UNIVERSITY OF CALIFORNIA, SAN DIEGO

Graph Varieties

A dissertation submitted in partial satisfaction of the

requirements for the degree

Doctor of Philosophy

in

Mathematics

by

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2002

The dissertation of Jeremy Leander Martin is approved, and it is acceptable in quality and form for publication on microfilm:

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2002

To my wife, Jennifer Wagner, the point in my variety.

TABLE OF CONTENTS

Signature Page . . . iii

Dedication . . . iv

Table of Contents . . . v

List of Figures . . . vii

List of Tables . . . viii

Acknowledgements . . . ix

Vita and Publications . . . x

Abstract of the Dissertation . . . xi

Introduction . . . 1

1 The Geometry of Graph Varieties . . . 5

1.1 Definitions . . . 5

1.2 The Picture Space and Picture Variety of a Graph . . . 7

1.2.1 Pictures and the Picture Space . . . 7

1.2.2 The Affine Picture Space . . . 8

1.2.3 Cellules . . . 9

1.3 Combinatorial Rigidity Theory . . . 12

1.4 Equations Defining $\tilde{\mathcal{V}}(G)$ . . . 17

1.5 Geometric Properties of $\mathcal{X}(G)$ and $\mathcal{V}(G)$ . . . 26

2 The Slope Variety of the Complete Graph 31

2.1 Introduction 31

2.2 A Lower Bound for the Degree of $\tilde{\mathcal{S}}(K_n)$ 32

2.2.1 The Recurrence 32

2.2.2 Decreasing Planar Trees 36

2.3 Tree Polynomials of Wheels 46

2.3.1 Preliminaries 46

2.3.2 Special Spanning Trees of Wheels 47

2.3.3 Leading Trees of Wheels 54

2.4 The Stanley-Reisner Complex 62

2.4.1 Preliminaries 62

2.4.2 The Structure of Facets 66

2.4.3 Admissible Trees 71

2.4.4 Counting the Facets 74

2.4.5 Shellability 75

2.5 The Main Theorem 82

2.6 The h-vector of $\Delta(n)$ 84

2.6.1 A Combinatorial Interpretation of h_k^n 86

Bibliography 88

LIST OF FIGURES

1.1 The 4-gon is not rigid 13

1.2 A pseudocircuit which is not a circuit 15

2.1 Examples of decreasing planar trees 38

2.2 The $*$ operation on decreasing planar trees 39

2.3 An example with $N = 9$ and $K = 7$ 40

2.4 An example of the "otherwise" case with $N = 9$ and $K = 5$ 41

2.5 Wheels 46

2.6 Special spanning trees of wheels 50

2.7 A facet F of $\Delta(6)$ 72

2.8 The admissible tree T corresponding to F 72

2.9 $\phi(T^1)$ and $\phi(T^2)$ 74

2.10 $\psi(T)$ and $\phi(T)$ 75

2.11 $\mathcal{L}(T)$ 76

LIST OF TABLES

2.1 Some values of $d(n,k)$ 45

2.2 Some values of h_k^n 86

2.3 Matchings on [6] 86

ACKNOWLEDGEMENTS

This is where, I imagine, most writers of doctoral dissertations in mathematics insert something about wanting to thank all the great people who have helped them through graduate school, but not having enough room in the margin to list everyone. I haven't been able to resist the temptation.

First, my thesis advisor, Mark Haiman. It has been an honor and privilege to be his student. I especially want to thank Mark and Ellen Rose for hosting me during several trips to Berkeley over the last few years. I have learned from many other excellent teachers at UCSD: Adriano Garsia, Linda Rothschild, Adrian Wadsworth, and Nolan Wallach. I also appreciate the work of the superb staff in the Department of Mathematics, in particular Lois Stewart and Zelinda Collins. I wouldn't be a mathematician today if not for the superb mathematics teachers I've had my whole life: Noam Elkies, Harriet Pollatsek, Ravi Vakil, Nantel Bergeron, Lee Carlson, Beverly Feulner, Judy Glickman, Jim O'Connor, Carol Cocker, John Romano, and Bonnie Bass. Thank you also to my friends and colleagues at UCSD, past and present, for all the softball games, late-night bridge games, and floors to crash on (and windows to crash into): Will Brockman, Lucy Hadden, Sally Picciotto, Glenn Tesler, Jean Steiner, Rob and Marcella Ellis, Eugene Hung, Roummel Marcia, Renée Brunelle, Ben Raphael, David Little, Eric Rowell, Cameron and Katie Parker, Jason Lee, Travis Kowalski, Dave Glickenstein, among many others. Nancy Ying, while not technically family, might as well be a blood relation for all she has done to preserve my sanity during my two years in Chicago.

My first and greatest teachers are my parents, Arthur and Elinor Martin. They and my brother Jonathan have been a great source of strength to me during the past six years, as have my parents-in-law, Dave and Pat Wagner. Finally, the person who has endured stacks of scratch paper on the coffee table and early-morning mumblings about graph theory, my wonderful wife and best friend, Jennifer Wagner (who two years ago went through all this herself). Thank you!

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PUBLICATIONS

Geometry of Graph Varieties. To appear, 2002.

On the Slope Variety of the Complete Graph. To appear, 2002.

Ruling out (160,54,18) Difference Sets in Some Nonabelian Groups. With J. Alexander, R. Balasubramanian, K. Monahan, H. Pollatsek, and A. Sen. *J. Combin. Designs* 8: 221-231, 2000.

ABSTRACT OF THE DISSERTATION

Graph Varieties

by

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Doctor of Philosophy in Mathematics

University of California San Diego, 2002

Professor Nolan R. Wallach, Chair

We study configuration varieties parametrizing plane pictures $\mathbf{P}$ of a given graph G, with vertices v and edges e represented respectively by points $\mathbf{P}(v) \in \mathbb{P}^2$ and lines $\mathbf{P}(e)$ connecting them in pairs. Three such varieties naturally arise: the *picture space* $\mathcal{X}(G)$ of all pictures of G; the *picture variety* $\mathcal{V}(G)$, an irreducible component of $\mathcal{X}(G)$; and the *slope variety* $\mathcal{S}(G)$, essentially the projection of $\mathcal{V}(G)$ on coordinates m_e giving the slopes of the lines $\mathbf{P}(e)$. In practice, we most often work with affine open subvarieties $\tilde{\mathcal{X}}(G)$, $\tilde{\mathcal{V}}(G)$, $\tilde{\mathcal{S}}(G)$, in which the points $\mathbf{P}(v)$ lie in an affine plane and the lines $\mathbf{P}(e)$ are nonvertical.

We prove that the algebraic dependence matroid of the slopes is in fact the *generic rigidity matroid* $\mathcal{M}(G)$ studied by Laman *et. al.* [12], [8]. For each set of edges forming a circuit in $\mathcal{M}(G)$, we give an explicit determinantal formula for the polynomial relation among the corresponding slopes m_e. This polynomial enumerates decompositions of the given circuit into complementary spanning trees. We prove that precisely these "tree polynomials" cut out $\mathcal{V}(G)$ in $\mathcal{X}(G)$ set-theoretically. We also show how the full component structure of $\mathcal{X}(G)$ can be economically described in terms of the rigidity matroid, and show that when $\mathcal{X}(G) = \mathcal{V}(G)$, this variety has Cohen-Macaulay singularities.

We study intensively the case that G is the complete graph K_n. Describing $\mathcal{S}(K_n)$ corresponds to the classical problem of determining all relations among the slopes of the $\binom{n}{2}$ lines connecting n general points in the plane. We prove that the tree polynomials form a Gröbner basis for the affine variety $\tilde{\mathcal{S}}(K_n)$ (with respect to a particular term order). Moreover, the facets of the associated Stanley-Reisner simplicial complex $\Delta(n)$ can be described explicitly in terms of the combinatorics of decreasing planar trees. Using this description, we prove that $\Delta(n)$ is shellable, implying that $\mathcal{S}(K_n)$ is Cohen-Macaulay for all n. Moreover, the Hilbert series of $\tilde{\mathcal{S}}(K_n)$ appears to have a combinatorial interpretation in terms of perfect matchings.

Introduction

The theory of *configuration varieties* lies at the intersection between algebraic geometry and combinatorics. A configuration variety is defined as an algebraic subset of a product of Grassmannian varieties, defined by containment conditions among the individual factors. A hallmark of the theory is that the geometric structure of a configuration variety can often be described very explicitly in terms of the combinatorics of the underlying containment relations.

The simplest configuration variety, and the building block for all others, is the *Grassmannian* $\mathrm{Gr}(n, r)$, the set of all r-dimensional subspaces of affine n-space over a field $\mathbf{k}$ (typically algebraically closed). More generally, a configuration variety may be defined as follows. Let

$$X = \mathrm{Gr}(n, d_1) \times \ldots \times \mathrm{Gr}(n, d_s).$$

A configuration variety is then a set $Y \subset X$ defined by a set of conditions of the form $H_i \subset H_j$ on points $(H_1, \ldots, H_s) \in X$. Perhaps the best-known example is the (complete) *flag variety* Fl_n, defined by letting $s = n$ and $d_i = i$ for all i, and imposing the conditions $H_1 \subset H_2 \subset \cdots \subset H_n$. A classical application of combinatorics to algebraic geometry of Grassmannian and flag varieties is the *Schubert calculus*, which relates the intersection theory on Grassmannians to partitions and the symmetric group (see, e.g., [6]).

Much attention has been focused on configuration varieties that can be described in terms of the action of an algebraic group A. The flag variety Fl_n is perhaps the seminal example: The general linear group $GL_n(\mathbf{k})$ acts transitively on Fl_n, and the stabilizer of a given point is isomorphic to the Borel subgroup B of upper triangular invertible matrices, so Fl_n may be regarded as the quotient space $GL_n(\mathbf{k})/B$. An important advantage of the algebraic-group approach is that one can often avoid considering the explicit equations defining the configuration variety; instead, geometric questions about the configuration variety can often be reduced to combinatorial questions about the Weyl group of A. However, this approach has yielded strong results only when the containment relations satisfy certain conditions. For instance, Lakshmibai and Magyar study configuration varieties arising from the chamber family of the reduced decomposition of a Weyl group element [11], [13].

This dissertation aims to extend the theory of configuration varieties in a different direction. Specifically, we study algebraic varieties that parametrize plane pictures $\mathbf{P}$ of a given *graph* G, with vertices v and edges e represented respectively by points $\mathbf{P}(v) \in \mathbb{P}^2$ and lines $\mathbf{P}(e)$ connecting them in pairs.

Three such varieties naturally arise. First of all, there is the *picture space* $\mathcal{X}(G)$ of all pictures of G. Usually, $\mathcal{X}(G)$ is not irreducible. It is therefore natural to restrict attention to a second variety, namely the irreducible component of $\mathcal{X}(G)$ containing as a dense set those pictures in which the points $\mathbf{P}(v)$ are all distinct. This most generic component of the picture space is called the *picture variety* $\mathcal{V}(G)$. As we shall see, $\mathcal{V}(G)$ is cut out in $\mathcal{X}(G)$ purely by equations relating the slopes of the lines $\mathbf{P}(e)$. The crucial matter for the whole study is to understand the relations among these slopes. This leads us to consider the *slope variety* $\mathcal{S}(G)$, which is essentially the projection of $\mathcal{V}(G)$ on coordinates m_e giving the slopes of the lines $\mathbf{P}(e)$. In practice, we most often work with affine open subvarieties $\tilde{\mathcal{X}}(G)$, $\tilde{\mathcal{V}}(G)$, $\tilde{\mathcal{S}}(G)$, in which the points $\mathbf{P}(v)$ are required to lie in the affine plane $\mathbb{A}^2$ and the lines $\mathbf{P}(e)$ are required to be nonvertical.

In Chapter 1, we consider the features of varieties associated with an arbitrary graph G. We shall see that the *generic rigidity matroid* $\mathcal{M}(G)$ studied by Laman *et. al.* [12], [8] makes a somewhat surprising appearance here as the algebraic dependence matroid of the slopes. For each set of edges forming a circuit in the matroid $\mathcal{M}(G)$, we can write down an explicit determinantal formula for the essentially unique polynomial relation among the corresponding slopes m_e. We prove that precisely these relations cut out $\mathcal{V}(G)$ in $\mathcal{X}(G)$ set-theoretically. We also show how the full component structure of $\mathcal{X}(G)$ can be economically described in terms of the rigidity matroid, and show that when $\mathcal{X}(G) = \mathcal{V}(G)$, this variety has Cohen-Macaulay singularities.

The slope relation induced by each circuit in $\mathcal{M}(G)$ turns out to be a very remarkable polynomial. All its terms are square-free, and they have a beautiful combinatorial interpretation in terms of decompositions of the given circuit into complementary spanning trees. We conjecture that these "tree polynomials" should cut out $\mathcal{V}(G)$ scheme-theoretically as well as set-theoretically. We further suspect that they may always form a *universal Gröbner basis* for the ideal of the slope variety, and moreover, that both $\mathcal{S}(G)$ and $\mathcal{V}(G)$ are always Cohen-Macaulay.

In Chapter 2, we study intensively the case where G is the complete graph K_n. (The problem of describing the slope variety $\mathcal{S}(K_n)$ is of a very classical kind: it is exactly the problem of determining all relations among the slopes of the $\binom{n}{2}$ lines connecting n general points in the plane.) In this case, the tree polynomials form a Gröbner basis for the affine variety $\tilde{\mathcal{S}}(K_n)$ (with respect to a particular term order). Moreover, the facets of the associated Stanley-Reisner simplicial complex $\Delta(n)$ can be described explicitly in terms of the combinatorics of decreasing planar trees. This description leads to a proof that $\Delta(n)$ is shellable (with respect to at least one term ordering), which means that the variety $\mathcal{S}(K_n)$ is Cohen-Macaulay for all n. Moreover, the Hilbert series of $\tilde{\mathcal{S}}(K_n)$ appears to have a combinatorial interpretation in terms of perfect matchings.

When we first embarked upon the study of graph varieties, before obtaining the results indicated above, we already had some reasons to think they might be interesting. Since these reasons remain relevant, let us mention them briefly. Graph varieties provide the simplest non-trivial examples not fitting into Lakshmibai and Magyar's framework. Furthermore, for $G = K_n$, the graph variety $\mathcal{V}(G)$ is a blowdown of the Fulton-Macpherson *compactification of configuration spaces* [7], which desingularizes it. For general G, the same relation holds between $\mathcal{V}(G)$ and the DeConcini-Procesi *wonderful model of subspace arrangements* [4]. We expect that $\mathcal{V}(G)$ should not only be Cohen-Macaulay but should have rational singularities. This would be equivalent to a cohomology vanishing theorem for certain line bundles on the wonderful model, raising an important question for further study.

Chapter 1

The Geometry of Graph Varieties

1.1 Definitions

We work over an algebraically closed field $\mathbf{k}$. Affine and projective n-space over $\mathbf{k}$ are denoted by $\mathbb{A}^n$ and $\mathbb{P}^n$ respectively; the Grassmannian variety of all r-dimensional subspaces of $\mathbf{k}^n$ is denoted by $\mathrm{Gr}(n, r)$.

The symbol $\uplus$ denotes a disjoint union. If n is a positive integer, we put

$$[n] = \{1, 2, \ldots, n\}.$$

A *graph* G is a pair (V, E), where $V = V(G)$ is a finite set of *vertices* and $E = E(G)$ is a set of *edges*, or unordered pairs of distinct vertices $\{v, w\}$. We frequently abbreviate $\{v, w\}$ by vw when no confusion can arise (for instance, when the vertices are one-digit positive integers). The vertices v, w are the *endpoints* of the edge vw. If $G' = (V', E')$ is a graph with $V' \subset V$ and $E' \subset E$, we say G' is a *subgraph* of G and write $G' \subset G$. For $W \subset V$, we define

$$\begin{aligned} K(W) &= \{vw \mid v, w \in W,\ v \neq w\}, \\ E(W) &= E \cap K(W), \\ G|_W &= (W, E(W)) \qquad\qquad \text{(the \textit{induced subgraph} on } W). \end{aligned}$$

The *complete graph on* V is the graph $(V, K(V))$. We write K_n for the complete graph on $[n]$. For $E \subset E$ and $v \in V$, we define the *valence* of v with respect to E to be

$$\mathrm{val}_E(v) = |\{e \in E \mid v \in e\}|$$

and the *support* or *vertex support* of E to be

$$V(F) = \{v \in V \mid \mathrm{val}_E(v) > 0\}.$$

If $v \in V(F)$ then we say that F is *supported at* v.

For vertices $v_1, \dots, v_s$, we write

$$(v_1, \dots, v_s) = \{v_1v_2,\ v_2v_3,\ \dots,\ v_{s-1}v_s\} \subset E.$$

If the v_i are all distinct, then $(v_1, \dots, v_s)$ is called a *path*. If $v_1, \dots, v_{s-1}$ are distinct and $v_1 = v_s$, then $(v_1, \dots, v_s)$ is called a *polygon* or $(s-1)$-*gon*. A polygon is more usually called a "cycle" or "circuit," but we wish to reserve these words for other uses.

G is *connected* if every pair of vertices are joined by a path, and is a *forest* if at most one such path exists for every pair. A connected forest is called a *tree*. A *spanning tree* of G (or of V) is a tree $T \subset E$ with $V(T) = V$. A *connected component* of G is a maximal connected subgraph; every graph has a unique decomposition into connected components (where some components may be isolated vertices).

A *partition* of a finite set V is a set $\mathcal{A} = \{A_1, \dots, A_s\}$ of pairwise disjoint subsets of V whose union is V. We write $\sim_{\mathcal{A}}$ for the equivalence relation on V

whose equivalence classes are the A_i's. The sets A_i are called the *blocks* of $\mathcal{A}$. We distinguish two extreme cases, the *discrete partition* $\mathcal{D}_V$, all of whose blocks are singletons, and the *indiscrete partition* $\mathcal{I}_V$, which has only one block. Finally, if $\mathcal{A}$ and $\mathcal{B}$ are partitions of V, then we say that $\mathcal{A}$ *refines* $\mathcal{B}$, written $\mathcal{A} \preceq \mathcal{B}$, if every block of $\mathcal{A}$ is contained in some block of $\mathcal{B}$.

1.2 The Picture Space and Picture Variety of a Graph

1.2.1 Pictures and the Picture Space

Throughout this section, we consider a graph $G = (V, E)$ with $|V| = n$ and $|E| = r$. We define

$$\mathrm{Gr}(G) = \left(\prod_{v \in V} \mathrm{Gr}(3,1) \right) \times \left(\prod_{e \in E} \mathrm{Gr}(3,2) \right). \tag{1.1}$$

We may regard $\mathrm{Gr}(3,1)$ and $\mathrm{Gr}(3,2)$ as the sets of points and lines, respectively, in the projective plane $\mathbb{P}^2$. For $\mathbf{P} \in \mathrm{Gr}(G)$, we write $\mathbf{P}(v)$ and $\mathbf{P}(e)$ for the projections of $\mathbf{P}$ on the indicated factors in (1.1).

Definition 1.2.1. A *picture* of G is a point $\mathbf{P} \in \mathrm{Gr}(G)$ such that $\mathbf{P}(v) \in \mathbf{P}(e)$ whenever $v \in e$. The *picture space* $\mathcal{X}(G)$ is the set of all pictures of G. The picture space is Zariski-closed in $\mathrm{Gr}(G)$, since the condition $\mathbf{P}(v) \in \mathbf{P}(e)$ may be expressed as an equation in homogeneous coordinates.

Note that $\mathcal{X}(G) \cong \mathcal{X}(G_1) \times \mathcal{X}(G_2) \times \dots$, where the G_i are the connected components of G.

1.2.2 The Affine Picture Space

Fix homogeneous coordinates

$$[a_0 : a_1 : a_2]$$

on $\mathbb{P}^2$, identifying $\mathbb{A}^2$ with the points for which $a_0 \neq 0$ and giving affine coordinates on $\mathbb{A}^2$

$$x = a_1/a_0, \ y = a_2/a_0.$$

The equations defining $\mathcal{X}(G)$ in homogeneous coordinates are somewhat awkward to work with. However, all the geometric information we will require about $\mathcal{X}(G)$ can be recovered from the following affine open subset of it, on which the defining equations assume a more manageable form.

Definition 1.2.2. The *affine picture space* $\tilde{\mathcal{X}}(G)$ is the open subvariety of $\mathcal{X}(G)$ consisting of pictures $\mathbf{P}$ such that all points $\mathbf{P}(v)$ lie in $\mathbb{A}^2$ and no line $\mathbf{P}(e)$ is parallel to the y-axis.

Note that $\tilde{\mathcal{X}}(G)$ is open and dense in $\mathcal{X}(G)$, and that $\mathcal{X}(G)$ is covered by finitely many copies of $\tilde{\mathcal{X}}(G)$. In addition $\tilde{\mathcal{X}}(G)$ has affine coordinates

$$\begin{aligned} (x_v, y_v): &\qquad v \in V, \\ (m_e, b_e): &\qquad e \in E, \end{aligned} \tag{1.2}$$

where m_e and b_e denote respectively the slope and y-intercept of the line $\mathbf{P}(e)$. Thus $\tilde{\mathcal{X}}(G)$ is the vanishing locus (in $\mathbb{A}^{2n+2r}$, identified with an open subset of $\mathrm{Gr}(G)$) of the ideal generated by the equations

$$\begin{aligned} y_v &= m_e x_v + b_e, \\ y_w &= m_e x_w + b_e, \end{aligned} \tag{1.3}$$

for each $e = vw$. We may eliminate the variables b_e from (1.3), obtaining r equations

$$(y_v - y_w) = m_e(x_v - x_w). \tag{1.4}$$

We may also eliminate the variables y_v. For every polygon $P = (v_1, \ldots, v_s, v_1)$ of G, we sum the equations (1.4) over the edges of P, obtaining the equation

$$L(P) = \sum_{i=1}^{s} m_{e_i}(x_{v_i} - x_{v_{i+1}}) = 0. \tag{1.5}$$

where $e_i = v_i v_{i+1}$ and the indices are taken modulo s. Given a solution $(\mathbf{m}, \mathbf{x})$ of the equations (1.5), we may choose one y-coordinate arbitrarily and use (1.3) and (1.4) to recover the coordinates y_v and b_e. Putting

$$\begin{aligned} R_G &= \mathbf{k}[m_e \mid e \in E], \\ R'_G &= \mathbf{k}[m_e, x_v \mid e \in E, v \in V], \end{aligned} \tag{1.6}$$

we see that $\tilde{\mathcal{X}}(G) \cong \mathbb{A}^1 \times X$, where X is the subscheme of $\operatorname{Spec} R'_G \cong \mathbb{A}^{|V|+|E|}$ defined set-theoretically by the equations (1.5).

1.2.3 Cellules

There is a natural decomposition of $\mathcal{X}(G)$ into locally closed irreducible nonsingular subvarieties, which we call *cellules*. The decomposition is somewhat analogous to the decomposition of a flag variety into Schubert cells.

Definition 1.2.3. Let $\mathcal{A} = \{A_1, \ldots, A_s\}$ be a partition of V. The *cellule* of $\mathcal{A}$ in $\mathcal{X}(G)$ is the quasiprojective subvariety

$$\mathcal{X}_{\mathcal{A}}(G) = \{\mathbf{P} \in \mathcal{X}(G) \mid \mathbf{P}(v) = \mathbf{P}(w) \text{ iff } v \sim_{\mathcal{A}} w\}. \tag{1.7}$$

Unlike a Schubert cell, a cellule $\mathcal{X}_{\mathcal{A}}(G)$ is not isomorphic to an affine space. It is, however, a smooth fiber bundle. Consider an edge $e = vw$ with $v \sim_{\mathcal{A}} w$. Given the point $\mathbf{P}(v) = \mathbf{P}(w)$, the space of possiblities for the line $\mathbf{P}(e)$ is a copy of $\mathbb{P}^1$ (one of the q factors appearing in (1.8) below). On the other hand, if $v \not\sim_{\mathcal{A}} w$, then the coordinates of $\mathbf{P}(e)$ are determined uniquely by those of $\mathbf{P}(v)$ and $\mathbf{P}(w)$.

Therefore $\mathcal{X}_{\mathcal{A}}(G)$ has the bundle structure

$$\begin{array}{ccc} (\mathbb{P}^1)^q & \to & \mathcal{X}_A(G) \\ & & \downarrow \\ & & U \end{array} \tag{1.8}$$

where $q = |\{vw \in E \ \mid \ v \sim_A w\}|$, and $U = \{(p_1, \ldots, p_s) \in (\mathbb{P}^2)^s \ \mid \ p_i \neq p_j$ for $i \neq j\}$.

It follows immediately from (1.8) that

$$\dim \mathcal{X}_{\mathcal{A}}(G) = 2s + |\{vw \in E \ \mid \ v \sim_{\mathcal{A}} w\}|\,. \tag{1.9}$$

Definition 1.2.4. Let $G = (V, E)$ and $\mathbf{P} \in \mathcal{X}(G)$. $\mathbf{P}$ is called *generic* if no two of the points $\mathbf{P}(v)$ coincide. The *discrete cellule* $\mathcal{V}^\circ(G)$ is defined as the set of all generic pictures. Note that $\mathcal{V}^\circ(G) = \mathcal{X}_{\mathcal{D}}(G)$, where $\mathcal{D}$ is the discrete partition of V (the partition into singleton sets). The *picture variety* of G is

$$\mathcal{V}(G) = \overline{\mathcal{V}^\circ(G)}.$$

By the preceding discussion, $\mathcal{V}(G)$ is an irreducible component of $\mathcal{X}(G)$ of dimension $2|V|$. The *affine picture variety* of G is defined as

$$\tilde{\mathcal{V}}(G) = \mathcal{V}(G) \cap \tilde{\mathcal{X}}(G).$$

Remark 1.2.5. For $G = (V, E)$ and $W \subset V$, the *coincidence locus* of W is defined as

$$C_W = C_W(G) = \{\mathbf{P} \in \mathcal{X}(G) \ \mid \ \mathbf{P}(v) = \mathbf{P}(w) \text{ for all } v, w \in W\}. \tag{1.10}$$

Let G_0 be the graph with vertices $V = V(G)$ and no edges. We may regard $\mathcal{V}(G)$ as the simultaneous blowup of $(\mathbb{P}^2)^n = \mathcal{X}(G_0)$ along the coincidence loci $C_{\{v,w\}}$ for $vw \in E$. Indeed, the further blowup of $(\mathbb{P}^2)^n$ along all C_W, where $W \subset V$ is connected, is an instance of the "wonderful model of subspace arrangements" of DeConcini and Procesi [4]. This blowup is a desingularization of $\mathcal{V}(G)$. When G is the complete graph K_n, this is the "compactification of configuration space" of Fulton and MacPherson [7].

Note that the only cellule which is closed in $\mathcal{X}(G)$ is the *indiscrete cellule* $\mathcal{X}_{\mathcal{I}}(G)$, where $\mathcal{I}$ is the indiscrete partition of V (the partition with just one block).

Example 1.2.6. Let $G = K_2$. Denote by $\mathcal{D}$ and $\mathcal{I}$ respectively the discrete and indiscrete partitions of $V = V(G) = [2]$. The picture space $\mathcal{X}(K_2)$ is the blowup of $\mathbb{P}^2 \times \mathbb{P}^2$ along the diagonal $\Delta = \{(p_1, p_2) \in \mathbb{P}^2 \times \mathbb{P}^2 \mid p_1 = p_2\}$. The blowup map

$$\alpha : \mathcal{X}(K_2) \to \mathbb{P}^2 \times \mathbb{P}^2$$

is just the projection on the vertex coordinates. The exceptional divisor $\alpha^{-1}(\Delta)$ is the indiscrete cellule, which has dimension 3. Since there are no partitions of V other than $\mathcal{D}$ and $\mathcal{I}$, the complement of $\mathcal{X}_{\mathcal{I}}(K_2)$ must equal the discrete cellule, which has dimension 4 and is dense in $\mathcal{X}(K_2)$. Thus $\mathcal{V}(K_2) = \mathcal{X}(K_2)$.

Example 1.2.7. In general, the picture space $\mathcal{X}(G)$ is not irreducible. The first example, and in many ways the fundamental one, is $G = K_4$. Denote by $\mathcal{D}$ and $\mathcal{I}$ respectively the discrete and indiscrete partitions of $V = V(G) = [4]$. By (1.9), we have

$$\dim \mathcal{V}^\circ(K_4) = 8 = \dim \mathcal{X}_{\mathcal{I}}(K_4),$$

so $\mathcal{X}_{\mathcal{I}}(K_4)$ is too big to be contained in the closure of $\mathcal{V}^\circ(K_4)$. Hence $\mathcal{V}(K_4) \neq \mathcal{X}(K_4)$. As we will show later, the irreducible components of $\mathcal{X}(K_4)$ are precisely $\mathcal{V}(K_4)$ and $\mathcal{X}_{\mathcal{I}}(K_4)$.

We will soon see that the polynomials defining $\tilde{\mathcal{V}}(G)$ as a subvariety of $\tilde{\mathcal{X}}(G)$ involve only the variables m_e. In order to study these polynomials in isolation, we define a third type of graph variety. As before, identify $\mathbb{A}^2$ with an open affine subset of $\mathbb{P}^2$.

Definition 1.2.8. Let U be the set of pictures $\mathbf{P} \in \mathcal{V}(G)$ such that all lines $\mathbf{P}(e)$ meet $\mathbb{A}^2$; that is, no $\mathbf{P}(e)$ equals the line at infinity. Accordingly, every $\mathbf{P}(e)$ has a well-defined "slope" in $\mathbb{P}^1$. Forgetting all the data of $\mathbf{P}$ except the slopes gives a map

$$\phi : U \to (\mathbb{P}^1)^r. \tag{1.11}$$

We define the *slope variety* $\mathcal{S}(G)$ as the image of ϕ, and the *affine slope variety* as

$$\tilde{\mathcal{S}}(G) = \mathcal{S}(G) \cap (\mathbb{A}^1)^r.$$

An element $\mathbf{m} = (m_e \mid e \in E)$ of $\tilde{\mathcal{S}}(G)$ is called an *affine slope picture* of G.

Remark 1.2.9. Restricting ϕ to $\tilde{\mathcal{V}}(G)$ produces a map

$$\phi : \tilde{\mathcal{V}}(G) \twoheadrightarrow \tilde{\mathcal{S}}(G). \tag{1.12}$$

Note that every fiber of this map has dimension at least 3, because translation and scaling do not affect slopes of lines.

We will eventually show that $\tilde{\mathcal{S}}(G)$ is defined set-theoretically by the same ideal of R_G that cuts out $\tilde{\mathcal{V}}(G)$ as a subvariety of $\tilde{\mathcal{X}}(G)$. Our next goal is to identify this ideal, for which we will need tools from the theory of combinatorial rigidity.

1.3 Combinatorial Rigidity Theory

The behavior of graph varieties is governed in various ways by a certain combinatorial object, the *generic rigidity matroid*. Accordingly, we begin this section by sketching the elements of rigidity theory, collecting several facts which we will need later. (Our treatment here is necessarily brief; for a detailed exposition, we refer the reader to the monograph by Graver, Servatius and Servatius [8].) The main new result of this section, Theorem 1.3.5, describes the fundamental connection between the purely combinatorial theory of rigidity and the geometry of graph varieties. As a corollary, we obtain an alternative characterization of the generic rigidity matroid.

Let $G = (V, E)$ be a connected graph, and $\mathbf{P}$ a generic picture of G defined over $\mathbb{R}$. (For the sake of easy visualization, we abandon for the moment the requirement that the ground field be algebraically closed.) Imagine a physical model of **P** in which the vertices and edges are represented by "ball joints" and "rods" respectively. The rods are considered to be fixed in length, but are free to rotate about the joints in the plane of the picture. Intuitively, G is *length-rigid*, or simply *rigid*, if the physical realization of any generic picture of G will "hold its shape." More precisely, G is rigid if the distance between any two vertices in a generic picture is determined by the lengths of the edges in E. (This property is called "generic rigidity" in [8], as distinguished from other types of rigidity which we will not need here.)

For instance, let G be the 4-gon, i.e., $V(G) = [4]$ and $E(G) = 12, 23, 34, 41\}$. G is not rigid, since there are infinitely many incongruent rhombuses with equal side lengths.

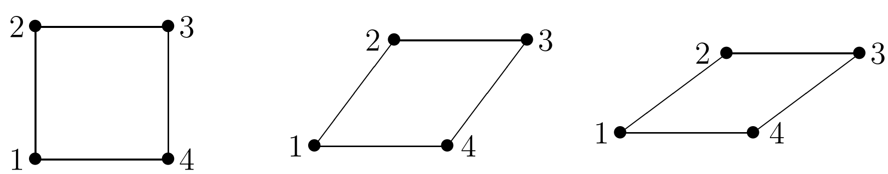


Figure 1.1: The 4-gon is not rigid

However, the graph $G' = (V, E \cup \{24\})$ is rigid, because a generic affine picture of G' is determined up to isometries of $\mathbb{R}^2$ by the lengths of its edges—that is, by the quantities $\sqrt{(x_v - x_w)^2 + (y_v - y_w)^2}$, where $vw \in E(G')$.

Definition 1.3.1. The *length-rigidity matroid* $\mathcal{M}(V)$ (called the *2-dimensional generic rigidity matroid* in [8]) is the algebraic dependence matroid on the squares of lengths of edges

$$(x_v - x_w)^2 + (y_v - y_w)^2, \qquad v, w \in V. \tag{1.13}$$

We may regard $\mathcal{M}(V)$ as a matroid on $K(V)$, associating the polynomial (1.13) with the edge vw. Accordingly, we say that a set of edges is independent in $\mathcal{M}(V)$, or *rigidity-independent*, if and only if the corresponding set of squared lengths is algebraically independent over $\mathbb{Q}$. Thus an edge set E is rigid if and only if E is a spanning set of $\mathcal{M}(V)$.

A fundamental result of rigidity theory is the following characterization of the bases and independent sets of $\mathcal{M}(V)$ ([8, Theorem 4.2.1], originally due to G. Laman). An edge set $E \subset K(V)$ is rigidity-independent if and only if

$$|F| \leq 2|V(F)| - 3 \text{ for all } F \subset E. \tag{1.14}$$

Furthermore, a rigidity-independent set E is a rigidity basis if and only if

$$|E| = 2|V| - 3. \tag{1.15}$$

In addition, E is a *rigidity circuit*—a minimal dependent set of $\mathcal{M}(V)$—if and only if $|E| = 2|V(E)| - 2$ and every proper subset F of E satisfies (1.14) [8, Theorem 4.3.1].

The rigidity circuits (called "rigidity cycles" in [8]) may be described another way. Define a *rigidity pseudocircuit* to be an edge set E equal to the edge-disjoint union of two spanning trees of $V(E)$. Then a rigidity circuit is a minimal rigidity pseudocircuit [8, Lemma 4.9.3 and Theorem 4.9.1].

Example 1.3.2. Let $r \geq 3$. The *k-wheel* is the graph with vertices $\{v_0, v_1, \ldots, v_k\}$ and edges

$$\{v_1v_2, v_2v_3, \ldots, v_kv_1\} \cup \{v_0v_1, v_0v_2, \ldots, v_0v_k\}.$$

The k-wheel is a rigidity circuit for all $k \geq 3$ [8, Exercise 4.13]. (In fact, the 3-wheel, which is isomorphic to K_4, and the 4-wheel are the only rigidity circuits on 5 or fewer vertices.) On the other hand, let G' be the following graph:

$E(G')$ is a rigidity pseudocircuit, since it is the disjoint union of the spanning trees $\{12, 15, 23, 34\}$ and $\{13, 14, 24, 25\}$, but it is not a rigidity circuit since $K_4 \subsetneq G'$.

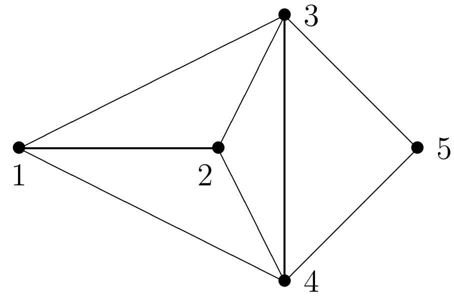


Figure 1.2: A pseudocircuit which is not a circuit

Definition 1.3.3. Let $G = (V, E)$ be a rigidity pseudocircuit. A *special spanning tree* of G is an edge set $T \subset E$ such that both T and $E \setminus T$ are spanning trees of V. The set of special spanning trees of G is denoted by $\mathrm{SST}(G)$. The pair S, T is called a *2-tree decomposition* of E (or of G).

The special spanning trees of a rigidity circuit will play a fundamental role in describing the equations which define $\tilde{\mathcal{V}}(G)$ and $\tilde{\mathcal{S}}(G)$.

Our local affine coordinates on $\tilde{\mathcal{X}}(G)$ measure the slopes of edges rather than their lengths, leading to an alternate notion of rigidity.

Definition 1.3.4. The *slope-rigidity matroid* $\mathcal{M}^s(V)$ on $K(V)$ is the algebraic dependence matroid on the rational functions

$$m_{vw} = \frac{y_w - y_v}{x_w - x_v}.$$

Theorem 1.3.5. *Let $G = (V, E)$, with $n = |V|$ and $r = |E|$. The following are equivalent:*

(i) *E is independent in $\mathcal{M}(V)$;*

(ii) *E is independent in $\mathcal{M}^s(V)$;*

(iii) $\tilde{\mathcal{S}}(G) \cong (\mathbb{A}^1)^r$;

(iv) $\mathcal{V}(G) = \mathcal{X}(G)$.

Proof. $(i) \implies (iv)$: Since $\mathcal{X}(G)$ is defined locally by $2r$ equations among $2n+2r$ coordinates, we have

$$\dim X \geq 2n = \dim \mathcal{V}(G)$$

for every irreducible component X of $\mathcal{X}(G)$. Therefore $\mathcal{V}^\circ(G)$ is dense in $\mathcal{X}(G)$ if and only if every other cellule has dimension $< 2n$.

Suppose E is rigidity-independent, hence satisfies (1.14). Let $\mathcal{A}$ be a partition of V which is not the discrete partition; we will show that $\dim \mathcal{X}_\mathcal{A}(G) < 2n$. The blocks of $\mathcal{A}$ may be numbered $A_1, \ldots, A_s$ so that

$$|A_1| = \cdots = |A_t| = 1 \qquad \text{and} \qquad |A_i| > 1 \text{ for } t < i \leq s.$$

We may rewrite the cellule dimension formula (1.9) as

$$\dim \mathcal{X}_\mathcal{A}(G) \;=\; 2s \;+\; \sum_{i=1}^{s} |K(A_i) \cap E|. \tag{1.16}$$

If $i \leq t$, then $K(A_i) = \emptyset$, while if $i > t$, then $|K(A_i) \cap E| \leq 2|A_i| - 3$ by (1.14). Hence

$$\begin{aligned} \dim \mathcal{X}_\mathcal{A}(G) \leq\; & 2s \;+\; \sum_{i=t+1}^{s} (2|A_i| - 3) \;=\; 2s + (2(n-t) - 3(s-t)) \\ =\; & 2n - s + t. \end{aligned} \tag{1.17}$$

Since $\mathcal{A}$ is not the discrete partition, we have $t < s$, so $\dim \mathcal{X}_\mathcal{A}(G) < 2n$ as desired.

$(iv) \implies (iii)$: No nonzero element of R_G vanishes on $\tilde{\mathcal{X}}(G)$, since the projection of the indiscrete cellule $\tilde{\mathcal{X}}_\mathcal{I}(G)$ on the second factor in (1.1) is surjective. On the other hand, every element of R_G that vanishes on $\tilde{\mathcal{S}}(G)$ vanishes on $\tilde{\mathcal{V}}(G)$. We conclude that if (iii) fails, then (iv) fails as well.

$(iii) \implies (ii)$: This is essentially the definition of the slope-rigidity matroid.

$(ii) \implies (i)$: Suppose that E is independent in $\mathcal{M}^s(V)$. Let $F \subset E$, and let H be the graph $(V(F), F)$. Then $\dim \tilde{\mathcal{V}}(H) = 2|V(F)|$, and the canonical surjection $\tilde{\mathcal{V}}(H) \to \tilde{\mathcal{S}}(H)$ has fiber dimension ≥ 3 (since translation and scaling do not affect slope), whence $\dim \tilde{\mathcal{S}}(H) \leq 2|V(F)| - 3$. On the other hand, F is independent in $\mathcal{M}^s(V)$ as well, so $\{m_f \mid f \in F\}$ is algebraically independent, and these variables form a system of parameters for $\tilde{\mathcal{S}}(G_F)$, so $|F| \leq 2|V(F)| - 3$. Therefore E satisfies Laman's condition (1.14) for independence in $\mathcal{M}(V)$. □

The equivalence of conditions (i) and (ii) implies the following:

Corollary 1.3.6. *For every vertex set V, the length-rigidity matroid $\mathcal{M}(V)$ and the slope-rigidity matroid $\mathcal{M}^s(V)$ are equal.*

1.4 Equations Defining $\tilde{\mathcal{V}}(G)$

Let $G = (V, E)$ be a connected graph. In this section, we explicitly construct an ideal $I = I_G$ defining the affine picture and affine slope varieties of G set-theoretically. The generators of I turn out to have a beautiful combinatorial description: their terms enumerate special spanning trees of the rigidity circuit subgraphs of G.

We begin with some computations which are most conveniently expressed in terms of the homology of G, considered as a 1-dimensional simplicial complex.

A *directed edge* of G is a symbol $[v, w]$, where $vw \in E$. An *orientation* of an edge $e = vw$ is chosen by putting either $e = [v, w]$ or $e = [w, v]$. In what follows, we fix an arbitrary orientation for each edge of G.

Let C be the free $\mathbb{Z}$-module on the directed edges of G, modulo the relations $[w, v] = -[v, w]$. (In homological terms, C is the set of 1-chains.) Each vertex

$v \in V$ may be regarded as a linear functional on C by the formula

$$v([x,y]) = \begin{cases} -1 & v = x \\ 1 & v = y \\ 0 & \text{otherwise.} \end{cases} \tag{1.18}$$

Let Z be the submodule of C on which all vertex functionals are zero (homologically speaking, the 1-cycles of G). Z is generated by the cycles

$$z(P) = \sum_{i=1}^{s} [v_i, v_{i+1}], \tag{1.19}$$

where $P = (v_1, \ldots, v_s, v_{s+1} = v_1)$ is a polygon of G.

The *support* $\mathrm{supp}(\gamma)$ of a chain $\gamma = \sum_{e \in E} c_e e$ is the set of edges e for which $c_e \neq 0$. Note that if $\gamma \in Z$ and $\mathrm{supp}(\gamma)$ is contained in some tree, then $\gamma = 0$.

Let T be a spanning tree of G and $S = E \setminus T$. For each edge $e \in S$ with $e = [v,w]$, the edge set $T \cup \{e\}$ contains a unique polygon of the form

$$P_T(e) = (v = v_0, v_1, \ldots, v_s = w, v), \tag{1.20}$$

to which corresponds the cycle

$$\begin{aligned} z_T(e) &= [v, v_1] + \cdots + [v_{s-1}, w] + [w, v] \\ &= -[v,w] + \sum_{i=0}^{s-1} [v_i, v_{i+1}] \\ &= -e + \sum_{f \in T} c^T_{ef} f, \end{aligned} \tag{1.21}$$

where $c^T_{ef} \in \{0, 1, -1\}$ for all f.

Note that the cycles $\{z_T(e) \mid e \in S\}$ generate Z for every spanning tree T. (If $\zeta = \sum_{e \in E} b_e e \in Z$, then let $\zeta' = \zeta + \sum_{e \in S} b_e z_T(e) = \sum_{f \in T} b'_e e$. Since $\mathrm{supp}(\zeta') \subset T$, we have $\zeta' = 0$.) Furthermore, there is an injective map of $\mathbb{Z}$-modules $C \to R'_G$ sending $[v,w]$ to $m_{vw}(x_v - x_w)$ for all directed edges $[v,w]$. The image of Z under this map is the $\mathbb{Z}$-module generated by the polynomials $L(P)$ defined in (1.5). Therefore, for every spanning tree T, the ideal defining $\tilde{\mathcal{X}}(G)$ is generated by the set $\{L(P_T(e)) \mid e \in S\}$.

Put $x_f = x_w - x_v$ for every edge $f = [v, w] \in T$. Let $e = [a, b] \in S$. Then

$$x_b - x_a \;=\; \sum_{f \in T} c^T_{ef} x_f, \tag{1.22}$$

so

$$L(P_T(e)) = \left(m_e \sum_{f \in T} c^T_{ef} x_f \right) - \left(\sum_{f \in T} c^T_{ef} m_f x_f \right) = \sum_{f \in T} c^T_{ef}(m_e - m_f) x_f. \tag{1.23}$$

We may summarize this calculation by a single matrix equation:

$$L_T \;=\; [L(P_T(e))] \;=\; M_T X_T \;=\; (D_S C_T - C_T D_T)\, X_T, \tag{1.24}$$

where

$$\begin{aligned}
L_T &= \text{the column vector } [L(P_T(e))]_{e \in S}\,; \\
C_T &= \left[c^T_{ef}\right]_{e \in S, f \in T}\,; \\
M_T &= \left[c^T_{ef}(m_e - m_f)\right]_{e \in S, f \in T}\,; \\
D_T &= \text{the diagonal matrix with entries } m_f,\ f \in T; \\
D_S &= \text{the diagonal matrix with entries } m_e,\ e \in S; \\
X_T &= \text{the column vector } [x_f]_{f \in T}\,.
\end{aligned} \tag{1.25}$$

Moreover, the defining equations (1.5) of $\mathcal{X}(G)$ are equivalent to the equation $L_T = 0$.

Example 1.4.1. Let $G = K_4$. Orient every edge $ij \in E(G)$ as $[i, j]$, where $i < j$. Let $T = \{[1, 2], [1, 3], [1, 4]\}$. For $e = [v, w] \notin T$, we have

$$P_T(e) = [v, w] + [w, 1] + [1, v] = [v, w] - ([1, w] - [1, v])\,,$$

so the coefficients c_{ef} defined in (1.21) are given by

$$\begin{array}{cc} & f \\ & \begin{array}{ccc} [1,2] & [1,3] & [1,4] \end{array} \\ e \;\begin{array}{c} [2,3] \\ [2,4] \\ [3,4] \end{array} & \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix} \end{array} \tag{1.26}$$

and the equations (1.5) are

$$\begin{aligned} L(P_{23}) &= m_{23}(x_3 - x_2) + m_{12}(x_2 - x_1) + m_{13}(x_1 - x_3) = 0, \\ L(P_{24}) &= m_{24}(x_4 - x_2) + m_{12}(x_2 - x_1) + m_{14}(x_1 - x_4) = 0, \\ L(P_{34}) &= m_{34}(x_4 - x_3) + m_{13}(x_3 - x_1) + m_{14}(x_1 - x_4) = 0. \end{aligned} \tag{1.27}$$

The linear change of variables $\bar{x}_v = x_v - x_1$ produces the matrix equation

$$M_T X_T = \begin{bmatrix} m_{12} - m_{23} & m_{23} - m_{13} & 0 \\ m_{12} - m_{24} & 0 & m_{24} - m_{14} \\ 0 & m_{13} - m_{34} & m_{34} - m_{14} \end{bmatrix} \begin{bmatrix} \bar{x}_2 \\ \bar{x}_3 \\ \bar{x}_4 \end{bmatrix} \tag{1.28}$$

and $M_T = D_S C_T - C_T D_T$.

If $G = (V, E)$ is a rigidity circuit, then M_T, C_T, and C_S are $(|V|-1) \times (|V|-1)$ square matrices.

Lemma 1.4.2. *Let $G = (V, E)$ be a rigidity circuit, $T \in \mathrm{SST}(G)$ a special spanning tree, and $S = E \setminus T$ (which is also a spanning tree of G). Then $C_T = C_S^{-1}$.*

Proof. The lemma is equivalent to the following: for all $e, g \in S$,

$$\sum_{f \in T} c^T_{ef}\, c^S_{fg} = \delta_{eg}, \tag{1.29}$$

where δ_{eg} is the Kronecker delta. Replacing each f on the right side of (1.21) with $f + z(S)f$, we see that

$$\zeta = e - \sum_{f \in T} c^T_{ef} \sum_{g \in S} c^S_{fg} g = e - \sum_{g \in S} g \sum_{f \in T} c^T_{ef} c^S_{fg} \in Z. \tag{1.30}$$

But $\text{supp}(\zeta) \subset S$, so $\zeta = 0$, implying (1.29). $\square$

Somewhat more generally, if G is a rigidity pseudocircuit and T, U are two spanning trees of G, then every member of the set $\{L(P_U(e)) \mid e \in E \setminus U\}$ may be expressed as an integer linear combination of the polynomials $\{L(P_T(e)) \mid e \in E \setminus T\}$, and vice versa. In addition $\{L(P_T(e)) \mid e \in E \setminus T\}$ is linearly independent (since each variable m_e, for $e \notin T$, appears in exactly one $L(P_T(e))$). Similarly, $\{L(P_U(e)) \mid e \in E \setminus U\}$ is linearly independent. Therefore

$$M_T X_T = B M_U X_U$$

for some invertible integer matrix B. In particular $\det B = \pm 1$, so $\det M_T$ is independent, up to sign, of the choice of T. Therefore, we are justified in defining

$$\tau(G) = \det M_T$$

up to sign. We call $\tau(G)$ the *tree polynomial* of G; the name is justified by the following result.

Theorem 1.4.3. *Let $G = (V, E)$ be a rigidity circuit, $|V| = n$, and $|E| = 2n - 2$.*

(i) *The polynomial $\tau(G)$ is homogeneous of degree $n-1$ and squarefree (that is, it is a sum of squarefree monomials). In addition, every coefficient of $\tau(G)$ is ± 1.*

(ii) *The monomials of $\tau(G)$ correspond to the special spanning trees of G, i.e.,*

$$\tau(G) = \sum_{T \in \mathrm{SST}(G)} \varepsilon(T) m_T. \tag{1.31}$$

where $\varepsilon(T) \in \{1, -1\}$ for each T and $m_T = \prod_{e \in T} m_e$.

(iii) *$\varepsilon(T) = (-1)^{n-1} \varepsilon(E \setminus T)$ for every $T \in \mathrm{SST}(G)$.*

(iv) *$\tau(G)$ vanishes on $\tilde{\mathcal{V}}(G)$.*

(v) *$\tau(G)$ vanishes on $\tilde{\mathcal{S}}(G)$.*

(vi) *$\tau(G)$ is irreducible.*

Proof. (i) By the construction of $\tau(G)$, each of its monomials is of degree $n-1$. Let $e \in E$. If $e \in T$ then m_e appears in only one column of M_T, while if $e \in S$ then m_e appears in only one row of M_T. It follows that $\tau(G)$ is squarefree.

(ii,iii) Let T be a spanning tree of G and $S = E \setminus T$. By the construction of $\tau(G)$ (in particular (1.24)), the monomials m_T and m_S appear in M_T with coefficients $\varepsilon(T) = (-1)^{n-1} \det C_T$ and $\varepsilon(S) = \det C_T$ respectively. If T and S are special, then $C_S = C_T^{-1}$ by Lemma 1.4.2; in particular $\det C_T = \det C_S = \pm 1$ since C_S and C_T have integer entries.

Now suppose that $U \subset E$ has cardinality $n-1$, but is not a special spanning tree. Then either U or $E \setminus U$ is not a tree; without loss of generality we may assume the former. We will show that the coefficients of m_U and $m_{E\setminus U}$ in $\tau(G)$ are both 0.

Let $A \subset U$ be a minimal set of edges such that $U \setminus A$ is a forest. Note that $A \neq \emptyset$. Let T be a spanning tree of G containing $U \setminus A$; then $T \cap (E \setminus U) \neq \emptyset$. Let $S = E \setminus T$ (so $S \supset A$). Construct the matrix M_T as in (1.24). Note that

$$c^T_{ab} = 0 \qquad \text{for all } a \in A,\ b \in T \setminus (U \setminus A) = T \cap (E \setminus U), \tag{1.32}$$

since the unique circuit of $T \cup \{a\}$ is contained in $U \setminus A \cup \{a\}$.

Let $a \in A$. By (1.32), every entry of the row of M_T corresponding to a is either zero or of the form $\pm(m_a - m_u)$, where $u \in U \setminus A$. In particular, no variable dividing $m_{E\setminus U}$ appears in this row, so the coefficient of $m_{E\setminus U}$ in $\tau(G)$ is 0. On the other hand, if $b \in T \setminus (U \setminus A) = T \cap (E \setminus U)$, we consider the column of M_T corresponding to b; by a similar argument, no variable dividing m_U appears in this column, so the coefficient of m_U in $\tau(G)$ is 0.

(iv) Since the generic affine pictures are dense in $\tilde{\mathcal{V}}(G)$, it suffices to show that $\tau(G)$ vanishes at each $\mathbf{P} \in \mathcal{V}^\circ(G) \cap \tilde{\mathcal{V}}(G)$. Indeed, $M_T(\mathbf{P})X_T(\mathbf{P}) = 0$ and $X_T(\mathbf{P}) \neq 0$, so $\tau(G) = \det M_T$ vanishes at $\mathbf{P}$.

(v) This is immediate from (iv) and the definition of $\tilde{\mathcal{S}}(G)$.

(vi) Suppose that $\tau(G) = f_1 \cdot f_2$. For every $e \in E$, we have

$$\deg_{m_e}(\tau(G)) = \deg_{m_e}(f_1) + \deg_{m_e}(f_2) = 1,$$

so $E = E_1 \cup E_2$, where $E_i = \{e \in E \mid \deg_{m_e}(f_i) = 1\}$. Let $G_i = (V, E_i)$. Since $\tilde{\mathcal{S}}(G)$ is by definition irreducible, either f_1 or f_2 vanishes on $\tilde{\mathcal{S}}(G)$ by part (v). Assume without loss of generality that f_1 vanishes on $\tilde{\mathcal{S}}(G)$. Then f_1 vanishes on $\mathcal{S}(G_1)$ as well via the natural surjection $\mathcal{S}(G) \to \mathcal{S}(G_i)$. By Theorem 1.3.5, E_1 must be rigidity-dependent, so $E_1 = E$ (because E contains no proper rigidity-dependent subset). Therefore $E_2 = \emptyset$ and the factorization of $\tau(G)$ is trivial. ☐

Remark 1.4.4. Given a connected graph $G = (V, E)$ with $|E| = 2|V| - 2$, not necessarily a rigidity circuit, and a spanning tree $T \subset E$, we may construct the matrix M_T and define $\tau(G) = \det M_T$ as before. If G is not a rigidity pseudocircuit, then the proof of Theorem 1.4.3 implies that $\tau(G) = 0$. If G is a rigidity pseudocircuit but not a circuit—say G contains a rigidity circuit $G' = (V', E')$ as a proper subgraph—then $\tau(G)$ is well-defined, up to sign, over all choices of T, and Theorem 1.4.3 goes through as before, with the exception that $\tau(G)$ is not irreducible. Indeed, let T' be a spanning tree of G' and $T \supset T'$ a spanning tree of G. Put $S = E \setminus T$ and $S' = E' \setminus T'$. Then the matrix M_T has the form

$$\begin{bmatrix} M_{T'} & 0 \\ * & * \end{bmatrix}$$

where the $|V'| - 1$ uppermost rows correspond to edges in S' and the $|V'| - 1$ leftmost columns correspond to edges in T'. It follows that $\tau(G')$ is a proper divisor of $\tau(G)$.

Example 1.4.5. Let $G = K_4$. Let T, M_T, X_T be as in Example 1.4.1. There are two kinds of spanning trees of G: paths (a, b, c, d), and "stars," such as T. The paths are special; the stars are not. There are $4!/2 = 12$ paths, and the sign of a path is given by the sign of the corresponding permutation in the symmetric group S_4, that is,

$$\tau(G) = \det M_T = -\frac{1}{2}\sum_{\sigma\in S_4} sgn(\sigma) m_{\sigma_1\sigma_2} m_{\sigma_2\sigma_3} m_{\sigma_3\sigma_4} \ . \tag{1.33}$$

On the other hand, if G' is the graph of Example 1.3.2 (a rigidity pseudocircuit which is not a circuit), then

$$\tau(G') = (m_{15} - m_{25})\, \tau(G).$$

Theorem 1.4.6. *Let $G = (V, E)$ be a graph. Let $I = I_G$ be the ideal of R_G generated by all tree polynomials $\tau(C)$, where $C \subset G$ is a rigidity circuit. Then:*

(i) *$\tilde{\mathcal{V}}(G)$ is the vanishing locus of IR'_G in $\tilde{\mathcal{X}}(G)$.*

(ii) *$\tilde{\mathcal{S}}(G)$ is the vanishing locus of I in $(\mathbb{A}^1)^r$.*

Proof. We may assume without loss of generality that G is connected, since every rigidity circuit is connected and $\tilde{\mathcal{V}}(G)$ is the product of the picture varieties of its connected components.

(i): Let Y be the vanishing locus of IR'_G in $\tilde{\mathcal{X}}(G)$. For every rigidity circuit $C \subset G$, the tree polynomial $\tau(C)$ vanishes on $\tilde{\mathcal{V}}(C)$ by Theorem 1.4.3, so it vanishes on $\tilde{\mathcal{V}}(G)$ as well. Hence $\tilde{\mathcal{V}}(G) \subset Y$.

We now establish the reverse inclusion, proceeding by induction on $n = |V|$. By Theorem 1.3.5, there is nothing to prove when E is rigidity-independent, in particular when $n \leq 3$.

Let $\mathbf{P} \in Y \cap \mathcal{X}_{\mathcal{A}}(G)$, where $\mathcal{A} = \{A_1, \ldots, A_s\}$ is a partition of V. If $\mathcal{A}$ is the discrete partition (that is, $s = n$), then there is nothing to prove.

Next, consider the case that $2 \leq s \leq n-1$. For each $i \in [s]$, let $G_i = G|_{A_i}$ be the restriction of G to the vertex set A_i (note that $|A_i| < n$). Also let

$$U = \bigcup_{\mathcal{B} \preceq \mathcal{A}} \tilde{\mathcal{X}}_{\mathcal{B}}(G) = \left\{ \mathbf{P}' \in \tilde{\mathcal{X}}(G) \;\mid\; \mathbf{P}'(v) \neq \mathbf{P}'(w) \text{ if } v \not\sim_{\mathcal{A}} w \right\}. \tag{1.34}$$

U is an open subset of $\tilde{\mathcal{X}}(G)$ containing $\mathbf{P}$, and there is a natural open embedding

$$\pi : U \to \prod_{i=1}^{s} \tilde{\mathcal{X}}(G_i).$$

Now $I_{G_i} \subset IR_G$ for every i. By induction, $\tilde{\mathcal{V}}(G_i)$ is the vanishing locus of I_{G_i} in $\tilde{\mathcal{X}}(G_i)$. Therefore

$$P \in \pi^{-1}\left(\prod_{i=1}^{s} \tilde{\mathcal{V}}(G_i) \right) . \tag{1.35}$$

The set on the right of (1.35) is irreducible and contains $\mathcal{V}^{\circ}(G)$ as an open, hence dense, subset. Therefore $\mathbf{P} \in \tilde{\mathcal{V}}(G)$, as desired.

Finally, suppose that $\mathcal{A}$ is the indiscrete partition of V. Thus $s = 1$ and $A_1 = V$. Fix a spanning tree T of G and let M_T be the matrix defined in (1.24). Recall that M_T is an $(r-n+1) \times (n-1)$ matrix, with rows indexed by $E \setminus T$ and columns indexed by T, and that $\tilde{\mathcal{X}}(G)$ is defined by the equations $M_T X_T = 0$, and that $X_T(\mathbf{P}) = 0$.

We claim that $M_T(\mathbf{P})$ has rank $< n-1$. If M_T has fewer than $n-1$ rows then there is nothing to prove. Otherwise, consider any $(n-1) \times (n-1)$ submatrix M' of M_T with rows indexed by the elements of some $S \subset E \setminus T$. Then $T \cup S$ does not satisfy (1.14), so it contains a rigidity circuit C, and $\tau(C)$ divides $\det M'$ by Remark 1.4.4, establishing the claim. Thus the nullspace of $M_T(\mathbf{P})$ contains a nonzero vector X'. For every $\lambda \in \mathbf{k}$, we have $(M_T)(\lambda X') = 0$, so there is a picture $\mathbf{P}_\lambda$ with the same slope coordinates as $\mathbf{P}$ and x-coordinates of vertices given by $\lambda X'$. The $\mathbf{P}_\lambda$ form an affine line in Y with $\mathbf{P}_0 = \mathbf{P}$. Moreover, if $\lambda \neq 0$, then $\mathbf{P}_\lambda \notin \mathcal{X}_{\mathcal{A}}(G)$, hence $\mathbf{P}_\lambda \in \tilde{\mathcal{V}}(G)$ as shown previously. Therefore $\mathbf{P}_0 = \mathbf{P} \in \tilde{\mathcal{V}}(G)$ as well.

(ii) Let Z be the vanishing locus of I in $(\mathbb{A}^1)^r$. It is immediate from the definition of the slope variety that $Z \supset \tilde{\mathcal{S}}(G)$. Now suppose that the tree polynomials vanish at some affine slope picture $\mathbf{m}$. Fix a spanning tree T of G and let X be a nullvector of the matrix $M_T(\mathbf{m})$. The data $\mathbf{m}$ and X define an affine line in $\tilde{\mathcal{X}}(G)$; by part (i) of the theorem, the line is contained in $\tilde{\mathcal{V}}(G)$. Therefore $\mathbf{m} \in \tilde{\mathcal{S}}(G)$. $\square$

In scheme-theoretic terms, we have proven that

$$\begin{aligned} \tilde{\mathcal{V}}(G) &\cong \mathbb{A}^1 \times \operatorname{Spec} R'_G/\sqrt{J_G} \qquad \text{and} \\ \tilde{\mathcal{S}}(G) &\cong \operatorname{Spec} R_G/\sqrt{I_G} \end{aligned} \tag{1.36}$$

as reduced schemes, where $J_G = I_G R'_G + (L(P))$. However, we do not know at this point whether every function vanishing on $\tilde{\mathcal{V}}(G)$ or $\tilde{\mathcal{S}}(G)$ lies in J_G or I_G respectively; i.e., whether these ideals are radical. In the special case that G is a rigidity cycle, the ideal I_G is radical because it is principal, generated by the irreducible polynomial $\tau(G)$. We will prove in Chapter 2 that I_G is radical when G is the complete graph K_n.

1.5 Geometric Properties of $\mathcal{X}(G)$ and $\mathcal{V}(G)$

In this section, we use the algebraic results of the previous sections to prove certain geometric facts about the picture space. First, we give a combinatorial condition which describes when one cellule of $\mathcal{X}(G)$ is contained in the closure of another cellule. Using this result, we can give a complete combinatorial description of the irreducible components of the picture space. Second, we prove that $\mathcal{V}(G)$ is Cohen-Macaulay when G is rigidity-independent.

Definition 1.5.1. Let $G = (V, E)$ be a graph, $F \subset E$, and $\mathcal{A}$ a partition of V. We say $\mathcal{A}$ *collapses* F if $V(F)$ is contained in some block of $\mathcal{A}$. In this case, the equations defining $\mathcal{X}(G)$ impose no restrictions on the slopes of the lines $\mathbf{P}(e)$ for pictures $\mathbf{P} \in \mathcal{X}_{\mathcal{A}}(G)$ and edges $e \in F$.

Theorem 1.5.2. *Let $G = (V, E)$ be a graph and $\mathcal{A}$ a partition of V. Then $\mathcal{X}_{\mathcal{A}}(G) \subset \mathcal{V}(G)$ if and only if no rigidity circuit of G is collapsed by $\mathcal{A}$.*

Proof. We first consider two special cases. If E is rigidity-independent, then $\mathcal{V}(G) = \mathcal{X}(G)$ by Theorem 1.3.5, so there is nothing to prove. Now suppose that E is a rigidity circuit. The only partition of V which collapses E is the indiscrete partition $\mathcal{I}$. The cellule dimension formula (1.9) gives $\dim \mathcal{X}_{\mathcal{I}}(G) = 2n = \dim \mathcal{V}^{\circ}(G)$, so $\mathcal{X}_{\mathcal{I}}(G) \not\subset \overline{\mathcal{V}^{\circ}(G)} = \mathcal{V}(G)$. On the other hand, if $\mathcal{A}$ is neither the discrete nor indiscrete partition, then $\dim \mathcal{X}_{\mathcal{A}}(G) < 2n$ by (1.17) (since the inequality (1.14) holds for every proper subset of a rigidity circuit). The indiscrete cellule is itself closed, and all components of $\mathcal{X}(G)$ have dimension $\geq 2n$, so it follows that $\mathcal{X}_{\mathcal{A}}(G) \subset \mathcal{V}(G)$.

We turn now to the general case. By Theorem 1.4.6, it is enough to prove that for every rigidity circuit C of G, $\tau(C)$ vanishes on $\tilde{\mathcal{X}}_{\mathcal{A}}(G)$ if and only if $\mathcal{A}$ does not collapse C. One direction is immediate: if $\mathcal{A}$ collapses C, then $\tau(C)$ does not vanish on $\tilde{\mathcal{X}}_{\mathcal{A}}(G)$ and consequently $\tilde{\mathcal{X}}_{\mathcal{A}}(G) \not\subset \tilde{\mathcal{V}}(G)$. On the other hand, suppose that $\mathcal{A}$ does not collapse C. Consider the natural map $\tilde{\mathcal{X}}_{\mathcal{A}}(G) \to \tilde{\mathcal{X}}(C)$. The image Z of this map does not intersect the indiscrete cellule of $\tilde{\mathcal{X}}(C)$. By the special case, $\tau(C)$ vanishes on Z, hence on $\mathcal{X}_{\mathcal{A}}(G)$. ∎

Given a graph $G = (V, E)$ and a partition $\mathcal{A} = \{A_1, \ldots, A_s\}$ of V, we define a graph $G/\mathcal{A}$ with vertices $\{A_i\}$ and edges

$$\big\{\{A_i, A_j\} \;\mid\; vw \in E \text{ for some } v \in A_i,\; w \in A_j\big\}.$$

Also, if $\mathcal{A}$ and $\mathcal{B}$ are partitions of V with $\mathcal{A} \preceq \mathcal{B}$, then we write $\mathcal{B}/\mathcal{A}$ for the partition on the blocks of $\mathcal{A}$ setting two blocks equivalent if both are subsets of the same block of $\mathcal{B}$.

Theorem 1.5.3. *Let $G = (V, E)$ be a graph, and let $\mathcal{A}, \mathcal{B}$ be partitions of V. Then $\mathcal{X}_{\mathcal{B}}(G) \subset \overline{\mathcal{X}_{\mathcal{A}}(G)}$ if and only if the following conditions hold:*

(a) $\mathcal{A} \preceq \mathcal{B}$*;*

(b) No rigidity circuit of $G/\mathcal{A}$ *is collapsed by* $\mathcal{B}/\mathcal{A}$*; and*

(c) If A_i *and* A_j *are distinct blocks of* $\mathcal{A}$ *contained in the same block of* $\mathcal{B}$*, then there is at most one edge of* E *with one endpoint in* A_i *and one in* A_j*.*

Proof. It is sufficient to prove the corresponding statement for the affine cellules $\tilde{\mathcal{X}}_{\mathcal{B}}(G) = \mathcal{X}_{\mathcal{B}}(G) \cap \tilde{\mathcal{X}}(G)$ and $\tilde{\mathcal{X}}_{\mathcal{A}}(G) = \mathcal{X}_{\mathcal{A}}(G) \cap \tilde{\mathcal{X}}(G)$.

Suppose that $\mathcal{X}_{\mathcal{B}}(G) \subset \overline{\mathcal{X}_{\mathcal{A}}(G)}$. If $v \sim_{\mathcal{A}} w$, the equation $\mathbf{P}(v) = \mathbf{P}(w)$ holds on $\mathcal{X}_{\mathcal{B}}(G)$, so $v \sim_{\mathcal{B}} w$, which establishes (a). For every rigidity circuit C of $G/\mathcal{A}$, the function $\tau(C)$ vanishes on $\mathcal{X}_{\mathcal{B}}(G)$, so $\mathcal{B}/\mathcal{A}$ cannot collapse C. Finally, if A_i and A_j are contained in the same block of $\mathcal{B}$ and e, e' form a counterexample to (c), then the equation $\mathbf{P}(e) = \mathbf{P}(e')$ holds on $\mathcal{X}_{\mathcal{A}}(G)$ but not on $\mathcal{X}_{\mathcal{B}}(G)$, a contradiction.

Now suppose that conditions (a), (b) and (c) hold. Let $E' = \{vw \in E \mid v \sim_{\mathcal{A}} w\}$, $U = \mathbb{A}^{|E'|}$, and

$$Z = \bigcup_{\mathcal{B} \succeq \mathcal{A}} \tilde{\mathcal{X}}_{\mathcal{B}}(G) = \left\{ \mathbf{P} \in \tilde{\mathcal{X}}(G) \;\middle|\; \mathbf{P}(v) = \mathbf{P}(w) \text{ if } v \sim_{\mathcal{A}} w \right\}. \tag{1.37}$$

Observe that an affine picture $\mathbf{P} \in Z$ is equivalent to a picture of $G/\mathcal{A}$, together with the slopes of the lines $\mathbf{P}(e)$ for $e \in E'$. That is, we have an isomorphism

$$\pi : Z \to \tilde{\mathcal{X}}(G/\mathcal{A}) \times U, \tag{1.38}$$

Restricting π to the cellules under consideration, we have a commutative diagram of quasiaffine varieties:

$$\begin{array}{ccccccc} \tilde{\mathcal{X}}_{\mathcal{A}}(G) & \subset & \overline{\tilde{\mathcal{X}}_{\mathcal{A}}(G)} & \subset & Z & \supset & \tilde{\mathcal{X}}_{\mathcal{B}}(G) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \tilde{\mathcal{V}}^{\circ}(G/\mathcal{A}) \times U & \subset & \tilde{\mathcal{V}}(G/\mathcal{A}) \times U & \subset & \tilde{\mathcal{X}}(G/\mathcal{A}) \times U & \supset & \tilde{\mathcal{X}}_{\mathcal{B}/\mathcal{A}}(G/\mathcal{A}) \times U \end{array} \tag{1.39}$$

where the vertical arrows are isomorphisms. From the diagram, $\tilde{\mathcal{X}}_{\mathcal{B}}(G) \subset \overline{\tilde{\mathcal{X}}_{\mathcal{A}}(G)}$ if and only if $\tilde{\mathcal{X}}_{\mathcal{B}/\mathcal{A}}(G/\mathcal{A}) \subset \tilde{\mathcal{V}}(G/\mathcal{A})$. By Theorem 1.5.2, this is equivalent to the condition that no rigidity circuit of $G/\mathcal{A}$ is collapsed by $\mathcal{B}/\mathcal{A}$, as desired. □

Remark 1.5.4. The notion of a pseudocircuit may be extended to multigraphs: a multigraph (V, E) is called a *pseudocircuit* if $|E| = 2|V| - 2$ and $|F| \leq 2|V(F)| - 2$ for all $\emptyset \neq F \subset E$ [8, p. 118]. For instance, a double edge is a pseudocircuit. In the previous theorem, we may consider $G/\mathcal{A}$ as a multigraph, in which the multiplicity of an edge $\{A_i, A_j\}$ is the number of edges in E with one endpoint in each of A_i and A_j. Then conditions (b) and (c) together are equivalent to the single condition that $\mathcal{B}/\mathcal{A}$ collapse no (multigraph) pseudocircuit of $G/\mathcal{A}$.

An immediate consequence of Theorem 1.5.3 is the following characterization of the irreducible components of $\mathcal{X}(G)$.

Theorem 1.5.5. *Let $G = (V, E)$. Then the irreducible components of $\mathcal{X}(G)$ are exactly the subvarieties $\overline{\mathcal{X}_{\mathcal{A}}(G)}$, where $\mathcal{A}$ is maximal with respect to the partial order described in Theorem 1.5.3.* □

We now investigate the Cohen-Macaulay property. Our main tool is the fact that if X is a Cohen-Macaulay scheme and Z is a "strongly Cohen-Macaulay" subscheme of X, then the blowup of X along Z is Cohen-Macaulay [9, Theorem 4.2] (see also [16]). One example of a strongly Cohen-Macaulay subscheme is a local complete intersection.

Lemma 1.5.6. *Let $G = (V, E)$, $e = vw \in E$, and $H = (V, E \setminus \{e\})$. Suppose that $\mathcal{V}(H)$ is Cohen-Macaulay and that $\mathcal{V}(H) \cap \mathcal{X}_{\mathcal{A}}(H)$ has codimension ≥ 2 in $\mathcal{V}(H)$ for all partitions $\mathcal{A}$ of V with $v \sim_{\mathcal{A}} w$. Then $\mathcal{V}(G)$ is Cohen-Macaulay.*

Proof. Let Z be the (possibly non-reduced) intersection $\mathcal{V}(H) \cap C_e(H)$, defined in local affine coordinates by the equations $x_v = x_w$, $y_v = y_w$. Z is defined locally by two equations, so each of its components has codimension ≤ 2. On the other hand, speaking set-theoretically, $C_e(H)$ is the union of cellules $\mathcal{X}_\mathcal{A}(H)$ with $v \sim_\mathcal{A} w$; by assumption, $C_e(H)$ has codimension ≥ 2, so Z does as well. Therefore Z is a local complete intersection in $\mathcal{V}(H)$, and $\mathcal{V}(G)$ is the blowup of $\mathcal{V}(H)$ along Z, so $\mathcal{V}(G)$ is Cohen-Macaulay. □

Theorem 1.5.7. *Let $G = (V, E)$, $e = vw \in E$, and $H = (V, E \setminus \{e\})$. If $\mathcal{V}(H)$ is Cohen-Macaulay and e is not contained in any rigidity circuit subgraph of G, then $\mathcal{V}(G)$ is Cohen-Macaulay.*

Proof. Let $\mathcal{A}$ be a partition of V with $v \sim_\mathcal{A} w$. The cellule $\mathcal{V}_\mathcal{A}(G) = \mathcal{X}_\mathcal{A}(G) \cap \mathcal{V}(G)$ has positive codimension in $\mathcal{V}(G)$. Since no rigidity circuit contains e, the equations defining $\mathcal{V}_\mathcal{A}(G)$ impose no constraints on the line $\mathbf{P}(e)$. Therefore

$$\mathcal{V}_\mathcal{A}(G) \cong \mathcal{V}_\mathcal{A}(H) \times \mathbb{P}^1.$$

In particular $\mathcal{V}_\mathcal{A}(H)$ has codimension ≥ 2 in $\mathcal{V}(H)$, since $\dim \mathcal{V}(G) = \dim \mathcal{V}(H) = 2|V|$. Thus $\mathcal{V}(G)$ is Cohen-Macaulay by Lemma 1.5.6.

Theorem 1.5.8. *Let $G = (V, E)$. If G is rigidity-independent, then $\mathcal{V}(G)$ is Cohen-Macaulay.* □

Proof. If $E = \emptyset$, the result is trivial since $\mathcal{V}(G) \cong (\mathbb{P}^2)^{|V|}$. Otherwise, we add one edge at a time, applying Theorem 1.5.7 at each stage. □

Chapter 2

The Slope Variety of the Complete Graph

2.1 Introduction

In this chapter, we study the graph varieties $\mathcal{V}(G)$ and $\mathcal{S}(G)$ in the special case that G is the complete graph K_n. Our results are summarized in Theorem 2.5.1.

First, we study the degree of the affine slope variety $\tilde{\mathcal{S}}(K_n)$. We set up a bivariate recurrence giving a lower bound on the degree of the "flattened slope variety" $\mathcal{S}(n,k)$, whose points correspond to pictures of K_n in which the vertices $1 \dots k$ are required to be collinear. We then show that this geometric recurrence is equivalent to a purely combinatorial one enumerating decreasing planar trees, allowing us to conclude that

$$\deg \tilde{\mathcal{S}}(K_n) \leq \frac{(2n-4)!}{2^{n-2}(n-2)!}. \tag{2.1}$$

It is possible to prove directly that the tree polynomials of all wheel subgraphs of K_n (cf. Example 1.3.2) generate the ideal $I_n = I_{K_n}$ of Theorem 1.4.6. In this chapter, we will prove a stronger statement: the tree polynomials of wheels are a Gröbner basis for I_n under a large class of term orderings.

We begin with a detailed examination of the tree polynomials of wheels and their leading monomials (fixing a convenient term order). We obtain an explicit combinatorial characterization of the initial ideal $\mathrm{in}(I_n)$ (Theorem 2.3.14) and of the maximal monomials not divisible by any such leading monomial (Theorem 2.4.8). There is a bijection between these monomials and the decreasing planar trees with $n-1$ nodes. The number of such trees is given by the right side of (2.1); therefore, the monomials are precisely the facets of the *Stanley-Reisner simplicial complex* $\Delta(n)$ of $\mathrm{in}(I_n)$.

The Stanley-Reisner theory provides a powerful combinatorial tool for determining geometric properties of an ideal [3], [17]. In the present case, the complex $\Delta(n)$ is pure and shellable, which implies that $\tilde{\mathcal{S}}(K_n)$ is Cohen-Macaulay. The proof of shellability allows us to give a recurrence for the h-vector of $\Delta(n)$, which appears to have an unexpected connection to the combinatorics of perfect matchings.

2.2 A Lower Bound for the Degree of $\tilde{\mathcal{S}}(K_n)$

2.2.1 The Recurrence

We derive a recurrence giving a lower bound for the degree of the affine slope variety $\tilde{\mathcal{S}}(K_n)$. We will make use of the following facts about degree:

1. Degree is defined (set-theoretically) as follows. If X is a d-dimensional algebraic subset of affine n-space, then a general hyperplane of codimension d will intersect X in finitely many points. The degree of X, $\deg X$, is then defined as the maximum number of intersection points.

2. If H is any hyperplane (of arbitrary codimension), then

$$\deg(X \cap H) \leq \deg X. \tag{2.2}$$

3. For all varieties X and Y, we have $\deg(X \times Y) = (\deg X)(\deg Y)$.

4. If $C_1, \ldots, C_k$ are the irreducible components of X, with $\dim C_i = \dim X$ for $1 \leq i \leq r$ and $\dim C_i < \dim X$ for $i > r$, then then

$$\deg X = \sum_{i=1}^{r} \deg C_i. \tag{2.3}$$

We will actually compute the degree of a slightly more general variety than $\tilde{\mathcal{S}}(K_n)$, the "flattened slope variety" defined by

$$\tilde{\mathcal{S}}(n,k) = \{P \in \tilde{\mathcal{S}}(K_n) \mid m_{ij}(P) = 0 \text{ for } 1 \leq i < j \leq k\}. \tag{2.4}$$

(Thus $\tilde{\mathcal{S}}(n,1) = \tilde{\mathcal{S}}(K_n)$, and $\tilde{\mathcal{S}}(n,n)$ consists of a single point.) We will derive a recurrence giving lower bounds $e(n,k)$ for the degrees of the varieties $\tilde{\mathcal{S}}(n,k)$. Note that for a picture in $\tilde{\mathcal{S}}(n,k)$, the first k vertices lie on a common horizontal line.

Our procedure is as follows. We intersect a sequence of hyperplanes with $\tilde{\mathcal{S}}(K_n)$ by requiring a sequence of linear forms to vanish, one at a time. At each stage, one of two things will happen. The first possibility is that the dimension drops by one; in this case, the degree either remains unchanged or decreases. The second possibility is that the algebraic set under consideration breaks into several components. Each of these can be shown to be isomorphic to a variety of the form $\tilde{\mathcal{S}}(i,j) \times \tilde{\mathcal{S}}(i',j')$. Summing the degrees of the components of maximal dimension then yields the desired recurrence.

We will make frequent use of the canonical surjection $\phi : \tilde{\mathcal{V}}(K_n) \to \tilde{\mathcal{S}}(K_n)$ defined in (1.12).

Lemma 2.2.1. $\dim \tilde{\mathcal{S}}(n,k) = \begin{cases} 0 & k = n \\ 2n-k-2 & k < n. \end{cases}$

Proof. When $k = n$, $\tilde{\mathcal{S}}(n,k)$ is a point. Otherwise, consider a picture $\mathbf{P} \in \phi^{-1}(\tilde{\mathcal{S}}(n,k))$. We have two degrees of freedom in choosing $\mathbf{P}(v_1)$, one for each of the points $\mathbf{P}(v_2), \ldots, \mathbf{P}(v_k)$ (since they must lie on the same horizontal line as $\mathbf{P}(v_1)$) and two for each of the points $\mathbf{P}(v_{k+1}), \ldots, \mathbf{P}(v_n)$. This totals $2n-k+1$. Subtracting 3 for translation and scaling gives $2n-k-2$ as claimed. □

Theorem 2.2.2. *Let $n \geq 2$ and $k \in [n]$. Then*

$$\deg \tilde{\mathcal{S}}(n,k) \geq e(n,k), \tag{2.5}$$

where $e(n,k)$ is defined recursively by

$$\begin{aligned} e(n,n) &= 1, \\ e(n,k) &= e(n,k+1) \ + \ e(n-1,k-1) \ + \\ &\sum_{w=1}^{k-1} \sum_{x=0}^{n-k-2} \binom{k-1}{w} \binom{n-k-1}{x} e(w+x+1,w) \ e(n-w-x,k+1-w). \end{aligned} \tag{2.6}$$

Proof. Let

$$\mathcal{S}'(n,k) = \{\mathbf{m} \in \mathcal{S}(n,k) \ | \ m_{k,k+1}(\mathbf{m}) = 0\}. \tag{2.7}$$

Note that

$$\dim \mathcal{S}'(n,k) = \dim \mathcal{S}(n,k) - 1 = 2n-k-3 \tag{2.8}$$

and that

$$\deg \mathcal{S}'(n,k) \leq \deg \mathcal{S}(n,k) \tag{2.9}$$

by (2.2). If $\mathbf{P} \in \phi^{-1}(\mathcal{S}'(n,k))$, then $\mathbf{P}(m_{k,k+1}) = 0$, so $\mathbf{P}(v_{k+1})$ must lie on the same horizontal line as do $\mathbf{P}(v_1), \ldots, \mathbf{P}(v_k)$. The next step is to break $\mathcal{S}'(n,k)$ into pieces depending on which vertices other than $\mathbf{P}(v_1)$ coincide with $\mathbf{P}(v_k)$ (again, for all $\mathbf{P} \in \phi^{-1}(\mathcal{S}'(n,k))$).

Let $W \subset [2,k]$, $X \subset [k+2,n]$, and $Y = [k+2,n] \setminus X$. Set $w = |W|$ and $x = |X|$ (so $|Y| = n-k-x-1$). Define

$$\begin{aligned} &\mathcal{S}'(n,k)_{W,X} \;= \\ &\quad \mathcal{S}'(n,k) \;\cap\; \phi\{\mathbf{P} \in \tilde{\mathcal{V}}(K_n) \;\mid\; \mathbf{P}(v_i) = \mathbf{P}(v_{k+1}) \text{ iff } i \in W \cup X\}. \end{aligned} \tag{2.10}$$

Case 1: $W = \emptyset$. In this case, it does not matter what X is, since the linear forms $m_{2,k+1}, \ldots, m_{k,k+1}$ must all vanish on $\mathcal{S}'(n,k)_{W,X}$. Therefore

$$\bigcup_X \mathcal{S}'(n,k)_{\emptyset,X} = \mathcal{S}(n,k+1). \tag{2.11}$$

Note that this set has dimension $2n-k-3$.

Case 2: $W \neq \emptyset$. Consider the map π_1 projecting $\mathcal{S}'(n,k)_{W,X}$ on the coordinates of edges with both endpoints in $W \cup X \cup \{k+1\}$. A typical point in the image is a slope picture of K_{w+x+1} in which every m_{ij} vanishes for $i,j \in W$. Therefore

$$\pi_1\left(\mathcal{S}'(n,k)_{W,X}\right) \cong \mathcal{S}(w+x+1,w). \tag{2.12}$$

Now consider the map π_2 projecting $\mathcal{S}'(n,k)_{W,X}$ on the coordinates of edges with both endpoints in $[n] \setminus (W \cup X) = [k+1] \setminus W \cup Y$. Note that the coordinates $m_{i,k+1}$ must all vanish for $i \notin w$, since for every $\mathbf{P} \in \phi^{-1}(\mathcal{S}'(n,k)_{W,X})$, the vertices $\mathbf{P}(v_i)$ and $\mathbf{P}(v_{k+1})$ lie on a common horizontal line but are not equal. Therefore, a typical point in the image is a slope picture of K_{n-w-x} in which every m_{ij} vanishes for $i,j \in [k+1] \setminus W$. Therefore

$$\pi_2\left(\mathcal{S}'(n,k)_{W,X}\right) \cong \mathcal{S}(n-w-x,k+1-w). \tag{2.13}$$

Combining (2.12) and (2.13) gives

$$\mathcal{S}'(n,k)_{W,X} \cong \mathcal{S}(w+x+1,w) \times \mathcal{S}(n-w-x,k+1-w). \tag{2.14}$$

The dimension of the right side of (2.14) is

$$\begin{cases} 2n-k-3+(w-k+1) & X=[k+2,n] \\ 2n-k-3 & \text{otherwise.} \end{cases}$$

So $\mathcal{S}'(n,k)_{W,X}$ is a maximal-dimensional component of $\mathcal{S}(n,k)$ unless

$$Y=\emptyset \quad \text{and} \quad W\neq[2,k]. \tag{2.15}$$

By (2.3), we can now obtain a lower bound for $\deg\mathcal{S}'(n,k)$ by summing the contributions from all components not excluded by (2.15). The same lower bound holds for $\deg\mathcal{S}(n,k)$ by (2.9).

The lower bound is given by the recurrence (2.6). The term $e(n,k+1)$ comes from (2.11), and the term $e(n-1,k-1)$ comes from the component $\mathcal{S}'(n,k)_{W,X}$ with $W=[2,k]$ and $X=[k+2,n]$. Finally, the double sum gives the contributions of the components $\mathcal{S}'(n,k)_{W,X}$ described in case 2, with $X\neq[k+2,n]$. The binomial coefficients count the number of ways to choose the sets W and X. □

2.2.2 Decreasing Planar Trees

The recurrence (2.6) of Theorem 2.2.2 has a combinatorial interpretation in terms of *decreasing planar trees*. We begin by defining decreasing planar trees and listing some of their basic properties, then show that they are enumerated by a recurrence equivalent to (2.6).

Definition 2.2.3. A *rooted planar tree* is a tree $T=(V,E)$ with the following additional structure.

(1) We designate a unique vertex $v\in V$ as the *root* of T, writing $v=\mathrm{rt}(T)$. For w,w' in T, we say that w is an *ancestor* of w' (equivalently, w' is a *descendant* of w) iff there is a path in E of the form $(\mathrm{rt}(T),\ldots,w,\ldots,w')$. If w is an ancestor of w' and $\{w,w'\}\in E$, then w is the *parent* of w' and w' is a *child* of w. The root

vertex has no parent; each other vertex has a unique parent. If two vertices have the same parent, they are said to be *siblings*. A vertex with no children is called a *leaf*; a vertex that is not a leaf is *internal*.

(2) For every $v \in V$, the set of children of v is equipped with a total ordering $>_v$. Typically, we think of the children as being arranged left to right.

We will frequently use the term "nodes" to refer to the vertices of a planar tree, in order to avoid confusion when the nodes are labeled with sets of vertices of K_n.

Definition 2.2.4. A *binary tree* is a rooted planar tree T in which every node has either 0 or 2 children.

Definition 2.2.5. Let V be a finite set and $>$ a total ordering of V (typically, $V \subset \mathbb{N}$). A rooted planar tree T on V is a *decreasing planar tree* if $v > w$ whenever v is an ancestor of w. (Note that there are no restrictions on the orderings $>_v$.) We set

$$\begin{aligned} \mathrm{Dec}(V) &= \{\text{decreasing planar trees on } V\}, \\ \mathrm{Dec}(n) &= \mathrm{Dec}([n]), \\ d(n) &= |\,\mathrm{Dec}(n)|. \end{aligned} \tag{2.16}$$

Note that $\mathrm{rt}(T) = \max(V)$ for all $T \in \mathrm{Dec}(V)$.

Note that $|\,\mathrm{Dec}(1)| = |\,\mathrm{Dec}(2)| = 1$. In addition, if $T = (V, E) \in \mathrm{Dec}(n-1)$, we may construct a tree $T' \in \mathrm{Dec}(n)$ by incrementing all node labels and attaching a new node labeled 1. The number of ways in which this can be done is $|V| + |E| = 2n-3$ (because 1 is either the first child of a vertex v, or the edge connecting 1 to its parent is immediately to the right of some edge e), so

$$|\,\mathrm{Dec}(n)| \;=\; (2n-3)(2n-5)\ldots(3)(1) \;=\; \frac{(2n-2)!}{(n-1)!\;2^{n-1}}. \tag{2.17}$$

A rooted planar tree on V can be represented by a diagram in which the children of each vertex v are placed immediately below their parents, ordered left to right according to $>_v$. In the figure below, the two trees on the right are not equal.

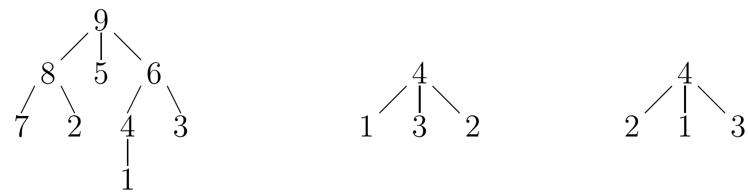


Figure 2.1: Examples of decreasing planar trees

We denote by $T|v$ the subtree of T consisting of the node v and all its descendants. The subtrees of $\mathrm{rt}(T)$ are called *principal subtrees*. We denote the parent of v by the symbol v^P, and its ith child by v^i. This notation can be iterated: for instance, v^{P2} is the second child of the parent of v.

Let v and w be vertices of a binary tree T. We say that w is a *left descendant* of v if

$$w \in \{v^1, v^{11}, \dots\},$$

and w is a *right descendant* of v if

$$w \in \{v^2, v^{22}, \dots\}.$$

Each set W of nodes of T has a well-defined supremum $\sup(W)$, the "youngest common ancestor" of its members.

The *largest leaf* of a decreasing planar tree T is

$$L(T) = \max\{v \in V \ \mid \ v \text{ is a leaf}\}.$$

Finally, we define an operation which combines two decreasing planar trees.

Definition 2.2.6. Let $T_1 \in \mathrm{Dec}(V_1)$ and $T_2 \in \mathrm{Dec}(V_2)$, with $V_1 \cap V_2 = \emptyset$ and $\max(V_1) < \max(V_2)$. The decreasing planar tree

$$T_1 * T_2$$

on $V_1 \cup V_2$ is constructed by attaching $\operatorname{rt} T_1$ as the leftmost child of $\operatorname{rt}(T_2)$.

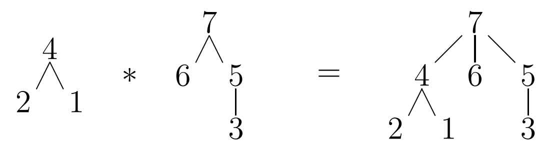


Figure 2.2: The $*$ operation on decreasing planar trees

Lemma 2.2.7. *For* $1 < K \leq N$,

$$d(N, K-1) = \sum_{a=1}^{K-1} \binom{K-1}{a} \, d(a,a) \; d(N-a, K-a). \tag{2.18}$$

Proof. The right side of (2.18) counts trees $T \in \operatorname{Dec}(N)$ of the form $T = T' * T''$, where

$$\begin{aligned} &V(T') = A \subseteq [K-1], \\ &|A| = a, \\ &L(T'') \leq K. \end{aligned} \tag{2.19}$$

Given such a tree T, form a tree $f(T)$ by detaching T' and reattaching it as the leftmost subtree of K. Thus $f(T) \in \operatorname{Dec}(N, K-1)$.

On the other hand, let $U \in \operatorname{Dec}(N, K-1)$. Note that K is an internal vertex of U, and all descendants of K are in $[K-1]$. Let U_1 be the leftmost subtree of K and form the tree $g(U) = U_1 * (U - U_1)$; i.e., detach U_1 and reattach it as the leftmost subtree of N. The result is a tree of type (2.19). The functions f and g are inverses, so we have a bijection which establishes (2.18). $\square$

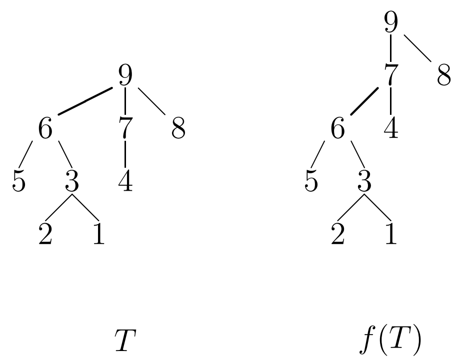


Figure 2.3: An example with $N = 9$ and $K = 7$

Lemma 2.2.8. *For* $1 < K \leq N$*,*

$$\sum_{a=1}^{K-1} \sum_{c=1}^{N-K-1} \binom{K-1}{a} \binom{N-K-1}{c} \; d(a+c, a) \; d(N-a-c, K-a) \;\; = \\ \sum_{w=0}^{K-2} \sum_{y=1}^{N-K-1} \binom{K-1}{w} \binom{N-K-1}{y} \; d(w+y, w+1) \; d(N-w-y, K-w-1). \tag{2.20}$$

Proof. The left side of (2.20) counts trees $T = T' * T'' \in \mathrm{Dec}(N)$ such that

$$\begin{aligned}
&V(T') = A \uplus C, \\
&V(T'') = B \uplus \{K\} \uplus D \uplus \{N\}, \\
&A \uplus B = [K-1], \\
&|A| = a, \qquad |B| = K-a-1, \qquad 1 \leq a \leq K-1, \\
&C \uplus D = [K+1, N-1], \\
&|C| = c, \qquad |D| = N-K-c-1, \qquad 1 \leq c \leq N-K-1, \\
&L(T') \in A, \\
&L(T'') \in B \cup \{K\}.
\end{aligned} \tag{2.21}$$

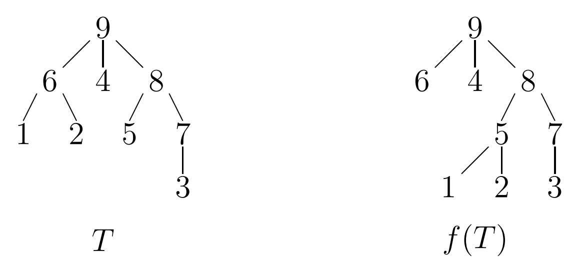
Meanwhile, the right side of (2.20) counts trees $U = U' * U'' \in \mathrm{Dec}(N)$ such that

$$\begin{aligned}
&V(U') = W \mathbin{\dot\cup} Y, \\
&V(U'') = X \mathbin{\dot\cup} \{K\} \mathbin{\dot\cup} Z \mathbin{\dot\cup} \{N\}, \\
&W \mathbin{\dot\cup} X = [K-1], \\
&|W| = w, \qquad |X| = K - w - 1, \qquad 0 \le w \le K-2, \\
&Y \mathbin{\dot\cup} Z = [K+1, N-1], \\
&|Y| = y, \qquad |Z| = N - K - y - 1, \qquad 1 \le y \le N-K-1, \\
&L(T') \in W \cup \{\min(Y)\}, \\
&L(T'') \in X.
\end{aligned} \tag{2.22}$$

Let $T = T' * T''$, where T' and T'' are as in (2.21). Define a tree $f(T)$ as follows. If K is not a leaf of T, then $f(T) = T$. Otherwise, detach all descendants of $\min(C)$ and reattach them to K in the same order. The resulting tree satisfies (2.22), with

$$\begin{aligned}
W &= A \setminus \{\text{descendants of } \min(C)\}, \\
X &= B \mathbin{\dot\cup} \{\text{descendants of } \min(C)\}, \\
Y &= C, \\
Z &= D.
\end{aligned} \tag{2.23}$$

$A = \{1, 2\}$
$B = \{3, 4\}$
$C = \{6\}$
$D = \{7, 8\}$



$W = \emptyset$
$X = \{1, 2, 3, 4\}$
$Y = \{6\}$
$Z = \{7, 8\}$

Figure 2.4: An example of the "otherwise" case with $N = 9$ and $K = 5$

Let $U = U' * U''$, where U' and U'' are as in type (2.22). Define a tree $g(U)$ as follows. If $\min(Y)$ is not a leaf of U, then $g(U) = U$. Otherwise, detach all descendants of K and reattach them to $\min(Y)$ in the same order. The resulting tree satisfies (2.21), with

$$\begin{aligned} A &= W \cup \{\text{descendants of } \min(Y)\}, \\ B &= X \setminus \{\text{descendants of } \min(Y)\}, \\ C &= Y, \\ D &= Z. \end{aligned} \tag{2.24}$$

Note that for all $T = T' * T'' \in \mathrm{Dec}(N)$ satisfying (2.21), $\min(C)$ is not a leaf of T, because $L(T') \in A$, so all leaves of T' are in A. Similarly, for all $U = U' * U'' \in DPT(N)$ satisfying (2.22), K is not a leaf of U, because $L(U'') \in X$, so all leaves of U' are in X. So if $f(T) = T$, then $g(f(T)) = T$, and if $g(U) = U$, then $f(g(U)) = U$. Meanwhile, in the "otherwise" cases, the two surgeries are inverse operations by definition. It follows that f and g are inverse functions, hence bijections, establishing (2.20). ☐

Lemma 2.2.9. *For* $1 < K \leq N$*,*

$$\begin{aligned} &d(N, K) = d(N-1, K) \\ &\quad + \sum_{a=1}^{K-1} \sum_{c=0}^{N-K-1} \binom{K}{a} \binom{N-K-1}{c} \, d(a+c, a) \, d(N-a-c, K-a) \end{aligned} \tag{2.25}$$

Proof. The set of trees in $\mathrm{Dec}(N, K)$ in which N has only one child is in bijection with $\mathrm{Dec}(N-1, K)$; the bijection is simply erasing node N. On the other hand, if $T \in \mathrm{Dec}(N, K)$ is a tree in which N has more than one child, then we may write

T as $T' * T''$, where

$$
\begin{aligned}
&V(T') = A \cupdot C, \\
&V(T'') = B \cupdot D \cupdot \{N\}, \\
&A \cupdot B = [K], \\
&|A| = a, \qquad |B| = K - a, \qquad 1 \le a \le K-1, \\
&C \cupdot D = [K+1, N-1], \\
&|C| = c, \qquad |D| = N - K - c - 1, \qquad 0 \le c \le N-K-1, \\
&L(T') \in A, \\
&L(T'') \in B,
\end{aligned}
\tag{2.26}
$$

and such trees are counted by the double sum in (2.25). □

Lemma 2.2.10. *For* $1 < K \le N$,

$$
\begin{aligned}
d(N,K) &= d(N-1,K) + d(N,K-1) \\
&\quad + \sum_{x=0}^{K-2} \sum_{y=0}^{N-K-1} \binom{K-1}{x} \binom{N-K}{y} \\
&\qquad \times\ d(N-K+x-y, x+1)\ d(K-x+y, K-x-1).
\end{aligned}
\tag{2.27}
$$

Proof. Using the Pascal relation

$$
\binom{K}{a} = \binom{K-1}{a} + \binom{K-1}{a-1},
\tag{2.28}
$$

we rewrite (2.25) as

$$
\begin{aligned}
&d(N,K) = d(N-1,K) \\
&+ \sum_{a=1}^{K-1} \sum_{c=0}^{N-K-1} \binom{K-1}{a} \binom{N-K-1}{c}\ d(a+c,a)\ d(N-a-c, K-a) \\
&+ \sum_{a=1}^{K-1} \sum_{c=0}^{N-K-1} \binom{K-1}{a-1} \binom{N-K-1}{c}\ d(a+c,a)\ d(N-a-c, K-a).
\end{aligned}
\tag{2.29}
$$

Breaking off the $c = 0$ term of the first sum, we obtain

$$\begin{aligned} d(N,K) &= d(N-1,K) \\ &+ \sum_{a=1}^{K-1} \binom{K-1}{a} d(a,a)\, d(N-a,K-a) \\ &+ \sum_{a=1}^{K-1} \sum_{c=1}^{N-K-1} \binom{K-1}{a} \binom{N-K-1}{c} d(a+c,a)\, d(N-a-c,K-a) \\ &+ \sum_{a=1}^{K-1} \sum_{c=0}^{N-K-1} \binom{K-1}{a-1} \binom{N-K-1}{c} d(a+c,a)\, d(N-a-c,K-a). \end{aligned} \tag{2.30}$$

Applying Lemma 2.2.7 to the single sum and Lemma 2.2.8 to the first double sum, and putting $a = z$, $c = y$ in the second double sum gives

$$\begin{aligned} d(N,K) &= d(N-1,K) + d(N,K-1) \\ &+ \sum_{z=0}^{K-2} \sum_{y=1}^{N-K-1} \binom{K-1}{z} \binom{N-K-1}{y} d(z+y,z+1)\, d(N-z-y,K-z-1) \\ &+ \sum_{z=1}^{K-1} \sum_{y=0}^{N-K-1} \binom{K-1}{z-1} \binom{N-K-1}{y} d(z+y,z)\, d(N-z-y,K-z). \end{aligned} \tag{2.31}$$

In the first double sum, put $x = z$ and $w = y$. Note that the summand is zero if $y = N-K$, so we may change the upper limit of summation to $N-K$. Meanwhile, in the second double sum, put $x = z-1$ and $w = y+1$. We obtain

$$\begin{aligned} d(N,K) &= d(N-1,K) + d(N,K-1) \\ &+ \sum_{x=0}^{K-2} \sum_{w=1}^{N-K} \binom{K-1}{x} \binom{N-K-1}{w} d(x+w,x+1)\, d(N-x-w,K-x-1) \\ &+ \sum_{x=0}^{K-2} \sum_{w=1}^{N-K} \binom{K-1}{x} \binom{N-K-1}{w-1} d(x+w,x+1)\, d(N-x-w,K-x-1). \end{aligned} \tag{2.32}$$

Now applying the Pascal identity (2.28) gives (2.25) as desired. □

Putting $e(i,j) = d(i-1, i-j)$ in (2.27) and setting

$$n = N+1, \qquad k = N-K+1, \qquad N = n-1, \qquad K = n-k,$$

we recover the recurrence (2.6). We have proved the following combinatorial lower bound for degree:

Theorem 2.2.11.

$$\deg \mathcal{S}(n,k) \geq d(n-1, n-k). \tag{2.33}$$

In particular, since $\mathcal{S}(n,1) = \tilde{\mathcal{S}}(K_n)$*, we have*

$$\deg \tilde{\mathcal{S}}(K_n) \geq d(n-1,n-1) = |\operatorname{Dec}(n-1)| = \frac{(2n-4)!}{2^{n-2}(n-2)!}. \qquad \square$$

As we will soon see, equality holds in (2.33). Some of the values of $d(n,k)$ are as follows:

Table 2.1: Some values of $d(n,k)$

	k							
	1, 2	3	4	5	6	7	8	9
$n=2$	**1**							
3	**1**							
4	**3**	1						
5	**15**	7	1					
6	**105**	57	15	1				
7	**945**	561	195	31	1			
8	**10395**	6555	2685	633	63	1		
9	**135135**	89055	40725	12105	1995	127	1	
10	**2027025**	1381905	684495	237555	52605	6177	255	1

2.3 Tree Polynomials of Wheels

In this and the following sections, we investigate the algebraic properties of the ideal I_{K_n} cutting out $\tilde{\mathcal{S}}(K_n)$ set-theoretically.

2.3.1 Preliminaries

Definition 2.3.1. Let $V = \{v_0, \ldots, v_k\} \subset [n]$. The *k-wheel* $W = W(v_0; v_1, \ldots, v_k)$ is the graph with vertices V and edges

$$E = \{v_0v_1,\ \ \ldots,\ \ v_0v_k\} \cup \{v_1v_2,\ \ \ldots,\ \ v_{k-1}v_k,\ \ v_kv_1) \ . \tag{2.34}$$

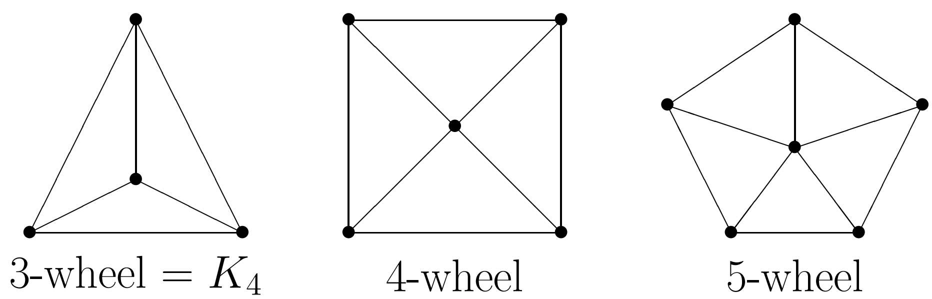


Figure 2.5: Wheels

Note that there are $2k$ equivalent notations for every k-wheel. The vertex v_0 is called the *center* of W, and $v_1, \ldots, v_k$ are its *spokes*. An edge between two spokes is called a *chord*; an edge between a spoke and the center is a *radius*. We denote the sets of chords and radii by $\mathrm{Ch}(W)$ and $\mathrm{Rd}(W)$ respectively. It is convenient to put $v_{k+1} = v_1$, so that

$$\mathrm{Ch}(W) = \{v_iv_{i+1} \ \ | \ \ i \in [k]\}.$$

2.3.2 Special Spanning Trees of Wheels

Let $W = W(v_0; v_1, \ldots, v_k) \subset K_n$, and let $\mathrm{SST}(W)$ be the set of special spanning trees of W. For all $i \in [k]$ and all $T \in \mathrm{SST}(W)$, we have

$$\mathrm{val}_T(v_i) = 1 \quad \text{or} \quad \mathrm{val}_T(v_i) = 2. \tag{2.35}$$

Note that not all spokes v_i have the same valence with respect to T. Moreover,

$$1 \;\leq\; \mathrm{val}_T(v_0) = 2k + 2 - \sum_{i=1}^{k} \mathrm{val}_T(v_i) \;\leq\; k - 1. \tag{2.36}$$

Lemma 2.3.2. *Let $T \in \mathrm{SST}(W)$ and $i, j \in [k]$. Then at least one of the four edges*

$$v_0v_i,\ v_0v_j,\ v_iv_{i+1},\ v_{j-1}v_j$$

lies in the complementary tree $\overline{T} = E(W) \setminus T$.

Proof. Suppose not. Let i, j be a counterexample such that $j - i$ is as small as possible. If necessary, we may reindex the spokes so that $i \leq j$. Note that

if $j - i = 0$, then $\mathrm{val}_T(v_i) = 3$;
if $j - i = 1$, then T contains the polygon (v_0, v_i, v_j);
if $j - i = 2$, then T contains the polygon (v_0, v_i, v_{i+1}, v_j).

In each case we have a contradiction. Now suppose $j - i > 2$. The chords $v_{i+1}v_{i+2}, \ldots, v_{j-2}v_{j-1}$ cannot all be in T, or else $(v_0, v_i, \ldots, v_j) \subset T$. Let k and l be the least and greatest indices, respectively, such that $v_kv_{k+1}, v_{l-1}v_l \notin T$ and $i < k \leq l < j$. Now $v_0v_k \notin T$, or else $(v_0, v_i, \ldots, v_k) \subset T$. For the same reason $v_0v_l \notin T$. But then the pair v_k, v_l constitutes a counterexample, and $|l - k| < |j - i|$, contradicting the choice of i and j. $\square$

Lemma 2.3.3. *Let $T \in \mathrm{SST}(W)$. At least one of the following statements is true:*

$$\text{For all } i \in [k],\ v_0v_i \in T \quad \text{iff} \quad v_iv_{i+1} \in \overline{T}, \quad \text{or}$$
$$\text{For all } i \in [k],\ v_0v_i \in T \quad \text{iff} \quad v_{i-1}v_i \in \overline{T}.$$

Proof. Suppose that both (2.3.3) and (2.3.3) fail. By Lemma 2.3.2, the only possibility is that

$$v_0v_i \in T, \quad v_iv_{i+1} \in T, \quad v_0v_j \in \overline{T}, \quad v_{j-1}v_j \in \overline{T} \tag{2.37}$$

for some i, j. Assume without loss of generality that $i < j$ (reversing the indexing if necessary). that $j - i \neq 1$, for then $v_iv_{i+1} \in T \cap \overline{T} = \emptyset$. Also, $j - i \neq 2$, for then $i + 1 = j - 1$ and either

$$\{v_0v_i,\ v_iv_{i+1},\ v_0v_{i+1}\} \subset T$$

or else

$$\{v_0v_j,\ v_{j-1}v_j,\ v_0v_{j-1}\} \subset \overline{T},$$

both of which are impossible. Therefore $j - i \geq 3$.

The chords $v_{i+1}v_{i+2}, \ldots, v_{j-2}v_{j-1}$ cannot all be in T, or else T would contain a polygon. Let h be the greatest number in $[i + 1, j - 2]$ such that

$$v_{h-1}v_h \in T.$$

The radii $v_0v_h, \ldots, v_0v_{j-1}$ must all lie in T to prevent $\overline{T}$ from containing a polygon. But then

$$\{v_0v_i,\ v_0v_h,\ v_iv_{i+1},\ v_{h-1}v_h\} \subset T,$$

violating Lemma 2.3.2. □

In this sense, each $T \in \mathrm{SST}(W)$ is oriented either "clockwise" or "counterclockwise." Note that T and $\overline{T}$ always have the same orientation; see, e.g., Example 2.3.5 below.

Proposition 2.3.4. *Let $d : [k] \to [2]$ be a nonconstant function. There exist exactly two special spanning trees of $W = W(v_0; v_1, v_2, \ldots, v_k)$ for which $\mathrm{val}(v_i) = d(i)$.*

Proof. Put $T = \mathrm{Rd}(W)$ in (1.24). The resulting matrix $M_{\mathrm{Rd}(W)}$ has the form

$$\begin{bmatrix} m_{v_0v_1} - m_{v_1v_2} & m_{v_1v_2} - m_{v_0v_2} & 0 & \cdots & 0 \\ 0 & m_{v_0v_2} - m_{v_2v_3} & m_{v_2v_3} - m_{v_0v_3} & \cdots & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ m_{v_0v_k} - m_{v_kv_1} & 0 & \cdots & \cdots & m_{v_kv_1} - m_{v_0v_1} \end{bmatrix} \qquad (2.38)$$

Taking the determinant, we obtain the formula

$$\tau(W) \;=\; \prod_{i=1}^{k}(m_{v_0v_i} - m_{v_iv_{i+1}}) \;+\; (-1)^{k-1}\prod_{i=1}^{k}(m_{v_iv_{i+1}} - m_{v_0v_{i+1}}), \qquad (2.39)$$

(where the indices of spokes are taken modulo k), or equivalently

$$\tau(W) \;=\; \prod_{i=1}^{k}(m_{v_0v_i} - m_{v_iv_{i+1}}) \;-\; \prod_{i=1}^{k}(m_{v_0v_{i+1}} - m_{v_iv_{i+1}}). \qquad (2.40)$$

No two monomials coming from the same product in (2.40) cancel. In addition, if T is a special spanning tree of W, then the monomial m_T cannot appear in both products in (2.40). Indeed, there is certainly a spoke v_i which is incident to both a chord and a radius in T; that is, either

$$\{v_0v_i, v_iv_{i+1}\} \subset T,$$

when m_T cannot appear in the first product in (2.40), or else

$$\{v_0v_i, v_{i-1}v_i\} \subset T.$$

when m_T cannot appear in the second product in (2.40).

The only cancellation that does occur involves the monomials $m_{\mathrm{Ch}(W)}$ and $m_{\mathrm{Rd}(W)}$. Each of these appears in both products in (2.40), once with coefficient $+1$

and once with coefficient -1, so it cancels. Accordingly, to enumerate the number of special spanning trees by the valences of spokes, we may substitute $z_i z_{i+1}$ for $m_{v_i v_{i+1}}$ and z_i for $m_{v_0 v_i}$ in (2.40) (where the z_i are indeterminates) and change all the $-$'s to $+$'s. This yields the expression

$$\prod_{i=1}^{k}(z_i + z_i z_{i+1}) \;+\; \prod_{i=1}^{k}(z_{i+1} + z_i z_{i+1}) \;-\; 2\left(\prod_{i=1}^{k} z_i + \prod_{i=1}^{k} z_i^2\right) \;=\; 2\sum_{d} z_i^{d(i)} \quad (2.41)$$

where the sum is taken over all nonconstant functions $d : [k] \to [2]$. □

Example 2.3.5. Let $d : [8] \to [2]$ be given by

$$d(4) = d(6) = d(7) = d(8) = 1,$$
$$d(1) = d(2) = d(3) = d(5) = 1.$$

By Proposition 2.3.4, there are exactly two special spanning trees T, S of $W = W(0; 1, 2, \ldots, 8)$ such that $\text{val}_T(i) = \text{val}_S(i) = d(i)$ for all $i \in [8]$. In the figures below, the edges in T and S are indicated by solid lines, and those in $\overline{T}$ and $\overline{S}$ by dotted lines.

$T = \{03, 05, 06, 07, 12, 23, 45, 81\}$ $\qquad$ $S = \{01, 05, 07, 08, 12, 23, 34, 56\}$

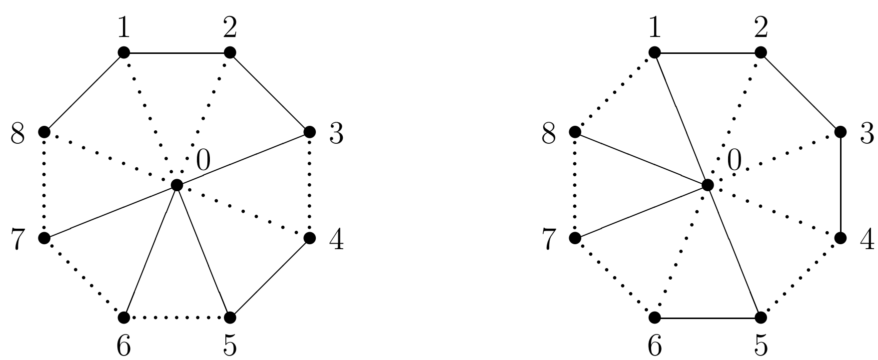


Figure 2.6: Special spanning trees of wheels

Observe that for all spokes v with $\{0, v\} \in T$, the chord $\{v-1, v\}$ may be in T, but $\{v, v+1\}$ is definitely not. In other words, all paths in T consisting of chords (for instance, $(3,2,1,8)$) proceed counterclockwise from a radius (in that case, 03). The same is true for $\overline{T}$. On the other hand, S and $\overline{S}$ are "oriented clockwise."

Let $d : [k] \to [2]$ be a nonconstant function. Define the *type* of a chord $v_i v_{i+1}$ with respect to d to be

$$t(v_i v_{i+1}) = \{d(i), d(i+1)\}. \tag{2.42}$$

For brevity we speak of "type-11", "type-12," and "type-22" chords. Note that there must be a positive even number of type-12 chords. E.g., for the function d defined in Example 2.3.5, we have

$$\begin{aligned} t\,(34) &= t\,(45) = t\,(56) = t\,(81) = 12, \\ t\,(12) &= t\,(23) = 22, \\ t\,(67) &= t\,(78) = 11. \end{aligned}$$

Lemma 2.3.6. *Let $i \in [k]$.*

(i) *If $t(v_i v_{i+1}) = 22$, then $v_i v_{i+1} \in T$.*

(ii) *If $t(v_i v_{i+1}) = 11$, then $v_i v_{i+1} \in \overline{T}$.*

Proof. If (i) is false, then the edges $v_{i-1}v_i$, $v_{i+1}v_{i+2}$, $v_0 v_i$, $v_0 v_{i+1}$ all lie in T, violating Lemms 2.3.2. If (ii) is false, those edges all lie in $\overline{T}$, again violating Lemma 2.3.2. □

Define the *type* of a radius $v_0 v_i$ to be

$$t(v_0 v_i) = \{d(i-1), d(i+1)\}. \tag{2.43}$$

As before, we will refer to "type-11" radii, etc. For the function d of Example 2.3.5, we have

$$\begin{aligned} t\,(01) &= t\,(03) = t\,(06) = t\,(08) = 12, \\ t\,(02) &= t\,(04) = 22, \\ t\,(05) &= t\,(07) = 11. \end{aligned}$$

Lemma 2.3.7. *Let $d : [k] \to [2]$ be nonconstant, and $i \in [k]$.*

(i) *If $t(v_0v_i) = 22$, then $v_0v_i \in \overline{T}$.*

(ii) *If $t(v_0v_i) = 11$, then $v_0v_i \in T$.*

Proof. The two statements are equivalent (just switch T and $\overline{T}$). We will prove (i). If $d(i) = 2$, then v_iv_{i+1}, $v_{i-1}v_i \in T$ by Lemma 2.3.6, so to have $\mathrm{val}_T(v_i) = 2$ we must have $v_0v_i \in \overline{T}$. On the other hand, suppose $d(i) = 1$ and $v_0v_i \in T$. Then v_iv_{i+1}, $v_{i-1}v_i \in \overline{T}$, and to have $\mathrm{val}_T(v_{i\pm 1}) = 2$ the edges $v_{i-2}v_{i-1}$, $v_{i+1}v_{i+2}$, v_0v_{i-1}, v_0v_{i+1} must all lie in T, violating Lemma 2.3.2. □

Suppose now that

$$d(i) = 1, \qquad d(i+1) = d(i+2) = \cdots = d(j) = 2, \qquad d(j+1) = 1.$$

Then $v_{i+1}v_{i+2}, \ldots, v_{j-1}v_j \in T$. If v_iv_{i+1} and v_jv_{j+1} lie in T, then $\{v_iv_{i+1}, \ldots, v_jv_{j+1}\}$ is a connected component of T, which is impossible. On the other hand, if both v_iv_{i+1} and v_jv_{j+1} lie in $\overline{T}$, then v_0v_{i+1} and v_0v_j lie in T; those two radii, together with $v_{i+1}v_{i+2}, \ldots, v_{j-1}v_j$, form a polygon in T, also a contradiction. Therefore, exactly one of the chords v_iv_{i+1}, v_jv_{j+1} lies in T. The same argument goes through if we switch "2" with "1" and T with $\overline{T}$. Indeed, if we traverse the circumference of W, ignoring type-11 and type-22 chords and coloring type-12 chords alternately red and blue, then T contains all the chords of one color and $\overline{T}$ contains all those of the other color.

If it is given which chords lie in T and which in $\overline{T}$, then each radius v_0v_i lies in T iff

$$d(i) - |T \cap \{v_{i-1}v_i, v_iv_{i+1}\}| = 1.$$

By the previous remark, requiring that a single type-12 chord lie in T or $\overline{T}$ suffices to determine T.

Alternatively, it suffices to specify whether a single type-12 radius v_0v_i lies in

T or $\overline{T}$. Without loss of generality $t(v_{i-1}v_i) = 11$ or 22 and $t(v_iv_{i+1}) = 12$. We know whether $v_{i-1}v_i \in T$, and the value of $d(i)$ determines whether $v_iv_{i+1} \in T$.

To summarize:

Proposition 2.3.8. *Let $d : [k] \to [2]$ be a nonconstant function, and T a spanning tree of $W(v_0; v_1, \ldots, v_k)$. Suppose that $\mathrm{val}_T(v_i) = d(i)$ for all i, and define the type of each edge as in (2.42) and (2.43). Then T can be described as follows: it consists of all type-22 chords, all type-11 radii, half the type-12 chords and half the type-12 radii.* □

Finally, we will need the following "exchange rules."

Lemma 2.3.9. *Let $T \in \mathrm{SST}(W)$. Suppose $v_{i-1}v_i, v_iv_{i+1} \in \overline{T}$, so that $v_0v_i \in T$ perforce. Assume without loss of generality that the path in $\overline{T}$ from v_i to v_0 passes through v_{i+1}. Then*

$$T \setminus \{v_0v_i\} \cup \{v_iv_{i+1}\}, \qquad \overline{T} \setminus \{v_iv_{i+1}\} \cup \{v_0v_i\}$$

is a 2-tree decomposition of W.

Proof. Since v_i is a leaf of T, it is clear that $T \setminus \{v_0v_i\} \cup \{v_iv_{i+1}\}$ is a spanning tree of W. Meanwhile, v_i and v_0 are in different components of $\overline{T} \setminus \{v_iv_{i+1}\}$, so $\overline{T} \setminus \{v_iv_{i+1}\} \cup \{v_0v_i\}$ is a tree. □

Lemma 2.3.10. *Let $T \in \mathrm{SST}(W)$. Suppose $v_{i-1}v_i, v_iv_{i+1} \in \overline{T}$, so that $v_0v_i \in T$ perforce. Assume without loss of generality that the path in $\overline{T}$ from v_i to v_0 passes through v_{i+1}. Then $v_0v_{i-1} \in T$, and*

$$T \setminus \{v_0v_{i-1}\} \cup \{v_{i-1}v_i\}, \qquad \overline{T} \setminus \{v_{i-1}v_i\} \cup \{v_0v_{i-1}\}$$

is a 2-tree decomposition of W.

Proof. v_{i-1} and v_i are in different components of $T \setminus \{v_0v_{i-1}\}$, and v_{i-1} and v_0 are in different components of $\overline{T} \setminus \{v_{i-1}v_i\}$, so $T \setminus \{v_0v_{i-1}\} \cup \{v_{i-1}v_i\}$ and $\overline{T} \setminus \{v_{i-1}v_i\} \cup \{v_0v_{i-1}\}$ are both spanning trees of W. □

Lemma 2.3.11. *Let $T \in \mathrm{SST}(W)$. Suppose $v_{i-1}v_i \in \overline{T}$ and $v_0v_i, v_iv_{i+1} \in T$, so that $v_0v_{i+1} \in \overline{T}$ perforce. Then:*

(i) *If v_0v_i is not the only radius in T, then*

$$T \setminus \{v_0v_i\} \cup \{v_{i-1}v_i\}, \qquad \overline{T} \setminus \{v_{i-1}v_i\} \cup \{v_0v_i\}$$

is a 2-tree decomposition of W.

(ii) *If v_0v_{i+1} is not the only radius in $\overline{T}$, then*

$$T \setminus \{v_iv_{i+1}\} \cup \{v_0v_{i+1}\}, \qquad \overline{T} \setminus \{v_0v_{i+1}\} \cup \{v_iv_{i+1}\}$$

is a 2-tree decomposition of W.

Proof. (i) v_i is a leaf of $\overline{T}$, so $\overline{T} \setminus \{v_{i-1}v_i\} \cup \{v_0v_i\}$ is a spanning tree of W. If the path from v_{i-1} to v_i in T does not go through v_0, then it must be $\mathrm{Ch}(W) \setminus \{v_{i-1}v_i\}$. But then T contains at least $k-1$ chords and 2 radii, which is impossible. Therefore v_{i-1} and v_i lie in different components of $T \setminus \{v_0v_i\}$, and $T \setminus \{v_0v_i\} \cup \{v_{i-1}v_i\}$ is a tree.

(ii) The path from v_0 to v_{i+1} in T is just (v_0, v_i, v_{i+1}), so v_0 and v_{i+1} are in different components of $T \setminus \{v_iv_{i+1}\}$. If the path from v_i to v_{i+1} in $\overline{T}$ does not go through v_0, then it must be $\mathrm{Ch}(W) \setminus \{v_iv_{i+1}\}$, which again is impossible. So v_i and v_{i+1} are in different components of $\overline{T} \setminus \{v_0v_{i+1}\}$. Therefore $T \setminus \{v_iv_{i+1}\} \cup \{v_0v_{i+1}\}$ and $\overline{T} \setminus \{v_0v_{i+1}\} \cup \{v_iv_{i+1}\}$ are trees. ☐

2.3.3 Leading Trees of Wheels

We now consider the problem of determining the leading monomial of the tree polynomial of a wheel, using the tools just developed. We fix the following ordering on the variables m_{ij}:

$$m_{12} > m_{13} > m_{14} > \ldots > m_{1n} > m_{23} > \ldots \tag{2.44}$$

Many of our results hold more generally for orderings "compatible" with the numbering of vertices, in the sense that

$$v' < v'' \implies m_{vv'} > m_{vv''} \tag{2.45}$$

for all $v, v', v'' \in [n]$.

We now extend $>$ to a term ordering on R_n. This can be done in many ways; the two most important cases are as follows:

1. **Graded lexicographic order**, in which

$$\prod_{i,j} m_{ij}^{a_{ij}} >_{\text{lex}} \prod_{i,j} m_{ij}^{b_{ij}} \tag{2.46}$$

   iff either $\sum a_{ij} > \sum b_{ij}$, or else $a_{ij} > b_{ij}$ for the greatest edge ij (with respect to $>$) for which $a_{ij} \neq b_{ij}$. For instance,

$$m_{12}m_{13}m_{34} >_{\text{lex}} m_{12}m_{23}m_{24}, \tag{2.47}$$

   because m_{13} appears with degree 1 in the first monomial and 0 in the second.

2. **Reverse lexicographic order**, in which

$$\prod_{i,j} m_{ij}^{a_{ij}} >_{\text{rlex}} \prod_{i,j} m_{ij}^{b_{ij}}$$

   iff either $\sum a_{ij} > \sum b_{ij}$, or else $a_{ij} < b_{ij}$ for the least edge ij (with respect to $>$) for which $a_{ij} \neq b_{ij}$. For instance, the inequality (2.47) goes the other way with respect to $>_{\text{rlex}}$ because m_{34} appears with degree 1 in the first monomial and 0 in the second.

We order the edges of K_n according to (2.44):

$$\{1,2\} > \{1,3\} > \{1,4\} > \ldots > \{1,n\} > \{2,3\} > \ldots\ , \tag{2.48}$$

and regard the chosen ordering of monomials (such as graded-lex or reverse-lex) as an ordering of the subsets of $E(K_n)$ (equivalently, the subgraphs of K_n), associating an edge set $E \subset E(K_n)$ with the squarefree monomial

$$m_E = \prod_{e \in E} m_e. \tag{2.49}$$

Specifically, graded-lex order induces the following ordering on edge sets: for $E, F \subset E(K_n)$, $E >_{\text{lex}} F$ iff

$$\begin{aligned} \text{either} \quad & |E| > |F| \\ \text{or} \quad & |E| = |F| \quad \text{and} \quad \max(E \# F) \in E. \end{aligned} \tag{2.50}$$

where the symbol # denotes symmetric difference, and the maximum of an edge set is defined with respect to the ordering (2.48). Meanwhile, reverse-lex order induces the following ordering: $E >_{\text{rlex}} F$ iff

$$\begin{aligned} \text{either} \quad & |E| > |F| \\ \text{or} \quad & |E| = |F| \quad \text{and} \quad \min(E \# F) \in F, \end{aligned} \tag{2.51}$$

Given a wheel $W \subset K_n$, we wish to identify the *leading tree* $LT(W)$ of W, that is, the special spanning tree of W corresponding to the leading monomial of $\tau(W)$. We begin by computing the valence of each spoke of $LT(W)$. By Proposition 2.3.8, this will rule out all but two possibilities for the leading tree.

Proposition 2.3.12. *(1)* *If $v_0 \notin \{\min(V), \max(V)\}$, then for all $i \in [k]$,*

$$\text{val}_{LT(W)}(v_i) = \begin{cases} 1 & v_i > v_0 \\ 2 & v_i < v_0. \end{cases} \tag{2.52}$$

Hence $\text{val}_{LT(W)}(v_0) = r - 1$, where v_0 is the rth largest [equivalently, $(k+2-r)$th smallest] vertex of V.

(2) *If $v_0 = \min(V)$, then $\text{val}_{LT(W)}(v_0) = k - 1$.*

(3) *If $v_0 = \max(V)$, then $\text{val}_{LT(W)}(v_0) = 1$.*

Proof. (1) Suppose that $v_0 \notin \{\min(V), \max(V)\}$. Let T be a special spanning tree of W not satisfying (2.52); we will show that there exists a special spanning tree $T' \in \mathrm{SST}(W)$ with $m_{T'} > m_T$.

Suppose first that $\mathrm{val}_T(v_i) = 1$ for some $v_i < v_0$.

Case 1: $v_{i-1}v_i$, $v_iv_{i+1} \in \overline{T}$ and $v_0v_i \in T$. Without loss of generality we may assume that the path from v_i to v_0 in $\overline{T}$ passes through v_{i+1}. Now $v_0v_{i-1} \in T$, so the tree

$$T' = T \setminus \{v_0v_{i-1}\} \cup \{v_{i-1}v_i\}$$

is special by Lemma 2.3.10. Since $v_i < v_0$, we have $v_{i-1}v_i > v_0v_{i-1}$ and thus $T' > T$.

Case 2: v_0v_i, $v_iv_{i+1} \in \overline{T}$ and $v_{i-1}v_i \in T$. Then $v_0v_{i+1} \in T$ perforce. By Lemma 2.3.11, either

$$T' = T \setminus \{v_0v_{i+1}\} \cup \{v_iv_{i+1}\}$$

is special (hence $T' > T$) or else v_0v_{i+1} is the only radius in T, when

$$T = \mathrm{Ch}(W) \setminus \{v_iv_{i+1}\} \cup \{v_0v_{i+1}\}.$$

In this case, the tree

$$T \setminus \{v_jv_{j+1}\} \cup \{v_0v_{j+1}\}$$

is special for all $j \neq i$. If $T = LT(W)$, then we must have $v_0 > v_j$ for each $j \neq i$. Since $v_0 > v_i$ it follows that $v_0 = \max(v_i)$, which contradicts our assumption.

The proof that $\mathrm{val}_{LT(W)}(v_i) = 1$ for $v_i > v_0$ is analogous.

(2), (3) Suppose that $v_0 = \min(V)$ and $\mathrm{val}_S(v_0) < k - 1$ for some special spanning tree $S \in \mathrm{SST}(W)$. Then $\overline{S}$ contains at least two radii, say v_0v_i and v_0v_j. At least one of the chords $v_{i-1}v_i$, v_iv_{i+1} lies in S. If both do, then by Lemma 2.3.9, one of

$$S_1 = S \setminus \{v_{i-1}v_i\} \cup \{v_0v_i\}, \qquad S_2 = S \setminus \{v_iv_{i+1}\} \cup \{v_0v_i\}$$

is special. If $v_{i-1}v_i \in S$ and $v_iv_{i+1} \in \overline{S}$, then by Lemma 2.3.11 S_1 is special, since v_0v_i is not the only radius in $\overline{S}$. Now $S_1, S_2 > S$, so S cannot be the leading tree of W. The proof of (3) is analogous. □

The preceding result is valid for any term ordering that respects (2.45). Henceforth, however, we will consider only the case that the variables m_e are ordered by (2.44) and monomials by graded strict lexicographic order (2.46), (2.50). As it happens, the initial ideals with respect to $>_{\text{lex}}$ and $>_{\text{rlex}}$ have very similar descriptions. However, the Stanley-Reisner complex of I_n with respect to $>_{\text{lex}}$ is easier to describe. We shall investigate that case in detail, and merely state without proof the analogous results for leading trees with respect to reverse lexicographic order.

Let J_n be the ideal of R_n generated by all monomials $m_{LT(W)}$, where W ranges over all subwheels of K_n and the leading tree is taken with respect to $>_{\text{lex}}$.

Proposition 2.3.13. *Let* $W = W(v_0; v_1, \dots, v_k) \subset K_n$. *Put* $V = V(W) = \{v_0, v_1, \dots, v_k\}$.

(1) Suppose $v_0 = \min(V)$. *Reindex the spokes of* W *so that* $v_1 = \max\{v_1, \dots, v_k\}$ *and* $v_2 > v_k$. *Then*

$$LT(W) = \mathrm{Rd}(W) \setminus \{v_0v_1\} \cup \{v_kv_{k+1}\}. \tag{2.53}$$

(2) Suppose $v_0 = \max(V)$. *Reindex the spokes of* W *so that* $v_1v_2 = \min(\mathrm{Ch}(W))$ *and* $v_1 > v_2$. *Then*

$$LT(W) = \mathrm{Ch}(W) \setminus \{v_1v_2\} \cup \{v_0v_2\}. \tag{2.54}$$

Proof. (1) By Proposition 2.3.12 (2), $LT(W)$ contains exactly one chord. Since $v_0v_i > v_jv_{j+1}$ for all i, j, the unique radius not in $LT(W)$ must be

$$\min(\mathrm{Rd}(W)) = v_0v_1,$$

which implies (2.53) because $v_k v_{k+1} > v_1 v_2$.

(2) Define

$$T_i = \mathrm{Ch}(W) \cup \{v_0 v_i\} \setminus \{\min(v_{i-1}v_i, v_i v_{i+1})\}. \tag{2.55}$$

By Proposition 2.3.12 (3), $LT(W)$ contains exactly one radius, so $LT(W) = T_i$ for some i. Note that

$$T_1 = \mathrm{Ch}(W) \cup \{v_0 v_1\} \setminus \{v_1 v_2\} \tag{2.56}$$

and

$$T_2 = \mathrm{Ch}(W) \cup \{v_0 v_2\} \setminus \{v_1 v_2\}. \tag{2.57}$$

Therefore

$$\max(T_1 \# T_2) = v_0 v_2 \in T_2. \tag{2.58}$$

On the other hand, if $i > 2$, then

$$\begin{aligned} \max(T_i \# T_2) &= \max\{v_0 v_i,\ v_0 v_2,\ v_1 v_2,\ \min(v_{i-1}v_i, v_i v_{i+1})\} \\ &= \min(v_{i-1}v_i,\ v_i v_{i+1}) \in T_2. \end{aligned}$$

Therefore $LT(W) = T_2$, giving (2.54). □

In the case that $v_0 \notin \{\min(V), \max(V)\}$, we set

$$d(i) = \begin{cases} 1 & v_i > v_0 \\ 2 & v_i < v_0. \end{cases} \tag{2.59}$$

By Proposition 2.3.12 (1) we have $\mathrm{val}_{LT(W)}(v_i) = d(i)$. By Proposition 2.3.4, there are exactly two special spanning trees $T, T' \in \mathrm{SST}(W)$ satisfying these valence conditions, containing all type-22 chords and type-11 radii. Furthermore, $T \# T'$ is the set of type-12 edges (both chords and radii). Define the *critical edge* of W to be

$$ce(W) = \max(T \# T'); \tag{2.60}$$

by (2.50), whichever of T, T' contains $ce(W)$ is the leading tree of W.

We can now describe precisely the set of square-free monomials which appear as leading terms of wheels.

Theorem 2.3.14. *Let T be a tree with vertex support $V \subset [n]$. There exists a wheel $W \subset K_n$ such that $T = LT(W)$ (with respect to graded strict-lex order), if and only if T contains a path $(v_1, \ \ldots, \ v_k)$ satisfying the conditions*

$$\begin{aligned} & k \geq 4; \\ & \max\{v_1, \ldots, v_k\} = v_1; \\ & \max\{v_2, \ldots, v_k\} = v_k; \\ & v_2 > v_{k-1}. \end{aligned} \tag{2.61}$$

Proof. First, given a path P satisfying (2.61), we will construct a wheel W such that $LT(W) = P$. Second, we will show that for all W, the tree $LT(W)$ contains a path satisfying (2.61).

Suppose $P = (v_1, \ldots, v_k)$ satisfies (2.61). Let $W = W(v_k; v_1, v_2, \ldots, v_{k-1})$. By Proposition 2.3.12, $LT(W)$ is a path from v_k to v_1. The two possibilities for $LT(W)$ are P and

$$P' = (v_k, \ v_2, \ v_3, \ \ldots, \ v_{k-2}, \ v_{k-1}, \ v_1).$$

Now $v_1 > v_k > v_2 > v_{k-1}$, so

$$\begin{aligned} ce(W) &= \max(P \# P') \\ &= \max\{v_1v_2, \ v_{k-1}v_k, \ v_kv_2, \ v_{k-1}v_1\} \\ &= v_{k-1}v_k \in P \end{aligned}$$

so $P = LT(W)$.

For the second part of the proof, let $W = W(v_0; v_1, \ldots, v_k)$ and $T = LT(W)$. We will find a path $P \subset T$ satisfying (2.61).

Case 1: $v_0 = \min(V)$. Reindex the spokes of W so that $v_1 = \max\{v_1, \ldots, v_k\}$ and $v_2 > v_k$. Then by (2.53)

$$LT(W) = \mathrm{Rd}(W) \setminus \{v_0v_1\} \cup \{v_kv_{k+1}\}. \tag{2.62}$$

Since $v_0 < v_k < v_2 < v_1$, we may take

$$P = (v_1,\ v_k,\ v_0,\ v_2).$$

Case 2: $v_0 = \max(V)$. Reindex the spokes of W so that $v_1v_2 = \min(\mathrm{Ch}(W))$ and $v_1 > v_2$. Then $LT(W) = \mathrm{Ch}(W) \setminus \{v_1v_2\} \cup \{v_0v_2\}$ by (2.54). Let $v_j = \max\{v_1, \ldots, v_k\}$. Obviously $j \neq 2$. Also $j \neq 3$ (since $v_1v_2 < v_2v_3$, implying $v_1 > v_3$). Consider the path

$$P = (v_0,\ v_2,\ v_3,\ \ldots,\ v_j).$$

The two largest vertices of P are v_0 and v_j, and if $v_2 < v_{j-1}$ then $v_{j-1}v_j < v_1v_2$, a contradiction. So P satisfies (2.61).

Case 3: $v_0 \notin \{\min(V), \max(V)\}$. Let $e = ce(W)$.

Case 3a: $e \in \mathrm{Ch}(W)$. We may assume $e = v_1v_2$, $d(1) = 1$, $d(2) = 2$ (so $v_1 > v_0 > v_2$). Now $v_0v_2 > v_1v_2$, so $t(v_0v_2) \neq 12$; hence $t(v_0v_2) = 11$ and $v_0v_2 \in T$. Then $v_2v_3 \in \overline{T}$, and $d(3) = 1$ (otherwise $t(v_2v_3) = 22$ and $v_2v_3 \in T$, which is false). Then $t(v_2v_3) = 12$. Since v_1v_2 is the critical edge, we have $v_1v_2 > v_2v_3$ and $v_1 < v_3$. In particular $v_1 \neq \max(V)$. Let $v_j = \max(V)$, and let P be the path from v_1 to v_j in T. If $v_0v_j \in T$, then

$$P = (v_1,\ v_2,\ v_0,\ v_j),$$

which satisfies (2.61) because $v_j > v_1 > v_0 > v_2$. Otherwise,

$$P = (v_1,\ v_2,\ v_0,\ v_i,\ v_{i-1},\ \ldots,\ v_j).$$

for some i. Then

$$d(v_i) = d(v_{i-1}) = \cdots = d_(v_{j-1}) = 2,$$

so v_j and v_1 are respectively the largest and second largest vertices of P. Moreover, $t(v_{j+1}v_j) = 12$, and v_1v_2 is the critical edge, so $v_{j+1} > v_2$. So P satisfies (2.61).

Case 3b: $e \in \mathrm{Rd}(W)$. We may assume $e = v_0v_1$, $d(2) = 2$, $d(k) = 1$. Then $v_k > v_0 > v_2$ and $v_0v_1 < v_1v_2$; hence $t(v_1v_2) \neq 12$ and $d(1) = 2$. Let j be the smallest number in $[k]$ such that $d(j) = 1$, and let

$$P = \{v_{j-1}v_j,\ v_{j-2}v_{j-1},\ \ldots,\ v_1v_2,\ v_0v_1\}.$$

The chords $v_1v_2, \ldots, v_{j-2}v_{j-1}$ are all of type 22, hence lie in T. Therefore all radii

$$v_0v_2,\ v_0v_3,\ \ldots,\ v_0v_{j-1}$$

lie in $\overline{T}$ (otherwise T contains a polygon), so $v_{j-1}v_j \in T$ (because $d(j-1) = 2$). Thus $P \subset T$. Moreover, $j \geq 3$, so P contains at least three edges. Also, the endpoints of P, namely v_j and v_0, are respectively its largest and second largest vertex, and $v_{j-1} > v_1$ (else v_0v_{j-1} is the critical edge instead of v_0v_1). Thus P satisfies (2.61). □

The initial ideal with respect to reverse-lex order has an analogous description. We omit the proof, which is similar to that of Theorem 2.3.14.

Theorem 2.3.15. *Let T be a tree with vertex support $V \subset [n]$. There exists a wheel $W \subset K_n$ such that $T = LT(W)$ (with respect to reverse-lex order), if and only if T contains a path $P = (v_1,\ \ldots,\ v_k)$ satisfying the conditions*

$$\begin{aligned} &k \geq 4;\\ &\max\{v_1, \ldots, v_k\} = v_1;\\ &\max\{v_2, \ldots, v_k\} = v_k;\\ &v_2 < v_{k-1}. \end{aligned} \tag{2.63}$$

2.4 The Stanley-Reisner Complex

2.4.1 Preliminaries

We briefly summarize the definition of a simplicial complex and some related terminology. For more detail, see for instance [3, ch.5].

Definition 2.4.1. Let V be a finite set. A *simplicial complex on* V is a set Δ of subsets of V which contains as members all singleton sets $\{v\}$, $v \in S$, and with the property that if $F \in \Delta$ and $F' \subset F$, then $F' \in \Delta$. The elements of Δ are called *faces*. A maximal face is called a *facet*. The *dimension* of a face F is $\dim F = |F| - 1$, and the *dimension* of Δ is

$$\dim \Delta \;=\; \max\{\dim F \;\mid\; F \in \Delta\}. \tag{2.64}$$

Δ is a *simplex* if it has exactly one facet, and is *pure* if all its facets have the same cardinality. The *f-vector* $f(\Delta) = (f_0, f_1, \ldots)$ is defined by

$$\sum_i f_i t^i = \sum_{F \in \Delta} t^{\dim F} \tag{2.65}$$

and the *h-vector* $h(\Delta) = (h_0, h_1, \ldots)$ is defined by

$$h_j = \sum_{i=0}^{j} (-1)^{j-i} \binom{d-i}{j-i} f_{i-1} \tag{2.66}$$

[3, p. 213].

Recall the definitions

$$\begin{aligned} R_n &= \mathbf{k}[m_{i,j} \mid 1 \le i < j \le n], \\ I_n &= (\tau(G) \mid G \subset K_n \text{ a rigidity circuit}) \;\subset\; R_n, \\ J_n &= (m_{LT(W)} \mid W \subset K_n \text{ a wheel}) \;\subset\; \mathrm{in}(I_n), \end{aligned} \tag{2.67}$$

where $\mathrm{in}(I_n)$ denotes the initial ideal with respect to graded lexicographic order, as defined in (2.44) and (2.46).

We will prove that $J_n = \mathrm{in}(I_n)$, i.e., the tree polynomials of wheels form a Gröbner basis for I_n under graded lexicographic order. We devote this section to studying the ideal J_n. Our main tool in doing so is the following combinatorial object.

Definition 2.4.2. The *Stanley-Reisner complex* of J_n is the simplicial complex $\Delta(n)$ on $E(K_n)$ defined by

$$\Delta(n) = \{E \subset E(K_n) \ \mid \ m_E \notin J_n\}. \tag{2.68}$$

That is, an edge set E is a face of $\Delta(n)$ if and only if E contains no path of the form (2.61). For this reason, we refer to a path satisfying (2.61) as *forbidden*. The *Stanley-Reisner ring* of $\Delta(n)$ is

$$\tilde{R}_n = R_n / J_n. \tag{2.69}$$

If $V = \{v_1 < \cdots < v_n\}$ is a finite totally ordered set (typically, $V \subset \mathbb{N}$), then we may define a simplicial complex $\Delta(V)$ on $E(K(V))$, isomorphic to $\Delta(n)$, by replacing the edge ij with $v_i v_j$ for all i, j.

We will use the following facts about Stanley-Reisner complexes (see [17], [3]). First,

$$1 + \dim \Delta(n) = \dim \tilde{R}_n. \tag{2.70}$$

In addition, the Hilbert series of $\tilde{R}_n$ is given by the h-vector of $\Delta(n)$:

$$\mathrm{Hilb}_{\tilde{R}_n}(t) = \frac{h_0 + h_1 t + \dots}{(1-t)^{\dim \tilde{R}_n}}, \tag{2.71}$$

No path with fewer than 3 edges satisfies (2.61). Therefore $\Delta(2)$ is the (0-dimensional) simplex on $E(K_2)$, and $\Delta(3)$ is the (2-dimensional) simplex on $E(K_3)$. In particular

$$\dim \Delta(n) = 2n - 4 \qquad \text{for } n \leq 3. \tag{2.72}$$

Lemma 2.4.3. *Suppose $n \geq 4$. Let E be a polygon on $[n]$ not containing the edge $\{1, n\}$. Then E contains a forbidden path.*

Proof. We proceed by induction on n. If $n = 4$, then the only possibility is

$$E = \{12, 34, 42, 21\},$$

which contains the forbidden path $(4,2,1,3)$.

Now, suppose $n > 4$. If $\{1, n-1\} \in E$, then $(n-1, 1, \ldots, n) \subset E$ is forbidden. Otherwise, we may write without loss of generality

$$E = (v_1 = n,\ v_2,\ \ldots,\ v_i = n-1,\ \ldots,\ v_j = 1,\ \ldots,\ v_n).$$

In particular $i + 1 < j < n$, so $r = n - i + 1 \geq 4$ and

$$E' = (v_i,\ v_{i+1},\ \ldots,\ v_n).$$

is an r-gon. By induction, E' contains a forbidden path P'. If $\{n-1, v_n\} \notin P'$ then $P' \subset E$. Otherwise, $n-1$ is a terminal of P', and E contains the forbidden path

$$P' \setminus \{\{n-1, v_n\}\} \cup \{\{n, v_n\}\}. \qquad \square$$

In addition, we will need several facts about connected and 2-connected graphs. For details, see [2].

Let $G = (V, E)$ be a connected graph. A vertex $v \in V$ is called an *articulation point* if the graph $G - v = G|_{V \setminus \{v\}}$ is disconnected. In this case, v is said to *separate* w and x if w and x lie in different connected components of $G - v$. Equivalently, if $P \subset E$ is a path and $w, x \in V(P)$, then $v \in V(P)$. G is called *2-connected* if it has no articulation point.

If $G = (V, E)$ is connected and $v \in V$ is an articulation point, then G is the edge-disjoint union of $r \geq 2$ connected subgraphs

$$G_1 = (V_1, E_1),\ \ldots,\ G_r = (V_r, E_r),$$

where $E_i \cap E_j = \emptyset$ and $V_i \cap V_j = \{v\}$ for $i \neq j$. The subgraphs G_i are called the *v-articulation components of G.*

If $G = (V, E)$ is connected and $v \in V$ is not an articulation point, then we may define the *2-connected component of G containing v* to be the unique maximal

2-connected subgraph $G' = (V', E')$ of G containing v. We may realize G' as the intersection of all graphs G_a, where a ranges over all articulation points of G and G_a is the a-articulation component of G containing v. In this case E' is the union of all polygons supported at v. The subgraph G' is called the *2-connected component of G containing v*. If (V', E') and (V'', E'') are distinct 2-connected components of G, then either $V' \cap V'' = \emptyset$ or $V' \cap V'' = \{a\}$, where a is an articulation point of G. Moreover, G is the edge-disjoint union of its 2-connected components.

Menger's Theorem [2, p. 35] states that if v, w are vertices of a graph $G = (V, E)$ and $n \in \mathbb{N}$, then either there are $n + 1$ edge-disjoint paths between v and w, or else there is a set of n vertices in $V \setminus \{v, w\}$ whose deletion disconnects v and w. In the case $n = 1$, this becomes the statement that

$$\begin{array}{l}\text{if no polygon in } E \text{ is supported on both } v \text{ and } w,\\ \text{then there exists an articulation point separating } v \text{ and } w.\end{array} \tag{2.73}$$

2.4.2 The Structure of Facets

Theorem 2.4.4. *Let $n \geq 3$, F a facet of $\Delta(n)$, and $G = ([n], F)$. Then:*

(1) *G is connected.*

(2) *F contains the edges $\{1, n\}$ and $\{n-1, n\}$.*

(3) *The face*

$$\hat{F} = F \setminus \{\{1, n\}\} \tag{2.74}$$

is connected.

(4) *$\hat{F}$ has an articulation point $a \in \{2, 3, \ldots, n-1\}$ which separates 1 and n.*

(5)

$$F \;=\; \hat{F} \cup \{\{1,n\}\} \;=\; F^1 \cup F^2 \cup \{\{1,n\}\}, \qquad \text{where}$$

$$\begin{aligned} &V(F^1) \cup V(F^1) = [n], \\ &V(F^1) \cap V(F^2) = \{a\}, \\ &1 \in V(F^1), \\ &n \in V(F^2). \end{aligned} \tag{2.75}$$

(6) *The vertex 1 is not an articulation point of* $\hat{F}$ *or* F*.*

(7) *We may choose the articulation point* a *such that*

$$a = \max(V^1). \tag{2.76}$$

(8) F^1 *and* F^2 *are facets of* $\Delta(V^1)$ *and* $\Delta(V^2)$ *respectively.*

(9) $|F| = 2n-3$*. In particular,* $\Delta(n)$ *is pure of dimension* $2n-4$*, and by (2.70) we have*

$$\operatorname{codim} J_n = 2n - 3. \tag{2.77}$$

(10) F *is 2-connected.*

(11) *The decomposition (2.75) is unique.*

Proof. (1) Suppose F is disconnected. Let v, w be the largest vertices of their respective connected components. Then $F \cup \{\{v,w\}\} \in \Delta(n)$, so F is not a facet.

(2) No path satisfying (2.61) can contain either $\{1,n\}$ or $\{n-1,n\}$, so both those edges must lie in every maximal element of $\Delta(n)$.

(3) Suppose that $\hat{F}$ is disconnected. Let C be the connected component of $\hat{F}$ containing the edge $\{n-1,n\}$; then $1 \notin V(C)$ and in particular $\{1,n-1\} \notin F$. Let $F' = \hat{F} \cup \{\{1,n-1\}\}$. If F' contains a forbidden path P, then P must include

the edge $\{1, n-1\}$, but in that case P must be of the form $(n-1, 1, \ldots, v, n)$, and $P \setminus \{\{1, n-1\}\} \subset F$, which is impossible. So $F' \in \Delta(n)$. But then $F' \cup \{1, n\} = F \cup \{1, n-1\} \in \Delta(n)$, contradicting the hypothesis that F is a facet.

(4) Immediate from Lemma 2.4.3 and (2.73).

(5) Immediate from (4).

(6) Suppose that 1 is an articulation point of $\hat{F}$. Let $G^2 = (V^2, E^2)$ be the 1-articulation component of $\hat{F}$ containing the vertex n, and $G^1 = (V^1, E^1)$ the union of all the other 1-articulation components. Let

$$x = \max(V^1)$$

and

$$y = \min\{v \in V^2 \;\mid\; \{1, v\} \in \hat{F}\}.$$

Case 1: $x < y$. Suppose that $\hat{F} \cup \{\{x, y\}\}$ contains a forbidden path P. Then $\{x, y\} \in P$, since $\hat{F} \in \Delta(n)$ and $P \not\subset \hat{F}$. Both endpoints of P are $\geq y > x$, hence lie in V^2. Therefore, P is of the form

$$(b_r, \ldots, b_1, 1 = a_1, \ldots, a_s, x, y = c_1, \ldots, c_t)$$

with

$$\begin{aligned} &r, s, t \geq 1, \\ &a_i \in V^1, \\ &b_i, c_i \in V^2 \end{aligned}$$

and either

$$b_r = \max(V(P)), \quad c_t = \max_2(V(P)), \quad b_{r-1} > c_{t-1}$$

or

$$c_t = \max(V(P)), \quad b_r = \max_2(V(P)), \quad b_{r-1} < c_{t-1},$$

where $\max_2(S)$ denotes the second largest element of the set S. Note that $b_1 > y > x > 1$, so either $r > 1$ or $t > 1$. Hence the path

$$(b_r, b_{r-1}, \ldots, b_1, 1, y = c_1, \ldots, c_t) \subset \hat{F}$$

has at least four vertices, and is forbidden, which is a contradiction. Thus $F \cup \{x, y\} \in \Delta(n)$, contradicting the hypothesis that F is a facet.

Case 2: $x > y$. There is a path from y to x of the form

$$(y, 1, x_1, \ldots, x_r = x).$$

Let i be the least index such that $x_i > y$. If $i > 1$ then the subpath $(y, 1, x_1, \ldots, x_i)$ is forbidden, so $i = 1$ and $x_1 > y > 1$. Since 1 is not an articulation point of F^2, there exists a path in F^2 from y to n which is not supported at 1. Truncating this path at the first vertex greater than x_1 produces a path

$$(y = y_1, \ldots, y_r)$$

with $r \geq 2$ and $y_1, \ldots, y_{r-1} < x_1 < y_r$. Then

$$(x_1, 1, y, \ldots, y_r)$$

is a forbidden path in $\hat{F}$, which is impossible.

Hence 1 is not an articulation point of $\hat{F}$. The same is true for F, because adding an edge incident to 1 (namely $\{1, n\}$) cannot change whether 1 is an articulation point.

(7) We adopt the notation of part (5). We may take F^1 to be the 2-connected component of G supported at 1, and a the unique articulation point separating 1 and n. Let

$$m = \max(V(F^1));$$

in particular $m \geq a$. Suppose $m > a$. The edge set F^1 contains a polygon supported at both 1 and m, so $\{1, m\} \in F^1$ by Lemma 2.4.3. Deleting m does not disconnect F^1, so F^1 contains a path

$$P = (m, 1, \ldots, a).$$

On the other hand, there is a path in F^2 of the form

$$P' = (a = y_1, \ldots, y_r)$$

with $r \geq 1$ and $a, y_1, \ldots, y_{r-1} < m < y_r$. Then $P \cup P' \subset F$ is a forbidden path, which is impossible.

(8) Suppose that there is some edge $e \in K(V(F^1)) \setminus F^1$ such that

$$F^1 \cupdot \{e\} \in \Delta(V(F^1)).$$

Since F is a facet of $\Delta(n)$, the edge set $F \cup \{e\}$ must contain a forbidden path P. Certainly $\{1, n\} \notin P$, so $P \subset \hat{F} \cup \{e\}$. On the other hand, $P \not\subset F^1 \cupdot \{e\}$ and $P \not\subset F^2$. Since a is an articulation point of $\hat{F} \cup \{e\}$, the path P must be supported at a and have one terminal in each of $V(F^1) \setminus \{a\}$ and $V(F^2) \setminus \{a\}$. But this contradicts (2.61), since $a = \max(V(F^1))$. Therefore no such e exists. It follows that F^1 is a facet of $\Delta(V(F^1))$, as desired. The proof that F^2 is a facet of $\Delta(V(F^2))$ is analogous.

(9) We induct on n. The case $n = 3$ is trivial. If $n \geq 4$, then $|V^2| = n - |V^1| + 1$. By induction,

$$|F| \;=\; 1 + |F^1| + |F^2| \;=\; 1 + (2|V^1| - 3) + (2n - 2|V^1| - 1) \;=\; 2n - 3. \quad (2.78)$$

(10) We induct on n. The statement is trivial for $n = 2$ and $n = 3$. For the inductive step, we write

$$F = \hat{F} \cupdot \{\{1, n\}\} = F^1 \cupdot F^2 \cupdot \{\{1, n\}\}$$

as in (2.75). By induction F^1 and F^2 are 2-connected, so they are exactly the 2-connected components of $\hat{F}$, and F is 2-connected because the edge $\{1, n\}$ has one endpoint in each of $V(F^1)$ and $V(F^2)$.

(11) By (10), the face $\hat{F}$ has a unique articulation point, so F^1 must be the 2-connected component of $\hat{F}$ supported at 1 and F^2 must be the 2-connected component supported at n. $\square$

Theorem 2.4.5. *Let* $V^1, V^2 \subset \mathbb{N}$*, with*

$$\begin{aligned} &V^1 \cup V^2 = [n], 1 \in V^1 \setminus V^2, \\ &n \in V^1 \setminus V^2, \\ &V^1 \cap V^2 = \{\max(V^1)\}, \end{aligned} \tag{2.79}$$

and let F^1 *and* F^2 *be facets of* $\Delta(V^1)$ *and* $\Delta(V^2)$ *respectively. Then*

$$F = F^1 \mathbin{\dot\cup} F^2 \mathbin{\dot\cup} \{\{1, n\}\}$$

is a facet of $\Delta(n)$.

Proof. The vertex $a = \max(V^1)$ is an articulation point of $F^1 \cup F^2$. Suppose that F contains a forbidden path P. Note that $\{1, n\} \notin P$ and neither F^1 nor F^2 contains P as a subset, so P must have one terminal in each of $V^1 \setminus \{a\}$ and $V^2 \setminus \{a\}$. Thus P passes through a, which is impossible since $a = \max(V^1)$. So $F \in \Delta(n)$. In addition, $|F| = 2n - 3$ by (2.78), so F is a facet. □

2.4.3 Admissible Trees

Applying the decomposition (2.75) recursively allows us to represent a facet $F \in \Delta(V)$ by a binary tree $T = \mathbf{T}(F) = (\mathbf{V}, \mathbf{E})$. If $|V| = 2$, then $\mathbf{V} = \{V\}$ and $\mathbf{E} = \emptyset$. Otherwise, T is the tree with root V, left principal subtree $\mathbf{T}(F^1)$, and right principal subtree $\mathbf{T}(F^2)$. The nodes X of $\mathbf{T}(F)$ are labeled by subsets of $[n]$. They correspond to the edges e of F via the bijections

$$\begin{aligned} \varepsilon(X) &= \{\min(X), \max(X)\}, \\ \nu(e) &= \nu_T(e) = \varepsilon^{-1}(e). \end{aligned} \tag{2.80}$$

We extend this notation to sets of nodes and of edges, writing

$$\begin{aligned} \varepsilon(\{X_1, \ldots, X_r\}) &= \{\varepsilon(X_1), \ldots, \varepsilon(X_r)\}, \\ \nu_T(\{e_1, \ldots, e_r\}) &= \{\nu_T(e_1), \ldots, \nu_T(e_r)\}. \end{aligned} \tag{2.81}$$

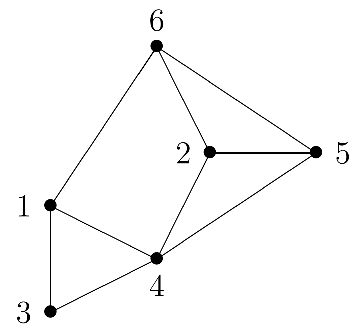


Figure 2.7: A facet F of $\Delta(6)$

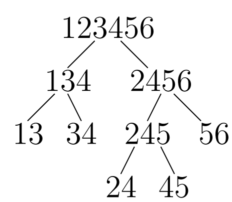


Figure 2.8: The admissible tree T corresponding to F

Example 2.4.6. Consider the facet $F \in \Delta(6)$ given by the following figure:

The corresponding admissible binary tree $\mathbf{T}(F)$ is

For instance, $\nu(26) = 2456$ and $\varepsilon(\{134, 45\}) = \{14, 45\} \subset F$.

A node X is a leaf if and only if it is labeled by a 2-element set. Otherwise,

$$\begin{aligned}
&1.\quad |X| > 2;\\
&2.\quad \min(X) \in X^1 \setminus X^2;\\
&3.\quad \max(X) \in X^2 \setminus X^1;\\
&4.\quad X^1 \cup X^2 = X;\\
&5.\quad X^1 \cap X^2 = \{\max(X^1)\}.
\end{aligned} \tag{2.82}$$

where X^1 and X^2 denote respectively the left and right children of X.

Definition 2.4.7. Let T be a binary tree and X a node of T. T is *admissible at* X if the pair (T, X) satisfies (2.82). T is *admissible* if it is admissible at each of its nodes.

Note that if T is admissible at X and $|X| > 2$, then

$$\max(X) > \max(X^1) \tag{2.83}$$

and

$$X^2 = X \setminus X^1 \cup \{\max(X^1)\}. \tag{2.84}$$

Theorem 2.4.8. *Let $T = (\mathbf{V}, \mathbf{E})$ be an admissible tree with root V. Then $\varepsilon(\mathbf{E})$ is a facet of $\Delta(V)$. Consequently, the function* $\mathbf{T}$ *is a bijection from facets to admissible trees.*

Proof. Since $|\varepsilon(\mathbf{E})| = 2|\mathbf{V}| - 3$, we need only show that no path $P = (v_1, \ldots, v_s) \subset \varepsilon(\mathbf{E})$ is forbidden. Let X_i be the node of T corresponding to the edge $\{v_i, v_{i+1}\}$, and Y the greatest common ancestor of the nodes $\nu_T(\{v_i, v_{i+1}\})$. If $Y = X_i$ for some i, then $\varepsilon(Y) = \{\min(V(P)), \max(V(P))\}$. No forbidden path whose support is contained in $V(P)$ can include this edge, so P is not forbidden. □

2.4.4 Counting the Facets

Let V be a set of positive integers, $|V| = n \geq 2$. Denote the sets of admissible binary trees on V and decreasing planar trees on V by the symbols $\mathrm{Adm}(V)$ and $\mathrm{Dec}(V)$ respectively, and define

$$\hat{V} = V \setminus \{\min(V)\}. \tag{2.85}$$

Given $T \in \mathrm{Adm}(T)$, define a tree $\phi(T)$ as follows.

If $n = 2$, then $\phi(T)$ is a tree with one node, labeled $\{\max(V)\}$, and no edges.

If $n > 2$, then construct a tree $\psi(T)$ by relabeling the nodes of $\phi(T^2)$ with the elements of $\hat{V} \setminus V(T^1)$, preserving their relative order. We then define

$$\phi(T) = \phi(T^1) * \psi(T), \tag{2.86}$$

where $*$ is as in Definition 2.2.6.

Example 2.4.9. Let F and T be as in Example 2.4.6. Then

$$\phi(T^1) = \begin{array}{c}4\\ |\\ 3\end{array} \quad , \qquad \phi(T^2) = \begin{array}{c}6\\ |\\ 5\\ |\\ 4\end{array} \quad ,$$

Figure 2.9: $\phi(T^1)$ and $\phi(T^2)$

and

$$\hat{V} \setminus V(T^1) = \{2,3,4,5,6\} \setminus \{3,4\} = \{2,5,6\}.$$

Thus to form $\psi(T)$, we change the labels of $\phi(T^2)$ from 4,5,6 to 2,5,6 respectively. That is,

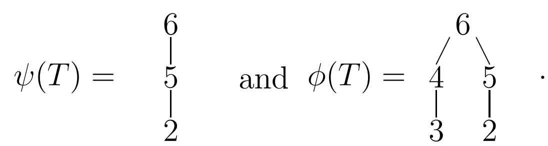


Figure 2.10: $\psi(T)$ and $\phi(T)$

Theorem 2.4.10. *The function* ϕ *is a bijection from* $\mathrm{Adm}(V)$ *to* $\mathrm{Dec}(\hat{V})$.

Proof. We proceed by induction on $|V|$. The base case $|V| = 2$ is obvious.

Now suppose $|V| > 2$. Let T and S be distinct elements of $\mathrm{Adm}(V)$. If $T^1 \neq S^1$, then $\phi(T)^1 \neq \phi(S)^1$, so $\phi(T) \neq \phi(S)$. If $T^1 = S^1$, then $T^2 \neq S^2$. Changing the labels of these two trees to $\hat{V} \setminus V(T^1) = \hat{V} \setminus V(S^1)$ does not change the fact that they are unequal, so $\psi(T) \neq \psi(S)$ and $\phi(T) \neq \phi(S)$. Therefore ϕ is injective.

For surjectivity, let $U \in \mathrm{Dec}(\hat{V})$. By induction, there exist admissible trees T and T' with $\phi(T) = U^1$ and $\phi(T') = U \setminus U^1$. Note that $|\,\mathrm{rt}(T)| + |\,\mathrm{rt}(T')| = |V| + 1$. Construct a new tree T'' by relabeling T' as follows: change $\mathrm{rt}(T')$ to the set $V \setminus \mathrm{rt}(T) \cup \{\max(\mathrm{rt}(T))\}$ and adjust all other labels accordingly. Then $U = \phi(T * T'')$. $\square$

Corollary 2.4.11. *(1) The simplicial complex* $\Delta(n)$ *has* $\dfrac{(2n-4)!}{2^{n-2}(n-2)!}$ *facets.*

(2) The ideal J_n *has degree* $\dfrac{(2n-4)!}{2^{n-2}(n-2)!}$.

2.4.5 Shellability

Let Δ be a pure simplicial complex. A linear ordering $F_1, \ldots, F_s$ of the facets of Δ is called a *shelling order* (and Δ is called *shellable*) if each F_i has a subset sh(F_i), called its *shelling set*, such that

$$\text{If } i, j \in [s] \text{ and } j < i, \text{ then } \operatorname{sh}(F_i) \not\subset F_j \tag{2.87}$$

and

$$\text{If } i \in [s] \text{ and } e \in \operatorname{sh}(F_i), \text{ then } F_i \setminus F_k = \{e\} \text{ for some } k \in [i-1]. \tag{2.88}$$

Let T be an admissible tree. We say that a node X of T is *firstborn* if it is the first child of its parent. In addition we define

$$\mathcal{L}(T) = \{\operatorname{rt}(T)\} \cup \{X \mid X \text{ is firstborn}\}, \tag{2.89}$$

and, for a facet F,

$$\mathcal{L}(F) = \varepsilon(\mathcal{L}(T)).$$

Example 2.4.12. Let F and T be as in Example 2.4.6. In the figure below, the nodse in $\mathcal{L}(T)$ are enclosed in boxes.

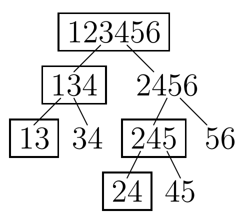
$T = \mathbf{T}(F) =$  ,

Figure 2.11: $\mathcal{L}(T)$

Thus $\varepsilon(\mathcal{L}(T)) = \{16, 25, 34, 24, 13\}$.

Lemma 2.4.13. *Let F be a facet of $\Delta(n)$ and $T = \mathbf{T}(F)$. Then $S = \varepsilon(\mathcal{L}(T))$ is a spanning tree of K_n.*

Proof. Since $|S| = n-1$, it suffices to prove that S contains no polygons. Suppose that $P = (v_1, \ldots, v_s) \subset S$. Let $V = V(P) = \{v_1, \ldots, v_s\}$, $A = \nu_T(P)$, and

$Y = \sup(A)$, so that $\varepsilon(Y) = \{\min(V), \max(V)\}$. If $Y \in A$, then by (2.83), Y is the only node in $\mathcal{L}(T) \cap T|Y$ such that $\varepsilon(Y)$ has $\max(Y)$ as an endpoint, which implies that P is not a polygon. On the other hand, if $Y \notin A$ then the existence of P contradicts Lemma 2.4.3. □

Definition 2.4.14. Define a total order on finite sets X, Y of positive integers as follows: $X > Y$ if either

$$\begin{aligned} \text{either} \quad & |X| > |Y| \\ \text{or} \quad & |X| = |Y| \quad \text{and} \quad \min(X \# Y) \in X. \end{aligned} \tag{2.90}$$

While this ordering is most convenient for our purposes, the only property really necessary for the proof of shellability is that $A \not\supset B$ whenever $A < B$.

Definition 2.4.15. Let T be a binary tree. The *traversal order* on the nodes of T is defined as follows: $X < Y$ if either X is a descendant of Y, or $X \in T|Z^2$ and $Y \in T|Z^1$, where Z is the youngest common ancestor of X and Y. The *traversal* $\mathrm{trav}(T)$ is the list of nodes of T, written in decreasing traversal order.

For example, if T is the tree of Example 2.4.12, we have

$$\mathrm{trav}(T) = (123456, 134, 13, 34, 2456, 245, 24, 45, 56).$$

Definition 2.4.16. Define a total order $<$ on facets F, G of $\Delta(n)$ as follows. Suppose

$$\begin{aligned} T = \mathbf{T}(F), \quad & \mathrm{trav}(T) = (X_1, \dots, X_N), \\ T' = \mathbf{T}(G), \quad & \mathrm{trav}(T') = (X'_1, \dots, X'_N). \end{aligned} \tag{2.91}$$

(where $N = 2n - 3$). Then $G < F$ if for some k we have

$$X_k < X'_k$$

and

$$X_i = X'_i \qquad \text{for } i < k.$$

Theorem 2.4.17. *The order $<$ of Definition 2.4.16 is a shelling order on the facets of $\Delta(V)$. The shelling sets* $\mathrm{sh}(F)$ *are defined (recursively) by the formula*

$$\mathrm{sh}(F) = \begin{cases} \emptyset & |V| \leq 3 \\ \mathrm{sh}(F^2) & V(F^1) = \{\min(V), \min_2(V)\} \\ \mathrm{sh}(F^2) \cup \mathcal{L}(F^1) & \text{otherwise,} \end{cases} \tag{2.92}$$

where $\min_2(V)$ *denotes the second smallest element of* V.

Proof. We will prove that the ordering on facets satisfies conditions (2.87) and (2.88).

Assume without loss of generality that $V = [n]$. Let F, G be facets of $\Delta(n)$, with $T = \mathbf{T}(F)$ and $U = \mathbf{T}(G)$. Suppose that $F > G$, and that $\mathrm{trav}(T)$ and $\mathrm{trav}(U)$ first differ in nodes X, Y respectively. That is, $X > Y$, and if $X' > X$ and $Y' > Y$ are nodes in the same position in their respective traversals, then $X' = Y'$. In particular, X^P and Y^P are labeled by the same set of integers, which must have cardinality at least 4 (otherwise $X = Y$). If $X^{P1} = Y^{P1}$ then $X^{P2} = Y^{P2}$ by (2.84). Therefore $X = X^{P1}$ and $Y = Y^{P1}$. In particular $E \subset \mathrm{sh}(F)$, where $E = \varepsilon(\mathcal{L}(T|X))$ is the set of firstborn descendants of X. If $\varepsilon(X) \notin E$, then X must consist of the two smallest elements of $X^P = Y^P$, which contradicts the assumption that $X > Y$. So $\varepsilon(X) \in E$.

We will show that $E \not\subset G$. If Z is a node of $U \setminus U|Y^P$, then $\varepsilon(Z)$ has an endpoint outside $X^P = Y^P$, so $\varepsilon(Z) \notin E$. Hence it suffices to prove that

$$E \not\subset \varepsilon(U|Y^P).$$

Suppose first that $\max(X) > \max(Y)$. In particular $\max(X) \notin Y$. Together with the fact that $\min(X) = \min(Y) \notin Y^{PP2}$, this implies that

$$\varepsilon(X) = \{\min(X), \max(X)\} \notin U|Y^P. \tag{2.93}$$

On the other hand, suppose that $\max(X) \leq \min(Y)$. Since $X > Y$, there exists a vertex $v \in X \setminus Y$. By Lemma 2.4.13, E is a spanning tree of the vertices in X. In particular, E contains a path E' from $\min(X) = \min(Y)$ to v. The vertex $\max(Y)$ is either a leaf of E (if $\max(X) = \max(Y)$) or not in $V(E)$ (if $\max(X) < \max(Y)$); in either case, $\max(Y) \notin V(E')$. Note that

$$v \in Y^{P2} \setminus Y \qquad \text{and} \qquad \min(Y) \in Y \setminus Y^{PP2}.$$

Since $\max(Y)$ is an articulation point of $\varepsilon(U|Y^P) \setminus \{\varepsilon(U)\}$ by Theorem 2.4.4, every path from $\min(Y)$ to v in $\varepsilon(U|Y^P)$ must either pass through $\max(Y)$ or include the edge $\varepsilon(Y^P)$. Since E' does neither of these things, we have $E \not\subset \varepsilon(U|Y^P)$ as desired.

We now prove that the order $<$ of Definition 2.4.16 satisfies condition (2.88). For a facet F of $\Delta(n)$ and an edge $e \in \mathrm{sh}(F)$, we must show that $\Delta(n)$ has a facet G satisfying

$$G < F \qquad \text{and} \qquad F \setminus G = \{e\}. \tag{2.94}$$

Let $T = \mathbf{T}(F)$ and and $X = \nu_T(e)$. We proceed by induction on n. There is nothing to prove if $n \leq 3$, since then $\Delta(n)$ has a unique facet, whose shelling set is empty.

Suppose first that $e \in \mathrm{sh}(F^2)$. By induction, the simplicial complex $\Delta(V(F^2))$ has a facet G^2 such that

$$G^2 < F^2 \qquad \text{and} \quad F^2 \setminus G^2 = \{e\}, \tag{2.95}$$

so the facet of $\Delta(n)$ given by

$$G = F^1 \cup G^2 \cup \{\{\min(V(F)), \max(V(F))\}\} \tag{2.96}$$

satisfies (2.94).

Now suppose that $e \in \mathcal{L}(F^1)$.

Case 1: X has children. Let

$$\begin{aligned} Y &= X^P, \\ E &= \varepsilon(T|Y). \end{aligned} \tag{2.97}$$

By Theorem 2.4.4, the face

$$E' = E^1 \setminus \{e\} \tag{2.98}$$

is 1-connected but not 2-connected (because it has $m = \max(X^1)$ as an articulation point). Let

$$F_1 = \varepsilon(T|X^1) \in \Delta(X^1), \tag{2.99}$$

and

$$F_2 = \varepsilon(T|X^2) \cup \varepsilon(T|Y^2) \in \Delta(X^2 \cup Y^2). \tag{2.100}$$

Let F_2' be a facet of $\Delta(X^2 \cup Y^2)$ which contains F_2'. (By a routine calculation, $|F_2| = 2|X^2 \cup Y^2| - 4$, so F_2' consists of F_2 and one additional edge by Theorem 2.4.4 (9).) Moreover, we have

$$X^1 \;\cup\; (X^2 \cup Y^2) = V, \tag{2.101}$$

and

$$X^1 \;\cap\; (X^2 \cup Y^2) = \{\max(X^1)\}, \tag{2.102}$$

so $G' = F_1 \cup F_2' \cup \{\varepsilon(Y)\}$ is a facet of $\Delta(Y)$. Let G be the facet of $\Delta(V)$ obtained by replacing $T|Y$ with $\mathbf{T}(G')$. Then $F \setminus G = \{e\}$. Moreover, the traversals of $\mathbf{T}(F)$ and $\mathbf{T}(G)$ first differ at the left child of Y, which is X in $\mathbf{T}(F)$ and X^1 in $\mathbf{T}(G)$. Now $X^1 \subsetneq X$, so $G < F$ in traversal order. Therefore G satisfies (2.94).

Case 2: X has no children. Define

$$X_0 = X, \tag{2.103}$$

$$X_i = (X_{i-1})^P \text{ for } i \geq 1, \tag{2.104}$$

$$s = \min\left\{t \geq 2 \;\mid\; (X_t)^1 = X_{t-1}\right\}, \quad \text{and} \tag{2.105}$$

$$Y = X_s. \tag{2.106}$$

In other words, Y is the youngest ancestor of X, other than X^P, such that the path from Y to X begins by moving down to the left.

Let $E = \varepsilon(T|Y)$. The face

$$E' = E^1 \setminus \{e\} = \varepsilon(T|Y^1) \setminus \{e\} \tag{2.107}$$

is 1-connected because E^1 is 2-connected by Theorem 2.4.4. We claim that E' is not 2-connected; specifically, that the vertex

$$m = \max(Y^1) \tag{2.108}$$

is an articulation point of E'. Define

$$E_i = \varepsilon(T|X_i) \setminus \{e\} \tag{2.109}$$

for $1 \leq i \leq s-1$ (so that $E_{s-1} = E'$ and $E_s = E \setminus \{e\}$). We will show by induction that for every i, the vertex m is an articulation point separating the endpoints of e, namely $\min(X)$ and $\max(X)$. If $i = 1$, then the vertex $\min(X)$ has valence 2 in E_1, so deleting e makes it into a leaf with stem m. Now suppose $1 < i < s$. Then

$$\begin{aligned} E_i &= \varepsilon(T|(X_i)^1) \ \cup \ \varepsilon(T|X_{i-1}) \ \cup \ \{\varepsilon(X_i)\} \setminus \{e\} \\ &= E_{i-1} \cup \varepsilon(T|(X_i)^1) \ \cup \ \{\varepsilon(X_i)\}. \end{aligned} \tag{2.110}$$

Suppose that $\varepsilon(T|(X_i)^1)$ contains a path P between the two m-articulation components of E_{i-1} containing $\min(X)$ and $\max(X)$ respectively. Since $\varepsilon(T|(X_i)^1)$ is 2-connected, we may assume that P is supported at $l = \min((X_i)^1) = \min(X_i)$. Let P' be a path in E_{i-1} from $\min(X)$ to $\max(X)$. Then $m \in V(P')$, and $Q = P \cup P'$ is a polygon supported at no fewer than four vertices. But $\min(V(Q)) = l$, $\max(V(Q)) = m$, and $\{l, m\} \notin Q$, which contradicts Lemma 2.4.3. Therefore no such P exists, and m is an articulation point of the face

$$E_i'' = E_{i-1} \ \cup \ \varepsilon(T|(X_i)^1). \tag{2.111}$$

The edge $\varepsilon(X_i)$ has m as an endpoint (since $m = \max(X_i) = m$), so adding it to E_i'' does not connect any two m-articulation components of E_i.

Finally,

$$\begin{aligned} E_s &= \varepsilon(T|X_{s-1}) \ \cup \ \varepsilon(T|Y^2) \ \cup \ \{\varepsilon(Y)\} \setminus \{e\} \\ &= E_{s-1} \ \cup \ \varepsilon(T|Y^2) \ \cup \ \{\varepsilon(Y)\}. \end{aligned} \tag{2.112}$$

We have already seen that m is an articulation point of E_{s-1}; in addition, m is the only vertex supporting both E_{s-1} and $\varepsilon(T|Y^2)$. Therefore $E_{s-1} \cup \varepsilon(T|Y^2)$ separates each pair of the vertices $\min(X)$, $\max(X)$ and $\max(Y)$. When we add the edge $\varepsilon(Y)$ to form E_s, the endpoints of X remain separated by m.

Define

$$F_1 = \text{the 2-connected component of } E' \text{ containing } \min(Y), \tag{2.113}$$

$$F_2 = (E' \setminus F_1) \cup \varepsilon(T|Y^2). \tag{2.114}$$

Note that

$$V(F_1) \subsetneq Y^1 = X_{s-1}, \tag{2.115}$$

$$V(F_1) \cup V(F_2) = Y, \quad \text{and} \tag{2.116}$$

$$V(F_1) \cap V(F_2) = \{m\}. \tag{2.117}$$

For $i = 1, 2$, let F_i' be a facet of $V(F_i)$ containing F_i. Then

$$G' = F_1' \cup F_2' \cup \{\varepsilon(Y)\}$$

is a facet of $\Delta(Y)$. As in Case 1, let G be the facet of $\Delta(V)$ obtained by replacing $T|Y$ with $\mathbf{T}(G')$. Then $F \setminus G = \{e\}$, and (2.115) implies that $G < F$ in traversal order. Therefore G satisfies (2.94). □

2.5 The Main Theorem

The following summarizes the facts we have proved about the slope variety and affine slope variety of K_n.

Theorem 2.5.1. *Let $n \geq 2$.*

(1) *The affine slope variety $\tilde{\mathcal{S}}(K_n)$ is defined scheme-theoretically by the ideal I_n. That is, I_n is reduced, hence $\tilde{\mathcal{S}}(K_n) \cong \operatorname{Spec} R_n/I_n$.*

(2) *The set*

$$\{\tau(W) \mid W \subset K_n \text{ is a wheel}\} \tag{2.118}$$

is a Gröbner basis for I_n with respect to graded lexicographic order.

(3) *Equality holds in the lower bounds established in (2.5); that is,* $\deg \mathcal{S}(n,k)$ *equals the number of decreasing planar trees on $[n-1]$ with largest leaf $\leq n-k$. In particular,*

$$\deg \mathcal{S}(K_n) = \frac{(2n-4)!}{2^{n-2}(n-2)!}.$$

(4) *The varieties $\tilde{\mathcal{S}}(K_n)$ and $\mathcal{S}(K_n)$ are Cohen-Macaulay.*

Proof. The variety $\tilde{\mathcal{S}}(K_n)$ is irreducible and reduced, hence is defined scheme-theoretically by the prime ideal $\sqrt{I_n}$. In addition, we have

$$J_n \subset \operatorname{in}(I_n) \subset \operatorname{in}(\sqrt{I_n}). \tag{2.119}$$

By (9) of Theorem 2.4.4 we have

$$\operatorname{codim} J_n = 2n-3 = \dim \mathcal{S}(K_n) = \operatorname{codim} \sqrt{I_n}$$

and by Theorems 2.4.11 and 2.2.11 we have

$$\deg J_n = \frac{(2n-4)!}{2^{n-2}(n-2)!} \leq \deg \sqrt{I_n}.$$

Since the ideal J_n is Cohen-Macaulay, it is unmixed. Therefore, any ideal strictly larger than J_n has either larger codimension or smaller degree. Combined with the previous two assertions, and the fact that codimension and degree are unchanged upon passing to the initial ideal [5, Thm. 15.26], we see that equality holds in (2.119). It follows that the tree polynomials of the wheels form a Gröbner basis for $\sqrt{I_n}$. In particular $I_n = \sqrt{I_n}$. To make the bounds in Theorem 2.2.11 sharp, equality must hold in (2.5). Finally, $\tilde{\mathcal{S}}(K_n)$ is Cohen-Macaulay because $\Delta(n)$ is shellable [3, Thm. 5.1.13]. Since Cohen-Macaulayness is a local condition, $\mathcal{S}(K_n)$ is Cohen-Macaulay as well. ∎

Using the computer algebra system *Macaulay* [1], it can be shown that for every $n \le 9$, the ideal I_n is generated by the polynomials

$$\{\tau(G) \ \mid \ G \subset K_n,\ G \cong K_4\}. \tag{2.120}$$

Equivalently, for every wheel W on 9 or fewer vertices, the polynomial $\tau(W)$ can be written in the form

$$\sum_{Q\subset[n],\ |Q|=4} a_Q \tau(K_Q), \tag{2.121}$$

with $a_Q \in R_n$. However, we have not found a formula for the coefficients a_Q. Note, by the way, that if the polynomials (2.120) do indeed generate I_n, then they are a *minimal* set of generators, because they are all homogeneous of degree 3 and no monomial appears in more than one generator.

2.6 The h-vector of $\Delta(n)$

When Δ is a shellable complex with facets $F_1, \ldots, F_s$, we have a combinatorial interpretation of the h-vector $h(\Delta) = (h_0, h_1, \ldots)$, and therefore of the Hilbert series of $\tilde{R}_n$ [3, Corollary 5.1.14]:

$$h_k = \left|\{i \in [s] \ \mid \ |\operatorname{sh}(F_i)| = k\}\right|. \tag{2.122}$$

Note that this is independent of the shelling order.

Accordingly, we can derive a recurrence defining $h(\Delta(n)) = (h_0^n, h_1^n, \ldots)$. It is convenient to work with the associated generating function

$$H_n(t) = h_0^n + h_1^n t + \ldots \ = \ \sum_{i=1}^{s} t^{F_i}. \tag{2.123}$$

Recall that the Hilbert series of the Stanley-Reisner ring of $\Delta(n)$, and therefore of $\tilde{R}_n$, is

$$\frac{H_n(t)}{(1-t)^{2n-3}}.$$

Theorem 2.6.1. *$h(\Delta(n))$ is defined recursively by*

$$\begin{aligned}
H_2(t) &= 1, \\
H_3(t) &= 1, \\
H_n(t) &= H_{n-1}(t) + (n-3)tH_{n-1}(t) \\
&\quad + \sum_{a=3}^{n-1} \binom{n-2}{a-1} t^{a-1} \frac{(2a-4)!}{2^{a-2}\,(a-2)!}\, H_{n-a+1}(t).
\end{aligned} \tag{2.124}$$

Therefore, for $n \geq 4$, the coefficients h_k^n are given by

$$h_k^n = h_k^{n-1} + (n-3)h_{k-1}^{n-1} + \sum_{a=3}^{n-1} \binom{n-2}{a-1} \frac{(2a-4)!}{2^{a-2}\,(a-2)!}\, h_{k-a+1}^{n-a+1}. \tag{2.125}$$

Proof. Let $s(F) = |\operatorname{sh}(F)|$. By the formula (2.92), we have

$$s(F) = \begin{cases} 0 & n \leq 3 \\ s(F^2) & V(F^1) = \{n-1, n\} \\ s(F^2) + |V(F^1)| - 1 & \text{otherwise} \end{cases} \tag{2.126}$$

The base cases $n = 2$ and $n = 3$ are obvious because the corresponding Stanley-Reisner complexes are simplices. Now suppose $n \geq 4$. Fix $A \subset [n-1]$ with $2 \leq a = |A| \leq n-2$. If $A = \{n-1, n\}$, then

$$\sum_{F:\ V(F^1)=A} t^{s(F)} = H_{n-1}(t). \tag{2.127}$$

Otherwise,

$$\sum_{F:\ V(F^1)=A} t^{s(F)} = t^{a-1}\ H_a(1)\ H_{n-a+1}(t). \tag{2.128}$$

Now $H_a(1)$ is just the number of facets in $\Delta(A)$. By (2.17) and Theorems 2.4.8 and 2.4.10, we have

$$H_a(1) = \frac{(2a-4)!}{2^{a-2}\,(a-2)!}. \tag{2.129}$$

Now summing over all possible A yields (2.124), and taking the t^k coefficients of both sides yields (2.125). □

2.6.1 A Combinatorial Interpretation of h_k^n

For small n, k, the values of h_k^n are as follows:

Table 2.2: Some values of h_k^n

n	k=0	1	2	3	4	5	6	7
2	1							
3	1							
4	1	1	1					
5	1	3	6	5				
6	1	6	21	41	36			
7	1	10	55	185	365	329		
8	1	15	120	610	2010	3984	3655	
9	1	21	231	1645	7980	25914	51499	47844

It appears that h_k^n equals the number of matchings on $[2n-4]$ containing k long pairs (i.e., pairs not of the form $\{i, i+1\}$). For example, the matchings on [6] are as follows:

Table 2.3: Matchings on [6]

k	Matchings on [6] with k long pairs
0	$\{12, 34, 56\}$
1	$\{12, 36, 45\}, \{16, 23, 45\}, \{14, 23, 56\}$
2	$\{12, 35, 46\}, \{13, 24, 56\}, \{13, 26, 45\}, \{15, 26, 34\}, \{15, 23, 46\}, \{16, 25, 34\}$
3	$\{13, 25, 46\}, \{14, 25, 36\}, \{14, 26, 35\}, \{15, 24, 36\}, \{16, 24, 35\}$

Kreweras and Poupard [10] gave several formulas for the number of matchings on $[2n-4]$ containing k short pairs. The numbers agree with ours up to $n=9$, but we have not yet been able to verify that our recurrences are equivalent to theirs, nor have we been able to establish a bijection between facets and matchings in which edges of $\mathrm{sh}(F)$ correspond to long pairs.

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