}). \quad (6.3)$$

It follows that

$$\Phi(\vec{\tau},\vec{z}) = e[Tr_{K/\mathbb{Q}}(m\lambda^2\tau+2m\lambda z)] \sum_{\alpha,\beta\in\delta_K^{-1}} c(\alpha,\beta)e[Tr_{K/\mathbb{Q}}(\alpha\tau+\beta(z+\lambda\tau+\mu))] =$$
$$\sum_{\alpha,\beta\in\delta_K^{-1}} c(\alpha,\beta)e[Tr_{K/\mathbb{Q}}([\alpha+\lambda^2 m+\lambda\beta]\tau+[\beta+2\lambda m]z)]. \quad (6.4)$$

Therefore

$$c(\alpha,\beta) = c(\alpha+\lambda^2 m+\lambda\beta, \beta+2\lambda m) = c(\alpha',\beta'). \quad (6.5)$$

Then since $4\alpha m-\beta^2 = 4\alpha' m-\beta'^2$, the Fourier coefficients depend only β mod $2m$ and on $4\alpha m-\beta^2$. □

6.3 Connection to vector valued modular forms

Assume we are given a Jacobi form over a totally real number field K with a Fourier expansion as above. The periodicity of the Fourier coefficients described in the previous section leads to a connection between Jacobi forms and vector valued modular forms just as in the classical case. Note that the Fourier coefficients of a Jacobi form for the real number field K may be expressed as

$$c(\alpha,\beta) = c_\mu(N) = c\left(\frac{N+\beta^2}{4m},\beta\right) \text{ for } \beta\equiv\mu \bmod 2m, \alpha = \frac{N+\beta^2}{4m}.$$

Note that we are allowed to assume $\beta^2-4\alpha m = -N \ll 0$ or $N=0$ by the earlier theorem. Extend the definition to all totally positive numbers (or zero) N by setting $c_\mu(N)=0$ if $N\not\equiv-\mu^2 \bmod 2m$. Now define the following functions

$$h_\mu(\vec{\tau}) = \sum_{N\geq\geq 0} c_\mu(N)e^{2\pi i Tr_{k/\mathbb{Q}}(N\tau/4m)} \quad (6.6)$$

$$\vartheta_{m,\mu}(\vec{\tau},\vec{z}) = \sum_{\rho\in\delta_K^{-1}\ \rho\equiv\mu\bmod 2m} e^{2\pi i Tr_{K/\mathbb{Q}}(\rho^2\tau/4m+\rho z)}. \tag{6.7}$$

Then we have

$$\Phi(\vec{\tau},\vec{z}) = \sum_{\alpha\in\delta_K^{-1},\ \alpha\geq\geq 0}\ \sum_{(\beta\in\mathcal{O}_K\ |\ \beta^2\leq 4\alpha m)} c(\alpha,\beta)e^{2\pi i Tr_{K/\mathbb{Q}}(\alpha\vec{\tau}+\beta\vec{z})} =$$

$$\sum_{\mu\bmod 2m}\ \sum_{(\rho\equiv\mu\bmod 2m)}\ \sum_{N\geq\geq 0} c_\mu(N)e\left[Tr_{K/\mathbb{Q}}\left(\frac{N+\rho^2}{4m}\tau+\rho z\right)\right] =$$

$$\sum_{\mu\bmod 2\mathbf{m}} h_\mu(\vec{\tau})\vartheta_{m,\mu}(\vec{\tau},\vec{z}). \tag{6.8}$$

Now it is easy to check that

$$\vartheta_{m,\mu}(\vec{\tau}+1,\vec{z}) = e^{2\pi i Tr_{K/\mathbb{Q}}(\frac{-\mu^2}{\mathbf{4m}})}\vartheta_{m,\mu}(\vec{\tau},\vec{z})$$

or more generally

$$\vartheta_{m,\mu}(\vec{\tau}+\gamma,\vec{z}) = e^{2\pi i Tr_{K/\mathbb{Q}}(\frac{-\mu^2\gamma}{\mathbf{4m}})}\vartheta_{m,\mu}(\vec{\tau},\vec{z}),\quad \forall\ \gamma\in\mathcal{O}_K \tag{6.9}$$

and therefore since $\Phi(\vec{\tau}+\gamma,\vec{z}) = \Phi(\vec{\tau},\vec{z})$, we get

$$h_\mu(\vec{\tau}+\gamma) = e^{-2\pi i Tr_{K/\mathbb{Q}}(\frac{-\mu^2\gamma}{4m})}h_\mu(\vec{\tau}),\quad \forall\ \gamma\in\mathcal{O}_K. \tag{6.10}$$

Now it remains to find a formula for $h_\mu(\frac{\vec{-1}}{\tau})$, (note that $\frac{\vec{-1}}{\tau} = (\frac{-1}{\tau_1},...\frac{-1}{\tau_n}))$ because results of Vaserstein, Cooke, and Liehl prove that the matrices

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix},\ \gamma\in\mathcal{O}_K$$

generate $Sl_2(\mathcal{O}_K)$, see [3],[4],[16],[23]. So knowing $h_\mu(\vec{\tau}+\lambda)$ and $h_\mu\left(\frac{\vec{-1}}{\tau}\right)$ gives the transformation properties of $h_\mu(\vec{\tau})$ for all $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in Sl_2(\mathcal{O}_K)$. In order to find $h_\mu(\frac{\vec{-1}}{\tau})$ we will create the theta functions $\vartheta_{m,\mu}(\vec{\tau},\vec{z})$ as the coefficients of a symplectic theta function and then specify the transformation formula for $\vartheta_{m,\mu}(\frac{\vec{-1}}{\tau},\frac{\vec{z}}{\tau})$ where the variables have the obvious meaning. Define the following matrices

$$Z = \begin{pmatrix} d_1(\frac{\vec{\tau}}{2m}) & d_1(\frac{\vec{z}}{2m}) \\ d_1(\frac{\vec{z}}{2m}) & d_1(\frac{\vec{\tau}'}{2m}) \end{pmatrix}$$

$$M_1 = \begin{pmatrix} 2m^{(1)} & 0 & ... \\ 0 & \ddots & \\ \vdots & & 2m^{(n)} \end{pmatrix}, \quad \mathcal{M} = \begin{pmatrix} M_1 & 0 \\ 0 & M_1 \end{pmatrix}.$$

Note that $m \gg 0$ because of the earlier theorem and therefore the matrix M_1 is positive definite. Therefore by selecting the imaginary parts of the τ' variables such that $y_j y_j' > v_j^2$ for all $1 \leq j \leq n$ using the notation introduced earlier, the variable Z is in the symplectic upper half space. Recall the matrix

$$\widehat{W} = \begin{pmatrix} W & 0 \\ 0 & W \end{pmatrix}, \quad W = (\omega_i^j)_{1 \leq i,j \leq n}, \quad \mathcal{O}_K = [\omega_1, ...\omega_n]\mathbb{Z}$$

as defined earlier. Also let $\vec{V} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathbb{C}^{2n}$ where $v_1, v_2 \in \mathbb{C}^n$. Let $\mu \in \mathcal{O}_K$ be a representative of a congruence class modulo m and let $\frac{\mu}{2m} = \sum_{j=1}^n b_j \omega_j$, where the $b_j \in \mathbb{Q}$ and let $v_1 = (b_j)_{1 \leq j \leq n}, v_2 = (0)$. Then set

$$\Theta_{m,\mu}\left(Z, \begin{pmatrix} 0 \\ \vec{V} \end{pmatrix}\right) = \sum_{r \in \mathbb{Z}^{2n}} e^{\pi i \, {}^t(\vec{V}+r) \, {}^t\widehat{W} \mathcal{M} Z \mathcal{M} \widehat{W} (\vec{V}+r)}.$$

Note that the presence of the matrix $\mathcal{M}$ changes the sum over rational integer vectors into a sum over the elements of the ideal $(2m)$ instead of $\mathcal{O}_K$ as in the first construction. So the sum of elements of the form ${}^t(\vec{V} + r)$ is actually a sum over the elements of $\mathcal{O}_K^2$ of the form $\begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix}$ where $\rho_2 \in \mathcal{O}_K$ and ρ_1 is equivalent to μ mod $2m$.

Expanded out this looks like

$$\Theta_{m,\mu}(\vec{\tau}, \vec{z}, \vec{\tau}', \begin{pmatrix} 0 \\ \vec{V} \end{pmatrix}) =$$

$$\sum_{\substack{\rho_1 \in \mathcal{O}_K, \rho_2 \in (2m), \\ \rho_1 \equiv \mu \bmod 2m}} e[Tr_{K/\mathbb{Q}}(\rho_1^2 \frac{\tau}{4m} + 2(\rho_1\rho_2)\frac{z}{4m})] e[Tr_{K/\mathbb{Q}}(\rho_2^2 \frac{\tau'}{4m})] \quad (6.11)$$

which again yields the Fourier expansion with respect to the τ'. The $\rho_2 = 2m$ Fourier coefficient is then $\vartheta_{m,\mu}(\vec{\tau}, \vec{z})$. By using the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ in $Sl_2(\mathbb{Z})$ embedded as in chapter 5 and considering the expansion of

$$\Theta_{m,\mu}\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \circ Z, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \circ \begin{pmatrix} 0 \\ \vec{V} \end{pmatrix}\right)$$

and the transformation formula for the symplectic theta function 5.2 (note that the vector notation $\vec{}$ has been dropped to ease the notation, however all of the τ, z, τ' should be considered as vectors),

$$\Theta_{m,\mu}\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \circ Z, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \circ \begin{pmatrix} 0 \\ \vec{V} \end{pmatrix}\right) = \Theta_{m,\mu}\left(\tau^{-1}, \frac{z}{\tau}, \tau' - \frac{z^2}{\tau}, \begin{pmatrix} -\vec{V} \\ 0 \end{pmatrix}\right)$$

$$= \sum_{\rho_1,\rho_2 \in \mathcal{O}_K} exp[\pi i Tr_{K/\mathbb{Q}}(\rho_1^2 \frac{-\tau^{-1}}{2m} + 2(\rho_1\rho_2)\frac{z}{2m\tau} - \rho_1 \frac{\mu}{2m})] exp[\pi i Tr_{K/\mathbb{Q}}(\rho_2^2 \frac{\tau' - \frac{z}{\tau}}{2m})]$$

$$= \chi \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} (\prod_{j=1}^{n} \frac{\tau_j}{2m^{(j)}})^{\frac{1}{2}} \theta \left(Z, \begin{pmatrix} 0 \\ \vec{V} \end{pmatrix}\right).$$

For ease of notation, let $\chi(\mathcal{S}) = \chi \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. By again examining the $\rho_2 = 2m$ coefficient and equating the different $\rho_2 = 2m$ coefficients, we get that

$$\vartheta_{m,\mu}\left(\frac{-1}{\tau}, \frac{z}{\tau}\right) = \chi(\mathcal{S})(\prod_{j=1}^{n} \frac{\tau_j}{2m^{(j)}})^{\frac{1}{2}} e^{2\pi i Tr_{K/\mathbb{Q}}(mz^2/\tau)} \sum_{\nu \bmod 2m} e[Tr_{K/\mathbb{Q}}(\frac{\nu\mu}{2m})]\vartheta_{m,\nu}(\vec{\tau}, \vec{z}). \tag{6.12}$$

Therefore since

$$\Theta_{m,\mu}\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \circ (Z, \begin{pmatrix} 0 \\ \vec{V} \end{pmatrix})\right) = \chi(\mathcal{S})\mathcal{N}(\tau)^k e^{2\pi i Tr_{K/\mathbb{Q}}(mz^2/\tau)} \Theta_{m,\mu}\left(Z, \begin{pmatrix} 0 \\ \vec{V} \end{pmatrix}\right),$$

we have

$$h_\mu\left(\frac{-1}{\tau}\right) = \chi(\mathcal{S})^{-1}\mathcal{N}(\tau)^k (\prod_{j=1}^{n} \frac{\tau_j}{2m^{(j)}})^{-\frac{1}{2}} \sum_{\nu \bmod 2m} exp(2\pi i Tr_{K/\mathbb{Q}}(\frac{\nu\mu}{2m}))h_\nu(\tau) \tag{6.13}$$

and we have now essentially proved the following correspondence

Theorem. *For a totally real number field K the correspondence between analytic Jacobi forms of weight k and index m and vector valued modular forms satisfying the transformation formulas 6.10,6.13 over K and bounded as $\tau_j \to \infty$, $\forall\, 1 \le j \le n$, of weight $k - \frac{1}{2}$*

given by

$$\Phi(\tau, z) = \sum_{\mu \bmod 2m} h_\mu(\tau)\vartheta_{m,\mu}(\tau, z) \longleftrightarrow (h_\mu(\tau))_{\mu \bmod 2m}$$

is an isomorphism between the spaces.

There is a small ambiguity which arises due to the fact that there is a choice of which square roots are taken, so in fact we need a metaplectic covering of the group $Sl_2(\mathcal{O}_K)$ to account for this. The metaplectic covering will not be pursued here except to note that a metaplectic covering of the group $Sl_2(\mathcal{O}_K)$ is a group that in essence contains group elements combined with a multiplier system evaluated at the group element. This functions much in the same way as the multiplier systems discussed earlier.

Chapter 7

General construction with quadratic forms

The initial example given in chapter 5 created Jacobi forms of weight $\frac{1}{2}$, index λ^2, and index vector (λ) for all λ in $\mathcal{O}_K$. We now present a more general construction which will create forms of arbitrary half integral weight, more general indexes in $\mathcal{O}_K$ and much more general index vectors. The forms which will be created here are going to be generalizations of the Jacobi theta functions for a positive definite quadratic form over the rational integers of the type that appear in [11] and more recently in section 5.1. For example, let Q be a positive definite quadratic form with entries in $\mathbb{Z}$ so for a in $\mathbb{Z}^n$, $a \neq 0$, $Q(a) = {}^t a Q a > 0$, and fix b in $\mathbb{Z}^n$ then define

$$\theta_{Q,b}(\tau, z) = \sum_{a \in \mathbb{Z}^n} e^{2\pi i ({}^t a Q a \tau + 2\, {}^t a Q b z)}.$$

The quadratic forms used in this construction will be required to be positive definite in all of the real conjugates and this is a significant restriction. For this general construction we are using the "$2\pi i$" version of the symplectic theta function which will force all of the transformation formulas onto a Γ_0 congruence subgroup instead of the Γ_θ type congruence subgroups. The use of the "πi" version is almost identical with the only difference being the subgroup of the symplectic group under which the functions transform.

There is a more general construction for theta functions of indefinite quadratic forms (i.e. not positive definite) over number fields but for this we simply refer the reader to [17], and [8] which shows how to create theta functions of indefinite quadratic forms. The techniques used in this paper may also be generalized to the indefinite quadratic forms with majorants by combining the methods of this paper and [17].

7.1 Quadratic forms

There are some facts about quadratic forms which will be useful in order to create Jacobi forms. A quadratic form is a symmetric function

$$Q : \mathbb{C}^n \times \mathbb{C}^n \longrightarrow \mathbb{C}$$

which will be represented as a matrix so

$$Q(\vec{a}, \vec{b}) = {}^t\vec{a}Q\vec{b}, \quad Q \in M_n(\mathbb{C}), \ \vec{a}, \vec{b} \in \mathbb{C}^n.$$

A symmetric quadratic form is such that

$$Q(\vec{a}, \vec{b}) = Q(\vec{b}, \vec{a})$$

so it is necessary that the matrix Q be symmetric, ${}^tQ = Q$. When Q is a matrix of real numbers, we call Q positive definite if

$$Q(\vec{a}) \equiv Q(\vec{a}, \vec{a}) > 0, \quad \forall \ \vec{a} \in \mathbb{C}^n.$$

Note that it is meaningless to talk about a quadratic form over the complex numbers being positive definite, however in the case of complex entries, a quadratic form will be required to have non-zero determinant. Given such a matrix there will always be an upper triangular matrix $L \in M_n(\mathbb{C})$ such that $Q = {}^tLL, \quad L = (l_{i,j})_{1 \le i,j \le n}$ with $l_{ij} = 0$ for $j < i$.

For this construction, the quadratic form Q is in $M_n(\mathcal{O}_K), \quad Q = (q_{s,t})_{1 \le s,t \le n}$ so all of the entries are integers in the number field. Since Q has entries in K, it is reasonable to define the conjugates $Q^{(j)} = (q_{s,t}^{(j)})_{1 \le s,t \le n}$ and denote the decomposition of each conjugate of the form $L^{[j]}$ so that $Q^{(j)} = {}^tL^{[j]}L^{[j]}$, also denote $L^{[r_1+r_2+j]} = \overline{L}^{[r_1+j]}$ to continue the notation and ordering from earlier. There is a significant restriction to the quadratic forms for this construction which is that the real conjugates of Q which are $Q^{(j)}, 1 \leq j \leq r_1$ must be positive definite. It is possible to do the construction without this restriction but this would force the introduction of majorants and even more notation.

7.2 The general theta functions

Given a quadratic form over $\mathcal{O}_K$ as in the previous section, we will create a theta function of the form for b in $\mathcal{O}_K^l$

$$\theta_{Q,b}(\vec{\tau},\vec{z}) = \sum_{a\in\mathcal{O}_K^l} e[(\sum_{j=1}^{r_1+r_2} {}^t a^{(j)}\, {}^t L^{[j]} \tau_j L^{[j]} a^{(j)} + 2\, {}^t a^{(j)}\, {}^t L^{[j]} z_j L^{[j]} b^{(j)})]. \tag{7.1}$$

These functions will be produced as before by creating a symplectic theta function and choosing the appropriate Fourier coefficients. This construction and the verification of the construction will prove the following theorem

Theorem. *Let K be an algebraic number field of degree $n = r_1 + 2r_2$ as usual. Let Q be a $l \times l$ quadratic form with entries in $\mathcal{O}_K$, such that $Q^{(j)} > 0$ for $1 \leq j \leq r_1$ (i.e. all of the real conjugates are positive definite), and let $\mathfrak{g}$ be the greatest common divisor of the principal minors of Q, and let $Q^{(j)} = {}^t L^{[j]} L^{[j]}$ for $1 \leq j \leq n$ then the function*

$$\Theta_{\frac{l}{2},\vec{m},\mathbf{m}}(\vec{\tau},\vec{z}) = \sum_{\vec{m},\mathbf{m}} \theta_{Q,b}(\vec{\tau},\vec{z})$$

is a Jacobi form of weight $\frac{l}{2}$, index $Q(b)$, and index vector Lb for the subgroup $\Gamma_0(4\delta_K \mathfrak{g}^{-2} det(Q)^2) \ltimes \mathcal{O}_K^2 \subseteq \Gamma^J(O_K)$. The sum over $\vec{m}, \mathbf{m}$ means to sum over all vectors $b \in \mathcal{O}_K^l$ such that $Q(b) = \mathbf{m}$ and $\overline{{}^t b\, {}^t L^{[j]}} L^{[j]} b = \overline{{}^t m^{(j)}} m^{(j)}$ for all $r_1 < j \leq r_1 + r_2$.

In the case where there is only one b in $\mathcal{O}_K^l$ in the sum then this is the natural generalization of the Jacobi theta function. However, for a general index pair $\{\mathbf{m}, \vec{m}\}$ all that can be shown is that the sum is over a finite number of b. Note δ_K is the different of K as in the earlier notation.

We begin by introducing notation let

$$d_l(a) = \begin{pmatrix} a & 0 & \dots & \\ 0 & a & & \\ \vdots & & \ddots & \\ & & & a \end{pmatrix}$$

which is an $l \times l$ matrix. Then for $\vec{\tau}$ in $\mathfrak{h}^{r_1} \times \mathfrak{h}^{(\mathbb{Q})r_2}$ and other similar variables

$$d_{l,1}(\vec{\tau}) = \begin{pmatrix} d_l(\tau_1) & 0 & \dots & \\ 0 & d_l(\tau_2) & & \\ \vdots & & \ddots & \\ & & & d_l(\tau_{r_1}) \end{pmatrix}$$

$$d_{l,2}(\vec{x}) = \begin{pmatrix} d_l(x_{r_1+1}) & 0 & \dots & \\ 0 & d_l(x_{r_1+2}) & & \\ \vdots & & \ddots & \\ & & & d_l(x_{r_1+r_2}) \end{pmatrix}$$

$$d_{l,3}(\vec{x}) = \begin{pmatrix} d_l(\bar{x}_{r_1+1}) & 0 & \dots & \\ 0 & d_l(\bar{x}_{r_1+2}) & & \\ \vdots & & \ddots & \\ & & & d_l(\bar{x}_{r_1+r_2}) \end{pmatrix}$$

where the notation $\tau_j = x_j + y_j\kappa$ for τ_j in $\mathfrak{h}^{(\mathbb{Q})}$ is used. Similarly define these matrices for the other variables $y_j, z_j, u_j, v_j, \tau'_j, x'_j, y'_j$. Now define the $l \times nl$ matrix

$$W^{(j)} = \begin{pmatrix} \omega_1^{(j)} & \dots & \omega_n^{(j)} & 0 & \dots & & & & & \\ 0 & \dots & 0 & \omega_1^{(j)} & \dots & \omega_n^{(j)} & 0 & \dots & & \\ & & & & & & \ddots & & & \\ & & & & & & & \omega_1^{(j)} & \dots & \omega_n^{(j)} \end{pmatrix}$$

and define the matrix $\widehat{W}$ and $\widehat{L}$ to be

$$\widetilde{W} = \begin{pmatrix} W^{(1)} \\ W^{(2)} \\ \dots \\ W^{(n)} \end{pmatrix}, \widehat{W} = \begin{pmatrix} \widetilde{W} & 0 \\ 0 & \widetilde{W} \end{pmatrix}, \ \widetilde{L} = \begin{pmatrix} L^{[1]} & 0 & \dots & \\ 0 & L^{[2]} & & \\ \vdots & & \ddots & \\ & & & L^{[n]} \end{pmatrix}, \widehat{L} = \begin{pmatrix} \widetilde{L} & 0 \\ 0 & \widetilde{L} \end{pmatrix},$$

where the $L^{[j]}$ were defined in the previous section. The variable for the symplectic theta function is defined similarly as

$$\widetilde{Z} = \begin{pmatrix} d_{l,1}(\vec{\tau}) & 0 & 0 & d_{l,1}(\vec{z}) & 0 & 0 \\ 0 & d_{l,2}(\vec{x}) & id_{l,2}(\vec{y}) & 0 & d_{l,2}(\vec{u}) & id_{l,2}(\vec{v}) \\ 0 & id_{l,3}(\vec{y}) & d_{l,3}(\vec{x}) & 0 & id_{l,3}(\vec{v}) & d_{l,3}(\vec{u}) \\ d_{l,1}(\vec{z}) & 0 & 0 & d_{l,1}(\vec{\tau}') & 0 & 0 \\ 0 & d_{l,2}(\vec{u}) & id_{l,3}(\vec{v}) & 0 & d_{l,2}(\vec{x}') & id_{l,2}(\vec{y}') \\ 0 & id_{l,2}(\vec{v}) & d_{l,3}(\vec{u}) & 0 & id_{l,3}(\vec{y}') & d_{l,3}(\vec{x}') \end{pmatrix}.$$

Now create the symplectic theta function

$$\Theta_{Sp,Q}(\widetilde{Z}) = \sum_{\vec{a} \in \mathbb{Z}^{2nl}} e^{2\pi i ({}^t\vec{a}\, {}^t\widehat{W}\, {}^t\widehat{L} \widetilde{Z} \widehat{L} \widehat{W} \vec{a})}, \tag{7.2}$$

but for this to be a symplectic theta function it must be verified that ${}^t\widehat{W}\, {}^t\widehat{L} \widetilde{Z} \widehat{L} \widehat{W}$ is in $\mathfrak{h}^{(2nl)}$ or at least that one can pick the extra upper half plane variables in order to put this matrix into the symplectic upper half space. The matrix is obviously symmetric by construction therefore it remains to show it has positive definite imaginary part. By the same reasoning as earlier consider the matrix $\widetilde{S}$

$$\widetilde{S} = \begin{pmatrix} I_{lr_1} & 0 & \cdots & & & \\ 0 & 0 & I_{lr_2} & & & \\ \vdots & I_{lr_2} & 0 & & & \\ & & & I_{lr_1} & & \\ & & & & 0 & I_{lr_2} \\ & & & & I_{lr_2} & 0 \end{pmatrix}$$

which has the same effect as before, it switches the complex conjugate rows of the matrix $L\widehat{W}$ which is denoted as $\overline{LW} = \widetilde{S} L W$. Therefore in order to show this matrix

$${}^t\widehat{W}\, {}^t\widehat{L} \widetilde{Z} \widehat{L} \widehat{W} \tag{7.3}$$

has positive definite imaginary part, note $\widetilde{S}^2 = I_{2nl}$ and consider

$${}^t\widehat{W}\, {}^t\widehat{L} \widetilde{Z} \widehat{L} \widehat{W} = {}^t\widehat{W}\, {}^t\widehat{L} \widetilde{S} \widetilde{S} \widetilde{Z} \widehat{L} \widehat{W} = \overline{{}^tW\, {}^tL} \widetilde{S} \widetilde{Z} L W$$

and if $\widetilde{S}\widetilde{Z}$ has positive definite imaginary part then so does 7.3 by the earlier lemma in chapter 5. The imaginary part of $\widetilde{S}\widetilde{Z}$ is

$$\begin{pmatrix} d_{l,1}(y) & & & d_{l,1}(v) & & \\ & d_{l,3}(y_1) & & & d_{l,3}(v_1) & d_{l,3}(u_2) \\ & & d_{l,2}(y_1) & & d_{l,2}(u_2) & d_{l,2}(v_1) \\ d_{l,1}(v) & & & d_{l,1}(y') & & \\ & d_{l,2}(v_1) & d_{l,3}(u_2) & & d_{l,3}(y_1') & \\ & d_{l,2}(u_2) & d_{l,3}(v_1) & & & d_{l,2}(y_1') \end{pmatrix}. \tag{7.4}$$

This matrix is just l copies of the matrix in 5.4 and since we can pick the y' and the y_1' variables so that 5.4 is positive definite, the same choices of y' and y_1' will force the above matrix to be positive definite.

Expanded out this symplectic theta function 7.2 looks like $\Theta_{Sp,Q}(\widetilde{Z}) = \Theta_{Sp,Q}(\vec{\tau}, \vec{z}, \vec{\tau}')$ and

$$\begin{aligned} \Theta_{Sp,Q}(\vec{\tau}, \vec{z}, \vec{\tau}') = & \sum_{a,b \in \mathcal{O}_K^l} e[(\sum_{j=1}^{r_1+r_2} {}^t a^{(j)}\, {}^t L^{[j]} \tau_j L^{[j]} a^{(j)} + 2\, {}^t a^{(j)}\, {}^t L^{[j]} z_j L^{[j]} b^{(j)})] \\ & e[(\sum_{j=1}^{r_1} {}^t b^{(j)}\, {}^t L^{[j]} \tau_j' L^{[j]} b^{(j)})] e[(\sum_{j=r_1+1}^{r_1+r_2} {}^t b^{(j)}\, {}^t L^{[j]} \tau_j' L^{[j]} b^{(j)})] \\ = \sum_{b \in \mathcal{O}_K^l} & \theta_{Q,b}(\vec{\tau}, \vec{z}) e[(\sum_{j=1}^{r_1} {}^t b^{(j)}\, {}^t L^{[j]} \tau_j' L^{[j]} b^{(j)})] e[(\sum_{j=r_1+1}^{r_1+r_2} {}^t b^{(j)}\, {}^t L^{[j]} \tau_j' L^{[j]} b^{(j)})] \end{aligned} \tag{7.5}$$

where

$$\theta_{Q,b}(\vec{\tau}, \vec{z}) = \sum_{a \in \mathcal{O}_K^l} e[(\sum_{j=1}^{r_1+r_2} {}^t a^{(j)}\, {}^t L^{[j]} \tau_j L^{[j]} a^{(j)} + 2\, {}^t a^{(j)}\, {}^t L^{[j]} z_j L^{[j]} b^{(j)})].$$

This last representation shows how the Jacobi theta functions $\theta_{Q,b}$ arise as the Fourier coefficients with respect to the τ' variables. However, there may be a number of vectors b in $\mathcal{O}_K^l$ which are all part of the same Fourier coefficient because they all produce the same index $Q(b)$. In particular this last representation should be written as

$$\sum_{m \in \mathcal{O}_K} \sum_{(b \in \mathcal{O}_K^l \mid Q(b)=m)} \theta_{Q,b}(\vec{\tau}, \vec{z}) e[\sum_{j=1}^{r_1} m^{(j)} \tau_j'] e[\sum_{j=r_1+1}^{r_2} {}^t b^{(j)}\, {}^t L^{[j]} \tau_j' L^{[j]} b^{(j)}]. \tag{7.6}$$

Some of these Jacobi theta functions will be separated from each other so that they will be shown to transform as Jacobi forms independent of the other Jacobi theta functions in the coefficeint. Some of these $\theta_{Q,b}$ will still be grouped together in the Fourier coefficient but it will be shown that they as a group transform correctly according to the definition of Jacobi forms over K with a given index vector.

Before we can fully state the transformation formulas of these Fourier coefficients it is necessary to examine the structure of the coefficient. The formula above 7.6 is a Fourier expansion with respect to the τ_j', for $1 \leq j \leq r_1$, and x_j', for $r_1+1 \leq j \leq r_1+r_2$ however the y_j', with $r_1+1 \leq j \leq r_1+r_2$ are not part of the expansion (because the form is not analytic). They are grouped in simply as a matter of notational convenience. However, it is possible to assume that the non-complex parts of the quaternionic variables, the y_j' are part of the Fourier expansion at least in order to show the transformation formulas. It was already noted that all of the Jacobi theta functions with index vectors b such that $Q(b) = \mathbf{m}$ are summed to make the $\mathbf{m}$th coefficient. It is possible to separate these into classes based on the associated index pair $(Q(b), \overline{{}^tb\,{}^tL}Lb)$. Since there are only a finite number of vectors b which produce the same index there is a smallest $\overline{{}^tb\,{}^tL}Lb$ because these are all real numbers. The sum of these Jacobi theta functions with the same index pair can then be separated from the rest of the Fourier coefficient by letting the y_j' for $r_1+1 \leq j \leq r_1+r_2$ goto infinity. This will force the theta functions associated to the larger $\overline{{}^tb\,{}^tL}Lb$ to goto zero faster. Now that this portion of the Fourier coefficient is separated it may be shown to be a Jacobi form over K of weight $\frac{l}{2}$, index index $Q(b)$ and index vector Lb where b is the vector associated to any one of the theta functions in the class. Similarly once this class of Jacobi theta functions is shown to be a Jacobi form one can take the next smallest $\overline{{}^tb\,{}^tL}Lb$ and repeat the process thus separating out all of the different classes of theta functions.

7.3 Transformation formulas

Now it is necessary to show this sum over a class of Jacobi theta functions transforms correctly. As before, it is necessary to extend the group embeddings from the first example so that the embedded group will act on $\widetilde{Z}$. We generalize the earlier definition of $\check{d}(\alpha)$ to $\check{d}_l(\alpha)$, for α in $\mathcal{O}_K$

$$\check{d}_l(\alpha) = \begin{pmatrix} d_l(\alpha^{(1)}) & 0 & \dots & 0 \\ 0 & d_l(\alpha^{(2)}) & & \\ \vdots & & \ddots & \\ 0 & & & d_l(\alpha^{(n)}) \end{pmatrix} = \begin{pmatrix} d_{l,1}(\alpha) & & \\ & d_{l,2}(\alpha) & \\ & & d_{l,3}(\alpha) \end{pmatrix}.$$

Then the embeddings of the elements of $\Gamma^J(\mathcal{O}_K)$ are:

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in Sl_2(\mathcal{O}_K) \hookrightarrow \begin{pmatrix} \check{d}_l(\alpha) & 0 & \check{d}_l(\beta) & 0 \\ 0 & I_{nl} & 0 & 0 \\ \check{d}_l(\gamma) & 0 & \check{d}_l(\delta) & 0 \\ 0 & 0 & 0 & I_{nl} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

$$[\lambda, \mu] \in \mathcal{O}_K^2 \hookrightarrow \begin{pmatrix} I_{nl} & 0 & 0 & \check{d}_l(\mu) \\ \check{d}_l(\lambda) & I_{nl} & \check{d}_l(\mu) & \check{d}_l(\lambda\mu) \\ 0 & 0 & I_{nl} & \check{d}_l(-\lambda) \\ 0 & 0 & 0 & I_{nl} \end{pmatrix} = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix}.$$

Set $T = \widehat{L}\widehat{W}$. It should be noted that the actions of $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \circ (\vec{\tau}, \vec{z})$ is the same as the action of

$$\begin{pmatrix} {}^tTA\,{}^tT^{-1} & {}^tTBT \\ T^{-1}C\,{}^tT^{-1} & T^{-1}DT \end{pmatrix} \circ {}^tT\widetilde{Z}T$$

on the $\vec{\tau}, \vec{z}$ variables, and similarly for the embedded $[\lambda, \mu]$ matrix. In fact the actions are exactly the same as those in chapter 5 in 5.6-5.9, since these embedding are simply l copies of the embeddings in chapter 5. So, by introducing a quadratic form the actions have not changed but it is still necessary to check that this matrix

$$\begin{pmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{C} & \widetilde{D} \end{pmatrix} = \begin{pmatrix} {}^tTA\,{}^tT^{-1} & {}^tTBT \\ T^{-1}C\,{}^tT^{-1} & T^{-1}DT \end{pmatrix} \tag{7.7}$$

is in $Sp_{4nl}(\mathbb{Z})$ or particularly $\Gamma_0^{(4nl)}(4)$, whenever $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ is in a subgroup of $Sl_2(\mathcal{O}_K)$ and similarly for the embedded $[\lambda, \mu]$ matrix.

In order to check that this embedded group matrix is actually in $Sp_{4nl}(\mathbb{Z})$ consider the entries of the matrices $\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D}$ which with a little computation are all traces of elements in K. One fact which is helpful here is that the entries of Q^{-1} are given in terms of determinants of minors divided by the determinant of Q. Therefore let $\mathfrak{g}$ be the ideal generated by the principal minors of Q then all of the entries of Q^{-1} are divisible by $\mathfrak{g}\det(Q)^{-1}$. The entries in $\widetilde{A}, \widetilde{D}$ are all traces of elements from the ideal generated by δ_K^{-1} the traces of these elements are by definition all in $\mathbb{Z}$. The entries of $\widetilde{B}$ are all traces of elements in an ideal contained in the integers O_K so they are rational integers. The entries of the matrix $\widetilde{C}$ are all traces of elements from the ideal $\delta_K^{-2}\mathfrak{g}^2(\det(Q))^{-2}$ and therefore in order for all of these entries to be rational integers we require that γ be in $\delta_K \det(Q)^2\mathfrak{g}^{-2}$. Now in order for this matrix to be in $\Gamma_0^{(4nl)}(4)$ it is also necessary that all of the entries of $\widetilde{C}$ be divisible by 4, therefore if γ is in $4\delta_K\det(Q)^2\mathfrak{g}^{-2}$ then the embedded matrix is in $\Gamma_0^{(4nl)}(4)$.

So far, the construction has produced a symplectic theta function which has combinations of Jacobi theta functions of a quadratic form Q over the field K as its Fourier coefficients and these may be separated into classes based on the index pair. In the last section it was shown that the embedded $\Gamma^J(\mathcal{O}_K)$ acted correctly on the embedded version of the space $\mathcal{H}$, and it is easy to check that the actions on the individual τ_j, z_j, τ_j' are the same as in section 5.3. It is also easy to check that $\det(C\widetilde{Z} + D) = \mathcal{N}(c\tau + d)^l$ as previously defined. Therefore since $\Theta_{Sp,Q}$ is a symplectic modular form, by using 5.2 we see that for all $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_0^{(4nl)}(4)$,

$$\Theta_{Sp,Q}(M \circ \widetilde{Z}) = \chi(M)\det(C\widetilde{Z} + D)^{\frac{1}{2}}\Theta_{Sp,Q}(\widetilde{Z}).$$

Expanding this we have,

$$\sum_{a\in\mathbb{Z}^{2nl}} e[2\pi i\, {}^t a\, {}^t T(M \circ \widetilde{Z})Ta] = \chi(M)\mathcal{N}(c\tau + d)^{\frac{l}{2}} \sum_{a\in\mathbb{Z}^{2nl}} exp(2\pi i\, {}^t a\, {}^t T\widetilde{Z}Ta).$$

We expand out the right hand side of this equation as in the previous calculations, and look at the Fourier expansion 7.6 as in chapter 5. Since each

$$\tau_j' \longrightarrow \tau_j' - \frac{\gamma^{(j)} z_j^2}{\gamma^{(j)}\tau_j + \delta^{(j)}}, \quad \forall \;\; 1 \le j \le r_1 \tag{7.8}$$

$$\tau'_j \longrightarrow \tau'_j - (u_j + \overline{v}_j\kappa)(\gamma^{(j)}\tau_j + \delta^{(j)})^{-1}\gamma^{(j)}(u_j + v_j\kappa), \quad \forall \;\; r_1 < j \le r_1 + r_2, \tag{7.9}$$

We get the exponential part of the transformation formulas by matching the Fourier coefficients in 7.6, and by replacing τ' as in 7.8, 7.9 where the index for the Jacobi theta function is given by ${}^t b\, {}^t LLb = Q(b)$ and the index vector is now given by (Lb).

Similarly, the embedded $[\lambda, \mu] \in \mathcal{O}_K^2$ matrix takes

$$\tau_j \longrightarrow \tau_j, \quad z_j \longrightarrow z_j + \tau_j\lambda^{(j)} + \mu^{(j)}, \; \forall \;\; 1 \le j \le r_1 + r_2$$

$$\tau'_j \longrightarrow \tau'_j + \lambda^{(j)2}\tau_j + 2\lambda^{(j)}z_j + \lambda^{(j)}\mu^{(j)} \;\; \forall \;\; 1 \le j \le r_1$$

$$\tau'_j \longrightarrow \tau'_j + \lambda^{(j)}\tau_j\lambda^{(j)} + 2\lambda^{(j)}z_j + \lambda^{(j)}\mu^{(j)}, \;\; \forall \;\; r_1 + 1 \le j \le r_1 + r_2.$$

Also the action of

$$\begin{pmatrix} \widehat{A} & \widehat{B} \\ \widehat{C} & \widehat{D} \end{pmatrix} = \begin{pmatrix} {}^tTA'\,{}^tT^{-1} & {}^tTB'T \\ T^{-1}C'\,{}^tT^{-1} & T^{-1}D'T \end{pmatrix}$$

on ${}^tT\widetilde{Z}T$ is the same as the action of $[\lambda, \mu]$ on the variables $(\vec{\tau}, \vec{z})$. It is trivial to see that $\det(C'\widetilde{Z} + D') = 1$ and it is known that $\chi(M) = 1$ in this case, and therefore we get the transformation formulas by the same arguement as above, i.e. by matching the Fourier coefficients and pulling out the transformation formulas for these Fourier coefficients. So this calculation has verified that the mth Fourier coefficient is in fact Jacobi forms of weight $\frac{l}{2}$, index $Q(b) = m$, and index vector Lb.

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