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Affine rings of low GK dimension

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The dissertation of Jason Pierre Bell is approved, and it is acceptable in quality and form for publication on microfilm:

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To my parents and my brothers.

TABLE OF CONTENTS

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ABSTRACT OF THE DISSERTATION

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We consider algebras of low GK dimension. We give a new, completely combinatorial proof that a finitely generated domain of GK dimension 1 must be a finite module over its center (Theorem 2.4.2). We also show that the monic localization of a polynomial ring over a left Noetherian ring is a Jacobson ring (Theorem 2.3.28). We show that any subfield of the quotient ring of a finitely graded non-PI Goldie algebra of GK dimension 2 over a field F must have transcendence degree at most 1 over F (Theorem 3.3.19). In the fourth chapter we give counter-examples to several questions in ring theory. We construct a prime affine algebra of GK dimension 2 that is neither primitive nor PI; we construct a prime affine algebra of GK dimension 3 that has non-nil Jacobson radical; and we construct a primitve affine algebra of GK dimension 3 with center that is not a field;

Chapter 1

Preliminaries

1.1 Gelfand Kirillov dimension

We begin by providing background information to the reader. Many of the proofs included in this thesis are standard arguments which can be found in many algebra texts. Nevertheless, we include these arguments for the convenience of the reader. Unless otherwise stated, all rings are assumed to have a multiplicative identity.

Let F be a field and let R be a finitely generated noncommutative F-algebra; that is, there exist elements $r_1, \ldots, r_m \in R$ such that any element in R can be expressed as a (noncommutative) polynomial over F in $r_1, \ldots, r_m$. We call such an algebra an *affine algebra* over F, or, for short, an *affine F-algebra*. Let V be a finite dimensional F-vector subspace of R satisfying:

1. $1 \in V$;

2. V generates R as an F-algebra.

We call such a subspace *a generating subspace*. Given a basis $\{1, v_1, \ldots, v_\ell\}$ for V, we denote by V^n the subspace of R consisting of all polynomials in $v_1, \ldots, v_\ell$ of degree at most n. In 1966 Gelfand and Kirillov introduced an isomorphism-invariant on rings which now bears their names. We define the *Gelfand Kirillov* *dimension* of R to be:

$$\limsup_{n\to\infty} \log\left(\dim\, V^n\right)\Big/\log(n). \tag{1.1.1}$$

We use the abbreviation GK dimension for Gelfand Kirillov dimension and denote it by $\mathrm{GKdim}(R)$. It is important to note that although it appears that the GK dimension is dependent on the choice of generating subspaces V, it is in fact independent of this choice. We prove this now.

Proposition 1.1.1 *GK dimension is independent of the choice of generating subspace.*

Proof. Let R be an affine F-algebra and let V and W be two generating subspaces and let $\mathrm{GKdim}_V(R)$ and $\mathrm{GKdim}_W(R)$ denote the values which occur when we use V and W respectively to compute the GK dimension of R as defined in item (1.1.1). Notice

$$R \;=\; \bigcup_{n=0}^{\infty} V^n \;=\; \bigcup_{n=0}^{\infty} W^n$$

and hence there exists j such that $V \subseteq W^j$. It follows that $V^n \subseteq W^{jn}$ and hence

$$\begin{aligned}\log\left(\dim\, V^n\right)\Big/\log(n) \;&\le\; \log\left(\dim\, W^{nj}\right)\Big/\log(n)\\ &=\; \left(1+\log(j)/\log(n)\right)\log\left(\dim\, W^{nj}\right)\Big/\log(nj).\end{aligned}$$

Notice j is fixed and so

$$1+\log(j)/\log(n)\to 1 \qquad \text{as } n\to\infty.$$

Hence taking the limsup of both sides as n goes to infinity we see that

$$\mathrm{GKdim}_V(R) \;\le\; \mathrm{GKdim}_W(R).$$

By symmetry, the reverse inequality holds and the result follows. ∎

In general, when R is not affine, we define

$$\mathrm{GKdim}(R) \;=\; \sup\left\{\mathrm{GKdim}(S)\;\Big|\; S\subseteq R,\; S \text{ affine}\right\}. \tag{1.1.2}$$

Given an affine algebra R and a finitely generated right R-module M, we define

$$\mathrm{GKdim}(M_R) \;=\; \limsup_{n\to\infty}\; \log\left(\dim\, WV^n\right)/\log n, \tag{1.1.3}$$

where V is a generating subspace for R and W is a finite subspace of M which generates M as an R-module. As before, the value of $\mathrm{GKdim}(M_R)$ is independent of the choices of V and W. When either R is not affine or M is not finitely generated, we define

$$\begin{aligned} &\mathrm{GKdim}(M_R) \qquad (1.1.4)\\ =\;& \sup\left\{\mathrm{GKdim}(M'_S)\;\middle|\; S\subseteq R,\ S \text{ affine},\ M'\subseteq M,\ M' \text{ finitely generated}\right\}. \end{aligned}$$

Before giving examples, we make some remarks about GK dimension.

Remark 1.1.2 *Let R be an affine F-algebra and let V be a generating subspace. If there exist positive constants C_1 and C_2 such that*

$$C_1 n^\alpha \;\le \dim\, V^n \;\le C_2 n^\alpha$$

for all n sufficiently large, then R has GK dimension α.

Remark 1.1.3 *Let R and S be affine F-algebras with generating subspaces V and W respectively. If there exist $\alpha \ge 0$, an integer m, and a positive constants C such that*

$$\dim\, V^{mn} \;\ge\; Cn^\alpha \dim\, W^n$$

for all n sufficiently large, then

$$\mathrm{GKdim}(R) \;\ge\; \mathrm{GKdim}(S) + \alpha.$$

If

$$\dim\, V^{mn} \;\le\; Cn^\alpha \dim\, W^n$$

for all n sufficiently large, then

$$\mathrm{GKdim}(R) \;\le\; \mathrm{GKdim}(S) + \alpha.$$

Remark 1.1.4 *Let R be an F-algebra. The following are true:*

- $\mathrm{GKdim}(S) \leq \mathrm{GKdim}(R)$ *whenever S is either a subring or a homomorphic image of R.*
- $\mathrm{GKdim}(R_1 \times R_2) = \max\{\mathrm{GKdim}(R_1), \mathrm{GKdim}(R_2)\}$.
- $\mathrm{GKdim}(R_1 \otimes_F R_2) \leq \mathrm{GKdim}(R_1) + \mathrm{GKdim}(R_2)$.
- *If $0 \to M'' \to M \to M' \to 0$ is a short exact sequence of right R-modules, then* $\mathrm{GKdim}(M_R) = \max\big(\mathrm{GKdim}(M'_R), \mathrm{GKdim}(M''_R)\big)$.
- *If $M \cong \oplus_{i=1}^n M_i$ as right R-modules, then*

$$\mathrm{GKdim}(M) = \max_{1 \leq i \leq n} \mathrm{GKdim}\big((M_i)_R\big).$$

Proofs of these statements can be found in Chapter 3 of [19] and Proposition 5.1 of [19].

Example 1.1.5 *Let $d \geq 2$ and let $A = F\{x_1, \ldots, x_d\}$ be the free F-algebra on d variables; i.e., A is the algebra consisting of all "noncommutative" polynomials in $x_1, \ldots, x_d$. Then the GK dimension of A is infinite.*

Proof. Let V be the F-vector space spanned by $1, x_1, \ldots, x_d$. Notice that a basis for V^n is given by all words in $x_1, \ldots, x_d$ of length at most n. A word of length n has n letters and for each letter we have d possible choices. Hence the number of words in $x_1, \ldots, x_d$ of length n is d^n. It follows that

$$\dim V^n = 1 + d + d^2 + \cdots + d^n \geq d^n.$$

Hence

$$\log\Big(\dim V^n\Big)\Big/\log(n) \geq n\log(d)/\log n \to \infty \qquad \text{as } n \to \infty. \qquad \blacksquare$$

An algebra such as this, in which $\dim V^n \geq d^n$ for some $d > 1$, is said to grow *exponentially*. An algebra of exponential growth has infinite GK dimension, but the converse is not true. Here is an example.

Example 1.1.6 *Let $R = F\{x, y\}$ be the free algebra on two generators and let*

$$I \;=\; \left(y^2, yx^iyx^jy \;\middle|\; i > j\right).$$

Then R/I has infinite GK dimension but does not have exponential growth.

Proof. Notice that R/I is an affine monomial algebra with an F-basis given by

$$\left\{x^jyx^{i_1}yx^{i_2}y\cdots x^{i_\ell}yx^k \;\middle|\; i_1 \leq i_2 \leq \cdots \leq i_\ell\right\} \cup \left\{x^jyx^k\right\} \cup \left\{x^j\right\}.$$

Notice

$$V \;:=\; F + Fx + Fy$$

is a generating subspace and

$$V^{2n} \;\supseteq\; \text{Span}\left\{yx^{i_1}yx^{i_2}\cdots yx^{i_\ell} \;\middle|\; i_1 + \cdots + i_\ell = n,\; 1 \leq i_1 \leq i_2 \leq \cdots \leq i_\ell\right\}.$$

Hence

$$\dim V^{2n} \;\geq\; p(n), \tag{1.1.5}$$

the number of partitions of n. We also have

$$\begin{aligned} V^n \;\subseteq\; & \text{Span}\left\{x^jyx^{i_1}y\cdots x^{i_\ell}yx^k \;\middle|\; i_1 + \cdots + i_\ell \leq n,\; 1 \leq i_1 \leq \cdots \leq i_k,\; j,k \leq n\right\} \\ & + \text{Span}\left\{x^jyx^k \;\middle|\; j,k \leq n\right\} \;+\; \text{Span}\left\{x^j \;\middle|\; j \leq n\right\}. \end{aligned}$$

Thus

$$\dim V^n \;\leq\; (n+1)^2 \sum_{j=0}^{n} p(j) + (n+1)^2 + (n+1) \;\leq\; 3(n+1)^3 p(n). \tag{1.1.6}$$

From Rademacher's formula for partitions of a number n [26] we have

$$p(n) \;\sim\; \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{2n/3}\right).$$

From equations (1.1.5) and (1.1.6), R has infinite GK dimension and does not grow exponentially. ∎

An algebra that does not have exponential growth is said to have *subexponential growth*. An algebra with finite GK dimension is said to have *polynomially bounded growth*.

Example 1.1.7 *The Weyl algebra*

$$W(F) \;=\; F\{x,y\}/(xy-yx-1)$$

has GK dimension 2.

Proof. The relation $yx = xy - 1$ allows us to write any word in x and y as a linear combination of elements of the form $x^i y^j$. Let $V = F + Fx + Fy$. The vector space V is a generating subspace. Moreover,

$$V^n \subseteq \text{Span}\left\{x^i y^j \;\middle|\; i,j \le n\right\}$$

and hence dim $V^n \le (n+1)^2$. On the other hand,

$$V^{2n} \supseteq \text{Span}\left\{x^i y^j \;\middle|\; i,j \le n\right\}$$

and so V^{2n} has dimension at least $(n+1)^2$. We see that $W(F)$ has GK dimension 2 by Remark 1.1.2. ∎

The Weyl algebra can be though of in terms of differentiation operators. Notice that the operator $(d/dt)t - t(d/dt)$ is the identity operator. We can therefore think of y as being multiplication by the indeterminate t and x being differentiation with respect to t.

Example 1.1.8 *Let R be a finite dimensional F-algebra. Then* $\text{GKdim}(R) = 0$.

Proof. Take $V = R$. Note that V is a generating subspace for R and $V^n = V$ for all n. Hence dim $V^n =$ dim R for all n and so by item (1.1.1) we see that R has GK dimension zero. ∎

In fact, the converse is true for affine algebras.

Proposition 1.1.9 *Let F be a field and let A be an F algebra. If A has GK dimension 0, then every affine subalgebra of A is finite dimensional over F; moreover, if A is affine and infinite dimensional over F, then* $\mathrm{GKdim}(A) \geq 1$.

Proof. Let $B \subseteq A$ be an affine subalgebra with generators $b_1, \ldots, b_m$. Take

$$V = Fb_1 + \cdots Fb_m + F.$$

If V^{n+1} strictly contains V^n for all $n \geq 1$, then we must have $\dim V^n \geq n$ by induction for all $n \geq 1$. It follows that B has GK dimension at least 1 by Remark 1.1.2 and so A must have GK dimension at least 1. Hence if A has GK dimension less than 1, every affine subalgebra must be finite dimensional over F. It follows that every affine subalgebra has GK dimension 0, and so A, too, has GK dimension 0. ∎

Proposition 1.1.10 *Suppose R is an F-algebra of GK dimension α. Then $R[t]$ has GK dimension $\alpha + 1$.*

Proof. Let V be a generating subspace for R. Then $W = V + Ft$ is a generating subspace for $R[t]$. Notice that since t commutes, we have that

$$W^n \subseteq V^n + V^n t + \cdots + V^n t^n$$

and hence $\dim W^n \leq (n+1) \dim V^n$. On the other hand if $m = \lfloor n/2 \rfloor$, then

$$W^n \supseteq V^m + V^m t + \cdots V^m t^m$$

and so $\dim W^n \geq (n/2) \dim V^m$. It follows from Remark 1.1.3 that $R[t]$ has GK dimension equal to $\mathrm{GKdim}(R) + 1 = \alpha + 1$. ∎

Corollary 1.1.11 *Let $A = F[x_1, \ldots, x_d]$ be a commutative polynomial ring in d variables. Then the GK dimension of A is d.*

Proof. By Example 1.1.8, F has GK dimension zero as an F-algebra. Using Proposition 1.1.10 and arguing by induction we see that A has GK dimension d. ∎

Another important fact about GK dimension is given by looking at "large" subalgebras of an algebra. We make this more precise. Given F-algebras A and B with $B \subseteq A$, we say that A *is a finite module* over B (or a *finite* B-module) if there exist $a_1, \ldots, a_m \in A$ such that $A = Ba_1 + \cdots + Ba_m$. We have the following useful proposition.

Proposition 1.1.12 *Let A and B be affine F-algebras and suppose that B is a subalgebra of A and that A is a finite B-module. Then A and B have the same GK dimension.*

Proof. Fix a generating subspace V of B. Write $A = Ba_1 + \cdots Ba_m$. Let $W = V + Fa_1 + \cdots + Fa_m$. Notice that $BW = A \supseteq W^2$ and hence there exists a finite dimensional subspace U of B such that $UW \supseteq W^2$. By enlarging U if necessary, we may assume that U is a generating subspace for B. It follows that

$$U^n W \supseteq W^n.$$

Notice that

$$\dim W^n \ \leq \ \dim U^n W \leq \dim U^n \cdot \dim W.$$

Thus

$$\dim U^n \geq \frac{1}{\dim W} \dim W^n.$$

Since W is a generating subspace for A and U is a generating subspace for B we see that the GK dimension of B is at least as large as the GK dimension of A. On the other hand, B is a subalgebra of A and hence its GK dimension cannot exceed the GK dimension of A. The result follows. ∎

We note that in commutative algebra there is another notion of dimension called the *Krull dimension* after Wolfgang Krull.

Definition 1.1.13 *Given a commutative algebra A, we define the* Krull dimension of A *to be*

$$\sup\left\{n \;\middle|\; \text{there exists a chain } P_0 \subsetneq \cdots \subsetneq P_n,\ P_0, \ldots, P_n \text{ prime ideals}\right\}.$$

We denote the Krull dimension by $\mathrm{Kdim}(A)$.

A related concept is that of *transcendence degree*

Definition 1.1.14 *Given a commutative algebra A over a field F, we define the* transcendence degree *of A to be the cardinality of the largest F-algebraically independent subset of A.*

We now give some important theorems about Krull dimension.

Theorem 1.1.15 *(Noether's normalization theorem) Let A be a commutative affine F-algebra of Krull dimension n. Then there exists a subalgebra $B \cong F[x_1, \ldots, x_n]$ with A a finite module over B.*

Proof. See [12] page 283 Theorem 13.3. ∎

Corollary 1.1.16 *Suppose A is a commutative affine F-algebra. Then* $\mathrm{Kdim}(A) = \mathrm{GKdim}(A)$.

Proof. Let m denote the Krull dimension of A. By Theorem 1.1.15 we have that there exists an A-subalgebra B such that

1. A is a finite B-module, and
2. $B \cong F[t_1, \ldots, t_m]$.

By Proposition 1.1.12 we have that A and B have the same GK dimension and by Corollary 1.1.11 we have that the GK dimension of B is equal to $m = \mathrm{Kdim}(A)$. The result follows. ∎

This corollary shows that GK dimension is a natural noncommutative analogue of Krull dimension. For non-affine commutative algebras it is not necessarily the case that the values of the Krull dimension and the Gelfand Kirillov dimension coincide. We give an example to show this.

Example 1.1.17 *Let F be a field and let $A = F(t_1, t_2, t_3, \cdots)$ be a purely transcendental extension of F of infinite transcendence degree. Then the Krull dimension of A is 0 and the GK dimension of A is infinite.*

Proof. Notice the only prime ideal of A is (0) and hence A has Krull dimension 1. Let m be a positive integer. Notice that $F[t_1, t_2, t_3 \cdots, t_m] \subseteq A$ and hence A has GK dimension at least m. Since m is arbitrary, we see that $\mathrm{GKdim}(A) = \infty$.

There is another property of GK dimension that shows that it is not always well-behaved; namely, the fact that the GK dimension of an algebra need not be an integer.

Theorem 1.1.18 *(Warfield [32]) For any real number $\alpha \geq 2$ there exists an F-algebra A_α having GK dimension α.*

Proof. It suffices to prove the claim for $2 < \alpha < 3$ by Proposition 1.1.10, because if A has GK dimension α, then $A[t_1, \ldots, t_d]$ has GK dimension $\alpha + d$. Let $2 < \alpha < 3$ and let $R = F\{x, y\}$. For each j, let $\mathcal{S}_j(\alpha)$ denote the set of all ordered pairs (i, k) such that $0 \leq i, k \leq j^{\frac{\alpha-1}{2}}$. Notice that $\mathcal{S}_j(\alpha)$ has cardinality $(\lfloor j^{\frac{\alpha-1}{2}} \rfloor + 1)^2$. We therefore obtain the estimates

$$\sum_{j=1}^{n} \mathrm{Card}(\mathcal{S}_j(\alpha)) \;\geq\; \sum_{j=1}^{n} j^{\alpha-1} \;=\; n^\alpha/\alpha + \mathrm{O}(n^{\alpha-1}) \tag{1.1.7}$$

and

$$\begin{aligned} \sum_{j=1}^{n} \mathrm{Card}(S_j(\alpha)) &\leq \sum_{j=1}^{n} (j^{\frac{\alpha-1}{2}} + 1)^2 \\ &= \sum_{j=1}^{n} \left(j^{\alpha-1} + 2j^{\frac{\alpha-1}{2}} + 1 \right) \\ &= n^\alpha/\alpha + \mathrm{O}(n^{\alpha-1}) + \mathrm{O}(n^{\frac{\alpha+1}{2}}). \end{aligned} \tag{1.1.8}$$

Consider the ideal

$$I_\alpha \;:=\; (y)^3 + \sum_{(i,k)\notin \mathcal{S}_j(\alpha)} Rx^iyx^jyx^kR.$$

Notice I_α is an ideal generated by monomials and $A_\alpha := R/I_\alpha$ has a basis as an F-vector space given by

$$\left\{x^iyx^jyx^k \;\middle|\; (i,k)\in \mathcal{S}_j(\alpha)\right\} \cup \left\{x^iyx^j\right\} \cup \left\{x^i\right\}.$$

Let V be the F-vector space spanned by $\{1,x,y\}$. Notice that

$$V^{5n} \supseteq \text{Span}\left\{x^iyx^jyx^k \;\middle|\; j\le n, (i,k)\in \mathcal{S}_j(\alpha)\right\}.$$

Hence the dimension of V^{5n} is at least

$$\sum_{j=1}^{n} \text{Card}(\mathcal{S}_j(\alpha)) = n^\alpha + \mathrm{O}(n^{\alpha-1}).$$

Hence A_α has GK dimension at least α. On the other hand,

$$V^n \subseteq \sum_{j\le n,(i,k)\in\mathcal{S}_j(\alpha)} Fx^iyx^jyx^k + \sum_{i,j\le n} Fx^iyx^j + \sum_{i=0}^{n} Fx^i.$$

Thus the dimension of V^n is at most

$$\sum_{j=0}^{n} \text{Card}(\mathcal{S}_j(\alpha)) + (n+1)^2 + (n+1) \;=\; n^\alpha + \mathrm{O}(n^2).$$

Hence the GK dimension of A_α is at most α. The result follows. ∎

We have now seen that for any

$$\alpha \in \{0\}\cup\{1\}\cup[2,\infty] \tag{1.1.9}$$

there exists an algebra having GK dimension α.

Thus we have almost completely determined the set of possible values which can occur as the GK dimension of an algebra. In fact, the set given in item (1.1.9) is the complete set of values. To deduce this we must show that there cannot be an affine algebra with GK dimension strictly between 1 and 2. This fact was first proved by Bergman [7] in 1978. To prove this we will need to prove some facts about words occurring in semigroups. This is the topic of the next section.

1.2 Words and semigroups

We use the approach of [19] in giving a proof of Bergman's gap theorem. Throughout this section R will denote an affine F-algebra generated by $1 = r_1, \ldots, r_m$ and V will denote the generating subspace

$$Fr_1 + \cdots + Fr_m.$$

For the remainder of this section, we shall use the notation

$$f(n) \;=\; \dim V^n/V^{n-1} \qquad \text{and} \qquad F(n) \;=\; \dim V^n. \tag{1.2.10}$$

Notice that

$$F(n) \;=\; f(0) + \cdots + f(n). \tag{1.2.11}$$

Let $\mathbf{Y}$ denote the free semigroup on $y_1, \ldots, y_m$; that is

$$\mathbf{Y} \;=\; \langle y_1, \ldots, y_m \rangle. \tag{1.2.12}$$

We put a degree lexicographical order on the words in Y by declaring

$$y_1 \;<\; y_2 \;<\; \cdots < \; y_m.$$

We have a semigroup homomorphism Φ from $\mathbf{Y}$ into R given by naturally extending the map which sends y_i to r_i for $1 \leq i \leq m$. We recursively construct a subset of elements of $\mathbf{Y}$. Let

$$\mathcal{M}_1 = \{y_1, \ldots, y_m\}.$$

Now assume that $\mathcal{M}_n$ has been constructed and consists of words of length n. List all words of length $n+1$ in lexicographical order. Starting form the beginning, remove each word w for which $\Phi(w)$ is an F-linear combination of words $\Phi(w')$ with $w' < w$. We define $\mathcal{M}_{n+1}$ to be the collection of words remaining at the end of this procedure. Define

$$\mathcal{M} \;=\; \bigcup_{n=1}^{\infty} \mathcal{M}_n. \tag{1.2.13}$$

We have the following easy lemma.

Lemma 1.2.1 *The following are true:*

1. $\Phi(\mathcal{M}) = \{\Phi(w) \mid w \in \mathcal{M}\}$ *is an F-basis for R.*
2. *Any subword of a word in $\mathcal{M}$ is again in $\mathcal{M}$.*

Proof. See Lemma 2.2 page 15 of [19]. ∎

From this lemma it follows that $f(n) = |\mathcal{M}_n|$.

Definition 1.2.2 *We say that a word $w = x_1 \cdots x_\ell \in Y$ is n-*periodic *if $x_i = x_{i+n}$ for $1 \leq i \leq \ell - n$.*

Lemma 1.2.3 *Suppose that a word $w = x_1 \ldots x_n \in Y$ is periodic with minimal period m. Suppose also that there exist i and j with $0 \leq i < j \leq n - m + 1$ such that*

$$x_{i+1}x_{i+2}\cdots x_{i+m-1} \;=\; x_{j+1}x_{j+2}\cdots x_{j+m-1}.$$

Then $m \mid (i-j)$.

Proof. See Lemma 2.3 of [19]. ∎

The next lemma is the key result that Bergman needed to estimate the growth of $f(n)$.

Lemma 1.2.4 *Suppose that $f(d) \leq d$ for some natural number d, then for $n \geq 2d$ any $w \in \mathcal{M}_n$ has the form*

$$w = w_1 w_2 w_3,$$

where w_2 is m-periodic for some $m \leq d$ and $\mathrm{length}(w_2) \geq d + m$ *and both w_1 and w_3 have length at most $d - m$.*

Proof. We prove this claim by induction on n. Suppose first that $n = 2d$. Write $w = x_1 x_2 \cdots x_{2d}$. Notice that w has $d+1$ subwords of length d and hence two of them must be equal, say

$$x_i x_{i+1} \cdots x_{i+d-1} \;=\; x_j x_{j+1} \cdots x_{j+d-1},$$

with $1 \leq i < j \leq d+1$. Let

$$w_2 \;=\; x_i x_{i+1} \cdots x_{j+d-1}$$

and let $m = j - i$. Note that $m \leq d$. Notice also that for $k \leq d$, the k^{th} position of w_2 is $x_{i+k-1} = x_{j+k-1}$. But x_{j+k-1} is the $(m+k)^{\text{th}}$ position of w_2 and hence w_2 is m-periodic. Taking

$$w_1 \;=\; x_1 \cdots x_{i-1} \qquad \text{and} \qquad w_3 \;=\; x_{j+d} \cdots x_{2d}$$

we see that $w = w_1 w_2 w_3$ and the conditions on the lengths of w_1 and w_3 are satisfied. Hence the conclusion of the statement of the lemma is true when $n = 2d$. Now suppose that the claim is true for all $n \leq N$ and consider the case $n = N+1$. Let $w \in \mathcal{M}_{N+1}$. We write $w = x_1 w'$, with $w' \in \mathcal{M}_N$. By the inductive hypothesis, we see

$$w' = u_1 u_2 u_3,$$

where u_2 is m-periodic for some $m \leq d$ and has length at least $d+m$ and u_1 and u_3 both have length at most $d-m$. Without loss of generality m is the minimal period of u_2. If $\text{length}(u_1) < d-m$, then the claim follows by taking

$$w_1 \;=\; x_1 u_1, \qquad w_2 \;=\; u_2, \qquad w_2 \;=\; u_3.$$

Hence we may assume that u_1 has length $d-m$. Our goal is to show that the last letter of u_1 is the m^{th} letter of u_2. By using the same reasoning as employed when proving the base case, $x_1 x_2 \cdots x_{2d}$ has two identical subwords of length d, say

$$x_i \cdots x_{i+d-1} \;=\; x_j \cdots x_{j+d-1},$$

with $1 \leq i < j \leq d+1$. Since $m \leq d$, Lemma 1.2.3 gives that $m \mid (j-i)$. The word

$$v \;:=\; x_i x_{i+1} \cdots x_{j+d-1}$$

is periodic with period $j-i$. Moreover, since u_2 begins at position $d-m+2$ and hence the last $m+j-2$ terms of v overlap with u_2. Let v' denote the last $m+j-2$

letters of v. Then v' is m-periodic. Since $i < j$ and $m \mid (j - i)$ and $j \leq d + 1$, we have $i \leq d + 1 - m$. From the above remarks,

$$w_2 \;:=\; x_i \cdots x_{d-m+1} u_2$$

is m-periodic. Taking

$$w_1 \;=\; x_1 \cdots x_{i-1} \qquad \text{and} \qquad w_3 \;=\; u_3,$$

we see the decomposition

$$w \;=\; w_1 w_2 w_3$$

satisfies the conditions in the statement of the theorem. The conclusion follows by induction. ∎

Proposition 1.2.5 *If $f(d) \leq d$ for some d then $f(n) \leq (2d+1)F(d)^3$ for all $n > 2d$.*

Proof. Notice that if $f(d) \leq d$, then by Lemma 1.2.4 any word of length $n \geq 2d$ can be expressed in the form $w_1 w_2 w_3$, where w_2 is periodic of length $m \leq d$ and w_1 and w_2 have length at most $d - m$. Since, w_1 and w_3 have length less than d, we conclude that there are at most $F(d)^2$ possible choices of (w_1, w_3). Now let u denote the first m letters of w_2. Notice that w_2 is completely determined by its length and the word u. Since there are at most d possible values for m, we have that there are at most $F(d)$ possible values of u. Also,

$$n \;\geq\; \text{length}(w_2) \;=\; n - \text{length}(w_1) - \text{length}(w_3) \;\geq\; n - 2d,$$

so there are at most $2d + 1$ possible lengths of w_2. Hence there are at most $(2d + 1)F(d)$ possible choices of w_2. Combining these facts, there are at most $(2d+1)F(d)^3$ choices for w. This completes the proof. ∎

In fact, it is possible to prove that if $f(d) \leq d$ for some d, then $f(n) \leq d^3$ for all $n \geq d$, but we do not need such a strong estimate.

Theorem 1.2.6 *(Bergman [7])* $\mathrm{GKdim}(R) \notin (1,2)$*; moreover if R has GK dimension 1, then R has linear growth and if R has GK dimension 2, then* $\dim V^n \geq n^2/2$ *for any generating subspace V.*

Proof. If $f(d) \leq d$ for some d, then by Proposition 1.2.5

$$f(n) \ \leq \ (2d+1)F(d)^3$$

for all $n \geq 2d$. Let

$$C = \max\{f(0), \ldots, f(2d-1), (2d+1)F(d)^3\}.$$

Then $f(n) \leq C$ for all n. Hence

$$\dim V^n \ = \ f(0) + \cdots + f(n) \ \leq \ C(n+1).$$

Thus R has linear growth. If, on the other hand, $f(n) > n$ for all n, then

$$\begin{aligned} \dim V^n = f(0) + \cdots + f(n) \ &\geq \ 1 + 2 + \cdots + (n+1) \\ &= \ (n+1)(n+2)/2. \end{aligned}$$

Hence $\dim V^n \geq n^2/2$ and so we see R has GK dimension 2. ∎

Corollary 1.2.7 *Let R be an F-algebra. Then*

$$\mathrm{GKdim}(R) \ \in \ \{0,1\} \cup [2,\infty];$$

moreover, for every $\alpha \in \{0,1\} \cup [2,\infty]$ there exists an F-algebra with

$$\mathrm{GKdim}(R) = \alpha.$$

Proof. By Theorem 1.2.6 and Proposition 1.1.9 any affine algebra has GK dimension lying in the set $\{0,1\} \cup [2,\infty]$. Since

$$\mathrm{GKdim}(R) \ = \ \sup\{\mathrm{GKdim}(S) \mid S \subseteq R \text{ is affine}\}$$

and $\{0,1\} \cup [2,\infty]$ is a closed set, we conclude that

$$\mathrm{GKdim}(R) \;\in\; \{0,1\} \cup [2,\infty].$$

By Proposition 1.1.18, for any α with $2 < \alpha < \infty$ there exists an F-algebra R with GK $\dim(R) = \alpha$. By Proposition 1.1.10 the F-algebras F, $F[t]$, $F[t_1, t_2]$, and $F[t_1, t_2, \ldots]$ have GK dimensions zero, 1, 2, and infinity respectively. This completes the proof. ∎

Bergman's work also gives us the following result.

Theorem 1.2.8 *Let R be an F-algebra of GK dimension 1. Then there exists $r \in R$ that is not algebraic over F.*

Proof. Suppose that R is algebraic over F. Since R has GK dimension 1, we have that $f(d) \leq d$ for some d. Hence every element of V^n can be expressed as a linear combination of words in $r_1, \ldots, r_m$ of length at most $2d$ and words of the form $w_1 w_2 w_3$, where w_1 and w_3 have length at most d and w_2 is periodic with minimal period at most d. Since the set of all words of length at most d is a finite set, there exists an integer M such that every word of length at most d satisfies a polynomial over F of degree M. We claim that any element of R can be expressed as a linear combination of words of length at most $(M+3)d$. To see this, it suffices to show that if $w_1 w_2 w_3$ has length greater than $(M+3)d$, then it is reducible. Suppose there exists a word $w = w_1 w_2 w_3$ of length at least $(M+3)d$ that is irreducible. We choose such a w with minimal length greater than $(M+3)d$. Since w_1 and w_3 have length at most d, we see that w_2 has length at least $(M+1)d$. Since w_2 has period at most d, we see that $w_2 = u^M u'$, where u is an initial subword of w_2. By assumption

$$u^M \;=\; \sum_{j=0}^{M-1} c_j u^j$$

and so

$$w_1 w_2 w_3 \;=\; \sum_{j=0}^{M-1} w_1 u^j u' w_3.$$

Thus w is reducible, a contradiction. It follows that every element of R can be expressed as a linear combination of words of length at most $(M+3)d$, and so $R = V^{(M+3)d}$, which gives that R is finite dimensional. ∎

Theorem 1.2.9 *(Pappacena [24]) Suppose $f(d) \leq d/C$ for some positive integer d and positive constant C. Then for any $w \in \mathcal{M}_n$, $n \geq d$ can be expressed as $w_1 w_2 w_3$ for some w_1, w_2, w_3 with w_2 periodic with period at most* $\text{length}(w_2)/(C+1)$.

Proof. Let u_i denote the subword of w of length d beginning at position i for $1 \leq i \leq (d+C)/C$. Since $f(d) \leq d/C$, we conclude that $u_j = u_k$ for some j and k with $1 \leq j < k \leq (d+C)/C$. We have

$$u_j \;=\; x_j x_{j+1} \cdots x_k x_{k+1} \cdots x_{j+d-1}$$

and

$$u_k \;=\; x_k x_{k+1} \cdots x_{j+d-1} x_{j+M} \cdots x_{k+d-1}.$$

Let

$$w_2 \;=\; x_j x_{j+1} \cdots x_{k+M-1}.$$

Comparing the i^{th} position of u_j and u_k we conclude that

$$x_{j+i-1} = x_{k+i-1}$$

for $1 \leq i \leq d$. Hence w_2 is periodic with period $k-j$; moreover, since $1 \leq j < k \leq (d+C)/C$, we conclude that $1 \leq k-j \leq d/C$. Notice that $\text{length}(w_2) = (k+d-1) - j + 1 = d + (k-j) \geq (C+1)(k-j)$. This completes the proof. ∎

Theorem 1.2.10 *Let V be a subspace of $M_m(F)$ containing the identity matrix. Then $V^{n+1} = V^n$ for all natural numbers $n \geq N = \lfloor 2n^{3/2} \rfloor$.*

Proof. If $f(d) > d/m$ for all $d < 2m^{3/2}$, then we have that

$$\dim V^N \;=\; f(0) + \cdots + f(N) \;\geq\; (0 + 1 \cdots + N)/n \geq N^2/2m \geq m^2.$$

But $M_m(F)$ has dimension m^2 and thus we conclude that V^N is the full matrix ring and hence $V^{n+1} = V^n$ for all $n \geq N$. If, on the other hand, $f(d) \leq d/m$ for some $d < N$, then by Theorem 1.2.9 any element of V^N can be expressed as a linear combination of elements of the form $w_1 w_2^m w_3$ with length$(w_1 w_2^m w_3) \leq N$. By the Cayley-Hamilton theorem w_2^m can be expressed as an F-linear combination of smaller powers of w_2 and hence $w_1 w_2^n w_3$ is in fact in V^{N-1}. It follows that $V^N = V^{N-1}$, and so $V^{n+1} = V^n$ for all $n \geq N$. ∎

Chapter 2

Structure theory

2.1 Structure theory for Artinian rings

To understand affine algebras of GK dimension zero, it is necessary to introduce the concept of an Artinian ring.

Definition 2.1.1 *A ring R is said to be* left Artinian *(respectively* right Artinian*), if R satisfies the descending chain condition on left (resp. right) ideals; that is, for any chain of left (resp. right) ideals*

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$$

there is some n such that

$$I_n = I_{n+1} = I_{n+2} = \cdots .$$

A ring that is both left and right Artinian is said to be Artinian.

A related concept is that of a *Noetherian ring.*

Definition 2.1.2 *A ring R is said to be* left Noetherian *(respectively* right Noetherian*) if R satisfies the ascending chain condition on left (resp. right) ideals. Just as in the Artinian case, we declare a ring to be* Noetherian *if it is both left and right Noetherian.*

An equivalent definition for a left Artinian ring is that every non-empty collection of left ideals has a minimal element (when ordered under inclusion). Similarly, a left Noetherian ring can be defined as a ring in which every non-empty collection of left ideals has a maximal element. A left Noetherian ring has the property that every left ideal is generated by a finite number of elements as a left R-module. In the commutative case the relation between the Artinian property and the Noetherian property is the following.

Theorem 2.1.3 *A commutative ring R is Artinian if and only if it is Noetherian and has Krull dimension zero.*

Proof. See [12] page 75 Theorem 2.14. ∎

In the noncommutative case, the relationship is not so simple. A theorem of Hopkins (see [16]) shows that Artinian rings are Noetherian just as in the commutative case; however, the relationship one might expect, namely that the GK dimension of an Artinian ring should be zero when considered an algebra over its center, does not hold. (See for example 6.6.18 on page 205 of [22].) It is an open problem whether an affine Artinian ring is necessarily finite dimensional and whether an Artinian ring can have GK dimension lying strictly between zero and infinity when considered as an algebra over its center.

It is clear that any finite dimensional F-algebra is Artinian, since a descending chain of left (or right) ideals is a descending chain of F-vector spaces. Thus an affine algebra of GK dimension zero is Artinian.

We shall now develop the Artin-Wedderburn theory of Artinian rings. To do this we must first introduce the concept of a *primitive ring*. Given a ring R, we say that a left R-module M is *faithful* if

$$r \cdot M = 0 \qquad \text{implies} \qquad r = 0.$$

We say that M is *simple* if it is nonzero and has no proper nonzero submodules. If R has no nonzero proper ideals, then we say R is a *simple ring*.

Definition 2.1.4 *A ring R is said to be (left)* primitive *if it has a faithful, simple left R-module. An ideal P of R is primitive, if R/P is a primitive ring and a ring R is said to be* semiprimitive *if* (0) *is the intersection of primitive ideals of R.*

One can also define the idea of right primitivity. Bergman [8] has constructed examples of rings that are left primitive but not right primitive. We give some examples of primitive rings.

Example 2.1.5 *A simple ring is primitive.*

Proof. Notice if R is a simple ring and $\mathcal{M}$ is a maximal left ideal of R, then $M := R/\mathcal{M}$ is simple as a left R-module. If $r \cdot M = 0$, then $(RrR) \cdot M = 0$. Since R has no nonzero proper ideals and $1 \cdot M \neq 0$, we conclude that $r = 0$. Thus M is also faithful as a left R-module. Hence R is primitive. ∎

Example 2.1.6 *Let D be a division ring. Then $M_n(D)$ is simple and hence primitive.*

Notice that $n \times 1$ vectors with entries in D is a left $M_n(D)$ module that is faithful and simple.

Example 2.1.7 *A commutative ring R is primitive if and only if it is a field.*

Proof. If R is a field, then it is primitive by Example 2.1.5. If R is primitive, then it has a faithful simple module M. Notice that since M is simple,

$$M \cong R/\mathcal{M}$$

for some maximal ideal $\mathcal{M}$ of R. But $\mathcal{M} \cdot M = 0$ and since M is faithful, we conclude that $\mathcal{M} = (0)$. Thus R is a field. ∎

Example 2.1.8 *Let F be a field of characteristic zero. Then the Weyl algebra, $W(F)$, is simple and hence primitive.*

Proof. Let

$$T_x(r) \;=\; xr - rx \qquad \text{and} \qquad T_y(r) \;=\; yr - ry \qquad \text{for } r \in W(F). \tag{2.1.1}$$

By induction we have

$$y^j x \;=\; xy^j - jy^{j-1}$$

for all $j \geq 0$ and hence

$$T_x(x^i y^j) \;=\; x^i(jy^{j-1}). \tag{2.1.2}$$

Similarly

$$T_y(x^i y^j) \;=\; -(ix^{i-1})y^j. \tag{2.1.3}$$

Let I be a nonzero ideal in $W(F)$. Let $a \in I$ be nonzero. We can write

$$a \;=\; \sum_{i,j} \beta_{i,j} x^i y^j$$

for some constants $\beta_{i,j} \in F$ with only finitely many of the $\beta_{i,j}$ nonzero. Let

$$N \;=\; \max\big\{i \mid \beta_{i,j} \neq 0 \text{ for some } j\big\}.$$

Notice that

$$T_y^N(a) \;=\; (-1)^N \sum_j \beta_{N,j} N! y^j \;\in\; I.$$

Let $\tilde{N}$ denote the largest value of j such that $\beta_{N,j} \neq 0$. Then $T_x^{\tilde{N}}(T_y^N(a)) = (-1)^N \beta_{N,\tilde{N}} N!\tilde{N}! \neq 0$ is an element of I and hence $I = W(F)$. ∎

Recall that $W(F)$ can be thought of as the ring of operators $F[t, d/dt]$. Let $V = F[t]$ and turn V into an $F[t, d/dt]$-module by endowing it with the natural action; i.e.,

$$t \cdot p(t) \;=\; tp(t), \tag{2.1.4}$$

$$\big(d/dt\big) \cdot p(t) \;=\; p'(t). \tag{2.1.5}$$

We claim that V is a faithful, simple module for $W(F)$ when F has characteristic zero. To see that it is simple, notice that for any polynomial $p(t) = p_m t^m + \cdots p_0 \in$

V of degree m, we have

$$\frac{1}{m!p_m} t^j (d^m/dt^m) \cdot p(t) \ = \ t^j$$

and hence any element of V generates V as a $W(F)$-module. To see that V is faithful, note that if a nonzero element $r \in W(F)$ annihilates V, then the two-sided ideal generated by r must also annihilate V. But $W(F)$ is simple and so 1 must annihilate V implying that V is the zero module. Thus V is indeed a faithful, simple $W(F)$-module.

The Jacobson density theorem is one of the most useful results for studying primitive rings. We give a proof of this result, but first we need a result due to Schur. Given a ring R and a left R-module M, we denote by

$$\mathrm{End}_R(M) \tag{2.1.6}$$

the ring of all R-module homomorphisms from M to M with multiplication given by composition of maps. Notice that M is a left $\mathrm{End}_R(M)$-module, with action given by $f \cdot x = f(x)$.

Lemma 2.1.9 *(Schur's lemma) If M is a simple R-module, then* $\mathrm{End}_R(M)$ *is a division ring.*

Proof. Let $f : M \to M$ be a nonzero homomorphism. Notice that the kernel of f is a submodule of M and is hence either (0) or M by the simplicity of M. Since f is nonzero, we have that the kernel is trivial and so f is injective. Notice that the image of f is a nonzero submodule of M and hence f is surjective. Thus f is a bijection. Take g to be the inverse of f. Notice that if $x, y \in M$ and $r \in R$, then

$$f(g(x+y) - g(x) - g(y)) = f(g(x+y)) - f(g(x)) - f(g(y)) = x + y - x - y = 0.$$

Since f is injective, we conclude that $g(x+y) = g(x) + g(y)$. Similarly, $g(rx) = rg(x)$, and so we see that g is an R-module homomorphism. ∎

Theorem 2.1.10 *(Jacobson density theorem) Let R be a primitive ring with a faithful simple module M. Let $D = \mathrm{End}_R(M)$. Then R is dense in $\mathrm{End}_D(M)$; that is, given a D-linearly independent subset of M, $\{x_1, \ldots, x_n\}$, and another subset of M, $\{y_1, \ldots, y_n\}$, of the same size there exists an element $r \in R$ such that $rx_i = y_i$ for $1 \leq i \leq n$.*

Proof. We prove this theorem by induction on n. Notice that when $n = 1$, the result is true since M is faithful. Suppose the claim is true for $n < m$ and consider the case $n = m$. Notice that $R(x_1, \ldots, x_{m-1}) \cong M \oplus M \oplus \cdots \oplus M$, where there are $m - 1$ copies of M appearing on the right hand side. If there exists an $r \in R$ such that $rx_i = 0$ for $i < m$ and $rx_m \neq 0$, then we are done, since rx_m generates M as an R-module. Thus we may assume that $rx_m = 0$ whenever $rx_i = 0$ for $i = 1, \ldots, m - 1$. It follows that we have a well-defined surjective map

$$\Phi : M^{m-1} \;\cong\; R(x_1, \ldots, x_{m-1}) \to M,$$

given by $(rx_1, \ldots, rx_{m-1}) \mapsto rx_m$. Let $f_j : M \to M^{(m-1)}$ be defined by

$$f_j(x) \;=\; (0, \ldots, 0, x, 0, \ldots, 0)$$

for $1 \leq j < m$. We have that $\Phi \circ f_j : M \to M$ is an element $\delta_j \in D$ for $1 \leq j < m$. Notice that for $j < m$, $\Phi \circ f_j(x_j) = r_j x_m$, where $r_j \in R$ satisfies

$$r_j x_i \;=\; \begin{cases} x_j & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

Thus $(r_1 + \cdots + r_{m-1} - 1)x_i = 0$ for $1 \leq i \leq m$. It follows that $(r_1 + \cdots + r_{m-1} - 1)x_m = 0$ and so

$$(r_1 + \cdots + r_{m-1})x_m = x_m.$$

Thus

$$\begin{aligned} \delta_1 x_1 + \cdots + \delta_{m-1} x_{m-1} &= \Phi \circ f_1(x_1) + \cdots + \Phi \circ f_{m-1} x_{m-1} \\ &= (r_1 + \cdots + r_{m-1})x_m \\ &= x_m. \end{aligned}$$

This contradicts the fact that $\{x_1, \ldots, x_m\}$ is a D-linearly independent subset of M. The result follows. ∎

An immediate corollary of this is the following.

Theorem 2.1.11 *Let R be a primitive, left Artinian ring. Then*

$$R \;\cong\; M_n(D)$$

for some division ring D.

Proof. Let M be a faithful, simple R-module, and let $D = \mathrm{End}_R(M)$. Notice that if M is infinite dimensional over D, then we can find a countably infinite D-linearly independent subset $\{x_1, x_2, \ldots\} \subseteq M$. Let

$$I_n \;=\; \left\{r \in R \;\middle|\; rx_i = 0 \text{ for } 1 \leq i \leq n\right\}.$$

Notice that

$$I_1 \;\supseteq\; I_2 \;\supseteq\; \cdots$$

is a descending chain of left ideals. By the density theorem there exists $r \in R$ such that $rx_1 = \cdots = rx_{n-1} = 0$ and $rx_n \neq 0$ and so $I_{n-1} \neq I_n$ for all $n \geq 2$. It follows that if M is infinite dimensional over D, then R is not Artinian. Hence we have that M is finite dimensional over D, say this dimension is n. Then $M \cong D \oplus D \oplus \cdots \oplus D$, where there are n copies of D. Hence R is a dense subring of $\mathrm{End}_D(M) \cong M_n(D)$. The only dense subring of $M_n(D)$ is the ring itself and so the result follows. ∎

In commutative algebra the concept of a prime ideal plays an important role. We now give the definition of a prime ideal in the noncommutative case. Given a ring R, we say that an ideal P is a *prime* ideal if whenever $aRb \subseteq P$ we have that either a or b is an element of P. Equivalently, P is prime if whenever I and J are ideals such that $IJ \subseteq P$, we necessarily have either I or J is contained in P. Notice this definition coincides with the definition of a prime ideal in a commutative ring. We say that a ring is *prime* if (0) is a prime ideal. Finally, we say that a ring is *semiprime* if (0) is the intersection of prime ideals in the ring. Equivalently, a semiprime ring is a ring with no nonzero nilpotent ideals.

Proposition 2.1.12 *A primitive ring is prime.*

Proof. Let R be a primitive ring and let M be a faithful simple R-module. Suppose $aRb = 0$. Then $aRbM = 0$. Suppose $b \neq 0$. Then there exists $m \in M$ such that $bm \neq 0$ since M is faithful. Since M is simple, we have $RbM = M$. Thus $aM = 0$ and so $a = 0$. Hence either a or b is zero and so R is prime. ∎

A prime ring need not be primitive. For example, take $R = F[t]$. Since R is a domain, it is prime. By Example 2.1.7, R is not primitive. Nevertheless, in the Artinian case a prime ring is indeed primitive. We prove this result now.

Proposition 2.1.13 *Let R be a prime left Artinian ring. Then R is primitive.*

Proof. Let L be a minimal nonzero left ideal in R. Notice L is a simple left R-module by the minimality of L. We claim also that L is a faithful R-module. To see this, suppose that there exists some $a \in R$ such that $aL = 0$ and choose $b \neq 0 \in L$. Then $aRb = 0$ and since R is prime we conclude that $a = 0$. Hence L is faithful. Thus R is primitive. ∎

Combining this result with Theorem 2.1.11 we see that a prime Artinian ring is isomorphic to matrices over a division ring. Hence every prime ideal in an Artinian ring is maximal, as the quotient is a simple ring. We now consider semiprime Artinian rings.

Proposition 2.1.14 *A left Artinian ring has only finitely many prime ideals.*

Proof. Suppose that $\{P_i \mid i \geq 1\}$ is an infinite set of distinct prime ideals. Notice that the descending chain

$$P_1 \supseteq P_1 \cap P_2 \supseteq \cdots$$

must eventually terminate and hence there exists an n such that

$$P_n \supseteq P_1 \cap P_2 \cap \cdots \cap P_{n-1}.$$

Letting $I_j = P_j + P_n$ for $1 \leq j \leq n$, we see that

$$P_n = I_1 \cap \cdots \cap I_{n-1}$$

and since P_n is a prime ideal, we conclude that $I_j = P_n$ for some j. Hence $P_j \subseteq P_n$. Since P_j is maximal, we have that $P_j = P_n$, which contradicts the assumption that $\{P_i \mid i \geq 1\}$ is a distinct set of primes. The result follows. ∎

We have just seen that an Artinian ring R has only finitely many primes, say $P_1, \ldots, P_n$. If R is also semiprime, then

$$(0) = \bigcap_{i=1}^{n} P_i.$$

Notice that by Theorem 2.1.11 and Proposition 2.1.13, a prime ideal in an Artinian ring is maximal. Thus P_i and P_j are comaximal for $i \neq j$; that is, $P_i + P_j = R$ for $i \neq j$. We can now employ the Chinese remainder theorem, which we quickly state.

Theorem 2.1.15 *(Chinese remainder theorem) Let $I_1, \ldots, I_n$ be pairwise comaximal ideals in a ring R. Then*

$$R \Big/ \left(\bigcap_{i=1}^{n} I_i \right) \cong \prod_{i=1}^{n} R/I_i.$$

Proof. See Proposition 2.2.1 on page 162 of [28] ∎

By this theorem we have

$$R \cong R \Big/ \left(\bigcap_{i=1}^{n} P_i \right) \cong \prod_{i=1}^{n} R/P_i.$$

Thus we have the following theorem.

Theorem 2.1.16 *(Artin-Wedderburn) A semiprime left Artinian ring is a finite product of matrix rings over division algebras.*

We have now completely determined the structure of semiprime Artinian rings. We continue our study of Artinian rings by introducing the concept of the Jacobson radical.

Definition 2.1.17 *Given a ring R we define the* Jacobson radical*, $J(R)$ to be*

$$J(R) \; = \; \bigcap \mathcal{M},$$

where the intersection is taken over all maximal right ideals $\mathcal{M}$ of R.

We now give some equivalent expressions for the Jacobson radical of a ring.

Theorem 2.1.18 *The following are equal to $J(R)$:*

1. $\{a \in R \mid 1 + ra \text{ is invertible for all } r \in R\}$;
2. $\{a \in R \mid 1 + ar \text{ is invertible for all } r \in R\}$;
3. $\bigcap\{P \mid P \text{ right primitive}\}$;
4. $\bigcap\{P \mid P \text{ left primitive}\}$;
5. $\bigcap\{\mathcal{M} \mid \mathcal{M} \text{ right maximal}\}$;
6. $\bigcap\{\mathcal{M} \mid \mathcal{M} \text{ left maximal}\}$.

Proof. See Chapter 1 of [15]. ∎

The Jacobson radical provides a measure of how "nice" a ring is. Notice that any nilpotent ideal is contained in the Jacobson radical. To see this, let N be a nilpotent ideal of a ring R and let $a \in N$ and $r \in R$. We have $(ra)^n = 0$ for some n and hence

$$(1 + ra + \cdots + (ra)^{n-1})(1 - ra) = (1 - ra)(1 + ra + \cdots + (ra)^{n-1}) = 1 - (ra)^n = 1$$

and so by the first expression for the Jacobson radical given in Theorem 2.1.18 we have $a \in J(R)$. Generally speaking, rings with nilpotent Jacobson radical tend to be better behaved than those without a nilpotent Jacobson radical. A related idea is that of being *Jacobson*. We say that a ring R is *Jacobson* if every prime homomorphic image has nil Jacobson radical. We now give some examples of rings and their Jacobson radicals.

Example 2.1.19 *Let F be a field and let R be a commutative, affine F-algebra; then $J(R)$ is nilpotent and hence R is Jacobson.*

Proof. See Theorem 4.19 on page 132 of [12]. ∎

Example 2.1.20 *A primitive ring has zero Jacobson radical.*

Proof. A primitive ring has the property that (0) is a primitive ideal. By the third condition of Theorem 2.1.18 we have that $J(R) = (0)$. ∎

Notice that we therefore have by Examples 2.1.8 and 2.1.6 that $W(F)$ and $M_n(D)$ have zero Jacobson radical for any field F of characteristic zero and any division ring D.

Example 2.1.21 *Let $R((x))$ denote the ring of* Laurent power series over *R; i.e.,*

$$R((x)) \;=\; \Big\{ \sum_{j=-M}^{\infty} b_j x^j \;\Big|\; b_j \in R \text{ for } j \geq -M, M \in \mathbb{Z} \Big\}.$$

If R has no nonzero nil ideals, then $J\big(R((x))\big) = (0)$.

Proof. Given a nonzero element

$$\beta = \sum_{j \geq -M} b_j x^j \in R((x))$$

with $b_{-M} \neq 0$, we call b_{-M} the *initial coefficient* of β. We define the initial coefficient of the zero Laurent series to be zero. Given an ideal $I \subseteq R((x))$ we denote by I_0 the ideal in R consisting of the initial coefficients of elements of I. Notice that if $I \subseteq J$ are ideals in $R((x))$, then $I = J$ if and only if $I_0 = J_0$. Note that if $\mathcal{M}$ is a maximal right ideal in R, then $\mathcal{M}R((x))$ is a maximal right ideal in $R((x))$. Hence we have that

$$J\big(R((x))\big) \subseteq J(R)R((x)).$$

Suppose that $\alpha \in J\big(R((x))\big)$ is nonzero. By multiplying α by an appropriate power of x if necessary, we may assume that

$$\alpha = \sum_{k\geq 0} a_k x^k$$

with $a_k \in J(R)$ for $k \geq 0$ and $a_0 \neq 0$. Notice that

$$a_0 + x - \alpha = x((1-a_1) + a_2 x + \cdots)$$

is a unit since x and $(1-a_1)$ are units. Hence

$$(a_0 + x)R((x)) + \alpha R((x)) = R((x)).$$

Since α is in the Jacobson radical of $R((x))$, we conclude that $(a_0 + x)R((x)) = R((x))$. Thus there exists

$$\beta = \sum_{k\geq -M} b_k x^k$$

such that $(a_0 + x)\beta = 1$. Since a_0 is not a unit, we see that $M > 0$. Computing coefficients we find that $a_0 b_{-M} = 0$ and

$$a_0 b_i + b_{-i+1} = \begin{cases} 0 & \text{if } i > -M \text{ and } i \neq 0, \\ 1 & \text{if } i = 0. \end{cases}$$

From these equations we find that $a_0^{\ell+1} b_{-M+\ell} = 0$ for $0 \leq \ell \leq M-1$ and $a_0^M b_0 = a_0^{M-1}$. Hence $a_0^{M-1}(a_0 b_0 - 1) = 0$. But $a_0 \in J(R)$ and thus $a_0 b_0 - 1$ is invertible and a_0 is nilpotent. We have now shown that for any

$$\alpha = \sum_{k\geq -M} a_k x^k \in J\big(R((x))\big)$$

with $a_{-M} \neq 0$ we have that a_{-M} is nilpotent. Hence $J\big(R((x))\big)_0$ is a nil ideal in R. Since R has no nonzero nil ideals we see that that $J\big(R((x))\big) = (0)$. ∎

Example 2.1.22 *Let $R = F[t]_{(t)}$. Then $J(R) = (t)$.*

Proof. The ideal (t) is the unique maximal ideal. ∎

Notice that $F[t]_{(t)}$ has non-nil Jacobson radical. In Chapter 4 we shall see an example of an affine ring with non-nil Jacobson radical. A clever counting argument due to Amitsur shows that over an uncountable field F an affine algebra has nil Jacobson radical.

Theorem 2.1.23 *(Amitsur [1]) Let R be a countably infinite or finite dimensional algebra over an uncountable field F. If $1 - \lambda r$ is invertible for uncountably many values of $\lambda \in F$, then r is algebraic over F.*

Proof. Let $\mathcal{S} \subseteq F$ be the set of all nonzero $\lambda \in F$ such that $1 - \lambda r$ is invertible in R. If $\mathcal{S}$ is uncountable, then

$$\{(1 - \lambda r)^{-1} \mid \lambda \in \mathcal{S}\}$$

is a linearly dependent set. Hence there exist $\lambda_1, \ldots, \lambda_k \in F - \{0\}$ and nonzero constants $\beta_1, \ldots, \beta_k \in F$ such that

$$\sum_{i=1}^{k} \beta_i (1 - \lambda_i r)^{-1} \;=\; 0.$$

Let $p[x] \in F[x]$ be the polynomial

$$\left(\prod_{i=1}^{k} (1 - \lambda_i x)\right)\left(\sum_{i=1}^{k} \beta_i (1 - \lambda_i x)^{-1}\right).$$

Notice $p(\lambda_1^{-1}) = c_1 \prod_{j=2}^{k} (1 - \lambda_j/\lambda_1)$ is nonzero and hence p is a nonzero polynomial. By construction $p(r) = 0$ and so r is algebraic. ∎

Corollary 2.1.24 *Let R be an affine algebra over an uncountable field F. Then $J(R)$ is nil.*

Proof. Let $a \in J(R)$. By Theorem 2.1.18 we have $1 - \lambda a$ is invertible for all $\lambda \in F - \{0\}$. Hence a satisfies some polynomial $p(x)$. Write $p(x) = x^m q(x)$ with $q(0) \neq 0$. Notice $q(a) = q(0) + a'$ where $a' \in J(R)$. Hence $q(a)$ is invertible. It follows that a^m is zero. Since a is arbitrary, we see $J(R)$ is nil. ∎

In the case that R is Artinian, we show that $J(R)$ is in fact nilpotent.

Theorem 2.1.25 *Let R be a left Artinian ring. Then $J(R)$ is nilpotent.*

Proof. We first show $J(R)$ is nil. Suppose there exists an element $a \in J(R)$ that is not nil. Then $0 \notin \{1, a, a^2, \cdots\}$ and so by Zorn's lemma we can choose an ideal P maximal with respect to the property that

$$P \cap \{1, a, a^2, \cdots\} \;=\; \varnothing.$$

We claim P is primitive. If $P = IJ$ for some ideals I and J properly containing P, then by maximality of P, we conclude that $a^m \in I$ and $a^n \in J$ and hence

$$a^{m+n} \subseteq IJ \subseteq P,$$

a contradiction. Thus P is prime. Proposition 2.1.13 gives that P is primitive and hence contains $J(R)$ by Theorem 2.1.18. This contradicts the fact that $a \notin P$. Thus $J(R)$ is nil. Notice

$$J(R) \supseteq J(R)^2 \supseteq J(R)^3 \supseteq \cdots$$

is a descending chain and hence must terminate. Thus there exists some n such that $J(R)^n = J(R)^{2n}$. Let $L = J(R)^n$ and choose $a \in L$ such that La is minimal. Since $L^2 = L$, we have $L^2 a = La$. Hence there is a $b \in L$ such that $Lba \neq (0)$. Notice $Lba \subseteq La$. By minimality of La we conclude that $Lba = La$. Thus $xba = ba$ for some $x \in L$. It follows that $x^m ba = ba$ for all positive integers m. But $x \in L \subseteq J(R)$ and hence x is nilpotent. This is a contradiction, since ba is nonzero. It follows that $J(R)$ is nilpotent. ∎

2.2 Rings satisfying a polynomial identity

An important concept in studying rings of low GK dimension is the notion of a *polynomial identity* for a ring.

Definition 2.2.1 *We say that an F-algebra A* satisfies a polynomial identity *(PI) if there exists a nonzero polynomial $p(t_1, \ldots, t_m)$ with coefficients in F such that $p(a_1, \ldots, a_m) = 0$ for all choices $a_1, \ldots, a_m \in A$. We say that such a ring is PI.*

Example 2.2.2 *Any commutative F-algebra satisfies the identity $t_1t_2 - t_2t_1$.*

Example 2.2.3 *(Wagner's identity) The ring of 2×2 matrices over a commutative F-algebra satisfies the identity $t_3(t_1t_2 - t_2t_1)^2 - (t_1t_2 - t_2t_1)^2 t_3$.*

Proof. Let $\mathbf{A}$ and $\mathbf{B}$ be 2×2 matrices over some commutative ring. Let $\mathbf{C} = \mathbf{AB} - \mathbf{BA}$. By the Cayley-Hamilton theorem

$$\mathbf{C}^2 \;=\; \mathrm{tr}(\mathbf{C})C - \det(\mathbf{C})\mathbf{I_2},$$

where $\mathbf{I_2}$ is the identity matrix. Since $\mathbf{C}$ has trace zero, we see that $\mathbf{C}^2$ is a scalar matrix and hence commutes with any other matrix. The result follows. ∎

Another example of PI algebras comes from *skew polynomial rings*. Given a ring R with an automorphism $\sigma : R \to R$, we construct the ring

$$R[z; \sigma]. \tag{2.2.7}$$

This ring consist of all elements of the form

$$\sum_{i=0}^{n} r_i z^i$$

with $r_1, \ldots, r_n \in R$ and multiplication given by

$$(rz^i)(r'z^j) = r\sigma^i(r')z^{i+j}.$$

Similarly, we define the skew Laurent polynomial ring $R[z, z^{-1}; \sigma]$.

Example 2.2.4 *Let A be a commutative F-algebra with an automorphism σ of finite order which fixes F. Let z be an indeterminate. Then $A[z;\sigma]$ and $A[z,z^{-1};\sigma]$ are PI.*

Proof. Use Proposition 1.6.25 on page 96 of [28]. ∎

A useful fact is that a ring satisfying a polynomial identity of degree n satisfies a *multilinear* polynomial identity of degree n.

Proposition 2.2.5 *If A satisfies a PI of degree n, then A satisfies a multilinear PI of degree n.*

Proof. Given a nonzero polynomial $f(x_1,\ldots,x_d)$ we define the *weight* of f to be the ordered pair (m,k), where

$$m \;=\; \max\{\deg_{x_i}(f) \mid 1 \leq i \leq d\}$$

and

$$k \;=\; \mathrm{Card}\big\{i \mid x_i \text{ has degree } m \text{ in } f\big\}.$$

A lexicographic order is defined on the weights of nonzero polynomials.

Suppose that the conclusion of the proposition is not true. Let R be a ring satisfying a polynomial identity of degree n that does not satisfy a multilinear identity of degree n. Fix such an n and among all polynomial identities of degree n satisfied by R. Choose $f(x_1,\ldots,x_d)$ of minimal weight. If f has weight $(1,d)$, then $d=n$ and f is multilinear. Hence we may assume that f has weight larger than $(1,d)$. Thus there exists $i \leq d$ such that x_i has degree $m \geq 2$ in f and f has weight (m,k) for some $k<d$. Without loss of generality $i=1$. Notice that

$$f(x_1+x_{d+1},x_2,\ldots,x_d) - f(x_1,\ldots,x_d) - f(x_{d+1},x_2,\ldots,x_d)$$

is nonzero and has degree smaller than m in the variables x_1 and x_{d+1}. Hence the polynomial $g(x_1,\ldots,x_{d+1})$ has smaller weight than f, a contradiction. ∎

This proposition is very useful, because it shows that the property of satisfying a polynomial identity is preserved under tensor products.

Corollary 2.2.6 *Let F be a field and R an F-algebra satisfying a multilinear identity f. Then $R \otimes_F A$ satisfies f for every commutative F-algebra A.*

Proof. This follows easily from the basic properties of tensor products. ∎

The Weyl algebra $W(F)$ for a field F of characteristic zero is an example of an algebra that does not satisfy a polynomial identity. This fact will be proved later using a theorem of Kaplansky. A direct proof is given now.

Proposition 2.2.7 *Let F be a field of characteristic zero. Then the Weyl algebra $W(F)$ does not satisfy a polynomial identity.*

Proof. Recall that $W(F)$ can be thought of as the ring of operators $F[t, d/dt]$ and recall that $F[t]$ is a left $W(F)$-module with action given by equations (2.1.4) and (2.1.5). Suppose $W(F)$ satisfies a homogeneous multilinear polynomial

$$p(t_1, \ldots, t_n) \;=\; t_n \cdots t_1 + \sum_{\substack{\sigma \in \mathcal{S}_n \\ \sigma \neq 1}} \beta_\sigma t_{\sigma(n)} \cdots t_{\sigma(1)}$$

of degree n. Take $t_i = t^i\left(d^{i-1}/dt^{i-1}\right)$ for $1 \leq i \leq m$. Notice that

$$t_j \cdots t_1 \cdot 1 \;=\; 1!\,2! \cdots (j-1)!\, t^j \;\neq\; 0$$

for $1 \leq j \leq n$. If σ is a non-trivial permutation in $\mathcal{S}_n$, then we can find the smallest index j such that $\sigma(j) \neq j$. Notice that $\sigma(j) > j$ since $\sigma(i) = i$ for $i < j$. It follows that

$$\begin{aligned} t_{\sigma(j)} \cdots t_{\sigma(1)} \cdot 1 \;&=\; t_{\sigma(j)}\left(t_{j-1} \cdots t_1 \cdots 1\right) \\ &=\; t^{\sigma(j)}\left(dt^{\sigma(j)-1}/dt^{\sigma(j)-1}\right) \cdot (t^{j-1}) \\ &=\; 0 \qquad \text{since } \sigma(j) > j. \end{aligned}$$

Hence

$$t_{\sigma(n)} \cdots t_{\sigma(1)} \cdot 1 \;=\; \begin{cases} (n-1)! \cdots 1!t^n & \text{if } \sigma \text{ is trivial,} \\ 0 & \text{otherwise.} \end{cases}$$

It follows that $p(t_1, \ldots, t_n) \cdot 1 \neq 0$ and hence $W(F)$ cannot be PI. ∎

Definition 2.2.8 *We define the* n^{th} standard identity

$$S_n(t_1, \ldots, t_n) \;=\; \sum_{\sigma \in \mathcal{S}_n} \operatorname{sgn}(\sigma) t_{\sigma(1)} \cdots t_{\sigma(n)}.$$

We show that an affine algebra of GK dimension zero satisfies one of the standard identities.

Theorem 2.2.9 *Let A be a finite dimensional F-algebra. Then A satisfies S_n, where $n = \dim\ A - 1$.*

Proof. Let $a_1, \ldots, a_n \in A$. There is a non-trivial dependence relation among $a_1, \ldots, a_n$. If $S_n(a_1, \ldots, a_n) = 0$, then $S_n(a_{\sigma(1)}, \ldots, a_{\sigma(n)}) = 0$ for all $\sigma \in \mathcal{S}_n$, and hence we may assume that $a_n = \beta_1 a_1 + \cdots \beta_{n-1} a_{n-1}$. Since S_n is multilinear, we have

$$S_n(a_1, \ldots, a_n) \;=\; \sum_{j=1}^{n-1} \beta_j S_n(a_1, \ldots, a_{n-1}, a_j).$$

Notice that $\mathcal{S}_n$ is the disjoint union of the alternating group $\mathcal{A}_n$ and $\mathcal{A}_n$ times the transposition (j, n). Hence we have

$$\begin{aligned} & S_n(t_1, \ldots, t_{n-1}, t_n) \\ = & \sum_{\sigma \in \mathcal{A}_n} t_{\sigma(1)} \cdots t_{\sigma(n)} - t_{\sigma(1)} \cdots t_{\sigma(j-1)} t_{\sigma(n)} t_{\sigma(j+1)} \cdots t_{\sigma(n-1)} t_{\sigma(j)}. \end{aligned}$$

By setting $t_i = a_i$ for $1 \leq i < n$ and $t_n = a_j$, $S_n(a_1, \ldots, a_{n-1}, a_j) = 0$. The result follows. ∎

Thus affine rings of GK dimension zero satisfy a polynomial identity. In particular, $M_n(F)$ satisfies the identity S_{n^2+1}. In fact, a theorem of Amitsur and Levitzki says that this result can be greatly improved.

Theorem 2.2.10 *Let F be a field. Then $M_n(F)$ satisfies the identity S_m for all $m \geq 2n$.*

Proof. See Theorem 14 on page 20 of [10]. ∎

We show that this is the best possible result.

Proposition 2.2.11 *$M_n(F)$ does not satisfy a nonzero polynomial identity of degree less than $2n$.*

Proof. Suppose $M_n(F)$ satisfies a PI of degree less than $2n$. Then we may assume that $M_n(F)$ satisfies a PI of degree $2n-1$. Moreover, we may assume that this PI is homogeneous and multilinear by Proposition 2.2.5. Hence we suppose that $M_n(F)$ satisfies

$$p(t_1, \ldots, t_{2n-1}) \;=\; t_1 \cdots t_{2n-1} + \sum_{\sigma \in \mathcal{S}_{2n-1}} \beta_\sigma t_{\sigma(1)} \cdots t_{\sigma(2n-1)}.$$

Let $e_{i,j}$ denote the matrix with a 1 in the i,j entry and zeros elsewhere. Take $t_i = e_{i,i}$ for odd i between 1 and $2n-1$ and take $t_i = e_{i-1,i}$ for even i between 1 and $2n-1$. Notice that $t_1 \cdots t_{2n-1} = e_{1,2n-1} \neq 0$. If σ is a non-trivial permutation, then $\sigma(i) > \sigma(i+1)$ for some i and so $t_{\sigma(i)} t_{\sigma(i+1)} = e_{\sigma(i),m_i} e_{\sigma(i+1),m_{i+1}}$, where $m_j = \sigma(j)$ if j is odd and $m_j = \sigma(j)+1$ if j is even. $m_i \geq \sigma(i) > \sigma(i+1)$ and so $e_{\sigma(i),m_i} e_{\sigma(i+1),m_{i+1}} = 0$. It follows that $t_{\sigma(1)} \cdots t_{\sigma(2n-1)} = 0$ for all non-trivial permutations σ. Hence

$$p(t_1, \ldots, t_{2n-1}) \;=\; e_{1,2n-1} \neq 0,$$

contradicting the fact that p is a PI for $M_n(F)$. ∎

One of the fundamental theorems in PI theory is the characterization of primitive PI rings given by Kaplansky.

Lemma 2.2.12 *Let R be a primitive ring with faithful simple module M satisfying a polynomial identity of degree n. Then*

$$R \;\cong\; M_m(D)$$

for some $m < \lfloor n/2 \rfloor$, where $D = \mathrm{End}_R(M)$.

Proof. By the Jacobson density theorem, R is a dense subring of the endomorphism ring of some D-vector space V. Suppose V has dimension greater than $\lfloor n/2 \rfloor$

over D. Then we can find elements $x_1, \ldots, x_k$ of V that are linearly independent over D with $k > \lfloor n/2 \rfloor$. Let

$$S \;=\; \left\{ r \in R \;\middle|\; rx_i \in \mathrm{Span}_D\{x_1, \ldots, x_k\} \text{ for } 1 \leq i \leq k \right\}$$

and

$$I \;=\; \left\{ r \in R \;\middle|\; rx_i = 0 \text{ for } 1 \leq i \leq k \right\}.$$

Here, I is an ideal of S and by the density theorem $S/I \cong M_k(D)$. Let Z denote the center of D. Notice $M_k(Z)$ is a subring of a factor ring of a subring of R and hence must satisfy the same polynomial identity as R. But $M_k(Z)$ does not satisfy a polynomial of degree less than $2k \geq n+1$ by Theorem 2.2.10, a contradiction. It follows that V has dimension at most $\lfloor n/2 \rfloor$. Hence R is a dense subring of $\mathrm{End}_D(V) \cong M_m(D)$ for some $m \leq \lfloor n/2 \rfloor$. It follows that

$$R \;\cong\; M_m(D). \blacksquare$$

Lemma 2.2.13 *Let D be a division ring with center Z and maximal subfield K. Then $D \otimes_Z K$ is primitive with simple faithful module D.*

Proof. Notice D is a $D \otimes_Z K$-module via the action

$$(\delta \otimes \beta) \cdot x \;=\; \delta x \beta.$$

Clearly D is simple. To see that it is also faithful, suppose

$$r \;=\; \delta_1 \otimes \beta_1 + \cdots + \delta_n \otimes \beta_n$$

annihilates D. Choose r with n minimal. Notice

$$(\delta_1^{-1} \otimes \beta_1^{-1})r$$

is also an annihilator and hence we may assume without loss of generality that $\delta_1 = \beta_1 = 1$. Notice

$$(\delta \otimes 1)r - r(\delta \otimes 1) \;=\; \sum_{i=2}^{n} (\delta\delta_i - \delta_i\delta) \otimes \beta_i$$

is also an annihilator for all $\delta \in D$. By the minimality of n we conclude that $\delta_2, \ldots, \delta_n \in Z$. But then

$$\begin{aligned} r &= \sum_{i=1}^{n} \delta_i \otimes_Z \beta_i \\ &= \sum_{i=1}^{n} 1 \otimes (\delta_i \beta_i) \qquad \text{since } \delta_i \in Z \\ &= 1 \otimes \Big(\sum_{i=1}^{n} \delta_i \beta_i \Big). \end{aligned}$$

Hence $r = 1 \otimes \beta$ for some $\beta \in K$. But

$$(1 \otimes \beta) \cdot 1 = \beta$$

and hence if $1 \otimes \beta$ is an annihilator of D, then $\beta = 0$. Thus D is a simple module. ∎

Theorem 2.2.14 *Let D be a division algebra with center Z and maximal subfield K. Suppose D satisfies a polynomial identity of degree n. Then*

$$D \otimes_Z K \;\cong\; M_m(K)$$

for some $m \leq \lfloor n/2 \rfloor$. In particular, $[D : Z] \leq \lfloor n/2 \rfloor^2$.

Proof. Let K be a maximal subfield of D. Notice that if an element $x \in D$ commutes with every element of K, then by maximality it is an element of K, for otherwise we could adjoin x to K and get a larger field. Next observe that

$$\mathrm{End}_{D \otimes_Z K}(D) \;\cong\; K,$$

because if $f : D \to D$ is a $D \otimes_Z K$-module, then

$$f(\delta) = f((\delta \otimes 1) \cdot 1) = \delta f(1)$$

and so f is determined by its value at 1. Taking $\beta \in K$, we see

$$\beta f(1) = f(\beta) = f((1 \otimes \beta) \cdot 1) = (1 \otimes \beta) \cdot f(1) = f(1)\beta$$

and so $\beta f(1) = f(1)\beta$ for all $\beta \in K$. So $f(1) \in K$ by our earlier remark. Thus we have a $D \otimes_Z K$-module homomorphism from $\mathrm{End}_{D\otimes_Z K}(D)$ into K; moreover, this map is non-trivial, since the identity endomorphism gets mapped to $1 \in K$. Since these are simple modules we conclude that this map is an isomorphism. Now $D \otimes_Z K$ is a primitive ring satisfying a polynomial identity. Hence

$$D \otimes_Z K \;\cong\; M_m(K)$$

where $m \leq \lfloor n/2 \rfloor$. But basic properties of tensor products give that

$$m^2 \;=\; [D \otimes_Z K : K] \;=\; [D : Z].$$

The result follows. ∎

Corollary 2.2.15 *(Kaplansky [18]) Let R be a primitive ring satisfying a PI of degree n. Then $R \cong M_m(D)$, where $m \leq \lfloor n/2 \rfloor$ and D has dimension at most $\lfloor n/2 \rfloor^2$ over its center.*

Proof. This follows immediately from Lemma 2.2.12 and Corollary 2.2.15. ∎

This theorem shows that the notions of being primitive and satisfying a polynomial identity are, in some sense, not compatible. This theorem can be applied to give an easy prove of Proposition 2.2.7. Notice that if

$$z \;=\; \sum_{i,j} \beta_{i,j} x^i y^j$$

is central, then

$$T_x(z) = \sum_{i,j} \beta_{i,j} j x^i y^{j-1} = 0,$$

where T_x is as in equation (2.1.1). Hence $\beta_{i,j} = 0$ if $j \neq 0$. Similarly, $\beta_{i,j} = 0$ if $i \neq 0$. Thus $z = \beta_{0,0} \in F$ and so $W(F)$ is infinite dimensional over its center. By Kaplansky's theorem a primitive, PI ring is finite dimensional over its center and so we conclude that $W(F)$ is not PI.

In fact, a theorem of Rowen (see Theorem 6.1.28 on page 99 of [29]) shows that any ideal of a semiprime PI ring intersects the center of the ring. Primitive rings on the other hand do not generally have large centers. Theorem 2.1.23 of Amitsur shows—at least over an uncountable field—that affine primitive rings have center equal to a field.

Proposition 2.2.16 *Let R be a primitive affine ring over an uncountable field F. Then the center of R is a field.*

Proof. Let M be a faithful simple R-module. Let $v \in M$ be nonzero. Observe that $\mathrm{End}_R(M)$ embeds in M via the map

$$\Phi \mapsto \Phi(v).$$

This map is injective since M is simple. Since M is a homomorphic image of R, we conclude that M is at most countably infinite dimensional over F. Hence $\mathrm{End}_R(M)$ is at most countably infinite dimensional over F. Notice also that the center of R embeds in $\mathrm{End}_R(M)$ by sending each central element $z \in R$ to the homomorphism Φ_z defined by

$$\Phi_z(v) = z \cdot v.$$

Since $\mathrm{End}_R(M)$ is a division ring Φ_z has an inverse for each central element $z \in R$. Let $z \in R$ be central and not an element of F. We have that

$$1 - \lambda\Phi_z$$

is invertible for all $\lambda \in F$ and hence Φ_z is algebraic by Theorem 2.1.23. Since $\mathrm{End}_R(M)$ is a division ring, Φ_z satisfies a polynomial with nonzero constant term, say

$$0 \;=\; 1 + \beta_1\Phi_z + \cdots \beta_m(\Phi_z)^m.$$

Hence

$$0 \;=\; 1 + \beta_1 z + \cdots + \beta_m z^m.$$

Notice that z is therefore invertible with inverse given by $-(\beta_1 + \cdots + \beta_m z^{m-1})$. It follows that the center of R is a field. ∎

Having described primitive PI rings, the following theorem is useful.

Theorem 2.2.17 *Let R be a prime affine PI ring. Then $J(R) = (0)$.*

Proof. See Theorem 13.10.3 on page 484 of [22]. ∎

Hence any prime PI ring has the property that (0) is the intersection of primitive ideals and we have an embedding

$$R \hookrightarrow \prod_{P \text{ primitive}} R/P.$$

Thus the study of prime PI rings can often be reduced to the study of primitive PI rings, which are matrix rings over division algebras by Kaplansky's theorem.

2.3 Goldie's Theorem

We now look at the idea of creating the quotient of a ring. In the commutative case, it is easy to construct the "field of fractions" of a given domain. The noncommutative case it is not so easy and, indeed, is sometimes not even possible. The key notion in constructing the quotient of a ring was given by Ore.

Definition 2.3.1 *Given a ring R and a multiplicatively closed subset $\mathcal{S}$ of R containing 1, we say that $\mathcal{S}$ is* left Ore *if given $s \in \mathcal{S}$ and $r \in R$ there exist $s' \in \mathcal{S}$ and $r' \in R$ such that*

$$r's \;=\; s'r \qquad \text{or equivalently} \qquad Rs \cap \mathcal{S}r \;\neq\; (0). \tag{2.3.8}$$

Right Ore subsets are defined analogously. Intuitively, one thinks of the elements of $\mathcal{S}$ as being invertible and condition (2.3.8) as being

$$rs^{-1} = (s')^{-1}r'.$$

Our ultimate goal is to create a quotient ring of R in which every element can be expressed as $s^{-1}r$ for some $r \in R$ and $s \in \mathcal{S}$. In the commutative case, it is possible to do this as long as $\mathcal{S}$ has no zero divisors. The corresponding notion in noncommutative algebra is that of regularity.

Definition 2.3.2 *We say that an element $r \in R$ is* left regular *(resp.* right regular*) if $r'r \neq 0$ (resp. $rr' \neq 0$) for all $0 \neq r' \in R$. An element that is both left and right regular is said to be* regular.

Here are two important examples of regular elements.

Example 2.3.3 *Let A be a domain. Then any nonzero element of A is regular.*

Example 2.3.4 *Let R be a prime ring. Then any nonzero central element $z \in R$ is regular.*

Proof. Suppose $rz = 0$ for some nonzero $r \in R$. Since z is central, we have $rRz = 0$, which contradicts the fact that R is prime. Hence z is left regular. Similarly, z is right regular. ∎

The notion of regularity is especially important when studying rings of finite GK dimension. As the next result shows, it allows one to proceed inductively when proving statements about rings of finite GK dimension.

Theorem 2.3.5 *Let R be an F-algebra of finite GK dimension and let I be an ideal in R which contains a regular element. Then*

$$\operatorname{GKdim}(R) \geq 1 + \operatorname{GKdim}(R/I).$$

Proof. Let $r \in I$ be regular and let V be a generating subspace for R which contains r. Notice that $V + I$ is a generating subspace for R/I. Choose a subset

$\mathcal{W}_n$ of V^n such that the image of $\mathcal{W}_n$ in R/I is a basis for $V^n + I$. By the definition of $\mathcal{W}_n$ and the regularity of r,

$$\left\{ r^i w \mid 0 \leq i < n, w \in \mathcal{W}_n \right\} \subseteq V^{2n}$$

is a linearly independent subset. It follows that

$$\dim V^{2n} \geq n \dim \operatorname{Span}(\mathcal{W}_n) = n \dim (\overline{V}^n).$$

Using Remark 1.1.3, we deduce that $\operatorname{GKdim}(R) \geq 1 + \operatorname{GKdim}(R/I)$. ∎

Regularity is also important in creating noncommutative quotient rings. As it turns out, if $\mathcal{S}$ is left Ore and consists of regular elements, then it is possible to "localize" at $\mathcal{S}$. We make this more precise.

Theorem 2.3.6 *Let R be a ring and let $\mathcal{S} \subseteq R$ be a left Ore subset of R consisting of regular elements. Then there exists a ring $Q_{\mathcal{S}}(R)$ such that:*

- *every element of $s \in \mathcal{S}$ is invertible in $Q_{\mathcal{S}}(R)$;*
- *every element of $Q_{\mathcal{S}}(R)$ can be expressed as $s^{-1}r$ for some $r \in R$ and $s \in \mathcal{S}$.*

Proof. See 2.1.3 on page 41 of [22]. ∎

We are most interested in the case that $\mathcal{S}$ is the set of all regular elements of R. In this case, we shall let $Q(R)$ denote the ring $Q_{\mathcal{S}}(R)$.

If an element $r \in R$ is not left regular, the set

$$\{x \in R \mid xr = 0\}$$

is a nonzero left ideal which we call the *left annihilator* of r. More generally, given a subset $\mathcal{X}$ of R we define the *left annihilator* of $\mathcal{X}$ to be

$$\{x \in R \mid x\mathcal{X} = 0\}$$

and we denote it by $\ell(\mathcal{X})$. Right annihilators are defined analogously and we denote the right annihilator of a subset $\mathcal{X}$ of R by $\mathrm{r}(\mathcal{X})$.

Goldie's theorem shows that under certain conditions, the regular elements of a semiprime ring R form a left Ore set and the resulting quotient ring is a semiprime Artinian ring. Thus by the Artin-Wedderburn theorem it is a finite product of matrix rings over division rings. We now give two definitions.

Definition 2.3.7 *A ring R is said to be (left)* Goldie *if:*

1. *R satisfies the ascending chain condition on left annihilators;*
2. *R contains no infinite direct sum of nonzero left ideals.*

Remark 2.3.8 *If R is a Goldie ring, then R satisfies the descending chain condition on right annihilators.*

Definition 2.3.9 *A left ideal I of R is said to be* essential *if it intersects any nonzero left ideal non-trivially.*

The significance of the second condition in the definition of a Goldie ring is that it shows that for any nonzero left ideal $I \subseteq R$ there exists a left ideal $J \subseteq R$ such that $I \cap J = (0)$ and $I + J$ is essential. We give some examples of Goldie rings.

Example 2.3.10 *A left Noetherian ring is a Goldie ring.*

Example 2.3.11 *Let A be a commutative domain. Then A is a Goldie ring.*

Proof. Condition 1 of Definition 2.3.7 is satisfied. If I and J are nonzero ideals of A, then $IJ \subseteq I \cap J$ is a nonzero ideal. Hence every ideal of A is essential and the second condition of Definition 2.3.7 is satisfied. ∎

More generally, we have the following result.

Example 2.3.12 *A prime PI ring is a Goldie ring.*

Proof. See Corollary 13.6.6 on page 465 of [22]. ∎

A theorem of Jategaonkar shows that under certain growth conditions, noncommutative domains are also Goldie.

Theorem 2.3.13 *(Jategaonkar [17]) Let F be a field and let A be an F-domain of subexponential growth. Then A is Goldie.*

Proof. Let a, b be nonzero elements of A. We shall show that $Aa \cap Ab \neq (0)$. Consider the F-subalgebra of A generated by a and b. This algebra cannot be the free algebra since A has subexponential growth. Hence there is some polynomial $f(a, b) = 0$. We can write this as $0 = \alpha + ra + r'b$ for some $r, r' \in A$ and $\alpha \in F$ with $(r, r') \neq (0, 0)$. Multiplying on the left by a, we see

$$0 = (ar + \alpha)a + ar'b.$$

It follows that $ar'b \in Ab \cap Aa$. If $ar'b = 0$, then we have $r' = 0$ and $0 = \alpha + ra$. If $\alpha = 0$, then $ra = 0$, which contradicts the fact that $(r, r') \neq (0, 0)$. Therefore $Aa = A$ and so $Aa \cap Ab = Ab \neq (0)$. Thus every left ideal of A is essential and so the second condition of Definition 2.3.7 is satisfied. Since A is a domain, we have that the first condition is satisfied, too. ∎

In particular, if A is a domain with finite GK dimension, then A is Goldie and has a quotient ring $Q(A)$. By Goldie's theorem, this quotient ring is of the form

$$Q(A) \cong M_n(D),$$

where D is a division algebra. Since every nonzero element of $Q(A)$ is of the form $b^{-1}a$ with a and b nonzero elements of A, we see that $Q(A)$ is a domain and hence $Q(A)$ is a division algebra. The following example shows that a domain with exponential growth need not be Goldie.

Example 2.3.14 *The free algebra $F\{x, y\}$ is not a Goldie ring.*

Proof. This algebra is a domain and so the first condition of Definition 2.3.7 is satisfied. The second condition, however, is not satisfied. To see this, notice that

$Rx \cap Ry = (0)$ and since y is regular, we have the sum of left ideals

$$Rx + Rxy + Rxy^2 + \cdots$$

is direct. ∎

Goldie's theorem is the following:

Theorem 2.3.15 *Let S be a semiprime Goldie ring. Then $Q(S)$ exists and is semiprime and left Artinian; moreover if S is prime, then $Q(S)$ is prime.*

We shall prove this theorem in the case that S is prime ring.

Lemma 2.3.16 *Let R be a prime Goldie ring and let s be a left regular element of R. Then Rs is an essential left ideal.*

Proof. Let I be a nonzero left ideal. Suppose that $I \cap Rs = (0)$. We claim the sum

$$I + Is + Is^2 + \cdots$$

is direct. To see this, suppose that

$$r_0 + r_1 s + \cdots + r_n s^n \;=\; 0$$

with $r_0, \ldots, r_n \in I$, $r_n \neq 0$, and with n minimal. Notice $r_0 \in I \cap Rs$ and hence must be zero. Thus

$$(r_1 + r_2 s + \cdots + r_n s^{n-1})s \;=\; 0.$$

Since s is left regular, we conclude that

$$r_1 + r_2 s + \cdots + r_n s^{n-1} \;=\; 0.$$

This contradicts the minimality of n. Hence $I \cap Rs \neq (0)$ and so Rs is essential. ∎

Lemma 2.3.17 *Let R be a prime Goldie ring and let s be a left regular element of R. Then s is regular.*

Proof. Suppose s is not right regular. Then $\mathrm{r}(s)$ is nonzero and hence we can find a right annihilator ideal L which is minimal among nonzero right annihilators contained in $\mathrm{r}(s)$. Since R is prime, $L^2 \neq 0$ and hence we can find $r \in L$ such that $rL \neq (0)$. Since s is left regular, Rs is essential. Thus there exists $x \in R$ such that $xr \in Rs$ with $0 \neq xr$. Notice that $xr\mathrm{r}(s) = 0$. Thus

$$r\big(\mathrm{r}(s)\big) \;\subseteq\; L \cap \mathrm{r}(x). \tag{2.3.9}$$

Now $L \cap \mathrm{r}(x)$ is a right annihilator as it is an intersection of two right annihilator ideals. Moreover,

$$r\big(\mathrm{r}(s)\big) \supseteq rL \neq (0),$$

and hence $L \cap \mathrm{r}(x)$ is nonzero by equation (2.3.9). By the minimality of L, $L \cap \mathrm{r}(x) = L$. Since $r \in L$, it follows that $xr = 0$, a contradiction. Thus s is right regular and hence regular. ∎

Lemma 2.3.18 *Let R be a semiprime Goldie ring. Then R satisfies the descending chain condition on left annihilators and the ascending chain condition on right annihilators.*

Proof. See Lemma 7.2.2 of [15] and use Remark 2.3.8. ∎

Proposition 2.3.19 *Let R be a prime Goldie ring and let J be an essential left ideal of R. Then J contains a regular element.*

Pick $s \in J$ such that s has minimal left annihilator. We claim that s is regular. If it is not regular, there exists a nonzero left ideal I such that $Rs \cap I = (0)$. Since J is essential, we have $J \cap I$ is nonzero. Replacing I by $I \cap J$, we may assume that I is contained in J. Let $r \in I$. If $a \in R$ is a left annihilator of $s - r$, then $as = ar$ and hence $as \in Rs \cap J = (0)$. Thus we have that a is a left annihilator of both s and $r \in I \subseteq J$. By the minimality of the left annihilator of s, we conclude that $\ell(r) \supseteq \ell(s)$ for all $r \in I$. Hence

$$\ell(s)I \;=\; (0).$$

Pick nonzero $b \in I$. Then Then $b'Rb = 0$ for all $b' \in \ell(s)$. Since R is prime, we conclude that $\ell(s) = 0$ and so s is left regular. It follows from Lemma 2.3.17 that s is regular. ∎

Theorem 2.3.20 *Let R be a prime Goldie ring. Then the set $\mathcal{S}$ of regular elements of R is a left Ore set.*

Proof. Let $s \in \mathcal{S}$ and let r be a nonzero element in R. We must show that $Rs \cap \mathcal{S}r$ is non-empty. Notice that Rs is essential by Lemma 2.3.16. Let

$$J = \left\{ x \in R \;\middle|\; xr \in Rs \right\}.$$

Notice that J is a left ideal. If L is any nonzero left ideal, then either $Lr = (0)$, in which case $L \subseteq J$; or, $Lr \neq (0)$, in which case $Lr \cap sR$ is nonzero, since sR is essential. It follows that $L \cap J$ is nonzero for any nonzero left ideal L and hence J is essential. Thus J contains a regular element s'. Hence $rs' = sr'$ for some $r' \in R$. This shows that the Ore conditions are satisfied. ∎

Theorem 2.3.21 *(Goldie's theorem) Let R be a prime Goldie ring. Then $Q(R)$ is a prime left Artinian ring.*

Proof. Let I be a left ideal in $Q(R)$. We have that $I \cap R$ is a left ideal in R. Since R is Goldie there exists a left ideal $J \subseteq R$ such that $J \cap (I \cap R) = (0)$ and $(I \cap R) + J$ is essential in R. By Proposition 2.3.19 the left ideal $(I \cap R) + J$ has a regular element s. It follows that

$$1 = s^{-1}s = e + e'$$

for some $e \in I$ and $e' \in JQ(R)$. Hence $e = e^2 + ee'$, or equivalently

$$e - e^2 = ee'.$$

Notice that $I \cap Q(R)J = (0)$ and $e - e^2 \in I$ and $ee' \in Q(R)J$. It follows that $e^2 = e$ and $ee' = 0$. Any $r \in I$ can be expressed as $r = re + r(1 - e)$. Since $r - re \in I$ and

$R(1-e) \in JQ(R)$, we conclude that $r = re$. Thus $I = Q(R)e$. Notice that

$$\ell\big(\mathrm{r}(e)\big) \;=\; \ell\big((1-e)Q(R)\big) \;=\; Q(R)e$$

and hence every left ideal of $Q(R)$ is the left annihilator of a right annihilator of an idempotent. Suppose

$$I_1 \;\supseteq\; I_2 \;\supseteq\; \cdots \tag{2.3.10}$$

is a descending chain of left ideals in $Q(R)$. Then there exist $e_1, e_2, \ldots \in Q(R)$ such that

$$I_n \;=\; \ell\big(\mathrm{r}(e_n)\big)$$

for all $n \geq 1$. Let $J_n \;=\; \mathrm{r}(e_n)$. Then we have an ascending chain of right ideals

$$J_1 \;\subseteq\; J_2 \;\subseteq\; \cdots.$$

Write $e_n = s_n^{-1} r_n$ with $r_n, s_n \in R$ and s_n regular. Then

$$\mathrm{r}(e_n) \;=\; \mathrm{r}(r_n).$$

Observe that

$$\mathrm{r}(r_n) \;=\; \{x \in R \mid r_n x = 0\}Q(R)$$

and hence by Lemma 2.3.18 the descending chain appearing in item (2.3.10) must eventually terminate. It follows that $Q(R)$ is left Artinian. To see that $Q(R)$ is prime, suppose that $aQ(R)b = 0$ for some nonzero $a, b \in Q(R)$. Write $a = s_1^{-1}a'$ and $b = s_2^{-1}b'$ with $a', b' \in R$ and $s_1, s_2 \in S$. Let $r \in R$ be arbitrary. Then

$$\begin{aligned} 0 \;&=\; a(rs_2)b \\ &=\; (s_1^{-1}a')(rs_2)(s_2^{-1}b') \\ &=\; s_1^{-1}a'rb'. \end{aligned}$$

Hence $a'Rb' = 0$, which contradicts the fact that a and b are nonzero. It follows that $Q(R)$ is a prime left Artinian ring. $\blacksquare$

We conclude this section on Goldie's theorem by giving an application of the ideas which have appeared in this chapter.

Given a ring R, we can form the polynomial ring $R[x]$. By the Hilbert basis theorem, $R[x]$ is left Noetherian whenever R is left Noetherian. Suppose R is left Noetherian and consider the set $\mathcal{S} \subseteq R[x]$ consisting of monic polynomials in R. We have the following fact.

Proposition 2.3.22 *(Resco, Small, Stafford [27]) The set $\mathcal{S}$ consists of regular elements of $R[x]$ and has the property that for any ideal $I \subseteq R$, $r(x) \in R[x]I$, and $s(x) \in \mathcal{S}$, there exist $r_1(x) \in R[x]I$ and $s_1(x) \in \mathcal{S}$ such that*

$$r_1(x)s(x) \;=\; s_1(x)r(x).$$

Proof. Let $s(x) = x^n + s_{n-1}x^{n-1} + \cdots + s_0 \in \mathcal{S}$. Suppose $s(x)(q_m x^m + \cdots + q_0) = 0$, with $q_m \neq 0$. Looking at the leading coefficient of x^{n+m} in the product, we see that $q_n = 0$, a contradiction. Hence $s(x)$ is regular. Now let I be an ideal in R, let $s(x) \in \mathcal{S}$ and let $r(x) \in R[x]I$. Consider the left ideal

$$J \;=\; \{a(x) \in R[x] \mid a(x)r(x) \in R[x]Is(x)\}.$$

We have an injection

$$\iota : R[x]/J \;\to\; R[x]I/\big(R[x]Is(x)\big)$$

given by $a(x) + J \mapsto a(x)r(x) + R[x]s(x)$. Since $s(x)$ is a monic polynomial, $R[x]I/\big(R[x]Is(x)\big)$ is a finitely generated left R-module. Since R is left Noetherian and $R[x]/J$ is a submodule of $R[x]/R[x]s(x)$, $R[x]/J$ is also finitely generated as a left R-module. Let $p_1(x), \ldots, p_n(x)$ span $R[x]/J$ as a left R-module. Choose a monic polynomial $q(x)$ of degree greater than the degrees of $p_1(x), \ldots, p_n(x)$. Then there is a polynomial $s_1(x) \in J$ and $r_1, \ldots, r_n \in R$ such that

$$q(x) \;=\; s_1(x) + r_1 p_1(x) + \cdots + r_n p_n(x).$$

Since $q(x) - s_1(x)$ has degree strictly smaller than the degree of $q(x)$, we see $s_1(x)$ has the same degree and same leading coefficient as $q(x)$. Thus $s_1(x)$ is a monic polynomial in J. We see that $s_1(x)r(x) = r_1(x)s(x)$ for some polynomial $r_1(x) \in R[x]I$. The claim follows. ∎

The following corollary is immediate.

Corollary 2.3.23 *$\mathcal{S}$ is left Ore and for any ideal $I \subseteq R$, $Q_{\mathcal{S}}(R[x])I$ is an ideal in $Q_{\mathcal{S}}(R[x])$.*

It follows that we can form the quotient ring $Q_{\mathcal{S}}(R[x])$. We denote this ring by $R\langle x\rangle$. We shall show that $R\langle x\rangle$ is a Jacobson ring. To do this, we require a few simple facts about Noetherian rings. The theorem we now give is due to Noether.

Theorem 2.3.24 *Let R be a left Noetherian ring. Then there exist prime ideals $P_1, \ldots, P_n$ of R such that $P_1 P_2 \cdots P_n = (0)$.*

Proof. Suppose not. Then we can choose an ideal I of R maximal with respect to the property that it does not contain a finite product of prime ideals. Evidently, I is not prime and hence there exist ideals J_1 and J_2 properly containing I such that $J_1 J_2 \subseteq I$. By maximality, $J_1 \supseteq P_1 P_2 \cdots P_m$ for some prime ideals $P_1, \ldots, P_m$ and $J_2 \supseteq Q_1 Q_2 \cdots Q_\ell$ for some prime ideals $Q_1, \ldots, Q_\ell$. Then $I \supseteq P_1 P_2 \cdots P_m Q_1 \cdots Q_\ell$, a contradiction. The result follows. ∎

Corollary 2.3.25 *Let R be a left Noetherian ring and let I be an ideal of R. Then there are finitely many prime ideals minimal above I.*

Proof. By Theorem 2.3.24 there exist prime ideals $P_1, \ldots, P_n$ of R containing I such that $I \supseteq P_1 \cdots P_n$. Let Q be a prime ideal minimal above I. Then

$$(P_1 + Q) \cdots (P_n + Q) \subseteq Q.$$

Since Q is prime, we see that $P_i + Q = Q$ for some i. Since P_i and Q are primes containing I and Q is a prime minimal above I, we see that $Q = P_i$. It follows that there are only finitely many primes minimal above I. ∎

Proposition 2.3.26 *(Levitzki [21]) Let R be a left Noetherian ring and let $P_1, \ldots, P_n$ be the minimal primes. Then $P_1 \cap P_2 \cdots \cap P_n$ is nilpotent.*

Proof. Let
$$J \;=\; \bigcap_{i=1}^{n} P_i.$$
Let $x \in J$. Observe that if x is not nil, then by either the Noetherian hypothesis or Zorn's lemma we can find an ideal P maximal with respect to the property that it does not intersect the set $\{1, x, x^2, \cdots\}$. It is easy to check that P must be prime. By assumption, $P \supseteq P_i$ for some i and hence $x \in J \subseteq P$, contradicting the fact that $P \cap \{1, x, x^2, \cdots\} \;=\; \varnothing$. Hence J is a nil ideal. We now show that J is nilpotent. Let I be the largest nilpotent ideal in R. Passing to R/I, we can assume that R has no nonzero nilpotent ideals. Choose nonzero $a \in J$ with maximal right annihilator. Then $\mathrm{r}(xa) = \mathrm{r}(a)$ for all $x \in R$ with $xa \neq 0$ by maximality. Since xa is nil, we have $(xa)^n = 0$ for some n. Choose n minimal. Then $\mathrm{r}\big((xa)^{n-1}\big) = \mathrm{r}(a)$. Since xa is in the right annihilator of $(xa)^{n-1}$, we see $axa = 0$ for all $x \in R$. Thus RaR is a nilpotent, contradicting the fact that R has no nonzero nilpotent ideals. ∎

We shall now answer a question of Resco, Small, Stafford (see [27]) by showing that when R is left Noetherian, $R\langle x\rangle$ is Jacobson.

Lemma 2.3.27 *Let J be an ideal in $R\langle x\rangle$. Then there exists an ideal I in R and $n \geq 1$ such that*
$$R\langle x\rangle I^n \;\subseteq\; J \;\subseteq\; R\langle x\rangle I.$$

Proof. Let I_d be the ideal consisting of 0 and all leading coefficients of polynomials of degree d in $J \cap R[x]$. Notice that
$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \cdots$$
is an ascending chain of ideals in R and hence becomes stable as some point, say n. Let $I = I_n$. We have $J \subseteq R\langle x\rangle I$. Notice that any polynomial in $R[x]I$ is congruent

to a polynomial of degree less than n mod $J \cap R[x]$. Since R is left Noetherian, we in fact have that $R[x]I/(J \cap R[x])$ is a finitely generated R-module. Let $Q_1, \ldots, Q_k$ denote the primes of $R[x]$ minimal above $J \cap R[x]$. Suppose $R[x]I$ is not contained in Q_i. Since $R[x]/Q_i$ is prime and Goldie, we have that $R[x]I/Q_i$ contains a regular element, $a(x)$. Hence

$$R[x]/Q_i \cong R[x]a(x)/Q_i \hookrightarrow R[x]I/Q_i.$$

It follows that $R[x]/Q_i$ is finitely generated as an R-module as it is a submodule of a finitely generated R-module. Hence Q_i contains a monic polynomial. Thus I is either contained in Q_i, or Q_i contains a monic polynomial. By relabeling if necessary, we may assume that $R[x]I \not\subseteq Q_i$ for $1 \leq i \leq \ell$ and $R[x]I \subseteq Q_i$ for $\ell < i \leq k$. Choose monic polynomials $s_i(x) \in Q_i$ for $1 \leq i \leq \ell$ and let $s(x) = \prod_{i=1}^{\ell} s_i(x)$. Then

$$s(x)R[x]I \subseteq \bigcap Q_i.$$

Since $\bigcap Q_i$ is nilpotent mod $J \cap R[x]$, we have

$$s(x)^n R[x]I^n \subseteq (Q_1 \cdots Q_k)^n \subseteq I \cap R[x]$$

for some n. Since $s(x)^n$ is monic, it follows that $R\langle x\rangle I^n \subseteq J$. This completes the proof. ∎

Theorem 2.3.28 *Suppose R is a left Noetherian ring. Then $R\langle x\rangle$ is Jacobson.*

Proof. It suffices to prove that $R\langle x\rangle/P$ has trivial Jacobson radical for every prime ideal P of R. By Lemma 2.3.27 there exists an ideal I in R such that $R\langle x\rangle I^n \subseteq P \subseteq R\langle x\rangle I$. Let $Q_1, \ldots, Q_k$ be the primes of R minimal above I. We claim $R\langle x\rangle/R\langle x\rangle Q_i$ has Jacobson radical zero. To see this we argue by contradiction. Suppose we can find $\alpha = a_0 + a_1 x + \cdots a_m x^m$ with $a_m \notin Q_i$ and $m \geq 1$ with nonzero image in $J\Big(R\langle x\rangle/R\langle x\rangle Q_i\Big)$. Since Q_i is prime and R/Q_i is Goldie, we may assume that the image of a_m is regular in R/Q_i. There exists $\beta = p(x)^{-1}(\sum_{j=1}^{n} b_j x^j) \in R\langle x\rangle$ such that $\beta(1+\alpha) \equiv 1$ mod $R\langle x\rangle Q_i$. The leading coefficient of $p(x)\beta(1+\alpha) = a_m b_n$

cannot be in Q_i since a_m is regular mod Q_i and b_n is not in Q_i. Hence $a_m b_n \equiv 1 \bmod Q_i$. This says that $b_n\alpha$ is a unit mod Q_i, contradicting the fact that α is in the Jacobson radical of $R\langle x\rangle/R\langle x\rangle Q_i$. Thus the Jacobson radical of $R\langle x\rangle/P$ is contained in $R\langle x\rangle Q_i/P$ for all i. Since $\bigcap Q_i$ is nilpotent mod I, using Proposition 2.3.22 we see

$$J(R\langle x\rangle/P)^m \subseteq R\langle x\rangle I/P$$

for some m. We also have that $R\langle x\rangle I^n \subseteq P$ and hence $J\Big(R\langle x\rangle\Big/P\Big)^{nm} = (0)$. Since P is prime, we conclude that $R\langle x\rangle/P$ has Jacobson radical (0). ∎

2.4 Rings of GK dimension one

We now look at the work of Small and Warfield who proved that affine prime rings of GK dimension 1 satisfy a polynomial identity. Later, along with Stafford, they proved that semi-prime affine rings of GK dimension 1 satisfy a polynomial identity. We shall give a combinatorial proof of the Small-Warfield theorem for domains of GK dimension 1. We begin with a theorem of Borho and Kraft [11] which is especially useful when studying domains of finite GK dimension.

Theorem 2.4.1 *Let B be a subdomain of an affine F-domain A and suppose that* $\mathrm{GKdim}(A) < \mathrm{GKdim}(B) + 1 < \infty$. *Then $B^* = B \setminus \{0\}$ is a left Ore set in A and $Q(A)$ is a finite dimensional right vector space over $Q(B)$.*

Proof. Let W be a generating subspace for B. Let $b \in B$ and $a \in A$ be nonzero. To show that B^* is left Ore, we must show that $Ab \cap Ba \neq \{0\}$. If the intersection is zero, we claim that

$$Ba + Bab + Bab^2 + \cdots$$

is direct. Suppose $b_0a+b_1ab+\cdots b_nab^n = 0$, with $b_n \neq 0$ and n minimal. Then $b_0a \in Ba\cap Ab = \{0\}$. Hence $b_0 = 0$. Since b is regular, we have $b_1a+b_2ab+\cdots b_nab^{n-1} = 0$, contradicting the minimality of n. Letting $V = W + Fa + Fb$, we see that the sum

$$V^{2n} \supseteq W^n a + W^n ab + \cdots + W^n b^{n-1} a$$

is direct. We conclude that

$$\dim V^{2n} \geq n \dim W^n.$$

It follows from Remark 1.1.3 that A has GK dimension at least one greater than the GK dimension of B, a contradiction. Thus we see that B^* is indeed left Ore in A. Let U be a generating subspace for A. Notice that if $Q(B)A \subseteq Q(A)$ is infinite dimensional as a left $Q(B)$-vector space, then $Q(B)U^n$, when considered as a left $Q(B)$-vector space, must have dimension at least n. We have

$$(W+U)^{2n} \supseteq W^nU^n \supseteq W^nu_1 + \cdots + W^nu_n.$$

Moreover, since $u_1, \ldots, u_n$ are B-linearly independent, we see that

$$\dim\, (W+U)^{2n} \geq n \dim W^n.$$

Using Remark 1.1.3 once again, we have that A has GK dimension at least one greater than the GK dimension of B, a contradiction. Hence $Q(B)A$ is a finite dimensional left $Q(B)$-vector space. To finish the proof we only need to show that $Q(B)A = Q(A)$. Let $a \in A$. Right multiplication by a gives a $Q(B)$-linear map from $Q(B)A$ to itself. Since a is regular, the map is injective. Since $Q(B)A$ is finite dimensional over $Q(B)$ we conclude that the map is a vector space isomorphism. Thus there exists $a' \in AQ(A)$ such that $a'a = aa' = 1$ and so a is invertible in $Q(B)A$. It follows that $Q(B)A = Q(A)$. ∎

Theorem 2.4.2 *Let A be an affine F-domain of GK dimension 1. Then A is PI.*

Proof. By Theorem 1.2.8 A is not algebraic. Hence there exists an element $r \in R$ such that $F[r]$ has GK dimension 1. By Proposition 1.1.12 the quotient division ring of A, $Q(A)$ is a finite dimensional over $F(r)$ as a left $F(r)$-vector space. Let m denote the dimension of $Q(A)$ as a left $F(R)$-vector space. Notice that $Q(A)$ embeds in $M_m(F(r))$ by sending an element $a \in Q(A)$ to the map corresponding to right multiplication by a. Regarding $M_m(F(r))$ as an $F(r)$-algebra, we see that it satisfies the identity S_{2m} by the Amitsur-Levitzki theorem. Hence $Q(A)$ also satisfies S_{2m} as it is a subring of $M_m(F(r))$. ∎

To completely understand domains of GK dimension 1 it is necessary to use *Tsen's theorem*. This theorem states that if F is an algebraically closed field; Z is a field extension of F of transcendence degree 1; and D is a division algebra that is finite dimensional over its center Z, then $D = Z$. We give a short proof of this theorem.

Definition 2.4.3 *We say that a field F is C_1 if for all $n > 1$, any homogeneous polynomial $f(t_1, \ldots, t_n)$ of degree $0 < d < n$ has a non-trivial solution in F^n.*

The following theorem shows the significance of C_1 fields.

Theorem 2.4.4 *Let D be a finite dimensional division algebra over its center Z. If Z is C_1, then $D = Z$.*

Proof. See Theorem 19.2 on page 370 of [25]. ∎

We give some examples of C_1 fields.

Example 2.4.5 *An algebraically closed field is C_1*

Proof. Let F be an algebraically closed field and let $f(t_1, \ldots, t_n)$ be a homogeneous polynomial of degree $d < n$. The polynomial $f(1, 1, \ldots, 1, x_n)$ is a polynomial in one variable and hence has a root $\alpha \in F$. We have $(1, 1, \ldots, 1, \alpha) \in F^n$ is a non-trivial solution. ∎

Example 2.4.6 *Any subfield of $\mathbb{R}$ is not C_1.*

Proof. The polynomial $t_1^2 + t_2^2 + t_3^2$ has only the trivial solution. ∎

Example 2.4.7 *A finite field is C_1.*

Proof. (Chevalley-Warning) Let F be a finite field of size $q = p^k$ where p is prime. To prove this we need the following claim.

Claim: For $0 \leq j < q-1$ we have

$$\sum_{\alpha \in F} \alpha^j = 0,$$

where 0^0 is defined to be one.

Proof. The claim is easily seen to be true when $j = 0$, so we may suppose that $0 < j < q-1$. Notice that the polynomial $t^j - 1$ has at most $j < q-1$ roots in F and hence there exists nonzero $\beta \in F$ such that $\beta^j \neq 1$. Multiplication by β is an isomorphism from F to F and hence

$$\sum_{\alpha \in F} \alpha^j \; = \; \sum_{\alpha \in F} (\beta\alpha)^j = \beta^j \sum_{\alpha \in F} \alpha^j.$$

The claim follows, since $\beta^j \neq 1$. ∎

Let $f(t_1, \ldots, t_n)$ be a homogeneous polynomial of degree $d < n$. Notice that $(1 - f^{q-1})(\alpha_1, \ldots, \alpha_n)$ is equal to 1 if $(\alpha_1, \ldots, \alpha_n)$ is a zero of f and is equal to 0 otherwise. Let $\mathcal{V}$ denote the set of zeros of f. We have

$$\sum_{(\alpha_1, \ldots, \alpha_n) \in F^n} (1 - f^{q-1})(\alpha_1, \ldots, \alpha_n) \; \equiv \; \mathrm{Card}(\mathcal{V}) \mod p.$$

But f^{q-1} is a linear combination of monomials of the form $t_1^{j_1} \cdots t_n^{j_n}$, where $j_1 + \cdots + j_n = d(q-1)$. Since $d < n$, we have $j_\ell < q-1$ for some ℓ and hence

$$\sum_{\alpha \in F} \alpha^{j_\ell} \; = \; 0. \tag{2.4.11}$$

Thus we see

$$\begin{aligned} \sum_{(\alpha_1, \ldots, \alpha_n) \in F^n} \alpha_1^{j_1} \cdots \alpha_n^{j_n} \; &= \; \prod_{k=1}^{n} \Big(\sum_{\alpha \in F} \alpha^{j_k} \Big) \\ &= \; 0 \qquad \text{by equation (2.4.11).} \end{aligned}$$

It follows that

$$\sum_{\alpha \in F^n} (1 - f^{q-1})(\alpha_1, \ldots, \alpha_n) \;=\; 0$$

and hence $\mathrm{Card}(\mathcal{V}) \equiv 0 \mod p$. But the trivial solution to f gives that $\mathcal{V}$ has size at least one. Hence there is a non-trivial solution to f and so F is C_1. ∎

We obtain a famous theorem of Wedderburn.

Corollary 2.4.8 *Let D be a finite division ring. Then D is commutative.*

Proof. Let Z denote the center of D. Since Z is finite, it is C_1 by Example 2.4.7. Hence $D = Z$. ∎

Lemma 2.4.9 *If F is C_1 then any finite extension of F is also C_1.*

Proof. Let L be a finite extension of F and let K be the separable closure of F in L and let $\beta_1, \ldots, \beta_m$ be a basis for K over F. Let $f(x_1, \ldots, x_n) \in K[x_1, \ldots, x_n]$ be a homogeneous polynomial of degree $d < n$. We make the substitution $x_i = \sum_{j=1}^m y_{i,j}\beta_j$ and let $g((y_{i,j}))$ denote the resulting polynomial. Let $\mathcal{S}$ denote the set of all field homomorphisms from K into the algebraic closure of K which fix F. We can extend $\sigma \in \mathcal{S}$ to a map on $K[y_{i,j}]$ by declaring that $\sigma(y_{i,j}) = y_{i,j}$ for all i, j. Define

$$h((y_{i,j})) \;:=\; \prod_{\sigma \in \mathcal{S}} \sigma(g).$$

Basic Galois theory gives that h is a homogeneous polynomial in F of degree dm and has mn variables. Hence it has a non-trivial solution. This gives a non-trivial solution to f in K^n. Thus K is C_1. Now, L is purely inseparable over K and therefore we have reduced to the case of showing that a finite purely inseparable extension of a C_1 field is C_1. We may, of course, assume that $L \neq K$ and therefore we have that F has characteristic p for some prime p. Induction shows that it is sufficient to prove the case that $K' = K(\alpha)$ with $\alpha^p \in K$. Let $f(x_1, \ldots, x_n) \in K'[x_1, \ldots, x_n]$ be a homogeneous polynomial of degree $d < n$. Making the substitution, $x_i = y_{i,0} + y_{i,1}\alpha + \cdots + y_{i,p-1}\alpha^{p-1}$ for $i = 1, \ldots, n$, we have f^p is a homogeneous polynomial in $\{y_{i,j} \mid 1 \leq i \leq n,\ 0 \leq j < p\}$ of degree dp and with coefficients of K. It follows that f has a non-trivial solution in $(K')^n$. ∎

Lemma 2.4.10 *Let F be an algebraically closed field. Suppose that $\Phi_1, \ldots, \Phi_d$ are non-constant homogeneous polynomials in $F[x_1, \ldots, x_n]$ with $d < n$. Then there is a non-trivial solution $\mathbf{a} \in F^n$ such that*

$$\Phi_1(\mathbf{a}) \;=\; \cdots \;=\; \Phi_d(\mathbf{a}).$$

Proof. This proof relies on results from commutative algebra; namely, the generalized Principal ideal theorem and the Nullstellensatz. A proof can be found in Proposition 19.3 on page 372 of [25]. ∎

Theorem 2.4.11 *Let F be an algebraically closed field and let K be a finitely generated field extension of F of transcendence degree 1. Then K is C_1.*

Proof. By Lemma 2.4.9, we may assume that $K = F(x)$. Let $f(t_1, \ldots, t_n)$ be a homogeneous form in $K[t_1, \ldots, t_n]$ of degree $d < n$. By multiplying f by a suitable polynomial in $F[x]$ we may assume that the coefficients of f are polynomials in $F[x]$. Let m denote the maximum of the degrees of the polynomials which occur as coefficients of f. Let ℓ be a positive integer and let

$$a_i \;=\; y_{i,0} + xy_{i,1} + \cdots + x^{\ell} y_{i,\ell}$$

for $1 \leq i \leq n$. Notice that

$$g\big((y_{i,j})\big) \;:=\; f(a_1, \ldots, a_n)$$

has degree at most $d\ell + m$ in x. Hence we can write

$$g\big((y_{i,j})\big) \;=\; \sum_{k=0}^{d\ell+m} \Phi_k\big((y_{i,j})\big) x^k,$$

where $\Phi_k \in F[\{y_{i,j}\}]$ is homogeneous for $1 \leq k \leq d\ell + m$. By Lemma 2.4.10 we have that $\Phi_0, \ldots, \Phi_{d\ell+m}$ have a common non-trivial zero whenever the number of variables exceeds $d\ell + m$. We have $n(\ell+1)$ variables and since $n > d$, $n(\ell+1) > d\ell + m$ for all ℓ sufficiently large. Hence there exists a non-trivial solution to g in $F^{n(\ell+1)}$. From this, we obtain a non-trivial solution to f in K^n. ∎

Corollary 2.4.12 *(Tsen's theorem) Let F be an algebraically closed field and let K be a finitely generated extension of F of transcendence degree 1. Then if D is a finite dimensional division algebra over K, then D is commutative.*

Proof. Use Theorem 2.4.4 and Theorem 2.4.11. ∎

This result allows us to describe domains of GK dimension 1 over algebraically closed fields.

Theorem 2.4.13 *(Small-Warfield [31]) An affine prime ring of GK dimension 1 is a finite module over its center and satisfies a PI.*

In fact, this theorem can be extended to semiprime affine rings [30]. A result of Artin and Tate shows that the center of an affine domain of GK dimension 1 is affine.

Theorem 2.4.14 *(Artin-Tate [3]) Let R be an affine F-algebra which is a finite module over its center Z. Then Z is an affine F-algebra.*

Proof. See Proposition 6.2.5 on page 115 of [29]. ∎

Theorem 2.4.15 *Let A be an affine F-domain of GK dimension 1 over an algebraically closed field F. Then A is commutative.*

Proof. Let Z denote the center of A. By the Small-Warfield theorem, A is a finite Z-module. Theorem 2.3.13 and Theorem 2.4.1 A has a ring of quotients $Q(A)$ that is finite dimensional over $Q(Z)$; moreover, $Q(Z)$ is a field extension of F of transcendence degree 1. Notice Z is a finite F-extension by Theorem 2.4.14 It follows that $Q(Z)$ is C_1 from Lemma 2.4.9. The center of $Q(A)$ contains $Q(Z)$ and is therefore a finite extension of $Q(Z)$. Since $Q(Z)$ is C_1 we see that the center of $Q(A)$ is C_1. Since $Q(A)$ is finite dimensional over its center, we conclude that $Q(A)$ is commutative by Tsen's theorem. Since A is a subring of the commutative ring $Q(A)$, we see that A is commutative. ∎

Thus we have seen that any affine domain of GK dimension 1 over an algebraically closed field F is in fact commutative. This is a special case of the following theorem of Small and Warfield

The estimates of Pappacena given in Theorem 1.2.9 give more information about rings of GK dimension 1. Notice that if R is an affine F-algebra and R/P is a finite-dimensional primitive homomorphic image of F, then

$$R/P \;\cong\; M_m(D)$$

for some m and some division ring D, by Kaplansky's theorem. Also, D is finite dimensional over F and hence $D = F$ if F is algebraically closed. In the case that F is algebraically closed, we call

$$R/P \;\cong\; M_m(F)$$

a *matrix image* of R of *degree* m. We now use Pappacena's estimates to find an upper bound for the degrees of the matrix images of an affine prime ring of GK dimension 1.

Theorem 2.4.16 *Let F be an algebraically closed field, let R be an F-algebra of GK dimension 1, and let V be a generating subspace for R. Then the degrees of the matrix images of R are bounded above by $4C^2$, where*

$$C \;=\; \sup\left\{\left(\dim V^n\right)/n\right\}.$$

Proof. Notice that C is finite by Theorem 1.2.6. Let

$$R/P \cong M_m(F)$$

be a matrix image of R of degree m. Notice that $\bar{V}$, the image of V in R/P is a generating set for R/P. By Theorem 1.2.10 we have

$$\dim V^{\lfloor 2m^{3/2} \rfloor} \;\geq\; m^2.$$

Thus

$$m^2/2m^{3/2} \;\leq\; m^2/\big(\lfloor 2m^{3/2} \rfloor\big) \;\leq\; C.$$

It follows that $m \leq 4C^2$. ∎

Corollary 2.4.17 *Let F be an algebraically closed field; let R be a prime F-algebra of GK dimension 1; and let V be a generating subspace for R. Then R satisfies a PI of degree at most $8C^2$, where*

$$C \;=\; \sup\Big\{\big(\dim V^n\big)/n\Big\}.$$

Proof. Notice that if S is an affine primitive ring of GK dimension less than or equal to 1, then it is PI and hence by Kaplansky's theorem

$$S \;\cong\; M_m(D)$$

for some division ring D which is finite dimensional over its center Z. By Theorem 2.4.14 Z is an affine F-algebra. But a field that is a finite F-algebra is necessarily a finite extension of F. Since F is algebraically closed we see that

$$S \;\cong\; M_m(F).$$

Thus

$$R/P \;\cong\; M_{m(P)}(F)$$

for some positive integer $m(P)$ which depends of P. By Theorem 2.4.16, $m(P) \leq 4C^2$ for all primitive ideals P. The Jacobson radical of R is (0) by Theorem 2.2.17 and hence (0) is an intersection of primitive ideals. We thus have an embedding

$$R \hookrightarrow \prod_{P \text{ primitive}} R/P \cong \prod_{P \text{ primitive}} M_{m(P)}(F).$$

Notice that $M_{m(P)}(F)$ is a subring of $M_{\lfloor 4C^2 \rfloor}(F)$ for all primitive ideals P. By the Amitsur-Levitzki theorem,

$$\prod_{P \text{ primitive}} R/P$$

satisfies the identity $S_{\lfloor 8C^2 \rfloor}$. Since R is a subring of this product, it must also satisfy this identity. The result follows. ∎

Chapter 3

Graded rings of low GK dimension

3.1 Goldie's theorem for graded rings

In this chapter we study graded rings, following the approach of [23]. A ring R is said to be *graded* if for each $n \in \mathbb{Z}$ there exist subspaces $R_n \subseteq R$ with the properties:

1. $R = \bigoplus_{n \in \mathbb{Z}} R_n$;
2. $R_n \cdot R_m \subseteq R_{n+m}$.

Given a decomposition satisfying properties 1. and 2. above we say that an element $r \in R_n$ is *homogeneous of degree n*. We say that a left (resp. right) ideal of a graded ring R is a *graded left ideal* (resp. *graded right ideal*) if it is generated by homogeneous elements.

Definition 3.1.1 *Let $R = \bigoplus R_n$ be a graded ring. We say that M is a* graded left R-module *if there exist linear spaces M_n for $n \in \mathbb{Z}$ such that*

$$M = \bigoplus_{n \in \mathbb{Z}} M_n$$

and $R_i \cdot M_j \subseteq M_{i+j}$ *for all* $i, j \in \mathbb{Z}$. *We say that a nonzero element of* M_n *is* homogeneous of degree n.

Example 3.1.2 *Let* F *be a field. Then* $R = F[t_1, \ldots, t_d]$ *is a graded ring.*

Proof. Take $R_n = (0)$ for $n < 0$ and take R_n to be the space of all homogeneous polynomials of degree n for $n \geq 0$. Then R is seen to be graded. ∎

Example 3.1.3 *Let* F *be a field and let* $R = F[x, y, z, t]/(x + x^2y + z^2 + t^3)$. *Then* R *is a graded ring in which* x, y, z *and* t *are homogeneous of degrees* 6, -6, 3, *and* 2 *respectively.*

It is non-trivial to show that the ring given in the preceding example is not isomorphic to a polynomial ring in three variables. Here is a noncommutative example of a graded algebra of GK dimension two.

Example 3.1.4 *Let* F *be a field and let* $R = F\{x, y\}/(xy - yx - x^2)$. *Then* R *is a graded ring with* x *and* y *homogeneous elements of degree one.*

Many of the results from the second chapter have graded analogues. We first introduce the notion of a *graded division ring.*

Definition 3.1.5 *A ring* Δ *is said to be a* graded division ring *if every nonzero homogeneous element is invertible.*

The homogeneous elements of degree zero in a graded division ring form a division ring.

Proposition 3.1.6 *Let* Δ *be a graded division ring. Then*

$$\Delta \cong D[z, z^{-1}; \sigma],$$

where D *is the division ring consisting of homogeneous elements of degree zero;* z *is a homogeneous element of minimum positive degree; and* σ *is an automorphism of* D.

Proof. Pick a homogeneous $z \in \Delta$ of minimum positive degree. For $\delta \in D$, we have $z\delta z^{-1} \in D$. Conjugation by z is thus an automorphism of D. It follows that $D[z, z^{-1}; \sigma]$ is a subring of Δ. Let $a \in \Delta$ be homogeneous. By the minimality of the degree of z, we see that the degree of z must divide the degree of a. Hence there is an integer k such that $\delta := az^{-k}$ is homogeneous of degree zero. Hence $a = \delta z^k \in D[z, z^{-1}; \sigma]$. Since every element of Δ is a linear combination of homogeneous elements, we conclude that

$$\Delta \subseteq D[z, z^{-1}; \sigma].$$

The result follows. ∎

Graded division rings are like ordinary division rings in many ways. For instance, given a graded division ring Δ and a left graded-Δ-module V, there exists a *graded-basis* for V; that is, there exists a subset $\mathcal{X}$ consisting of homogeneous elements of V such that any homogeneous element $v \in V$ has a unique expression

$$v \;=\; \sum_{i=1}^{n} \delta_i x_i$$

for some subset $\{x_1, \ldots, x_n\}$ of $\mathcal{X}$ and homogeneous elements $\delta_1, \ldots, \delta_n \in \Delta$. This shows graded-modules over graded-division rings behave much like vector spaces over ordinary division rings.

Definition 3.1.7 *We say a ring R is a* graded matrix ring *over a graded ring S. If*

$$R \;\cong\; M_m(S)$$

for some m. Given integers $d_1, \ldots, d_m$, we can grade R by declaring $(a_{i,j})_{1 \le i,j \le m}$ to be homogeneous of degree d if $a_{i,j}$ is homogeneous element of S of degree $d - d_j + d_i$ for $1 \le i, j \le m$.

Definition 3.1.8 *A graded ring R is said to be* left graded-Artinian *(respectively* right graded-Artinian*) if R satisfies the descending chain condition on graded left* *(resp. right) ideals; that is, any chain*

$$I_1 \supseteq I_2 \supseteq \cdots$$

of graded left (resp. right) ideals must eventually be constant. A graded ring that is both left and right graded-Artinian is said to be graded-Artinian.

As one might expect, a prime graded-Artinian ring is a graded matrix ring over a graded division ring. To show this, a graded analogue of the Jacobson density theorem is needed.

Given a graded ring R, we say that a left R-module is *graded-simple* if it is a graded module with no proper nonzero graded submodules. We say that a graded ring is (left) *graded-primitive* if it has a faithful graded-simple module. There is a graded analogue of Schur's lemma. If R is a graded ring and M is a graded-simple left R-module, then $\mathrm{End}_R(M)$ is graded with the homogeneous elements of degree n consisting of all homomorphisms $f : M \to M$ which map homogeneous elements of M of degree i to homogeneous elements of M of degree $i+n$ for all $i \in \mathbb{Z}$. With this grading, we have the following theorem.

Lemma 3.1.9 *(graded Schur's lemma) Let R be a graded ring with a graded-simple module M. Then $\mathrm{End}_R(M)$ is a graded division ring.*

Proof. Let f be a nonzero homogeneous element of $\mathrm{End}_R(M)$. Since f is homogeneous, both the image and kernel of f are graded R-modules. Since M is graded-simple, we conclude that f is both injective and surjective. Just as in the proof of Lemma 2.1.9 we see that f has an inverse in $\mathrm{End}_R(M)$. ∎

If R is a graded ring and M is a graded-simple left R-module, then, just as in the non-graded case, we can turn M into a left module over $\Delta := \mathrm{End}_R(M)$ via the action

$$f \cdot v = f(v).$$

Theorem 3.1.10 *(graded Jacobson density theorem) Let R be a graded-primitive ring, let M be a faithful graded-simple R-module, and let $\Delta = \mathrm{End}_R(M)$. Then* *given a finite Δ-linearly independent subset $\{x_1, \ldots, x_n\}$ of M consisting of homogeneous elements and a subset $\{y_1, \ldots, y_n\}$ of M of the same size consisting of homogeneous elements, there exists a homogeneous element $r \in R$ such that $rx_i = y_i$ for $1 \leq i \leq n$.*

Proof. The proof is similar to the proof of the Jacobson density theorem given in Theorem 2.1.10. ∎

We obtain the graded analogue of Proposition 2.1.13 as a corollary.

Corollary 3.1.11 *Let R be a prime left graded-Artinian ring. Then R is a matrix ring over a graded division ring.*

Proof. Just as in the ungraded case, we have that a prime left graded-Artinian ring is graded-primitive. Let M be a faithful graded-simple left R-module and let

$$\Delta \ = \ \mathrm{End}_R(M).$$

Just as in the proof of Theorem 2.1.11, there is a finite Δ-graded-basis $\{x_1, \ldots, x_n\}$ of M consisting of homogeneous elements. Thus

$$M \ \cong \ \Delta x_1 \oplus \cdots \oplus \Delta x_n.$$

Let d_i denote the degree of x_i for $1 \leq i \leq n$. Notice that

$$\mathrm{End}_R(M) \ \cong \ M_n(\Delta)$$

in which $(a_{i,j})$ is homogeneous element of degree d if $a_{i,j}$ is homogeneous of degree $d - d_j + d_i$ for all i and j. By the graded density theorem, we see that

$$R \ \cong \ M_n(\Delta).$$

The result follows. ∎

Having established the graded analogue of the Artin-Wedderburn theorem, we can now work toward obtaining a graded version of Goldie's theorem. We introduce some notation.

Notation 3.1.12 *Let R be a graded prime ring. If the set $\mathcal{S}$ of homogeneous regular elements of R is left Ore, we denote by $Q_{\mathrm{gr}}(R)$ the ring $Q_{\mathcal{S}}(R)$.*

Lemma 3.1.13 *Let R be a prime graded Goldie ring. Then any nonzero, graded left ideal I of R contains a non-nilpotent homogeneous element.*

Proof. Choose nonzero homogeneous $x \in I$ with maximal left annihilator. By maximality, $\ell(xrx) = \ell(x)$ for all homogeneous $r \in R$ such that $xrx \neq 0$. If rx is a nonzero homogeneous nilpotent element, then there is some $n > 1$ such that $(rx)^n = 0$ and $(rx)^{n-1} \neq 0$. It follows that $x(rx)^{n-2}$ is nonzero and $rxr \in \ell\big(x(rx)^{n-2}\big)$. Thus $rxr \in \ell(x)$. It follows that $(rx)^2 = 0$ for all homogeneous $r \in R$. Now suppose that $xrx \neq 0$ for some homogeneous $r \in R$. Then

$$r \;\in\; \ell(xrx) \;=\; \ell(x),$$

a contradiction. It follows that $xRx = 0$. Since R is prime, we conclude that $x = 0$, a contradiction. ∎

Theorem 3.1.14 *Let R be a graded ring. Then any essential graded left ideal has a homogeneous regular element.*

Proof. Let I be an essential left graded-ideal. We claim that there exists homogeneous elements $s_1, \ldots, s_n$ of I such that

$$\bigcap_{i=1}^{n} \ell(s_i) \;=\; (0). \tag{3.1.1}$$

We can find a homogeneous element $s_1 \in I$ that is not nilpotent. Choose a homogeneous element $s_1 \in I$ with maximal left annihilator. By maximality, we have

$$\ell(s_1^m) \;=\; \ell(s_1)$$

for all $m \geq 1$. If s_1 is left regular, we are done. If not

$$\ell(s_1) \cap I \neq (0)$$

and so we can find a non-nilpotent homogeneous element in the intersection. As before, we pick a non-nilpotent element s_2 in the intersection with maximal left annihilator. We have

$$\ell(s_2^m) \;=\; \ell(s_2)$$

for all $m \geq 1$. Observe that Rs_1+Rs_2 is direct. To see this, suppose that $as_1+bs_2 = 0$. Since s_2 is a left annihilator of s_1, right multiplying by s_1 gives $as_1^2 = 0$ and hence $as_1 = 0$. It follows that $bs_2 = 0$. If $\ell(s_1)\cap\ell(s_2) = (0)$ we are done. If not, then we can find an element $s_3 \in I \cap \ell(s_1) \cap \ell(s_2)$ such that the sum $Rs_1 + Rs_2 + Rs_3$ is direct. Continuing in this manner and using the fact that R is Goldie we see that there exists homogeneous non-nilpotent elements $s_1, \ldots, s_n \in I$ such that the sum

$$\sum_{i=1}^{n} Rs_i$$

is direct and equation (3.1.1) is satisfied. Since R is prime,

$$w \;=\; Rs_1^2 Rs_2^2 \cdots Rs_n^2 \;\neq\; (0).$$

Thus we can find homogeneous elements $r_2, \ldots, r_n \in R$ such that $s_1^2 r_2 s_2^2 r_2 \cdots r_n s_n^2$ is nonzero. Moreover, since every nonzero graded left ideal has a non-nilpotent homogeneous element, there is a homogeneous element r_1 such that

$$r_1 s_1^2 r_2 s_2^2 r_3 \cdots r_n s_n^2$$

is not nilpotent. Let

$$a_i \;:=\; (s_i r_{i+1} \cdots r_n s_n^2)(r_1 s_1^2 r_2 \cdots s_{i-1}^2 r_i s_i).$$

Since a_i^k is a subword of w^{k+1} and w is not nilpotent, a_i is non-nilpotent. Moreover, $a_i \in Rs_i$. By the maximality condition imposed upon the left annihilator of s_i, we conclude that $\ell(a_i) = \ell(s_i)$ for $1 \leq i \leq n$. Moreover, $Ra_i \subseteq Rs_i$ and thus

$$Ra_1 + \cdots + Ra_n$$

is direct. Hence

$$\ell(a_1 + \cdots + a_n) \;=\; \bigcap_{i=1}^{n} \ell(a_i) \;=\; \bigcap_{i=1}^{n} \ell(s_i) \;=\; (0).$$

It follows that $a_1 + \cdots + a_n$ is homogeneous and left regular. It follows from Lemma 2.3.17 that it is regular.

Theorem 3.1.15 *Let R be a graded Goldie ring. Then the set $\mathcal{S}$ of regular homogeneous elements of R is left Ore.*

Proof. The argument is just as in Theorem 2.3.20. ∎

Theorem 3.1.16 *(Goodearl-Stafford [14]) Let R be a prime, graded-Goldie ring. Then the set $\mathcal{S}$ of regular homogeneous elements is non-empty and left Ore and $Q_{\mathrm{gr}}(R)$ is a prime left graded-Artinian ring.*

Proof. The proof is just as in Theorem 2.3.21. ∎

We have now seen that a prime graded Goldie ring R has a graded quotient ring,

$$Q_{\mathrm{gr}}(R) \;\cong\; M_m(\Delta),$$

for some graded matrix ring over a graded division ring. For the remainder of the chapter, we shall use this fact to analyze subfields in the quotient rings of graded domains of finite GK dimension.

3.2 Graded Goldie rings of low GK dimension

We now look at finitely graded Goldie algebras of GK dimension 1. These are affine graded algebras in which, for each $n \geq 0$, the vector space consisting of homogenous elements of degree n is finite dimensional. We have seen that such an algebra is PI, but we can in fact show much more about the structure of such a ring.

Proposition 3.2.1 *Let $R = \bigoplus_{n=0}^{\infty} R_n$ be a finitely graded prime Goldie algebra of GK dimension 1 over an algebraically closed field F. Then $Q_{\mathrm{gr}}(R)$ is isomorphic to a graded matrix ring over $F[t, t^{-1}]$.*

Proof. We need the following claim.

Claim: There exists a constant $C > 0$ such that $\dim R_n \leq C$ for all n.

Proof. Let r be a regular homogeneous element of R. Let d denote the degree of r and let V be a generating subspace for R. Recall that there exists a constant C_0 such that

$$\dim V^n - \dim V^{n-1} \ \leq \ C_0$$

for all n. Let $\overline{V^n}$ denote the image of V^n in R/Rr. We have $r \in V^m$ for some m. Hence for n large,

$$\dim \left(\overline{V^n}\right) \ \leq \ \dim \left(V^n / V^{n-m} r\right) \ \leq \ \dim V^n - \dim V^{n-m} \ \leq \ C_0 m.$$

Since $R = \bigcup_n V^n$, we see R/Rr has dimension at most $C_0 m$ over F. Thus $R_n = R_{n-d} r$ for all n sufficiently large. Hence $\dim R_n = \dim R_{n-d}$ for all n sufficiently large. The claim now follows. ∎

By the graded version of Goldie's theorem, $Q_{\mathrm{gr}}(R)$ is a graded matrix ring over a graded division ring $D[t, t^{-1}; \sigma]$. Let x and y be homogeneous elements of R of the same degree with y regular. Let $s_1 = y$. Notice that $s_1(y^{-1}x) \in R$. By the Ore condition, there exists a regular homogeneous element $r_2 \in R$ such that $r_2(xy^{-1}x) \in R$. We define $s_2 = r_2 s_1$. In general, we can find a homogeneous regular element r_n such that $r_n\left(s_{n-1}(y^{-1}x)^{n-1}\right)y^{-1}x \in R$. We define $s_n = r_n s_{n-1}$. By construction, $s_n(y^{-1}x)^j \in R$ for $0 \leq j \leq n$; moreover, since $y^{-1}x$ is homogeneous of degree 0,

$$F + Fy^{-1}x + \cdots + F(y^{-1}x)^n \ \subseteq \ s_n^{-1} R_{m_n},$$

where m_n is the degree of s_n. It follows that

$$\dim \left(F + Fy^{-1}x + \cdots\right) \ \leq \ C,$$

where

$$C \;=\; \sup_n \; \dim R_n.$$

Thus the ring of homogeneous elements of degree zero is algebraic over F. Since D is a subring of this algebra, we see that D is algebraic over F and hence $D = F$. Since σ fixes F, we see that $D[t, t^{-1}; \sigma] = F[t, t^{-1}]$. The result follows. ∎

We now look at the work of Artin and Stafford. We have seen that affine algebras of GK dimension less than or equal to 1 are PI. Recall from Proposition 2.2.7 that a prime algebra of GK dimension 2 need not be PI. Artin and Stafford have shown that for graded prime rings of GK dimension 2, the ring is either primitive or PI. With the existence of a faithful simple module and results such as Jacobson's density theorem, primitive rings are much easier to study than general prime rings. This dichotomy theorem of Artin and Stafford is therefore quite useful. In fact, up to a finite dimensional vector space, Artin and Stafford completely describe graded domains of GK dimension 2. Recall from Theorem 2.3.13 that a graded domain of GK dimension 2 is necessarily Goldie. It follows that we can invert the homogeneous elements to obtain a quotient of the form

$$D[z, z^{-1}; \sigma],$$

where D is a division algebra over F and σ is an automorphism of D. Artin and Stafford discovered the following remarkable result about this division ring D.

Theorem 3.2.2 *Let A be a finitely graded F-domain with $2 \leq \mathrm{GKdim}(A) \leq 11/5$ and*

$$Q_{\mathrm{gr}}(A) \;\cong\; D[z, z^{-1}; \sigma].$$

Then D is a finite module over its center Z and Z is a finitely generated extension of F of transcendence degree 1 and A must have quadratic growth; moreover σ has finite order if and only if A is PI.

Proof. See Theorem 0.1 and Corollary 1.3 of [4]. ∎

By Tsen's theorem we see that the division algebra D occurring in Theorem 3.2.2 must be a field when F is algebraically closed.

3.3 Krull dimension

We have discussed the relationship between the Krull dimension from classical algebraic geometry and GK dimension. The obvious generalization of Krull dimension would be to look at the lengths of ascending chains of prime ideals. There is, however, a more useful analogue of Krull dimension which we now describe. This analogue relies on the notion of a *derivation* of a poset.

Definition 3.3.1 *Given a non-trivial poset $(\mathcal{P}, \leq)$, we define the* derivation *of $\mathcal{P}$ to be 0 if $\mathcal{P}$ satisfies the descending chain condition. If $\mathcal{P}$ does not satisfy the descending chain condition, we define its derivation to be the ordinal α where α is the smallest ordinal satisfying:*

For any descending chain $p_1 \geq p_2 \geq p_3 \geq \cdots$ in $\mathcal{P}$, the poset

$$\mathcal{P}_i \ := \ \{p \in \mathcal{P} \mid p_i \leq p \leq p_{i+1}\}$$

has derivation strictly less than α for all but finitely many i.

Example 3.3.2 *The poset $(\mathbb{Z}, \leq)$ has derivation 1.*

Proof. Let

$$a_1 \ \geq \ a_2 \ \geq \ a_3 \ \geq \ \cdots$$

be a descending chain of integers. Then for each $i \geq 1$, the poset consisting of all integers between a_{i+1} and a_i satisfies the descending chain condition and therefore has derivation 0. Since $\mathbb{Z}$ does not satisfy the descending chain condition, we see that $(\mathbb{Z}, \leq)$ has derivation 1. ∎

Example 3.3.3 *The poset $([0, 1], \leq)$ does not have a derivation.*

Proof. Suppose that $([0,1],\leq)$ has derivation α for some ordinal α. Let

$$x_1 \;>\; x_2 \;>\; x_3 \;>\; \cdots$$

be a strictly descending chain in $[0,1]$. Observe that for each $i \geq 1$ the poset $([x_{i+1},x_i],\leq)$ is isomorphic to $([0,1],\leq)$ as a poset and hence must have derivation α. We conclude that $([0,1],\leq)$ must have derivation at least $\alpha+1$, a contradiction. ∎

Proposition 3.3.4 *Let $(\mathcal{P},\leq)$ be a poset. If $\mathcal{P}$ satisfies the ascending chain condition then $(\mathcal{P},\leq)$ has a derivation.*

Proof. Given $p \leq q$ in $\mathcal{P}$, let $\mathcal{P}_{p,q}$ denote the subposet of $\mathcal{P}$ consisting of all $x \in \mathcal{P}$ with $p \leq x \leq q$. We first claim that $\mathcal{P}_{p,q}$ has a derivation for all $p,q \in \mathcal{P}$. Suppose this is not the case. Then choose p maximal with respect to the property that $\mathcal{P}_{p,q}$ does not have a derivation for some $q \geq p$. Now let

$$a_1 \;>\; a_2 \;>\; a_3 \;>\; \cdots$$

be a descending chain in $\mathcal{P}_{p,q}$. Then $a_i = p$ for at most one value of i and hence $\mathcal{P}_{p,q}$ must have derivation at most 1 greater than the supremum of the derivations of $\mathcal{P}_{p',q'}$ in which $p < p' < q' \leq q$, a contradiction. It follows that $\mathcal{P}_{p,q}$ has a derivation for all $p,q \in \mathcal{P}$ with $p \leq q$. Using the same reasoning as before, we see that $\mathcal{P}$ has a derivation that is at most 1 greater than the supremum of the derivations of $\mathcal{P}_{p,q}$ in which $p < q$, $p,q \in \mathcal{P}$. ∎

Definition 3.3.5 *Let R be a left Noetherian ring and let M be a left R-module M. We define the* (left) Krull dimension *of M to be the derivation of the poset consisting of R-submodules of M under inclusion and denote it by $\mathcal{K}(M)$. We define the* Krull dimension *of R to be the Krull dimension of R considered as a left R-module and denote it by $\mathcal{K}(R)$.*

Theorem 3.3.6 *Let R be a commutative Noetherian ring. Then*

$$\mathcal{K}(R) = \mathrm{Kdim}(R).$$

Proof. See Corollary 6.4.8 of [22]. ∎

We now give some results which show how Krull dimension behaves under taking twisted polynomial extension of rings, taking tensor products of rings, and inverting an Ore set in a ring R. We first need a few remarks about Noetherian rings.

Remark 3.3.7 *Let R be a left Noetherian F-algebra. Then the following rings are left Noetherian:*

- *$R[x;\sigma]$, where $\sigma : R \to R$ is an automorphism;*
- *$R \otimes_F K$, where K is a finitely generated field extension of F;*
- *$Q_{\mathcal{S}}(R)$, where $\mathcal{S}$ is an Ore set in R consisting of regular elements.*

Proof. See Theorem 1.2.9, Proposition 6.6.16, and Proposition 2.1.16 (iii) of [22]. ∎

We first analyze the behavior of Krull dimension when an Ore set is inverted.

Proposition 3.3.8 *Let R be a left Noetherian ring and let $\mathcal{S}$ be an Ore set in R consisting of regular elements. Then $\mathcal{K}(Q_{\mathcal{S}}(R)) \leq \mathcal{K}(R)$.*

Proof. Let $I \subsetneq J$ be left ideals in $Q_{\mathcal{S}}(R)$. Then $I \cap R \subseteq J \cap R$ are left ideals in R. Let $a \in J \setminus I$. Then $a = s^{-1}r$ for some $s \in \mathcal{S}$ and some $r \in R$. Observe that $sa \in J \cap R$. If $sa \in I \cap R$, then $a = s^{-1}(sa) \in I$, contradicting our assumption. It follows that we have an embedding of the poset of left ideals of $Q_{\mathcal{S}}(R)$ into the poset of left ideals of R. The result follows easily. ∎

We make the following conjecture about the behavior of Krull dimension.

Conjecture 3.3.1 *Let R be a left Noetherian ring that is not primitive and suppose $\mathcal{S}$ is an Ore set consisting of regular elements such that $Q_{\mathcal{S}}(R)$ is primitive. Then*

$$\mathcal{K}\big(Q_{\mathcal{S}}(R)\big) \;<\; \mathcal{K}(R).$$

Proposition 3.3.9 *Let R and S be left Noetherian rings. Suppose that S is a finite left R-module. Then $\mathcal{K}(S) \leq \mathcal{K}(R)$.*

Proof. See Lemma 6.5.3 of [22].

Proposition 3.3.10 *Let $R \subseteq S$ be left Noetherian rings. Suppose S is free as a right R-module. Then $\mathcal{K}(S) \geq \mathcal{K}(R)$. If, in addition, S is a finite left R-module, then $\mathcal{K}(S) = \mathcal{K}(R)$.*

Proof. Let $I \subsetneq J$ be left ideals in R. Then $SI \subseteq SJ$ are left ideals in S. We claim that $SI \neq SJ$. Let $a \in J \setminus I$. Suppose $a \in SI$. Let $\{s_\alpha \mid \alpha \in \mathcal{I}\}$ be a basis for S as a right R-module. Observe that $S \;=\; \bigoplus_{\alpha \in \mathcal{I}} s_\alpha R$ and therefore $SI \;=\; \bigoplus_{\alpha \in \mathcal{I}} s_\alpha I$. Write $1 \;=\; \sum_{j=1}^m s_{\beta_j} r_j$, where $r_1, \ldots, r_m \in R$ and $\beta_1, \ldots, \beta_m \in \mathcal{I}$. By assumption $a \in SI$ and hence we can write

$$a \;=\; \sum_{j=1}^{m} s_{\beta_j} r_j a \;=\; \sum_{i=1}^{n} s_{\alpha_i} b_i,$$

with $b_1, \ldots, b_n \in I$ and $\alpha_1, \ldots, \alpha_n \in \mathcal{I}$. Since $\{s_\alpha\}$ is a basis for S as a right R-module and $a \notin I$, we get a contradiction. Thus we may embed the poset of left ideals of R into the poset of left ideals of S. It follows that $\mathcal{R} \leq \mathcal{S}$. In the case that S is a finite left R-module, we have that $\mathcal{K}(S) \leq \mathcal{K}(R)$ by the preceding proposition. ∎

We now look at how Krull dimension behaves under taking skew Laurent extensions of a left Noetherian ring R. We first give a definition.

Definition 3.3.11 *Let R be a ring and let M be a left R-module. We say that M has* finite length *if there exist submodules*

$$M \;=\; M_0 \;\supseteq\; M_1 \;\supseteq\; M_2 \;\supseteq\; \cdots \;\supseteq\; M_n \;=\; (0)$$

such that M_i/M_{i+1} is simple as a left R-module for $0 \leq i < n$. If M does not have finite length, we say that M has infinite length.

Theorem 3.3.12 *Let R be a left Noetherian ring that is not Artinian and let σ be an automorphism of R. Suppose that every nonzero simple left $R[x, x^{-1}; \sigma]$-module has infinite length as an R-module. Then*

$$\mathcal{K}(R[x, x^{-1}; \sigma]) \ = \ \mathcal{K}(R).$$

Proof. See Theorem 6.6.10 of [22]. ∎

Corollary 3.3.13 *Let R be a prime left Noetherian ring that is not Artinian and is not primitive. Suppose that σ is an automorphism of R such that R has no nonzero proper ideals fixed by σ. Then*

$$\mathcal{K}(R) \ = \ \mathcal{K}(R[x, x^{-1}; \sigma]).$$

Proof. It suffices to show that $R[x, x^{-1}; \sigma]$ has no simple left modules of finite length over R. Suppose M is a simple left $R[x, x^{-1}; \sigma]$-module with finite length as a left R-module. Then there exist submodules

$$M = M_0 \supsetneq M_1 \supsetneq M_2 \supsetneq \cdots \supsetneq M_n \ = \ (0)$$

with M_i/M_{i+1} a simple R-module for $0 \leq i < n$. Since R is not primitive, there exists a nonzero ideal P_i which annihilates M_i/M_{i+1} for $0 \leq i < n$. Observe that $(P_{n-1} \cdots P_0)M = (0)$. Since R is prime, $P_{n-1} \cdots P_0 \neq (0)$. Thus M, when considered as an R-module, has some nonzero annihilator ideal I. Then

$$\sigma(I)M \ = \ (x^{-1}Ix)M \ = \ x^{-1}I(xM) \ \subseteq \ x^{-1}IM \ = \ x^{-1}(0) \ = \ (0).$$

Hence $\sigma(I) \subseteq I$. Similarly, $\sigma^{-1}(I) \subseteq I$. Hence $\sigma(I) = I$. ∎

Theorem 3.3.14 *Let K and L be finitely generated field extensions of a field F. Then $K \otimes_F L$ has Krull dimension equal to the minimum of* $\mathrm{trdeg}_F(K)$ *and* $\mathrm{trdeg}_F(L)$.

Proof. Notice that $K \otimes_F L$ is a finite free module over $K' \otimes_F L'$ for some purely transcendental extensions $K' \subseteq K$ and $L' \subseteq L$. Thus by Proposition 3.3.10, we may assume that K and L are purely transcendental extensions of F. Let $K = F(x_1, \ldots, x_n)$ and $L = F(y_1, \ldots, y_m)$. Without loss of generality $n < m$. Observe that $K \otimes_F L$ is obtained by inverting an Ore set in

$$F[x_1, \ldots, x_n] \otimes_F F(y_1, \ldots, y_m) \;\cong\; F(y_1, \ldots, y_m)[x_1, \ldots, x_n]$$

and hence has Krull dimension at most the Krull dimension of

$$S := F(y_1, \ldots, y_m)[x_1, \ldots, x_n].$$

Since this is a Noetherian commutative ring, we have that the Krull dimension of this ring is the same as $\mathrm{Kdim}(S)$ which is equal to n. Thus $K \otimes_F L$ has Krull dimension at most n. Observe that $P_i = (x_1 - y_1, \ldots, x_i - y_i)$ is a prime ideal of $K \otimes_F L$ for $1 \leq i \leq n$. Thus $\mathrm{Kdim}(F \otimes_F L) \geq n$. It follows that $\mathcal{K}(K \otimes_F L) \geq n$. We therefore obtain the desired result. ∎

Let A be a finitely graded Goldie algebra of GK dimension 2 over a field F. We can form the quotient ring $Q(A)$. Our main theorem of this section is the following.

Theorem 3.3.15 *Let A be a non-PI finitely graded domain of GK dimension 2 over a field F. Let K be a maximal subfield of $Q(A)$. Then K has transcendence degree at most 1 over F.*

We require a few simple lemmas.

Lemma 3.3.16 *Let D be a division algebra which is a finite module over its center Z. Suppose that Z is a finitely generated field extension of transcendence degree at least 1 over some field F. Let K be a finitely generated purely transcendental* *extension of F of transcendence degree at least 1. Then $M_n(D)\otimes_F K$ is prime and not primitive.*

Proof. Let $K = F(x_1,\ldots,x_n)$. Then $M_n(D)\otimes_F K$ can be obtained by inverting an Ore set of regular elements in the ring

$$M_n(D)\otimes_F F[x_1,\ldots,x_n] \;\cong\; M_n(D[x_1,\ldots,x_n]).$$

It follows that $M_n(D)\otimes_F K$ is prime as it is of the form $Q_{\mathcal{S}}(S)$ for some prime ring S and some Ore set of regular elements $\mathcal{S}$. Note that $M_n(D)\otimes_F K$ is PI by Corollary 2.2.6. Let $z\in Z$ be transcendental over F. Notice that $z\mathbf{I_n}-x_1$ generates a proper nonzero ideal in $M_n(D)\otimes_F K$. Kaplansky's theorem states that a primitive PI ring is simple. Using this result, we obtain the desired result. ∎

Lemma 3.3.17 *Let A be a finitely graded non-PI domain of GK dimension 2 over a field F. Write*

$$Q_{\mathrm{gr}}(A) \;=\; D[x,x^{-1};\sigma].$$

if $\alpha\in M_n(D)$ is central, then α is algebraic over F.

Proof. Suppose $\alpha = s^{-1}r\in D$ is fixed by σ and is not algebraic. Then D is a finite module over $F(\alpha)$ by Theorem 3.2.2. It follows that $D[x,x^{-1};\sigma]$ is a finite module over $F(\alpha)[x,x^{-1}]$. Thus $F(\alpha)[x,x^{-1}]$ and $D[x,x^{-1};\sigma]$ have the same GK dimension by Proposition 1.1.12. It follows from Theorem 2.4.1 that $Q(A) = Q(D[x,x^{-1};\sigma])$ is a finite dimensional vector space over $Q(F(\alpha)[x,x^{-1}]) = F(\alpha,x)$. Hence $Q(A)$ is isomorphic to a subring of a matrix ring over $F(\alpha,x)$. Thus $Q(A)$ is PI, a contradiction. ∎

Lemma 3.3.18 *Let A be a finitely graded non-PI domain of GK dimension 2 over a field F. Let*

$$L \;:=\; Z\big(Q_{\mathrm{gr}}(A)\big)\cap D$$

and let K be a finitely generated extension of L. Then $D\otimes_L K$ has no non-trivial ideals fixed by σ.

Proof. Suppose $0 \neq I$ is an ideal fixed by σ. Choose

$$a \;=\; \sum_{i=1}^{n} \alpha_i \otimes \beta_i \in I$$

with $\alpha_i \in D$ and $\beta_i \in K$ for $1 \leq i \leq n$ and choose a such that n is minimal. Since D is a division algebra, we may assume that $\alpha_1 = 1$ and that $\beta_1, \ldots, \beta_n$ are linearly independent over L. Observe that

$$(d \otimes 1)a - a(d \otimes 1) \;=\; \sum_{i=2}^{n} (d\alpha_i - \alpha_i d) \otimes \beta_i \;\in\; I$$

By the minimality of n, we see that α_i is in the center of D for $1 \leq i \leq n$. Further,

$$a - \sigma(a) \;=\; \sum_{i=2}^{n} \big(\alpha_i - \sigma(\alpha_i)\big) \otimes \beta_i \;\in\; I.$$

Thus α_i is in L for $1 \leq i \leq n$. It follows that

$$a \;= 1 \otimes \Big(\sum_{i=1}^{n} \alpha_i \beta_i \Big).$$

Thus a is a unit and so $I = Q_{\mathrm{gr}}(A)$. The result follows. ∎

Theorem 3.3.19 *Let A be a finitely graded non-PI domain of GK dimension 2 over a field F. Let K be a subfield of A. Then K has transcendence degree at most 1 over F.*

Proof. By Theorem 3.2.2, we can write

$$Q_{\mathrm{gr}}(A) \;=\; D[x, x^{-1}; \sigma],$$

where D is a finite module over its center and its center is a finitely generated extension of transcendence degree 1 over F. Let K be a maximal subfield of $Q(A)$. By Lemma 3.3.17,

$$Z\big(D[x, x^{-1}; \sigma]\big) \cap M_n(D)$$

is an algebraic extension of F, call it L. Then $K \supseteq L$. It suffices to prove that K has transcendence degree 1 over L. Suppose this is not the case. Then there exists a purely transcendental extension E of L such that $E \subseteq K$ and E has transcendence degree 2 over L. We have $E \otimes_L D[x, x^{-1}; \sigma]$ is prime and primitive by Lemma 3.3.16. Further, $E \otimes_L D$ has no nonzero proper ideals fixed by σ by Lemma 3.3.18. Thus $E \otimes_L Q_{\mathrm{gr}}(A)$ has the same Krull dimension as $E \otimes_L D$ by Corollary 3.3.13. Now $Q(A) \otimes_L E$ is a localization of $Q_{\mathrm{gr}}(A) \otimes_L E$ and hence we have the inequalities

$$\mathcal{K}(Q(A) \otimes_L E) \ \leq \ \mathcal{K}(Q_{\mathrm{gr}} \otimes_L E) \ \leq \ \mathcal{K}(D \otimes_L E). \tag{3.3.2}$$

Let Z denote the center of D. Observe that $\mathcal{K}(Z \otimes_L E) = \mathcal{K}(D \otimes_L E)$ by Proposition 3.3.10. Finally, note that $Z \otimes_L E$ is a subring of $Q(A) \otimes_L E$ and since $Q(A) \otimes_L E$ is free as a $Z \otimes_L E$-module, we see that the Krull dimension of $Z \otimes_L E$ is at most the Krull dimension of $Q(A) \otimes_L E$ by Proposition 3.3.10. Putting these inequalities together, we see

$$\mathcal{K}(E \otimes_L E) \ \leq \ \mathcal{K}(Q(A) \otimes_L E) \ \leq \ \mathcal{K}(Q_{\mathrm{gr}}(A) \otimes_L E) \ = \ \mathcal{K}(Z \otimes_L E). \tag{3.3.3}$$

Using Theorem 3.3.14, we see that

$$\mathrm{trdeg}_L(E) \ \leq \ \min\big(\mathrm{trdeg}_L(Z), \mathrm{trdeg}_L(E)\big).$$

Since Z has transcendence degree 1 over L, we have that the transcendence degree of E over L is at most 1, a contradiction. Thus K has transcendence degree at most 1 over L. Since L is an algebraic extension of F, we see that K has transcendence degree at most 1 over F. ∎

Chapter 4

Examples in finite GK dimension

4.1 Affinization theorems

We now give some examples of affine rings with low GK dimension that are poorly behaved. In 1981 Beĭdar [6] gave a construction of an affine, prime algebra with non-nil Jacobson radical, answering an old question of Amitsur. Beĭdar's construction was subsequently modified by Small. The key idea was to show that a countably-generated algebra that is not necessarily affine could appear as the corner of an affine algebra. We briefly describe Small's construction and then show how it can be modified to construct algebras of finite GK dimension.

Let C be a commutative ring. Given a prime, countably generated C-algebra T, we construct the *affinization* of R as follows. Let $R = C\{x, y\}$ and let

$$S \;=\; \begin{pmatrix} C + Ry & R \\ Ry & R \end{pmatrix}.$$

S is generated as a C-algebra by

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix}, \text{ and } \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}, \text{ where } a \in \{1, x, y\}. \quad (4.1.1)$$

Hence S is an affine C-algebra. $C + Ry$ is a free C-algebra on the infinitely many generators $\{x^i y | i \geq 0\}$. It follows that we have a surjective ring homomorphism

$$\Phi : C + Ry \ \to \ T. \tag{4.1.2}$$

Let

$$P \ = \ \ker(\Phi) \tag{4.1.3}$$

and let $e_{i,j}$ denote the matrix with a 1 in the (i,j) entry and zeros everywhere else. Notice P is a prime ideal. Observe that $Q' := S(e_{1,1}Pe_{1,1})S$ satisfies $e_{1,1}Q'e_{1,1} = Pe_{1,1}$. Using Zorn's lemma we can choose an ideal Q in S maximal with respect to the property that

$$e_{1,1}Qe_{1,1} \ = \ \begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix}. \tag{4.1.4}$$

By maximality, we have that Q is prime, since P is a prime ideal. We note that

$$Q \ \supseteq \ \begin{pmatrix} 0 & 0 \\ Ry & 0 \end{pmatrix} \begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & R \\ 0 & 0 \end{pmatrix} \ = \ \begin{pmatrix} 0 & 0 \\ 0 & RyPR \end{pmatrix}. \tag{4.1.5}$$

Similarly,

$$Q \ \supseteq \ PRe_{1,2}, \tag{4.1.6}$$

and

$$Q \ \supseteq \ RyPe_{2,1}. \tag{4.1.7}$$

Given an element $x \in S$, we denote by $\overline{x}$ the image of x in S/Q. The algebra S/Q has the property that

$$\overline{e_{1,1}}(S/Q)\overline{e_{1,1}} \ \cong \ T. \tag{4.1.8}$$

We call S/Q the *affinization of T with respect to* Φ and denote it by $\mathcal{A}(T, C; \Phi)$. We now state the main theorem of this chapter.

Theorem 4.1.1 *Let C be a commutative, affine F-domain and T a prime, countably generated prime C-algebra of GK dimension $\alpha < \infty$. Then there exists a homomorphism $\Phi : C + C\{x, y\}y \to T$ such that $\mathcal{A}(T, C; \Phi)$ has GK dimension between α and $\alpha + 2 + \mathrm{Kdim}(C)$. If $C = F$ then $\mathcal{A}(T, F; \Phi)$ has GK dimension precisely equal to $\alpha + 2$.*

Using this theorem we are able to give the following examples.

- An affine prime algebra of GK dimension 2 that is neither primitive nor PI.
- An affine algebra of GK dimension 3 whose Jacobson radical is not nil.
- A primitive affine algebra of GK dimension ≤ 4 that has center not equal to a field.

We first prove a lemma that will be necessary to obtain the upper bounds for our GK dimension estimates.

Lemma 4.1.2 *Let T be a countably generated C-algebra and let Φ, P, and Q be as in equations (4.1.2), (4.1.3), and (4.1.4). Suppose $\Phi : C + C\{x,y\}y \to T$ has the property that $\Phi\big(x^i y\big) = 0$ if $i \notin \mathcal{M}$ for some set $\mathcal{M} = \{m_1, m_2, \ldots\} \subseteq \mathbb{N}$ with the property that $m_{i+1} \geq 3m_i$; also, suppose that for any m, the set*

$$\big\{\Phi\big(x^i y\big) \;\big|\; i \geq m\big\}$$

spans T as a C-module. Then

$$e_{2,2} Q e_{2,2} \;=\; RyPRe_{2,2},$$

where $R = C\{x,y\}$.

Proof. Suppose that

$$u e_{2,2} \in e_{2,2} Q e_{2,2}, \qquad u \in R. \tag{4.1.9}$$

We have

$$u \;=\; p(x) + \sum_{i,j} x^i w_{i,j} x^j,$$

where $p(x) \in C[x]$ is a polynomial of degree, say d, and $w_{i,j} \in yR \cap Ry = yRy + Cy$ and at most finitely many of the $w_{i,j}$ are nonzero. Notice that

$$(x^k e_{1,2})(u e_{2,2})(x^\ell y e_{2,1}) \;=\; x^k u x^\ell y e_{1,1} \in e_{1,1} Q e_{1,1} \;=\; P e_{1,1}. \tag{4.1.10}$$

Hence

$$\Phi\big(x^{k+\ell}p(x)y\big) + \sum_{i,j} \Phi\big(x^{k+i}w_{i,j}x^{j+\ell}y\big) \ = \ 0.$$

Since $m_{i+1} - m_i \to \infty$ and $w_{i,j}$ is zero for all but finitely many pairs (i, j), we have that for (i_0, j_0) there exists some index N such that for any $\ell_1, \ell_2 \geq N$ we have

$$\Phi\big(x^{m_{\ell_1}-i_0+i}y\big) \ = \ 0 \qquad \text{for } i \neq i_0, \text{ and} \tag{4.1.11}$$

$$\Phi\big(x^{m_{\ell_2}-j_0+j}y\big) \ = \ 0 \qquad \text{for } j \neq j_0. \tag{4.1.12}$$

Using these equations and the fact that $w_{i,j} \in yR \cap Ry$, we see that for any $\ell_1, \ell_2 \geq N$, $(x^{m_{\ell_1}-i_0+i})w_{i,j}(x^{m_{\ell_2}-j_0+j}y) \in P$ whenever $(i, j) \neq (i_0, j_0)$. Hence

$$x^{m_{\ell_1}+m_{\ell_2}-i_0-j_0}p(x)y + x^{m_{\ell_1}}w_{i_0,j_0}x^{m_{\ell_2}}y \in P,$$

whenever $\ell_1, \ell_2 \geq N$. It follows that if ℓ_1 and ℓ_2 are sufficiently large, then

$$\begin{aligned} m_{\max(\ell_1,\ell_2)} &< m_{\ell_1} + m_{\ell_2} - i_0 - j_0 \\ &\leq m_{\ell_1} + m_{\ell_2} - i_0 - j_0 + d \\ &\leq 2m_{\max(\ell_1,\ell_2)} + d \\ &< 3m_{\max(\ell_1,\ell_2)} \leq m_{\max(\ell_1,\ell_2)+1}. \end{aligned}$$

Hence

$$\Phi\big(x^{m_{\ell_1}+m_{\ell_2}-i_0-j_0}p(x)y\big) \ = \ 0$$

for all ℓ_1, ℓ_2 sufficiently large. Thus we have that

$$x^{m_{\ell_1}}w_{i_0,j_0}x^{m_{\ell_2}}y \in P$$

for all ℓ_1, ℓ_2 sufficiently large. Since $w_{i_0,j_0} \in yR \cap Ry$ we can write $w_{i_0,j_0} = yv$ with $v \in C + Ry$. Then there exists a positive integer N' such that

$$0 = \Phi\big(x^{m_{\ell_1}}w_{i_0,j_0}x^{m_{\ell_2}}y\big) \ = \ \Phi\big(x^{m_{\ell_1}}y\big)\Phi\big(v\big)\Phi\big(x^{m_{\ell_2}}y\big)$$

for all $\ell_1, \ell_2 > N'$. Since

$$\big\{\Phi\big(x^{m_{\ell_1}}y\big) \mid \ell \geq N'\big\}$$

spans T as a C-module, letting ℓ_1 and ℓ_2 range independently over all natural numbers greater than N', we find that $T\Phi(v)T = 0$ and hence $v \in P$. It follows that $x^{i_0}w_{i_0,j_0}x^{j_0} \in RyPR$ for all i_0, j_0. Hence $u \equiv p(x) \mod RyPR$. It follows from equation (4.1.5) that $p(x)e_{2,2} \in Q$, and so from equation (4.1.10), with $p(x)$ replacing u, we have $p(x)x^iy \in P$ for all i. We claim that this implies that $p(x) = 0$. Suppose that $p(x) \neq 0$ and write $p(x) = p_0 + \cdots + p_dx^d$ with $p_d \neq 0$. Using the fact that $m_{i+1} - m_i \to \infty$, we see that for j sufficiently large $0 = \Phi(p(x)x^{m_j-d}y) = p_d\Phi(x^{m_j}y)$. Since the set $\{\Phi(x^{m_j}\}$, with j running over all positive integers greater than some given number, spans T as a C-module, we see that $0 = p_d$, a contradiction. Hence $p(x) = 0$ and so $u \in RyPR$. Thus $e_{2,2}Qe_{2,2} \subseteq RyPRe_{2,2}$. From equation (4.1.5) we have that $e_{2,2}Qe_{2,2} = RyPRe_{2,2}$. ∎

Theorem 4.1.3 *Let T be a prime, countably generated F-algebra of GK dimension $\alpha < \infty$ and given Φ, let P, and Q be as in equations (4.1.3), and (4.1.4). Then there exists a homomorphism $\Phi : F + F\{x, y\}y \to T$ such that $\mathcal{A}(T, F; \Phi)$ has GK dimension $\alpha + 2$.*

Proof. Let $R = F\{x, y\}$, let $V \subseteq S$ be the vector space spanned by the generating set given in item (4.1.1), and let

$$W \;=\; F + Fx + Fy \;\subseteq\; R. \tag{4.1.13}$$

One has

$$V^n \;\subseteq\; \begin{pmatrix} F + W^ny & W^n \\ W^ny & W^n \end{pmatrix}.$$

We shall construct a homomorphism that will give an affinization of finite GK dimension. Let $\mathcal{B} = \{1, u_1, u_2 \ldots\} \subseteq T$ have the property that $\{u_i \mid i \geq m\}$ spans T as a F-vector space for any m. For example, if $\{v_1, v_2, \ldots\}$ is a basis for T as a F-vector space, we can take

$$\mathcal{B} \;=\; \{1, v_1, v_1, v_2, v_1, v_2, v_3, \ldots, v_1, v_2, \ldots, v_n, v_1, v_2, \ldots, v_{n+1}, \ldots\}.$$

Define U_j to be the vector space spanned by the first $j+1$ elements of $\mathcal{B}$; that is,

$$U_j \;=\; F + Fu_1 + \cdots + Fu_j. \tag{4.1.14}$$

Let $\varepsilon > 0$. Since T has GK dimension α, we have

$$\limsup_{n\to\infty} \;\log\bigl(\dim\,(U_j)^n\bigr)\Big/\log n \;\leq\; \alpha.$$

Hence there exists a positive integer m_j such that

$$\dim\,(U_j)^n \;<\; n^{\alpha+\varepsilon} \qquad \text{for all } n \geq m_j.$$

By increasing m_j if necessary, we may assume that $m_j \geq 3m_{j-1}$ for all $j \geq 2$. We define

$$\Phi : F + F\{x,y\}y \;\to\; T$$

by

$$\Phi\bigl(x^i y\bigr) \;=\; \begin{cases} u_j & \text{if } i = m_j \text{ for some } j \geq 0, \\ 0 & \text{if } i \neq m_j \text{ for all } j \geq 0. \end{cases}$$

Consider

$$\dim \begin{pmatrix} F + W^n y & 0 \\ 0 & 0 \end{pmatrix}.$$

Suppose $m_j \;\leq\; n \;<\; m_{j+1}$. Then since $Pe_{1,1} \;=\; e_{1,1}Qe_{1,1}$, a word in $W^n y\overline{e_{1,1}}$ is determined by its behavior modulo P. Since $n < m_{j+1}$, we have

$$\Phi\bigl(W^n y\bigr) \;\subseteq\; (U_j)^n.$$

Hence

$$\dim W^n y\overline{e_{1,1}} \;\leq\; \dim\,(U_j)^n \;\leq\; n^{\alpha+\varepsilon}.$$

We now compute the dimension of $W^n e_{2,2}$. Notice anything in $\overline{e_{2,2}}A(T,F;\Phi)\overline{e_{2,2}}$ can be expressed as a linear combination of powers of x, elements of the form $x^i y x^j$, and elements of the form $x^i y w y x^j$, where w is a word in x and y. Hence anything in W^n is contained in the span of

$$\{1, x, x^2, \ldots, x^n\} \cup \{x^i y x^j \mid 0 \leq i,j \leq n\} \cup \{x^i y W^n y x^j \mid 0 \leq i,j \leq n\}.$$

The dimensions of $\mathrm{Span}\{1, x, x^2, \ldots, x^n\}$ and $\mathrm{Span}\{x^i y x^j \mid 0 \leq i, j \leq n\}$ are bounded above by $(n+1)$ and $(n+1)^2$ respectively. Now $RyPRe_{2,2} \subseteq Q$ and hence the image in $\mathcal{A}(T, F; \Phi)$ of an element of the form $x^i ywy x^j$ is completely determined by the behavior of wy mod P. As $\Phi\big(W^n y\big) \subseteq (U_j)^n$, we have

$$\dim\, x^i y W^n y x^j \overline{e_{2,2}} \;\leq\; \dim\, (U_j)^n \;\leq\; n^{\alpha+\varepsilon}.$$

Since i, j can assume any value between 0 and n, we have for $n \geq m_j$,

$$\dim\, W^n \overline{e_{1,2}} \;\leq\; (n+1) + (n+1)^2 + (n+1)^2 n^{\alpha+\varepsilon} \;=\; \mathrm{O}(n^{2+\alpha+\varepsilon}).$$

Let A denote the "diagonal" of $\mathcal{A}(T, F; \Phi)$ and let B denote the "upper-triangular part" of $\mathcal{A}(T, F; \Phi)$. We have just shown that

$$A \;\cong\; \big((F + Ry)/P\big) \oplus \big(R/RyPR\big)$$

has GK dimension at most $\alpha + 2 + \varepsilon$. Observe that $B = A + \overline{e_{1,2}} A$ and hence B has GK dimension at most $\alpha + 2 + \varepsilon$ by Proposition 1.1.12. Finally, note that

$$\mathcal{A}(T, F; \Phi) \;=\; B + B\big(y\overline{e_{2,1}}\big)$$

and thus

$$\mathcal{A}(T, F; \Phi)$$

has GK dimension at most $\alpha + 2 + \varepsilon$, again using Proposition 1.1.12. Since $\varepsilon > 0$ is arbitrary, we conclude that $\mathcal{A}(T, F; \Phi)$ has GK dimension at most $\alpha + 2$. On the other hand, since T has GK dimension α and $\mathcal{B}$ spans T, there exists j such that

$$\limsup_{n \to \infty} \big(\dim\, (U_j)^n\big)/n^{\alpha-\varepsilon} = \infty.$$

We have $V^{m_j n + 3n + 2} \supseteq W^{m_j n + 3n + 2} e_{2,2}$. Note

$$\begin{aligned} & W^{m_j n + 3n + 2} \\ \supseteq\;\; & \{x^k ywy x^\ell \mid 0 \leq k, \ell \leq n \text{ and } w \text{ is a word of length } \leq m_j n + n\}. \end{aligned}$$

By Lemma 4.1.2 $e_{2,2}Qe_{2,2} = RyPR$. Hence the dimension of

$$\mathrm{Span}\{x^k y W^{m_j n+n} y x^\ell \mid 0 \le k, \ell \le n\}$$

is just $(n+1)^2$ times the dimension of $W^{m_j n+n}y$ modulo the ideal P. Since $\Phi\big(W^{m_j}y\big) \supseteq U_j$ and $W^{m_j n+n}y \supseteq \big(W^{m_j}y\big)^n$, we have

$$\Phi\big(W^{m_j n+n}y\big) \;\supseteq\; \Phi\big(W^{m_j}y\big)^n \;\supseteq\; (U_j)^n.$$

Hence the dimension of $W^{m_j n+n}y$ mod P is at least dim $(U_j)^n$. But

$$\limsup_{n\to\infty} \; \big(\dim\, (U_j)^n\big)/n^{\alpha-\varepsilon} \;=\; \infty,$$

and so

$$\limsup_{n\to\infty} \; \big(\dim\, \overline{V^{m_j n+3n+2}}\big)/\big(m_j n + 3n + 2\big)^{\alpha-\varepsilon} \;=\; \infty.$$

Hence the GK dimension of $\mathcal{A}(T, F; \Phi) \ge 2 + \alpha - \varepsilon$. It follows that $\mathcal{A}(T, F; \Phi)$ has GK dimension precisely equal to $2 + \alpha$. ∎

Corollary 4.1.4 *Suppose T is a countably generated prime C-algebra, where C is an affine F-algebra. Then there exists $\Phi : C + C\{x, y\}y \to T$ such that*

$$2 + \alpha \;\le\; \mathrm{GKdim}\big(\mathcal{A}(T, C; \Phi)\big) \;\le\; 2 + \alpha + \mathrm{Kdim}(C).$$

Proof. Let $\{1, c_1, \ldots, c_m\}$ be a generating set for C as an F-algebra. By Theorem 4.1.1 there exists $\Phi : F + F\{x, y\}y \to T$ such that $\mathcal{A}(T, F; \Phi)$ has GK dimension $\alpha + 2$. We can extend Φ to a homomorphism from $C + C\{x, y\}$ onto T by declaring that $\Phi(c) = c$ for all $c \in C$. Since $\overline{e_{1,1}}\mathcal{A}(T, C; \Phi)\overline{e_{1,1}} \cong T$, we deduce that

$$\mathrm{GKdim}\big(\mathcal{A}(T, C; \Phi)\big) \;\ge\; \mathrm{GKdim}(T) \;=\; \alpha.$$

If V is the vector space spanned by the generating set given in item (4.1.1), then

$$\limsup_{n\to\infty} \log\big(\dim\, \overline{V^n}\big)/\log n \;=\; \alpha + 2. \tag{4.1.15}$$

Let $\mathbf{I_2} \in \mathcal{A}(T, C; \Phi)$ denote the identity matrix. We have

$$Y \ := \ \sum_{i=1}^{m} Fc_i\mathbf{I_2} + V$$

generates $\mathcal{A}(T, C, \Phi)$ as an F-algebra. Notice

$$V^n \ \subseteq \ Y^n \ \subseteq \ \Big(\sum_{i=1}^{m} Fc_i\mathbf{I_2}\Big)^n V^n. \tag{4.1.16}$$

Equation (4.1.15) tells us that the GK dimension of $\mathcal{A}(T, C, \Phi)$ is at least $\alpha + 2$. Also,

$$\begin{aligned} &\dim \Big(\sum_{i=1}^{m} Fc_i\mathbf{I_2}\Big)^n \ V^n & (4.1.17) \\ \leq \ &\left(\dim\Big(\sum_{i=1}^{m} Fc_i\mathbf{I_2}\Big)^n\right)\Big(\dim \ V^n\Big). \end{aligned}$$

Since the GK dimension of C is the same as the Krull dimension of C, we have that for any $\varepsilon > 0$

$$\dim \Big(\sum_{i=1}^{m} Fc_i\mathbf{I_2}\Big)^n \ \leq \ n^{\mathrm{Kdim}(C)+\varepsilon}$$

for all n sufficiently large. Combining this fact with equations (4.1.15), (4.1.16), and (4.1.17) we see that $\mathcal{A}(T, C; \Phi)$ has GK dimension at most $\alpha + 2 + K\mathrm{dim}\ C$. ∎

We now compute the center of $\mathcal{A}(T, C; \Phi)$.

Proposition 4.1.5 *Let Φ be as in Corollary 4.1.4. Then the center of $\mathcal{A}(T, C; \Phi)$ is $\{c\mathbf{I_2} \mid c \in C\}$.*

Proof. Let $R = C\{x, y\}$ and let

$$\mathbf{z} \ = \ \begin{pmatrix} z_{1,1} & z_{1,2} \\ z_{2,1} & z_{2,2} \end{pmatrix}$$

be a central element of $\mathcal{A}(T, C; \Phi)$. Then

$$\begin{pmatrix} 0 & z_{1,1} \\ 0 & z_{2,1} \end{pmatrix} \;=\; \mathbf{z}\overline{e_{1,2}} \;=\; \overline{e_{1,2}}\mathbf{z} \;=\; \begin{pmatrix} z_{2,1} & z_{2,2} \\ 0 & 0 \end{pmatrix}.$$

Hence

$$z_{2,1}\overline{e_{2,2}} \;=\; \overline{e_{2,2}}(\mathbf{z}\overline{e_{1,2}})\overline{e_{2,2}} \;=\; \overline{e_{2,2}}(\overline{e_{1,2}}\mathbf{z})\overline{e_{2,2}} \;=\; 0.$$

It follows that $z_{2,1}\overline{e_{1,2}} = \overline{e_{1,2}}(z_{2,1}\overline{e_{2,2}}) = 0$. Thus

$$\mathbf{z} \;=\; \begin{pmatrix} z_{1,1} & z_{1,2} \\ 0 & z_{2,2} \end{pmatrix}.$$

We also have

$$z_{1,2}\overline{e_{1,2}} + z_{2,2}\overline{e_{2,2}} \;=\; \mathbf{z}\overline{e_{2,2}} \;=\; \overline{e_{2,2}}\mathbf{z} \;=\; z_{2,2}\overline{e_{2,2}}.$$

Thus $z_{1,2}\overline{e_{1,2}} = 0$ and so

$$\mathbf{z} \;=\; \begin{pmatrix} z_{1,1} & 0 \\ 0 & z_{2,2} \end{pmatrix}.$$

Write

$$z_{2,2} \;=\; p(x) + \sum_{i,j} x^i w_{i,j} x^j \qquad \bmod RyPR,$$

with $p(x) \in C[x]$ and $w_{i,j} \in Ry \cap yR$ and only finitely many of the $w_{i,j}$ nonzero. We have

$$z_{2,2}x^m\overline{e_{2,2}} \;=\; \mathbf{z}(x^m\overline{e_{2,2}}) \;=\; (x^m\overline{e_{2,2}})\mathbf{z} \;=\; x^m z_{2,2}\overline{e_{2,2}}.$$

By Lemma 4.1.2, $z_{2,2}x^m - x^m z_{2,2} \in RyPR$ and so

$$z_{2,2}x^m x^n y - x^m z_{2,2} x^n y \in P$$

for all $m, n \geq 0$. By assumption, there exists a set $\mathcal{M} = \{m_1, m_2, \ldots\}$ with $m_{i+1} - m_i \to \infty$ and $\Phi\big(x^i y\big) = 0$ if $i \notin \mathcal{M}$. Just as in the proof of Lemma 4.1.2, we can take $n = m_{\ell_1} - i_0$ and $n' = m_{\ell_2} - j_0$; for ℓ_1, ℓ_2 sufficiently large, we have

$$z_{2,2}x^{n'}x^n y - x^{n'}z_{2,2}x^n y \;\equiv\; x^{m_{\ell_2}} w_{i_0,j_0} x^{m_{\ell_1}} y \;\; \bmod \;\; P.$$

Since $w_{i_0,j_0} \in Ry \cap yR$, there exists $v \in C + Ry$ such that $w_{i_0,j_0} = yv$. It follows that there exists a positive integer N' such that

$$0 \;=\; \Phi\big(x^{m_{\ell_2}}y\big)\Phi\big(v\big)\Phi\big(x^{m_{\ell_1}}y\big)$$

for all $\ell_1, \ell_2 > N'$. By letting ℓ_1 and ℓ_2 range independently over all natural numbers greater than N' we see that $R\Phi\big(v\big)R = 0$. Hence $v \in P$. It follows that $x^{i_0}w_{i_0,j_0}x^{j_0} \in RyPR$. Thus $z_{2,2} \equiv p(x)$ mod $RyPR$. We may therefore assume that $z_{2,2} = p(x)$ by item (4.1.5). Write $p(x) = p_0 + \cdots + p_d x^d$. We have

$$x^n yp(x)x^{n'}y - p(x)x^n yx^{n'}y \in P \qquad \text{for all } n, n' \geq 0. \tag{4.1.18}$$

Fix $i > 0$. We take $n = m_{\ell_1} - i$ and $n' = m_{\ell_2}$. Since $m_{i+1} - m_i \to \infty$, we have

$$\Phi\big(x^{m_\ell - j}y\big) \;=\; \Phi\big(x^{m_\ell - i + j}y\big) \;=\; 0$$

for $0 \leq j \leq d$, $j \neq i$, for all ℓ sufficiently large. Hence

$$\Phi\big(x^{m_{\ell_1}-i}yp(x)x^{m_{\ell_2}}y - p(x)x^{m_{\ell_1}-i}yx^{m_{\ell_2}}y\big) \;=\; -p_i\Phi\big(x^{m_{\ell_1}}y\big)\Phi\big(x^{m_{\ell_2}}y\big).$$

By allowing ℓ_1 and ℓ_2 to range independently over all sufficiently large numbers, we see that $p_i = 0$ for all $i > 0$. Hence $z_{2,2} = c \in C$. Now $c\mathbf{I_2}$ is central and hence

$$\mathbf{z} - c\mathbf{I_2} \;=\; (z_{1,1} - c)e_{1,1}$$

is central. But the nonzero elements of a prime ring are regular and $(z_{1,1} - c)e_{1,1}$ is annihilated by $e_{2,2}$. We conclude that $z_{1,1} = c$ and so $\mathbf{z} = c\mathbf{I_2}$. ∎

An important fact about the affinization of a C-algebra T is that T is primitive if and only if $\mathcal{A}(T, C; \Phi)$ is primitive. In fact, $T \cong e_{1,1}\mathcal{A}(T, C; \Phi)e_{1,1}$, and so this fact is just a consequence of the following proposition.

Proposition 4.1.6 *(Lanski, Resco, Small [20]) Let R be a prime ring with a nonzero idempotent e. Then R is primitive if and only if eRe is primitive.*

Proof. Suppose R is primitive. Let M be a faithful simple R-module. Notice that eM is an eRe-module. Note that if $ere \in eRe$ annihilates eM, then $0 = (ere)eM = (ere)M$ and so $ere = 0$. Hence eM is a faithful eRe-module. Suppose $ev \in eM$ is nonzero. Then since $ev \in M$ and M is simple as an R-module, we have $Rev = M$. Thus $(eRe)ev = eM$ and so we see that eM is a simple eRe-module. Hence eRe is primitive.

Suppose eRe is primitive. Then there is a maximal left ideal $\mathcal{M} \subseteq eRe$ that does not contain a nonzero two-sided eRe-ideal. Notice $I := R\mathcal{M} + R(1-e)$ has the property that $eIe = \mathcal{M}$. By Zorn's lemma we can find a left R-ideal $\mathcal{M}'$ that contains I and is maximal with respect to the property that $e\mathcal{M}'e = \mathcal{M}$. The ideal $\mathcal{M}'$ is necessarily a maximal left-ideal, because if $\mathcal{M}' + Rx$ properly contains $\mathcal{M}'$, then $eRe = \mathcal{M} + eRxe$; hence

$$e \;=\; v + erxe \tag{4.1.19}$$

for some $v \in \mathcal{M}$ and some $r \in R$. Since $v \in \mathcal{M} \subset eRe$, we have that $v = ve$ and so equation (4.1.19) gives that $(1 - v - erx)e = 0$. Thus $1 - v - erx \in R(1-e)$. It follows that

$$1 \in R\mathcal{M} + R(1-e) + Rx \;\subseteq\; \mathcal{M}' + Rx$$

and so $\mathcal{M}'$ is a maximal left ideal. Suppose $\mathcal{M}'$ contains a nonzero two-sided ideal J. Then eJe is a two-sided eRe-ideal contained in $\mathcal{M}$. Hence $eJe = 0$. But this is impossible because R is prime, J is nonzero, and e is nonzero. Thus $\mathcal{M}'$ cannot contain a nonzero two-sided ideal. Hence R is primitive. ∎

4.2 Applications of Affinization

We now give some applications of Theorem 4.1.1.

Example 4.2.1 *An affine prime F-algebra of GK dimension 3 with non-nil Jacobson radical for fields F with $|F| \leq \aleph_0$.*

Suppose $|F| \leq \aleph_0$. Beĭdar [6] first constructed an affine prime ring with non-nil Jacobson Radical; this example was subsequently modified by Small. We show that Small's construction can be further modified to give such an example with GK dimension 3. Note that $F[t]_{(t)}$ is countably infinite dimensional over F and has GK dimension 1 as an F-algebra. Hence there is a homomorphism $\Phi : F + F\{x, y\}y \to F[t]_{(t)}$ such that $A = \mathcal{A}(F[t]_{(t)}, F; \Phi)$ is an affine algebra with GK dimension 3. We have that

$$e_{1,1}J(A)e_{1,1} \;\supseteq\; J(e_{1,1}Ae_{1,1}) \;\cong\; J(F[t]_{(t)}) \;=\; tF[t]_{(t)}.$$

Hence A has non-nil Jacobson radical. This is much different from the situation when F is uncountable. In this case Corollary 2.1.24 shows that the Jacobson radical of any affine F-algebra is nil. Also, a prime affine ring of GK dimension less than or equal to 1 is PI by a Theorem 2.4.13 and hence has Jacobson radical (0) by Theorem 2.2.17. It is unknown whether the Jacobson radical of a prime, affine ring of GK dimension 2 is necessarily nil. We make the following conjecture.

Conjecture 4.2.1 *Suppose R is an affine F-algebra of GK dimension less than 3. Then $J(R)$ is nil. In particular, if R is also right Goldie then R is Jacobson.*

The affinization technique will probably not yield a counter-example to this conjecture, because it will only produce a ring of GK dimension less than 3 if the affinization is performed upon a ring of GK dimension zero; if R has GK dimension 0 then R is algebraic over F and hence $J(R)$ is nil.

Example 4.2.2 *A primitive F-algebra with GK dimension 3 and center equal to a polynomial ring for fields F with $|F| \leq \aleph_0$.*

We saw in Proposition 2.2.16 that over an uncountable field F, the center of an affine primitive F-algebra is a field. We show that this is not the case when F is not uncountable. Let F be a field with $|F| \leq \aleph_0$. Consider the field $F(t)$. This is a primitive countably generated $F[t]$-algebra. By Theorem 4.1.1 there exists a homomorphism $\Phi : F[t] + F[t]\{x, y\} \to F(t)$ such that $\mathcal{A}(F(t), F[t]; \Phi)$ has GK dimension between 3 and 4. (In fact by carefully choosing Φ we can ensure that it has GK dimension precisely 3.) By Proposition 4.1.5 we see that the center of this ring is the polynomial ring $F[t]\mathbf{I_2}$. Thus $\mathcal{A}(F(t), F[t]; \Phi)$ is a primitive affine F-algebra of GK dimension 3 with center equal to a polynomial ring.

Example 4.2.3 *An affine prime ring of GK dimension 2 that is neither PI nor primitive.*

Artin and Stafford [4] proved that an affine prime graded Goldie algebra of GK dimension 2 over an algebraically closed field is either primitive or PI. We show that this is not the case in general for prime rings of GK dimension 2.

We shall begin by giving a construction of a prime F-algebra of GK dimension 0 that is not primitive, and then we shall use affinization on this example. Let us create a free F-algebra on infinitely many generators $\{t_1, t_2, \dots\}$. We let $\mathcal{W}_n$ be the collection of all words of length n on $t_1, \dots, t_n$ and let

$$I \;=\; \left\langle \bigcup_{i=1}^{\infty} \mathcal{W}_i \right\rangle.$$

Remark 4.2.4 *Note that I is a monomial ideal and hence if $c_1, \dots, c_m$ are nonzero and $w_1, \dots, w_m$ are distinct words in $t_1, t_2, \dots$, then*

$$\sum_{i=1}^{m} c_i w_i \in I$$

only if $w_1, \dots, w_m \in I$.

Let

$$R \;=\; F\{t_1, t_2, \dots\}/I.$$

We prove two lemmas to establish the facts that we shall need.

Lemma 4.2.5 *R is not PI and has GK dimension 0.*

Proof. To see that R is not PI, suppose that R satisfies a multilinear identity

$$p(x_1, x_2, \dots, x_n) \;=\; x_1 x_2 \cdots x_n + \sum_{\substack{\sigma \in \mathcal{S}_n \\ \sigma \neq 1}} c_\sigma x_{\sigma(1)} \cdots x_{\sigma(n)}.$$

Consider $p(t_2, t_3, \ldots, t_{n+1})$. By Remark 4.2.4 we have that $t_2t_3 \cdots t_{n+1} \in I$. Since $t_1t_2 \cdots t_n$ is a word in I, we have that there is a word $w \in \mathcal{W}_i$ for some $i \leq n$ and words v and v' such that

$$t_1 \cdots t_n \;=\; vwv'.$$

Thus $w = t_it_{i+1} \cdots t_{i+j}$ for some i and j with $2 \leq i \leq i+j \leq n+1$. Since w has length $j+1$, we conclude that $w \in \mathcal{W}_{j+1}$. This is a contradiction, since t_{i+j} appears in $\mathcal{W}_{j+1}$ and $i + j \geq j + 2$. Hence R does not satisfy a polynomial identity.

To see that R has GK dimension 0, note that if $p_1, \ldots, p_m \in F\{t_1, t_2, \ldots\}$, then there exists some k such that

$$p_1, \ldots, p_m \in F\{t_1, \ldots, t_k\} \;\subseteq\; F\{t_1, t_2, \ldots\}.$$

Since

$$I \cap F\{t_1, \ldots, t_k\}$$

contains all words in $t_1, \ldots, t_k$ of length at least k, we conclude that

$$F\{t_1, \ldots, t_k\}/\big(I \cap F\{t_1, \ldots, t_k\}\big)$$

has dimension at most $1 + k + k^2 + \cdots + k^{k-1}$. Hence R has GK dimension 0. ∎

Lemma 4.2.6 *R is prime and $J(R) = \langle t_1, t_2, \ldots \rangle$. In particular, R is not primitive.*

Proof. Suppose $a, b \in F\{t_1, t_2, \ldots\}$ are such that

$$a\big(F\{t_1, t_2, \ldots\}\big)b \;\subseteq\; I$$

and $a, b \notin I$. It is no loss of generality to assume that a and b are words, by Remark 4.2.4. We then have $at_mb \in I$ for all m. Let $\text{length}(u)$ denote the length of a word u and choose

$$m \;>\; 1 + \text{length}(a) + \text{length}(b).$$

Since $at_m b$ is a word that is in I, there exists

$$w \in \bigcup_{i \geq 1} \mathcal{W}_i$$

and words v, v' such that $at_m b = vwv'$. If t_m does not occur in w, then w is a subword of either a or b, contradicting the fact that $a, b \notin I$. Hence t_m appears in the word w. It follows that

$$\text{length}(w) \;\geq\; m \;>\; 1 + \text{length}(a) + \text{length}(b) \;=\; \text{length}(at_m b).$$

This is a contradiction, and so R is prime. Let

$$r \;\in\; \langle t_1, t_2, \ldots \rangle \;\subseteq\; F\{t_1, t_2, \ldots\}.$$

Then there exists a k such that

$$r \;\in\; F\{t_1, \ldots t_k\} \;\subseteq\; F\{t_1, t_2, \ldots\}.$$

We have that r^k is a linear combination of words in $t_1, \ldots, t_k$, all of which have length at least k. Hence $r^k \in I$. It follows that the Jacobson radical of R is the ideal generated by the images of $\{t_i \mid i \geq 1\}$. ∎

From these lemmas we see that R is a prime ring of GK dimension zero and is neither PI nor primitive; moreover, R is countable dimensional as an F-algebra. Hence by Theorem 4.1.1 there exists a homomorphism $\Phi : F + F\{x, y\}y \to R$ such that $\mathcal{A}(R, F; \Phi)$ is a prime ring of GK dimension 2. Since R is not primitive, $\mathcal{A}(R, F; \Phi)$ is not primitive by Proposition 4.1.6. Also $\mathcal{A}(R, F; \Phi)$ is not PI, as R does not satisfy a polynomial identity.

Fisher and Snider (see Example 2.8 of [13]) give an example of a prime F-algebra of GK dimension 0 that has nonzero Jacobson radical. It can be shown that their example is not PI and hence we obtain via affinization another example of a prime affine ring of GK dimension 2 that is neither primitive nor PI. In both the example given here and the example that arises from [13], the degrees of the matrix images are bounded. We therefore repeat the following questions of Small.

Question 4.2.7 *Are the degrees of the matrix images of a prime, affine ring of GK dimension 2 necessarily bounded?*

Question 4.2.8 *Must an affine algebra with quadratic growth necessarily be either primitive or PI?*

We have seen that affinization gives a way of constructing examples in finite GK dimension. We note that using affinization one can also answer in the affirmative the following question of Kaplansky: does there exists an affine primitive algebraic algebra which is not of GK dimension 0. This example can be found in [5].

Bibliography

[1] S.A. Amitsur, Algebras over infinite fields. *Proc. Amer. Math. Soc.* **73** (1956), 35–48.

[2] S.A. Amitsur and L.W. Small, GK dimensions of corners and ideals. *J. Algebra* **133** (1990), no. 2, 244–248.

[3] E. Artin and J.T. Tate, A note on finite ring extensions. *J. Math. Soc. Japan* **3**, (1951), 74–77.

[4] M. Artin and J.T. Stafford, Noncommutative graded domains with quadratic growth. *Invent. Math.* **122** (1995), no. 2, 231–276.

[5] J.P. Bell and L.W. Small, A question of Kaplansky. *J. Algebra.* To appear.

[6] K.I. Beĭdar, Radicals of finitely generated algebras. *Uspekhi Mat. Nauk* **36** (1981), no. 6 (222), 203–204.

[7] G.M. Bergman, *A note on growth functions of algebras and semigroups*, mimeographed notes, University of California, Berkeley, 1978.

[8] G.M. Bergman, A ring primitive on the right but not on the left. *Proc. Amer. Math. Soc.* **15** (1964), 1000.

[9] G.M. Bergman, Gelfand-Kirillov dimensions of factor rings. *Comm. Algebra* **16** (1988), no. 12, 2555–2567.

[10] B. Bollobás, *Graph theory. An introductory course.* Springer-Verlag, New York-Berlin, 1979.

[11] W. Borho and H. Kraft, Über die Gelfand-Kirillov-Dimension. *Math. Ann.* **220** (1976), no. 1, 1–24.

[12] D. Eisenbud, *Commutative algebra. With a view toward algebraic geometry.* Graduate Texts in Mathematics, 150. Springer-Verlag, New York, 1995.

[13] J.W. Fisher and R.L. Snider, Prime Von Neumann regular rings and primitive group algebras. *Proc. Amer. Math. Soc.* **44** (1974), no. 2, 244–250.

[14] K.R. Goodearl and J.T. Stafford, The graded version of Goldie's theorem. *Algebra and its applications (Athens, OH, 1999)*, 237–240, Contemp. Math., 259. *Amer. Math. Soc., Providence, RI, 2000.*

[15] I.N. Herstein, *Noncommutative rings.* Reprint of the 1968 original. With an afterword by L.W. Small. Carus Mathematical Monographs, 15. Mathematical Association of America, Washington, DC, 1994.

[16] C. Hopkins, Rings with minimal condition for left ideals. *Ann. Math.* **40** (1939), 712–730.

[17] A.V. Jategaonkar, Ore domains and free algebras. *Bull. London Math. Soc.* **1** (1969), 45–46.

[18] I. Kaplansky, Rings with a polynomial identity. *Bull. Amer. Math. Soc.* **54** (1948), 575–580.

[19] G.R. Krause and T.H. Lenagan, *Growth of algebras and Gelfand-Kirillov dimension.* Revised edition. Graduate Studies in Mathematics, 22. American Mathematical Society, Providence, RI, 2000.

[20] C. Lanski, R. Resco, and L.W. Small, On the primitivity of prime rings. *J. Algebra* **59** (1979), no. 2, 395–398.

[21] J. Levitzki, Solution of a problem of G. Koethe. *Amer. J. Math.* **67** (1945), 437–442.

[22] J.C. McConnell and J.C. Robson, *Noncommutative Noetherian Rings.* John Wiley and Sons, New York, 1987.

[23] C. Năstăsescu and F. Van Oystaeyen, *Graded Ring Theory*, North-Holland Publishing Co., Amsterdam-New York-Oxford, 1982.

[24] C.J. Pappacena, An upper bound for the length of a finite-dimensional algebra. *J. Algebra* **197** (1997), no. 2, 535–545.

[25] R.S. Pierce, *Associative Algebras.* Graduate Texts in Mathematics, 88. Studies in the History of Modern Science, 9. Springer-Verlag, New York-Berlin, 1982.

[26] H. Rademacher, On the expansion of the partition function in a series. *Ann. of Math. (2)* **44** (1943), 416–422.

[27] R. Resco, L.W. Small, and J.T. Stafford, Krull and global dimensions of semiprime Noetherian PI-rings. *Trans. Amer. Math. Soc.* **274** (1982), no. 1, 285–295.

[28] L.H. Rowen, *Ring theory. Vol. I.* Pure and Applied Mathematics, 128. Academic Press, Inc., Boston, MA, 1988.

[29] L.H. Rowen, *Ring theory. Vol. II.* Pure and Applied Mathematics, 128. Academic Press, Inc., Boston, MA, 1988.

[30] L.W. Small, J.T. Stafford, and R.B. Warfield Jr., Affine algebras of Gelfand-Kirillov dimension one are PI. *Math. Proc. Cambridge Philos. Soc.* **97** (1985), no. 3, 407–414.

[31] L.W. Small and R.B. Warfield Jr., Prime affine algebras of Gelfand-Kirillov dimension one. *J. Algebra* **91** (1984), no. 2, 386–389.

[32] R.B. Warfield, Jr., The Gelfand-Kirillov dimension of a tensor product. *Math. Z.* **185** (1984), 441–447.