UNIVERSITY OF CALIFORNIA, SAN DIEGO

The Combinatorics of the Permutation Enumeration of Wreath Products between Cyclic and Symmetric Groups

A dissertation submitted in partial satisfaction of the
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by

Jennifer D. Wagner

Committee in charge:

Professor Jeffrey Remmel, Chair
Professor Adriano Garsia
Professor Te C. Hu
Professor Hans Wenzl
Professor S. Gill Williamson

2000

The dissertation of Jennifer D. Wagner is approved, and it is acceptable in quality and form for publication on microfilm:

Chair

University of California, San Diego

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For Jeremy

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VITA

1995	B. A., with honors, Phi Beta Kappa, Grinnell College
1995–1999	Teaching Assistant, Department of Mathematics, University of California, San Diego
1997	M. A., University of California, San Diego
1999-2000	Senior Teaching Assistant, Department of Mathematics, University of California, San Diego
2000	Ph. D., University of California San Diego

ABSTRACT OF THE DISSERTATION

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Professor Jeffrey Remmel, Chair

Brenti introduced a homomorphism $\xi : \Lambda \to \mathbf{Q}(x)$ defined on the the elementary symmetric functions by

$$\xi(e_k(X)) = \frac{(1-x)^k}{k!},$$

where Λ is the space of homogeneous polynomials in an infinite number of variables $X = (x_1, x_2, \ldots)$ which are constant under all permutations of these variables. He proved that the homomorphism ξ has the remarkable property that when it is applied to a homogeneous symmetric function $h_k(X)$, the result is the well-known Eulerian polynomial, which is also the generating function for the number of descents of a permutation. In addition, if ξ is applied to a power symmetric function, the result is a generating function for another permutation statistic.

Beck and Remmel used combinatorial interpretations of the transition matrices between bases of Λ to give combinatorial proofs of these and other related identities, including q-analogs. In addition, they used these combinatorial methods to develop an analog of Brenti's permutation enumeration for B_n, the hyperoctahedral group consisting of signed permutations.

In the dissertation we extend Brenti, Beck and Remmel's results to wreath

products $C_k \S S_n$ between cyclic and symmetric groups, which can be considered as groups of permutations signed with k^{th} roots of unity.

The key steps in our extension to $C_k \S S_n$ include the following.

- We develop the representation theory of $C_k \S S_n$ in an appropriate way, including the definition of a characteristic map from the class functions on $C_k \S S_n$ to a space of symmetric functions, and an extension of lambda-ring notation to take into account the complex signs.

- We determine combinatorial interpretations of the transition matrices between bases of the appropriate space.

- We define appropriate statistics on the elements or $C_k \S S_n$. Since there are a number of ways to define such statistics, we are forced to choose among several possible definitions.

- We use combinatorial methods to define an analog of ξ, which when applied to certain basis elements, gives the desired generating functions on elements of $C_k \S S_n$.

- We give combinatorial proofs of the desired identities. The proofs include interpretation of sums in terms of combinatorial objects, and the performance of involutions on the objects.

Introduction

In [3], Brenti introduced a homomorphism from the symmetric functions to polynomials of one variable over the rationals which, when applied to specific bases of the symmetric functions, gives generating functions for statistics on elements of S_n. This homomorphism, $\xi : \Lambda \longrightarrow \mathbf{Q}[x]$, is defined on the elementary symmetric functions as

$$\xi(e_n) = \frac{(1-x)^{n-1}}{n!}.$$

Brenti used algebraic methods to prove results such as the following.

$$n!\xi(h_n) = \sum_{\sigma \in S_n} x^{des(\sigma)},$$

where $des(\sigma)$ is the number of descents of the permutation σ, and

$$\frac{n!}{z_\lambda}\xi(p_\lambda) = \sum_{\sigma \in S_n(\lambda)} x^{e(\sigma)},$$

where $e(\sigma)$ is the number of excedances of σ and $S_n(\lambda)$ is the conjugacy class of S_n indexed by λ.

Beck and Remmel [2] gave combinatorial proofs of these and other related results, and gave q-analogs. Beck [1] then defined a similar map on a space of symmetric functions associated to B_n, and proved similar results. It is important to note that Beck's results for B_n and the q-analogs for both cases were possible only through understanding the combinatorial proofs for S_n.

In this text, we first state Beck and Remmel's results for S_n and B_n, then extend their ideas to determine similar results for wreath products $C_k \S S_n$ between cyclic and symmetric groups. In order to define the symmetric functions associated with $C_k \S S_n$, we must understand its representation theory. This is developed in Chapter 3, and uses the notation of extended λ-ring notation that is developed in Chapter 2. The combinatorial methods used in the permutation enumeration proofs depend on combinatorial interpretations of the transition matrices between bases of these symmetric functions, which are developed in Chapter 4. Finally, we develop the analog of Brenti, Beck and Remmel's results in Chapter 5.

Chapter 1

Permutation Enumeration of S_n and B_n

In this chapter, we will review the results of Brenti [3], Beck and Remmel [1], [2], which are the motivation for this work. We begin with some notation and definitions, then state most of the results, giving a few proofs to demonstrate Beck and Remmel's methods.

1.1 Preliminaries

Here we will give some notation and definitions which we will use in this chapter.

A *partition* λ of a positive integer n is a sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$, with $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_l$, such that $\lambda_1 + \lambda_2 + \cdots + \lambda_l = n$. If λ is a partition of n, we write $\lambda \vdash n$. A partition can be represented as a *Ferrers' diagram*, F_λ, which consists of left-justified rows of squares such that the rows, from top to bottom, have $\lambda_1, \lambda_2, \ldots, \lambda_l$ squares, respectively. The conjugate partition λ' is the partition whose Ferrers' diagram is the reflection of F_λ about the diagonal that extends northeast from the lower left corner. The Ferrers' diagrams $F_{(1,2,2,3,5)}$ and $F_{(1,2,2,3,5)'}$ are given in Figure 1.1.

A *tableau* of shape λ is a filling of F_λ with positive integers such that each cell

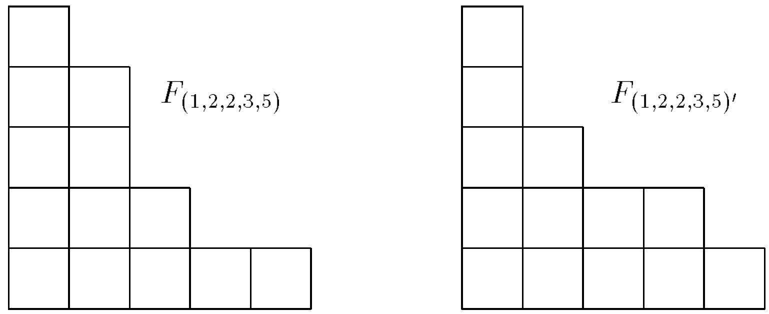

Figure 1.1: The Ferrers' diagrams $F_{(1,2,2,3,5)}$ and $F_{(1,2,2,3,5)'} = F_{(1,1,2,4,5)}$.

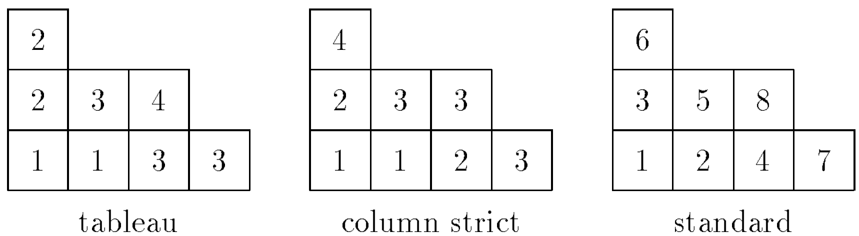

Figure 1.2: A tableau, column strict tableau, and standard tableau of shape $(1, 3, 4)$.

of the diagram is filled with exactly one integer, and the integers increase weakly left to right in rows and bottom to top in columns. A tableau is *column strict* if the integers increase strictly in columns. A tableau is *standard* if it is a column strict tableau filled with the integers $1, 2, \ldots, n$. Examples of tableaux, column strict tableaux, and standard tableaux are given in Figure 1.2. If T is a column strict tableau of shape λ, let $T_{i,j}$ be the integer filling of the cell in the i^{th} row and the j^{th} column. Then the weight of T, $w(T)$, is defined by

$$w(T) = \prod_{(i,j)\in F_\lambda} x_{T_{i,j}}.$$

A polynomial $P(x_1, x_2, \ldots, x_N)$ is *symmetric* if and only if $P(x_{\sigma_1}, x_{\sigma_2}, \ldots, x_{\sigma_N}) = P(x_1, x_2, \ldots, x_N)$ for all elements σ of the symmetric group S_N. Let $\Lambda_n = \Lambda_n(x_1, x_2, \ldots, x_N)$ be the set of all symmetric polynomials that are homogeneous of degree n. Let

$$\Lambda = \Lambda(x_1, x_2, \ldots, x_N) = \Lambda = \bigoplus_{n \geq o} \Lambda_n(x_1, x_2, \ldots, x_N).$$

There are six classical bases of Λ_n, which are indexed by partitions of n. We define these bases for $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) \vdash n$. The *monomial* basis $\{m_\lambda\}_{\lambda \vdash n}$ of $\Lambda_n$is given by

$$m_\lambda(x_1, x_2, \ldots, x_N) = \sum_{\substack{i_1, i_2, \ldots, i_N \\ r(i_1, i_2, \ldots, i_N) = \lambda}} x_1^{i_1} x_2^{i_2} \cdots x_N^{i_N},$$

where $r(i_1, i_2, \ldots, i_N)$ is a weakly increasing rearrangement of $i_1, i_2, \ldots, i_N$. The *power* basis $\{p_\lambda\}_{\lambda \vdash n}$ is defined by $p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_l}$, where

$$p_k(x_1, x_2, \ldots, x_N) = \sum_{i=1}^{N} x_i^k.$$

The *elementary* basis $\{e_\lambda\}_{\lambda \vdash n}$ is defined by $e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_l}$, where

$$e_k(x_1, x_2, \ldots, x_N) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq N} x_{i_1} x_{i_2} \cdots x_{i_k}.$$

The *homogeneous* basis $\{h_\lambda\}_{\lambda \vdash n}$ is given by $h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_l}$, where

$$h_k(x_1, x_2, \ldots, x_N) = \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq N} x_{i_1} x_{i_2} \cdots x_{i_k}.$$

The *Schur* basis $\{s_\lambda\}_{\lambda \vdash n}$ is given by

$$s_\lambda = \sum_{T \in CS_\lambda} w(T),$$

where CS_λ is the set of all column strict tableaux of shape λ. Finally, the *forgotten* basis $\{f_\lambda\}_{\lambda \vdash n}$ is the dual basis of the elementary basis with respect to the Hall inner product which is defined by declaring that

$$\langle m_\lambda, h_\mu \rangle = \delta_{\lambda,\mu},$$

where $\delta_{\lambda,\mu}$ is 1 if $\lambda = \mu$ and 0 otherwise.

Let S_n be the symmetric group on n elements, and let $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ be a permutation of S_n given in one-line notation (i.e. $\sigma(n) = \sigma_n$). We can then define a number of statistics on such a permutation. If $\sigma_i > \sigma_{i+1}$, then i is said to be a *descent* of σ. The number of descents of σ is $des(\sigma) = |\{i : \sigma_i > \sigma_{i+1}\}|$. If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) \vdash n$ we can also define the λ-*descents* of σ. We take σ in one-line notation, and break it into pieces of lengths $\lambda_1, \lambda_2, \ldots, \lambda_l$. We then only count the number of descents $\sigma_i > \sigma_{i+1}$ such that both i and $i+1$ occur within the same piece, then we denote the sum by $des_\lambda(\sigma)$. For example, if $\sigma = \begin{array}{cccccccc} 8 & 6 & 2 & 7 & 4 & 3 & 1 & 5 \end{array}$ and $\lambda = (1, 3, 4)$, we break σ into pieces [8][6 2 7][4 3 1 5]. Here, $des_\lambda(\sigma) = 3$ while $des(\sigma) = 5$.

If $\sigma_i > i$, then i is an *excedance* of σ. We denote by $e(\sigma) = |\{i : \sigma > i\}|$ the number of excedances of σ. In the above example, $e(\sigma) = 3$.

An *inversion* occurs whenever $i < j$ but $\sigma_i > \sigma_j$. The number of inversions is denoted by $inv(\sigma) = \sum_{i<j} \chi(\sigma_i > \sigma_j)$, where we use the notation $\chi(A) = 1$ if the statement A is true, and $\chi(A) = 0$ if A is false. In the above example, $inv(\sigma) = 7+5+1+4+2+1+0+0 = 20$. We can also define the inversion statistic on words other than permutations. Let $\mathcal{R}(1^{a_1}, 2^{a_2}, \ldots l^{a_l})$ denote the rearrangements of a_1 1's, a_2 2's,..., and a_l l's. If $\sum_{i=1}^{l} a_i = n$ and $r = r_1 r_2 \cdots r_n \in \mathcal{R}(1^{a_1}, 2^{a_2}, \ldots l^{a_l})$, then an inversion occurs whenever $i < j$ and $r_i > r_j$. The number of inversions, then, is $inv(r) = \sum_{i<j} \chi(r_i > r_j)$.

Let B_n be the hyperoctahedral group on n elements. There are two helpful ways of describing B_n. First, one can think of it as a Coxeter group with generators $\sigma_1, \sigma_2, \ldots \sigma_{n-1}$ and τ, and relations

$$\begin{aligned} \sigma_i^2 = \tau^2 &= 1, \quad i = 1 \ldots n-1, \\ (\sigma_i \sigma_{i+1})^3 &= 1, \quad i = 1 \ldots n-2, \\ (\sigma_i \sigma_j)^2 &= 1, \quad |i - j| \geq 2 \\ (\sigma_i \tau)^2 &= 1, \quad i = 1 \ldots n-2, \\ (\sigma_{n-1} \tau)^4 &= 1. \end{aligned}$$

The generators σ_i are, in fact, generators of the symmetric group, with $\sigma_i = (i, i+1)$ being the transposition of i and $i+1$ written in cyclic form for $i = 1, 2, \ldots, n-1$. The final generator is $\tau = (-n)$, that is, a generator that maps n to $-n$. Because of this, we can also think of B_n as the group of *signed permutations*. That is, if $\sigma \in B_n$,

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ \sigma_1 & \sigma_2 & \sigma_3 & \cdots & \sigma_n \end{pmatrix}$$

where $\sigma_i \in \{\pm 1, \pm 2, \ldots, \pm n\}$. As with the symmetric group, we can write elements of B_n in cycle notation with cycles of the form

$$\begin{pmatrix} i_1 & i_2 & \cdots & i_m \\ \epsilon_1 i_2 & \epsilon_2 i_3 & \cdots & \epsilon_m i_1 \end{pmatrix}.$$

We will usually write such cycles in one-line notation:

$$(\epsilon_m i_1, \epsilon_1 i_2, \cdots, \epsilon_{m-1} i_m).$$

Note that here, i_1 is mapped to $\epsilon_1 i_2$, i_2 is mapped to $\epsilon_2 i_3$, and so on; when determining where each i is to be sent, ignore the sign on it and only consider the sign on the element to which it is being mapped.

In order to define the necessary statistics on elements of B_n, we must first define an ordering and a partial ordering on the elements. Define the linear order Θ by the following.

$$1 <_\Theta 2 <_\Theta \cdots <_\Theta n <_\Theta \cdots <_\Theta -n <_\Theta \cdots <_\Theta -2 <_\Theta -1. \tag{1.1}$$

Define the partial order , by

$$1 \equiv -1 <_\Gamma 2 \equiv -2 <_\Gamma \cdots <_\Gamma n \equiv -n.$$

The ordering Θ is used because the number of inversions with respect to Θ corresponds to the length of an element of B_n when considered as a Coxeter group. Unfortunately, the inversions we are able to count here do not correspond to Θ, but to the partial order , .

Now we are ready to define some widely accepted statistics on elements of B_n. For $\sigma \in B_n$, let $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ where $\sigma_i \in \{\pm1, \pm2, \ldots \pm n\}$. Then if $\sigma_i >_\Theta \sigma_{i+1}$, i is a B_n-*descent* of σ. The number of B_n-descents is then denoted by $des_B(\sigma) = |\{i : \sigma_i >_\Theta \sigma_{i+1}\}|$, where $\sigma_{n+1} = n+1$. For $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) \vdash n$, the number of B_n λ-*descents* is defined in the following way. Write $\sigma \in B_n$ in one-line notation, then break it into pieces of lengths $\lambda_1, \lambda_2, \ldots, \lambda_l$. Then $des_{B,\lambda}(\sigma)$ counts only the B_n-descents such that i and $i+1$ lie in the same segment. A B_n-*inversion* of σ occurs whenever $i < j$ and $\sigma_i >_\Gamma \sigma_j$. We then denote the number of B_n-inversions by $inv(\sigma) = \sum_{i<j} \chi(\sigma_i >_\Gamma \sigma_j)$.

For $\sigma \in B_n$, let $\sigma = (\sigma_{1_1}\sigma_{1_2}\cdots\sigma_{1_{l(1)}})(\sigma_{2_1}\sigma_{2_2}\cdots\sigma_{2_{l(2)}})\cdots(\sigma_{k_1}\sigma_{k_2}\cdots\sigma_{k_{l(k)}})$ be in cycle notation. We say that a B_n-*descedance* occurs at the j^{th} position of the i^{th} cycle if either $1 \leq j < i \leq l(i)$ and $\sigma_{i_j} >_\Theta \sigma_{i_{j+1}}$, or $j = l(i)$ and $\sigma_{i_{l(i)}} >_\Theta \sigma_{i_1}$. If k is the number of cycles of σ, the number of B_n-descedances is denoted by

$$de_B(\sigma) = \sum_{i=1}^{k} \left(|j : \sigma_{i_j} >_\Theta \sigma_{i_{j+1}}, \quad 1 \leq j < l(i)| + \chi(\sigma_{i_{l(i)}} >_\Theta \sigma_{i_1}) \right).$$

For our study of the q-analogs of the results, we need notation for q-analogs of the factorial, binomial coefficient, and multinomial coefficient. These are defined by the following expressions.

$$[n] = 1 + q + q^2 + \cdots + q^{n-1},$$

$$[n]! = [n][n-1][n-2]\cdots[1],$$

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!},$$

$$\begin{bmatrix} n \\ k_1 k_2 \cdots k_l \end{bmatrix} = \frac{[n]!}{[k_1]![k_2]!\cdots[k_l]!}.$$

Finally, the following is a useful theorem regarding q-analogs, which is due to Carlitz [4].

Theorem 1.1. *Let* $\sum_{i=1}^{l} a_i = n$, *then*

$$\sum_{r \in \mathcal{R}(1^{a_1}, 2^{a_2}, \ldots, l^{a_l})} q^{inv(r)} = \begin{bmatrix} n \\ a_1 a_2 \ldots a_l \end{bmatrix},$$

where $\mathcal{R}(1^{a_1}, 2^{a_2}, \ldots, l^{a_l})$ *is the set of rearrangements of* a_1 *1's,* a_2 *2's,* $\ldots$*, and* a_l *l's.*

1.2 ξ Applied to Certain Bases of Λ

In this section we will state the results of Brenti and Beck and Remmel regarding the images of some symmetric function bases under the homomorphism ξ which is defined as follows.

Definition 1.2. The homomorphism $\xi : \Lambda \longrightarrow \mathbf{Q}[x]$ is defined on the elementary basis of Λ by

$$\xi(e_n) = \frac{(1-x)^{n-1}}{n!}$$

for $n \in \{1, 2, \ldots\}$, and by setting $\xi(e_0) = 1$.

Brenti [3] gives explicit expressions for $\xi(h_n)$, $\xi(p_n)$ and $\xi(p_\lambda)$, and the leading coefficient of $\xi(s_\lambda)$. Beck and Remmel [1] [2] give combinatorial interpretations for these expressions, then use the interpretations to find expressions for $\xi(h_\lambda)$, the coefficient of $(1-x)^{n-l(\lambda)}/n!$ in $\xi(s_\lambda)$, and q-analogs, which will be discussed in the next section.

1.2.1 A Combinatorial Interpretation of $n!\xi(h_n)$

First we consider the image of the basis of homogeneous symmetric functions under ξ. We have the following theorem due to Brenti. The proof given is due to Beck and Remmel.

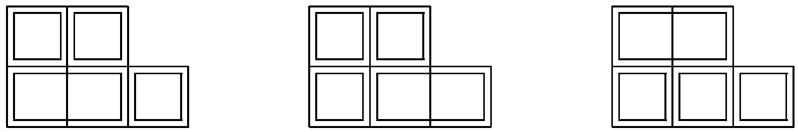

Figure 1.3: The (1,1,1,2)-brick tabloids of shape (2,3)

Theorem 1.3. *Let $\xi : \Lambda \longrightarrow \mathbf{Q}[x]$ be the ring homomorphism defined in Definition 1.2. Then*

$$n!\xi(h_n) = \sum_{\sigma \in S_n} x^{des(\sigma)}.$$

Proof. The proof depends on the fact, due to Eğecioğlu and Remmel [7], that one can express h_λ in terms of e_μ by

$$h_\lambda = \sum_{\mu \vdash n} (-1)^{n-l(\mu)} B_{\mu,\lambda} e_\mu \tag{1.2}$$

where $B_{\mu,\lambda}$ denotes the number of *μ-brick tabloids* of shape λ. To construct a μ-brick tabloid of shape λ, begin with the shape λ and fill it with bricks of sizes $\mu_1, \mu_2, \ldots, \mu_{l(\mu)}$ in such a way that each brick lies in exactly one row of λ. Figure 1.3 shows all three $(1,1,1,2)$-brick tabloids of shape $(2,3)$.

We use the special case of (1.2) with $\lambda = (n)$ to interpret $n!\xi(h_n)$. Multiplying by $n!$ and applying the homomorphism ξ to both sides gives

$$\begin{aligned} n!\xi(h_n) &= \sum_{\mu \vdash n} (-1)^{n-l(\mu)} B_{\mu,(n)} n! \xi(e_\mu) \\ &= \sum_{\mu \vdash n} (-1)^{n-l(\mu)} B_{\mu,(n)} n! \prod_{i=1}^{l(\mu)} \frac{(1-x)^{\mu_i}}{\mu_i!} \\ &= \sum_{\mu \vdash n} \binom{n}{\mu_1, \mu_2, \ldots, \mu_l} B_{\mu,(n)} (x-1)^{n-l(\mu)} \\ &= \sum_{\mu \vdash n} \sum_{T \in \mathcal{B}_{\mu,(n)}} \binom{n}{\mu_1, \mu_2, \ldots, \mu_l} (x-1)^{n-l(\mu)} \end{aligned} \tag{1.3}$$

where $\mathcal{B}_{\mu,(n)}$ is the set of μ-brick tabloids of shape λ, and if $\mu = (\mu_1, \mu_2, \ldots, \mu_l)$, then

$$\binom{n}{\mu_1, \mu_2, \ldots, \mu_l} = \frac{n!}{\prod_{i=1}^{l} \mu_i!}.$$

We will begin by showing that the right hand side of (1.3) is equivalent to a sum of signed, weighted combinatorial objects. That is, for some class of objects $\mathcal{O}_{h_n}$,

$$\sum_{\mu \vdash n} \sum_{T \in \mathcal{B}_{\mu,(n)}} \binom{n}{\mu_1, \mu_2, \ldots, \mu_l} (1-x)^{n-l(\mu)} = \sum_{o \in \mathcal{O}_{h_n}} w(o), \tag{1.4}$$

where $w(o)$ is the signed weight of the object o. Then we will define a sign-reversing, weight-preserving involution on these objects, the fixed points of which will express $\sum_{\sigma \in S_n} x^{des(\sigma)}$.

To define one of the objects, first select μ, a partition of n, and a μ-brick tabloid of shape (n), i.e. which has one row of length n. Use the multinomial coefficient to fill each brick of the tabloid with a decreasing sequence of integers from the set $\{1, 2, \ldots, n\}$ such that each integer appears exactly once in the tabloid. Use the term $(x-1)^{n-l(\mu)}$ to assign a sign and weight to each cell c of the tabloid as follows:

$$w(c) = \begin{cases} 1, & c \text{ is at the end of a brick,} \\ -1 \text{ or } x, & \text{otherwise.} \end{cases} \tag{1.5}$$

This takes into account the fact that there should be exactly one cell in each brick whose weight is not determined by one of the $n - l(\mu)$ terms of the form $(x-1)$. The signed weight of the object, o, is then defined to be $w(o) = \prod_{c \in o} w(c)$. Given this definition of these objects, it is clear that (1.4) holds. Figure 1.4 shows an example of such an object.

We now define a sign-changing, weight-preserving involution on the objects in $\mathcal{O}_{h_n}$. To perform the involution, begin checking from the left hand side of the tableau for the first occurrence of one of the following conditions. When one of these is found, perform the corresponding operation.

- If there is a decrease between the integer fillings of the last cell c of one brick

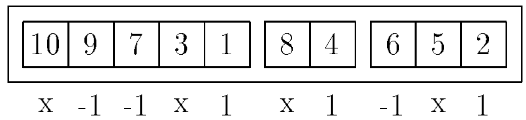

Figure 1.4: An example of the objects in $\mathcal{O}_{h_n}$.

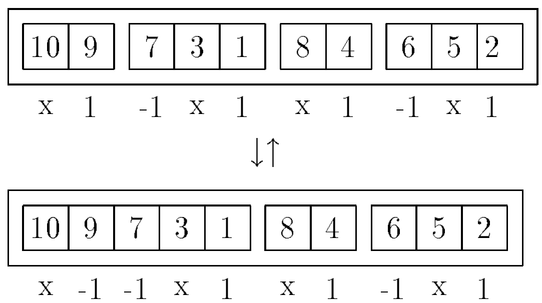

Figure 1.5: An example of the involution on $\mathcal{O}_{h_n}$.

and that of the first cell of the next, join the two bricks together, and change the weight of c from +1 to -1.

- If there is a cell c with weight -1, divide the brick after c and change the weight of c from -1 to +1.

Figure 1.5 gives an example of the involution.

Since only the sign of the weight on one cell is changed, it is clear that this is a sign-changing, weight-preserving involution. The fixed points of this involution are the filled μ-brick tableaux of shape (n) such that the following two properties hold:

- If c is a cell at the end of a brick, $w(c) = 1$. Otherwise, $w(c) = x$.

- The fillings are such that the integers decrease within bricks and increase between consecutive bricks.

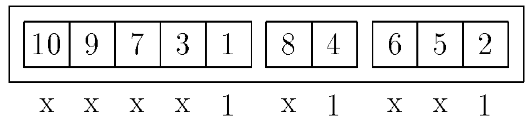

Figure 1.6: A fixed point of the involution on $\mathcal{O}_{h_n}$.

Figure 1.6 gives an example of a fixed point of the involution.

Consider the filling of such a μ-brick tableaux as a permutation of S_n. Then each descent of the permutation is weighted by x and every increase is weighted by 1. Moreover, there is exactly one object among the fixed points of the involution which has a given filling. Thus its weight is $des(\sigma)$, and the sum over all these objects is the generating function of S_n with respect to descents. ■

1.2.2 A Combinatorial Interpretation of $n!\xi(h_\lambda)$

If we consider now the image of the basis element h_λ, we obtain the following result.

Theorem 1.4. *Let $\xi : \Lambda(x) \longrightarrow \mathbf{Q}[x]$ be the homomorphism defined in Definition 1.2, and λ be a partition of n. Then*

$$n!\xi(h_\lambda) = \sum_{\sigma \in S_n} x^{des_\lambda(\sigma)}.$$

Proof. As in the previous proof, we begin by expressing the h_λ in terms of the e_μ's, and applying the homomorphism ξ. We then manipulate it in the same way to obtain the following expression.

$$n!\xi(h_\lambda) = \sum_{\mu \vdash n} \sum_{T \in \mathcal{B}_{\mu,\lambda}} \binom{n}{\mu_1, \mu_2, \ldots, \mu_l} (x-1)^{n-l(\mu)}.$$

We again express the right hand side of this expression as a sum of signed, weighted combinatorial objects. The objects in this set, $\mathcal{O}_{h_\lambda}$, are μ-brick tabloids of shape λ. As previously, each brick is filled in decreasing order with integers from

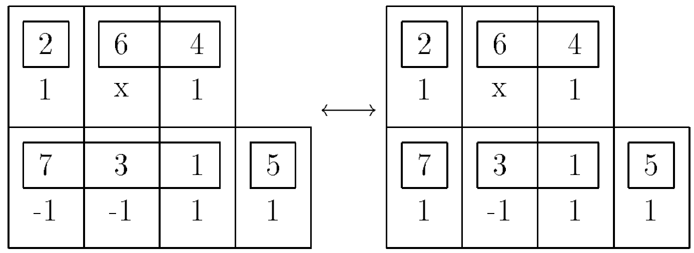

Figure 1.7: An example of the involution on $\mathcal{O}_{h_\lambda}$.

$[n]$, and each cell is weighted as in the previous proof. The weight of an object o is then defined by the product $\prod_{c \in o} w(c)$. This shows that we can write

$$\sum_{\mu \vdash n} \sum_{T \in \mathcal{B}_{\mu,\lambda}} \binom{n}{\mu_1, \ldots, \mu_{l(\mu)}} (x-1)^{n-l(\mu)} = \sum_{o \in \mathcal{O}_{h_\lambda}} w(o).$$

We again perform a sign-changing, weight-preserving involution, starting at the highest leftmost cell, and proceeding across each row from top to bottom, find the first occurance of one of the following conditions and perform the corresponding operation.

- If there is a decrease between the integer filling of the last cell c of one brick and the first cell of the next brick, and both bricks lie in the same row, join the two bricks together and change the weight of c from $+1$ to -1.

- If there is a cell $c$with weight -1, divide the brick after c and change the weight of c from -1 to $+1$.

Figure 1.7 gives an example of the involution.

Again, it is clear that this is a sign-changing, weight-preserving involution. Its fixed points are μ-brick tableaux of shape λ filled in such a way that the integer fillings increase between consecutive bricks in the same row and decrease within bricks, which are weighted so that the last cell of every brick receives weight 1 and all other cells receive weight x. However, we do not know what happens between rows.

Reading across the rows from top to bottom gives a permutation. The λ-descents of the permutation, that is, the descents which occur all within the same block when the permutation is divided into blocks of sizes $\lambda_1, \lambda_2, \ldots, \lambda_k$, are weighted by x. All of the cells which are the last in a λ-block or have an increase in the permutation are weighted by 1. Thus the weight of the tableaux is $x^{d_\lambda(\sigma)}$, and the sum over these objects is the generating function of the permutations of S_n with respect to λ-descents. ■

1.2.3 Other Results

Here we will state without proof the other results of Brenti, Beck and Remmel which are the result of applying the homomorphism ξ to bases of the symmetric functions. Their proofs are in the same spirit as the two given above. The following theorem was originally proved by Brenti, and given a combinatorial proof by Beck and Remmel.

Theorem 1.5. *Let $\xi : \Lambda \longrightarrow \mathbf{Q}[x]$ be the homomorphism defined in Definition 1.2, and λ be a partition of n. Then*

$$(n-1)!\xi(p_n) = \sum_{\sigma \in S_n((n))} x^{e(\sigma)},$$

and

$$\frac{n!}{z_\lambda}\xi(p_\lambda) = \sum_{S_n(\lambda)} x^{e(\sigma)},$$

where $S_n(\lambda)$ is the conjugacy class of S_n indexed by the partition λ, and $e(\sigma)$ is the number of excadences of the permutation σ.

In addition, Brenti gives an expression for the leading coefficient of $\xi(s_\lambda)$, and Beck and Remmel give the coefficient of $(1-x)^k$ in $\xi(s_\lambda)$. We will not state them here, as the definitions needed just to state them would take up too much space. The are given with proof in [1].

1.3 q-analogs for the S_n Case

In this section we give Beck and Remmel's q-analogs of the results stated in the previous section. Most of the proofs of these results are similar, so we only give the proof of the first result, stating the others without proof. We begin by defining a q-analog of ξ.

Definition 1.6. The homomorphism $\overline{\xi} : \Lambda \longrightarrow (\mathbf{Q}[q])[x]$ is defined on the elementary basis by

$$\overline{\xi}(e_n) = \frac{(1-x)^{n-1} q^{\binom{n}{2}}}{[n]!}$$

for $n \in \{1, 2, 3, \dots\}$, and $\overline{\xi}(e_0) = 1$.

If we apply $\overline{\xi}$ to h_n, the result is as follows.

Theorem 1.7. *Let ξ be as defined in Definition 1.6. Then*

$$[n]!\overline{\xi}(h_n) = \sum_{\sigma \in S_n} x^{des(\sigma)} q^{inv(\sigma)}$$

where $des(\sigma)$ is the number of descents of the permutation σ, and $inv(\sigma)$ is the number of inversions of σ.

Proof. As in the proof of Theorem 1.3, we begin by writing h_n in terms of the e_μ's.

$$h_n = \sum_{\mu \vdash n} (-1)^{n-l(\mu)} B_{\mu,(n)} e_\mu.$$

We then multiply both sides by $[n]!$ and apply $\overline{\xi}$.

$$\begin{aligned} [n]!\overline{\xi}(h_n) &= \sum_{\mu \vdash n} (-1)^{n-l(\mu)} B_{\mu,(n)} [n]! \overline{\xi}(e_\mu) \\ &= \sum_{\mu \vdash n} (-1)^{n-l(\mu)} B_{\mu,(n)} [n]! \prod_{i=1}^{l(\mu)} \frac{(1-x)^{\mu_i} q^{\binom{\mu_i}{2}}}{[\mu_i]!} \\ &= \sum_{\mu \vdash n} \sum_{T \in \mathcal{B}_{\mu,(n)}} \begin{bmatrix} n \\ \mu_1, \mu_2, \dots, \mu_l \end{bmatrix} q^{\sum_i \binom{\mu_i}{2}} (x-1)^{n-l(\mu)}. \end{aligned} \tag{1.6}$$

We will now interpret this as a sum of weighted combinatorial objects $o \in \mathcal{O}_{h_n,q}$. As in the proof of Theorem 1.3, we begin with single row tabloids filled with μ-bricks. We fill the bricks with the integers $1, 2, \ldots, n$ in the following way. Let $B_1, \ldots, B_l$ denote the bricks as they appear in order from left to right. Let $b_i = |B_i|$ for $i = 1, \ldots, l$ so $b_1, \ldots, b_l$ is a rearrangement of $\mu_1, \ldots, \mu_l$. Associate i's to each cell of brick B_i. For each rearrangement $r \in \mathcal{R}(1^{b_1}, 2^{b_2}, \ldots, l^{b_l})$, we create a permutation $\sigma(r)$ of n in the following way. Number the 1's from right to left, then the 2's, and so on. We then find the inverse permutation $\sigma^{-1}(r)$:

$$\begin{array}{rcccccccccccc} r & = & 1 & 3 & 2 & 1 & 3 & 3 & 1 & 2 & 1 & 3 & 3 \\ \sigma(r) & = & 4 & 11 & 6 & 3 & 10 & 9 & 2 & 5 & 1 & 8 & 7 \\ \sigma^{-1}(r) & = & 9 & 7 & 4 & 1 & 8 & 3 & 11 & 10 & 6 & 5 & 2 \end{array}$$

By the way we have constructed the permutation, we have blocks of decreasing integers which fit into the μ-bricks. Recall that by Theorem 1.1;

$$\begin{bmatrix} n \\ \mu_1, \mu_2, \ldots, \mu_l \end{bmatrix} = \sum_{r \in \mathcal{R}(1^{b_1}, 2^{b_2}, \ldots, l^{b_l})} q^{inv(r)}.$$

By the way we constructed $\sigma(r)$,

$$inv(\sigma^{-1}(r)) = inv(\sigma(r)) = inv(r) + \binom{b_1}{2} + \binom{b_2}{2} + \cdots + \binom{b_l}{2}.$$

We then have μ-brick tabloids of shape λ with the cells filled with the integers $1, 2, \ldots, n$ such that they decrease in bricks. We give an x-weight, $w_x(c)$ according to the rule (1.5). In addition, each cell also has a q-weight, $w_q(c)$ of q^{p_i} where p_i is the number of cells to the right of the cell c_i containing numbers which are lower than the integer contained in c_i. The weight of an object o is defined by $w(o) = \prod_{c \in o} w_x(c) w_q(c)$. This shows that we can write

$$\sum_{\mu \vdash n} \sum_{T \in \mathcal{B}_{\mu,(n)}} \begin{bmatrix} n \\ \mu_1, \ldots, \mu_{l(\mu)} \end{bmatrix} q^{\sum_i \binom{\mu_i}{2}} (x-1)^{n-l(\mu)} = \sum_{o \in \mathcal{O}_{h_n,q}} w(o).$$

Figure 1.8 gives an example of these objects.

10	9	7	3	1	8	4	6	5	2
xq^9	$-q^8$	$-q^6$	xq^2	1	xq^4	q^1	$-q^2$	xq^1	1

Figure 1.8: An example of an object in $\mathcal{O}_{h_n,q}$.

The involution on these objects is exactly the same as that in the proof of Theorem 1.3. Note that since we do not change the fillings of the tabloids, the q-weight does not change. As before, the x-weight changes only by sign. Thus the fixed points count the permutations of S_n with respect to the statistic $x^{des(\sigma)}q^{inv(\sigma)}$.

We will now state some of Beck and Remmel's other results regarding q-analogs. The proofs use similar methods to those above and are not given here. Other results may be found in [1].

Theorem 1.8. *Let λ be a partition of n, and let $\overline{\xi}$ be the homomorphism defined in Definition 1.6. Then*

$$[n]!\overline{\xi}(h_\lambda) = \sum_{\sigma \in S_n} x^{des_\lambda(\sigma)} q^{inv(\sigma)},$$

and

$$[n]!\overline{\xi}(p_n) = \sum_{k=1}^{n} q^{\binom{k}{2}} \frac{[n]!}{[k]!} k(x-1)^{k-1}\overline{\xi}(h_{n-k}),$$

where $des_\lambda(\sigma)$ is the number of λ-descents of σ, and $inv(\sigma)$ is the number of inversions of σ.

1.4 The Representation Theory of B_n

In order to examine the permutation enumeration of B_n, we must associate a space of symmetric functions to it and define an analog of ξ on that space of functions. Here, we review the representation theory of B_n as given by Stembridge [10] [9]. We follow the presentation of Beck [1] as a means of determining this space of symmetric functions. Later we will generalize the results for $C_k \S S_n$ following Beck's presentation.

1.4.1 Conjugacy Classes of B_n

Recall that we can write the elements of B_n in cycles of the form

$$\begin{pmatrix} i_1 & i_2 & \cdots & i_m \\ \epsilon_1 i_2 & \epsilon_2 i_3 & \cdots & \epsilon_m i_1 \end{pmatrix}$$

where $\epsilon_i = \pm 1$. We will usually write them in one line notation as

$$(\epsilon_m i_1, \epsilon_1 i_2, \ldots, \epsilon_{m-1} i_m)$$

with i_1 mapped to $\epsilon_1 i_2$, i_2 mapped to $\epsilon_2 i_3$ and so on. For example the element

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ -3 & 9 & -7 & 6 & 8 & 2 & 1 & 5 & -4 \end{pmatrix}$$

can be written in cycle notation as

$$\sigma = (1, -3, -7)(2, 9, -4, 6)(5, 8).$$

As a matter of convenience, we will replace the minus signs with bars over the numbers. In this case, the above element becomes

$$\sigma = (1, \overline{3}, \overline{7})(2, 9, \overline{4}, 6)(5, 8).$$

Now consider the product of the signs in each cycle. If the product of the signs is $+1$, we say that it is a *positive cycle*, or $+$ cycle. If the product of the signs is -1, we say that it is a *negative cycle*, or $-$ cycle. Conjugating an arbitrary element by one of the generators $\sigma_i = (i, i+1)$ preserves the sign inventory in each cycle, while conjugating by the generator $\tau = (-n)$ changes the sign of two entries in some cycle. Thus each conjugacy class must have a specified number of $+$ and $-$ cycles. Let $(\lambda, \mu) \vdash n$ denote a pair of partitions λ and μ such that $|\lambda| + |\mu| = n$. We then have the following proposition.

Proposition 1.9. *For some $(\lambda,\mu) \vdash n$, let $C_{(\lambda,\mu)}$ be the set of elements of B_n whose $+$ cycles have lengths $\lambda_1, \lambda_2, \ldots, \lambda_{l(\lambda)}$, and whose $-$ cycles have lengths $\mu_1, \mu_2, \ldots, \mu_{l(\mu)}$. Then the set of conjugacy classes of B_n is $\{C_{(\lambda,\mu)}\}_{(\lambda,\mu)\vdash n}$. Moreover,*

$$|C_{(\lambda,\mu)}| = \frac{2^n n!}{2^{l(\lambda)+l(\mu)} z_\lambda z_\mu},$$

where if $\lambda = (1^{a_1} 2^{a_2} \cdots n^{a_n})$, we define $z_\lambda = 1^{a_1} 2^{a_2} \cdots n^{a_n} a_1! a_2! \cdots a_n!$.

Our example above has two $+$ cycles of lengths 3 and 2 and one $-$ cycle of length 4 so it belongs to the conjugacy class $C_{(\lambda,\mu)}$ of B_9 where $\lambda = (3,2)$ and $\mu = (4)$.

1.4.2 The Characteristic Map, Inner Products and Dual Bases

For a statement A, let $\chi(A)$ be 1 if A is true and 0 if A is false. Then set $1_{(\lambda,\mu)} = \chi(\sigma \in C_{(\lambda,\mu)})$ as the indicator function for the conjugacy class indexed by (λ,μ). The set $\{1_{(\lambda,\mu)} \colon (\lambda,\mu) \vdash n\}$ is a then basis for the class functions $C(B_n)$ of the group B_n. We define the *characteristic map*,

$$ch : C(B_n) \longrightarrow \bigoplus_{k=0}^{n} \Lambda_k(X) \otimes \Lambda_{n-k}(Y),$$

by

$$ch \colon 1_{(\lambda,\mu)} \mapsto \frac{1}{z_\lambda z_\mu} p_\lambda(X) p_\mu(Y) \tag{1.7}$$

where $X = x_1, x_2, \ldots, x_N, Y = y_1, y_2, \ldots, y_N$ and $N \geq n$. From now on, we will denote the space $\bigoplus_{k=0}^{n} \Lambda_k(X) \otimes \Lambda_{n-k}(Y)$ by Λ_{B_n}, and we will let $\Lambda_B = \bigoplus_{n\geq 0} \Lambda_{B_n}$. The characteristic map is an analog of the Frobenius characteristic, $F \colon C(S_n) \longrightarrow \Lambda_n$, with $1_\lambda \mapsto \frac{1}{z_\lambda} p_\lambda(X)$, where 1_λ is the indicator function for the conjugacy class of S_n indexed by λ.

The usual inner product for any group G is given by

$$< \chi, \psi >_G = \frac{1}{|G|} \sum_{g\in G} \chi(g) \overline{\psi(g)}.$$

In B_n, then, we have

$$\begin{aligned}
\left\langle 1_{(\lambda,\mu)}, 1_{(\alpha,\beta)} \right\rangle_{B_n} &= \frac{1}{2^n n!} \sum_{\sigma \in B_n} 1_{(\lambda,\mu)}(\sigma) 1_{(\alpha,\beta)}(\sigma) \\
&= \frac{1}{2^n n!} \sum_{\sigma \in B_n} 1_{(\lambda,\mu)} \delta_{\lambda,\alpha} \delta_{\mu,\beta} \\
&= \frac{1}{2^n n!} |C_{(\lambda,\mu)}| \delta_{\lambda,\alpha} \delta_{\mu,\beta} \\
&= \frac{\delta_{\lambda,\alpha}\delta_{\mu,\beta}}{2^{l(\lambda)+l(\mu)} z_\lambda z_\mu}
\end{aligned}$$

where $\delta_{\lambda,\mu} = \chi(\lambda = \mu)$.

We would like to define an inner product $<,>_*$ on Λ_B in such a way that the above inner product is preserved under the characteristic map. We define it on the basis $\{p_\lambda(X)p_\mu(Y)\}$ as follows.

$$\left\langle \frac{p_\lambda(X)p_\mu(Y)}{z_\lambda z_\mu}, \frac{p_\alpha(X)p_\beta(Y)}{z_\alpha z_\beta} \right\rangle_* = \frac{\delta_{\lambda,\alpha}\delta_{\mu,\beta}}{2^{l(\lambda)+l(\mu)} z_\lambda z_\mu}. \tag{1.8}$$

Now fix some standard order of the pairs of permutations (λ, μ), and consider two bases $\{a_\lambda \overline{a}_\mu\}_{(\lambda,\mu)\vdash n}$ and $\{b_\lambda \overline{b}_\mu\}_{(\lambda,\mu)\vdash n}$ as the row vectors $< a_\lambda \overline{a}_\mu >_{(\lambda,\mu)\vdash n}$ and $< b_\lambda, \overline{b}_\mu >_{(\lambda,\mu)\vdash n}$. We say the two bases are *dual* if $< a_\lambda \overline{a}_\mu > \odot < b_\lambda \overline{b}_\mu >^T = I$. From (1.8),

$$\left\langle \frac{p_\lambda(X)p_\mu(Y)}{\sqrt{\frac{z_\lambda z_\mu}{2^{l(\lambda)+l(\mu)}}}}, \frac{p_\alpha(X)p_\beta(Y)}{\sqrt{\frac{z_\alpha z_\beta}{2^{l(\alpha)+l(\beta)}}}} \right\rangle_* = \delta_{\lambda,\alpha}\delta_{\mu,\beta}.$$

Thus the basis

$$\left\langle \frac{p_\lambda(x)p_\mu(Y)}{\sqrt{\frac{z_\lambda z_\mu}{2^{l(\lambda)+l(\mu)}}}} \right\rangle_{(\lambda,\mu)\vdash n}$$

is self dual.

Let $\Omega^{2n}(X, Y, \overline{X}, \overline{Y})$ denote the sum of the terms of degree $2n$ in

$$\prod_{i,j} \frac{1}{(1 - x_i \overline{x}_j)^2} \frac{1}{(1 - y_i \overline{y}_j)^2}.$$

Beck [1] gives proofs of the following two theorems which give a useful characterization of duality.

Theorem 1.10.

$$\sum_{(\lambda,\mu)\vdash n} \frac{p_\lambda(X)p_\mu(Y)}{\sqrt{\frac{z_\lambda z_\mu}{2^{l(\lambda)+l(\mu)}}}} \frac{p_\lambda(\overline{X})p_\mu(\overline{Y})}{\sqrt{\frac{z_\lambda z_\mu}{2^{l(\lambda)+l(\mu)}}}} = \Omega^{2n}(X,Y,\overline{X},\overline{Y}).$$

Theorem 1.11. *Let* $\{R_\lambda(X)\overline{R}_\mu(Y)\}_{(\lambda,\mu)\vdash n}$ *and* $\{Q_\lambda(\overline{X})\overline{Q}_\mu(\overline{Y}\}_{(\lambda,\mu)\vdash n}$ *be bases of* $\Lambda_{B_n}(X,Y)$. *Under the inner product* $<,>_*$, *these bases are dual if and only if*

$$\sum_{(\lambda,\mu)\vdash n} R_\lambda(X)\overline{R}_\mu(Y)Q_\lambda(\overline{X})\overline{Q}_\mu(\overline{Y}) = \Omega^{2n}(X,Y,\overline{X},\overline{Y}).$$

1.4.3 Lambda-Ring Notation

It is convenient to use lambda-ring notation to describe the irreducible characters of B_n. We define this notation on the power symmetric functions then extend the definition to the other bases. Let X and Y be a alphabets of variables. The following then define $p_r(X)$ where r is a nonnegative integer.

$$\begin{aligned} p_r(1) &= 1 \\ p_r(-X) &= -p_r(X) \\ p_r(X+Y) &= p_r(X)+p_r(Y) \end{aligned}$$

Then if $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \vdash n$, we define

$$p_\lambda(X) = p_{\lambda_1}(X)p_{\lambda_2}(X)\cdots p_{\lambda_k}(X).$$

We then use the Murgnaham-Nakayama Rule to extend the definition to the Schur functions.

$$s_\lambda(X) = \sum_{\mu\vdash n} \frac{\chi^\lambda_\mu}{z_\mu} p_\mu(X),$$

where χ^λ_μ is the irreducible character of S_n indexed by λ and evaluated at the conjugacy class indexed by μ. Given this definition, it is possible to show that

$$p_\lambda(X) = \sum_{\mu\vdash n} \chi^\lambda_\mu s_\mu(X).$$

Note that this definition also extends the lambda-ring notation to the elementary and homogeneous symmetric functions, since $e_r(X) = s_{(1^r)}(X)$ and $h_r(X) = s_{(r)}$, and $e_\lambda(X) = e_{\lambda_1}(X)e_{\lambda_2}(X)\cdots e_{\lambda_k}(X)$ and $h_\lambda(X) = h_{\lambda_1}(X)h_{\lambda_2}(X)\cdots h_{\lambda_k}(X)$.

Once we have these definitions, it is possible to prove the following identities. The proofs are given in Chapter 2.

$$s_\lambda(X+Y) = \sum_{\nu\subseteq\lambda} s_\nu(X)s_{\lambda/\nu}(Y),$$

$$s_\lambda(X-Y) = \sum_{\mu\subseteq\lambda}(-1)^{|\lambda/\nu|}s_\nu(X)s_{\lambda'/\nu'}(Y),$$

$$s_{\lambda/\mu}(-X) = (-1)^{|\lambda/\mu|}s_{\lambda'/\mu'}(X),$$

$$s_\lambda(XY) = \sum_{\mu,\nu\vdash n} K_{\lambda,\mu,\nu}s_\mu(X)s_\nu(Y)$$

where $K_{\lambda,\mu,\nu} = \sum_{\rho\vdash n}\frac{1}{z_\rho}\chi^\lambda_\rho\chi^\mu_\rho\chi^\nu_\rho$, and λ' is the conjugate partition of λ.

1.4.4 The Irreducible Characters of B_n

We begin our analysis of the irreducible characters of B_n by considering the one-dimensional characters. Let L be such a linear character. Then the characterization of B_n as Coxeter group gives us the following facts. First, $\sigma_i^2 = 1$ means $L(\sigma_i\sigma_i) = L(\sigma_i)L(\sigma_i) = L(\epsilon) = 1$ so $L(\sigma_i) = \pm 1$ for all i. Similarly, since $\tau^2 = 1$, $L(\tau) = \pm 1$. Finally, the relation $(\sigma_i\sigma_{i+1})^3 = 1$ gives us that $L(\sigma_i\sigma_{i+1}\sigma_i) = L(\sigma_{i+1}\sigma_i\sigma_{i+1})$ so $L(\sigma_i)L(\sigma_{i+1})L(\sigma_i) = L(\sigma_{i+1})L(\sigma_i)L(\sigma_{i+1})$ and $L(\sigma_{i+1}) = L(\sigma_i)$ for all $i \leq n-1$. Thus B_n has four linear characters, given in Table 1.1 as applied to elements of the conjugacy class indexed by (λ,μ).

Application of the characteristic map to these linear characters give the following results, written in λ-ring notation.

Theorem 1.12. *Let ch be the characteristic map defined in (1.7). Then*

$$ch(1_n) = h_n(X+\overline{X}), \tag{1.9}$$

Table 1.1: Linear characters of B_n applied at $C_{(\lambda,\mu)}$.

$L(\sigma_i)$	$L(\tau)$	character	applied at $C_{(\lambda,\mu)}$
1	1	1_n	1
-1	1	ϵ_n	$(-1)^{n-l(\lambda)-l(\mu)}$
1	-1	δ_n	$(-1)^{l(\mu)}$
-1	-1	$\delta_n\epsilon_n$	$(-1)^{n-l(\lambda)}$

$$ch(\epsilon_n) = e_n(X + \overline{X}), \tag{1.10}$$

$$ch(\delta_n) = h_n(X + \overline{X}), \tag{1.11}$$

and

$$ch(\delta_n\epsilon_n) = e_n(X + \overline{X}). \tag{1.12}$$

We prove (1.9) here. The proofs of (1.10), (1.11) and (1.12) are similar, with the proof of (1.11) appearing also in [1].

Proof.

$$\begin{aligned}
ch(1_n) &= ch\left(\sum_{(\lambda,\mu)\vdash n} 1_{(\lambda,\mu)}\right) \\
&= \sum_{(\lambda,\mu)\vdash n} ch(1_{(\lambda,\mu)}) \\
&= \sum_{(\lambda,\mu)\vdash n} \frac{p_\lambda(x)p_\mu(\overline{x})}{z_\lambda z_\mu} \\
&= \sum_{k=0}^{n}\left(\sum_{\lambda\vdash k}\frac{p_\lambda(x)}{z_\lambda}\right)\left(\sum_{\mu\vdash n-k}\frac{p_\mu(\overline{x})}{z_\mu}\right) \\
&= \sum_{k=0}^{n} h_k(x)h_{n-k}(\overline{x}) \\
&= h_n(X + \overline{X})
\end{aligned}$$

∎

We now turn our attention to the other irreducible characters of B_n. Note that if χ^γ is an irreducible character of S_n, we may regard it as a character of B_n by letting $\chi^\gamma(\sigma_i)$ be defined as for $\sigma_i \in S_n$ if σ_i is a generator, and letting $\chi^\gamma(\tau) = I_{f^\gamma}$ be the identity. We then have the following lemmas, which are proven in [1].

Lemma 1.13. *Let χ^γ be an irreducible character of S_n, and let δ be the linear character δ_n given in Table 1.1. Then*

$$ch(\chi^\gamma) = s_\gamma(X + \overline{X}),$$

and

$$ch(\delta\chi^\gamma) = s_\lambda(X - \overline{X}).$$

If χ is a character of a subgroup H of G, let $\chi \uparrow_H^G$ denote the character of G obtained by inducing χ to G. We then have the following characterization of the irreducible characters.

Theorem 1.14. *Let χ^λ and χ^μ be irreducible characters of S_k and S_{n-k}. Then the irreducible characters of B_n are $(\chi^\lambda \times (\delta\chi^\mu)) \uparrow_{S_k \times S_{n-k}}^{B_n}$, their characteristics are $s_\lambda(X + \overline{X})s_\mu(X - \overline{X})$, and their degrees are $\binom{n}{k} f^\lambda f^\mu$, where f^λ is the number of standard tableaux of shape λ.*

The proof appears in [1]. We provide a sketch of the proof. From Lemma 1.13, the statement about the characteristics is clear. To show irreducibility, it suffices to show that the basis $\left\langle s_\lambda(X + \overline{X})s_\mu(X - \overline{X})\right\rangle_{(\lambda,\mu)\vdash n}$ is self-dual. The degree is calculated by computing the inner product $\left\langle (\chi^\lambda \times (\chi^\mu\delta)) \uparrow_{S_k \times S_{n-k}}^{B_n}, 1_{(1^n,\emptyset)} \right\rangle_{B_n}$.

1.5 ξ_B Applied to Certain Bases of Λ_B

In this section we introduce a homomorphism ξ_B for the B_n case, an analog of ξ in the S_n case. We explore the results of applying this homomorphism to certain bases of $\Lambda_{B_n}(x, \overline{x})$, and find results analogous to the results for S_n. We begin by listing the bases of Λ_{B_n}.

Both of the following sets are self-dual bases of $\Lambda_{B_n}(x, \overline{x})$.

- $\langle p_\lambda(x)p_\mu(\overline{x})\rangle_{(\lambda,\mu)\vdash n}$,
- $\langle s_\lambda(X+\overline{X})s_\mu(X-\overline{X})\rangle_{(\lambda,\mu)\vdash n}$.

Using Jacobi-Trudi identities and dual bases, it can be shown that the following are also bases.

- $\langle h_\lambda(X+\overline{X})h_\mu(X-\overline{X})\rangle_{(\lambda,\mu)\vdash n}$,
- $\langle h_\lambda(X+\overline{X})e_\mu(X-\overline{X})\rangle_{(\lambda,\mu)\vdash n}$,
- $\langle e_\lambda(X+\overline{X})h_\mu(X-\overline{X})\rangle_{(\lambda,\mu)\vdash n}$,
- $\langle e_\lambda(X+\overline{X})e_\mu(X-\overline{X})\rangle_{(\lambda,\mu)\vdash n}$,
- $\langle m_\lambda(X+\overline{X})m_\mu(X-\overline{X})\rangle_{(\lambda,\mu)\vdash n}$,
- $\langle m_\lambda(X+\overline{X})f_\mu(X-\overline{X})\rangle_{(\lambda,\mu)\vdash n}$,
- $\langle f_\lambda(X+\overline{X})h_\mu(X-\overline{X})\rangle_{(\lambda,\mu)\vdash n}$,
- $\langle f_\lambda(X+\overline{X})f_\mu(X-\overline{X})\rangle_{(\lambda,\mu)\vdash n}$.

1.5.1 The Homomorphism ξ_B

We define an analog of ξ for the B_n case.

Definition 1.15. Define the homomorphism $\xi_B : \Lambda_B \to Q[x]$ on the elementary basis by

$$\xi_B(e_k(X+\overline{X})) = \frac{(1-x)^{k-1} + x(x-1)^{k-1}}{2^k k!},$$

and

$$\xi_B(e_k(X-\overline{X})) = \frac{(1-x)^{k-1} - x(1-x)^{k-1}}{2^k k!}$$

for $n \in \{1, 2, \dots\}$, and be setting $\xi_B(e_0(X+Y)) = \xi_B(e_0(X-Y) = 1$.

This is a definition on the entire space because $< e_\lambda(X+\overline{X})e_\mu(X-\overline{X}) >_{(\lambda,\mu)\vdash n}$ is a basis for Λ_B. It is important to note that this definition is suggested by the combinatorial proofs in the permutation enumeration of S_n. In fact, the homomorphism can be defined in a simpler form, but this definition suggests the weighting we will use in the combinatorial proofs for the B_n case.

1.5.2 Λ_{B_n}-Homogeneous Symmetric Functions Under ξ_B

To express $h_\lambda(X+\overline{X})h_\mu(X+\overline{X})$ in terms of $e_\alpha(X+\overline{X})e_\beta(X+\overline{X})$, we can express each part separately. That is, we can express $h_\lambda(X+\overline{X})$ in terms of $e_\alpha(X+\overline{X})$, and $h_\mu(X-\overline{X})$ in terms of $e_\beta(X-\overline{X})$. Thus we consider $\xi_B(h_\lambda(X+\overline{X}))$ and $\xi_B(h_\mu(X-\overline{X}))$ separately.

If we apply the homomorphism ξ_B to the basis elements $h_n(X+\overline{X})$ and $h_n(X-\overline{X})$, we achieve the following theorem, due to Beck [1].

Theorem 1.16. *If ξ_B is the homomorphism defined in Definition 1.15, then*

$$2^n n!\xi_B(h_n(X+\overline{X})) = \sum_{\sigma\in B_n} x^{des_B(\sigma)}, \tag{1.13}$$

and

$$2^n n!\xi_B(h_n(X-\overline{X})) = (1-x)^n, \tag{1.14}$$

where $des_B(\sigma)$ is the number of B_n-descents of σ.

Proof. We begin with the proof of (1.13). As in the S_n case, we begin by expressing $h_n(X+\overline{X})$ in terms of the $e_\alpha(X+\overline{X})$'s.

$$h_n(X+\overline{X}) = \sum_{\alpha\vdash n}(-1)^{n-l(\alpha)}B_{\alpha,(n)}e_\alpha(X+\overline{X}).$$

Then apply the homomorphism and multiply by $2^n n!$ to get

$$\begin{aligned}
2^n n!\xi_B(h_n(X+\overline{X})) &= \sum_{\alpha\vdash n}(-1)^{n-l(\alpha)}2^n n! B_{\alpha,(n)}\xi_B(e_\alpha(X+\overline{X})) \\
&= \sum_{\alpha\vdash n}(-1)^{n-l(\alpha)}2^n n! B_{\alpha,(n)}\prod_{i=1}^{l(\alpha)}\frac{(1-x)^{\alpha_i-1}+x(x-1)^{\alpha_i-1}}{2^{\alpha_i}\alpha_i!} \\
&= \sum_{\alpha\vdash n}\sum_{T\in\mathcal{B}_{\alpha,(n)}}\binom{n}{\alpha_1,\cdots,\alpha_l} \\
&\times \prod_{i=1}^{l(\alpha)}\left((x-1)^{\alpha_i-1}+x(1-x)^{\alpha_i-1}\right).
\end{aligned}$$

We will now show that the right side of this equation now corresponds to a sum of signed weighted objects $o\in\mathcal{O}_{Bh_n+}$. We begin with an α-brick tabloid of shape λ. The multinomial coefficient fills each brick with a decreasing sequence of integers such that exactly the set $\{1,2,\ldots,n\}$ is used to fill the tabloid. Each brick is also designated as either a *regular brick* or a *barred brick*. The weights of a cell c are defined as follows. If c is in a regular brick,

$$w(c)=\begin{cases}1, & c \text{ is at the end of a brick,}\\ -1 \text{ or } x, & \text{otherwise.}\end{cases}$$

This accounts for the $(x-1)^{\alpha_i-1}$ terms. If c is in a barred brick,

$$w(c)=\begin{cases}x, & c \text{ is at the end of a brick,}\\ 1 \text{ or } -x, & \text{otherwise.}\end{cases}$$

This accounts for the $x(x-1)^{\alpha_i-1}$ terms. The weight of an object o is defined by $w(o)=\prod_{c\in o}w(c)$. Thus we can write

$$\sum_{\alpha\vdash n}\sum_{T\in\mathcal{B}_{\alpha,(n)}}\binom{n}{\alpha_1,\ldots,\alpha_{l(\alpha)}}\prod_{i=1}^{l(\alpha)}\left((x-1)^{\alpha_i-1}+x(x-1)^{\alpha_i-1}\right)=\sum_{o\in\mathcal{O}_{Bh_n+}}w(o).$$

An example of these objects is given in Figure 1.9 .

We now perform an involution similar to those in the S_n case. Check from left to right in the tableau for the leftmost occurrence of one of the following, and perform the corresponding operation.

10	8	6	$\overline{9}$	$\overline{2}$	7	1	$\overline{12}$	$\overline{4}$	$\overline{3}$	$\overline{11}$	$\overline{5}$
x	-1	1	1	x	x	1	-x	1	x	-x	x

Figure 1.9: An example of the objects in $\mathcal{O}_{Bh_n+}$.

- If there is a decrease between the integer filling of the last cell c of a regular brick and that of the first cell of an adjacent regular brick, join the bricks together and change the weight of c from 1 to -1.
- If there is a decrease between the integer filling of the last cell c of a barred brick and that of the first cell of an adjacent barred brick, join the bricks together and change the weight of c from x to $-x$.
- If there is a cell c in a regular brick with weight -1, cut the brick after c and change the weight of c from -1 to 1.
- If there is a cell c in a barred brick with weight $-x$, cut the brick after c and change the weight of c from $-x$ to x.

This is a sign-changing, weight-preserving involution with fixed points with the following properties.

- The integer fillings decrease within each brick, and increase between adjacent regular bricks and between adjacent barred bricks.
- In regular bricks, the last cell has weight 1 and all other cells have weight x.
- In barred bricks, the last cell has weight x and all other cells have weight 1.

An example of such a fixed point is given in Figure 1.10 .

We now regard the sequence of integers as an element of B_n in one-line notation, with the integers in regular bricks as regular numbers and the integers in barred bricks as barred numbers. For the example in Figure 1.10, this is

$$\sigma = 10\ 8\ 6\ \overline{9}\ \overline{2}\ 7\ 1\ \overline{12}\ \overline{4}\ \overline{3}\ \overline{11}\ \overline{5}.$$

$$\begin{array}{|c|c|c|}\hline 10 & 8 & 6 \\ \hline \end{array}\ \begin{array}{|c|c|}\hline \overline{9} & \overline{2} \\ \hline \end{array}\ \begin{array}{|c|c|}\hline 7 & 1 \\ \hline \end{array}\ \begin{array}{|c|c|c|}\hline \overline{12} & \overline{4} & \overline{3} \\ \hline \end{array}\ \begin{array}{|c|c|}\hline \overline{11} & \overline{5} \\ \hline \end{array}$$

$$\begin{array}{cccccccccccc} \mathrm{x} & \mathrm{x} & 1 & 1 & \mathrm{x} & \mathrm{x} & 1 & 1 & 1 & \mathrm{x} & 1 & \mathrm{x} \end{array}$$

Figure 1.10: A fixed point of the involution on $\mathcal{O}_{Bh_n+}$.

Then with regard to the linear order for B_n defined in (1.1), each descent is weighted by x and each ascent is weighted by 1. Thus the weight of each fixed point is $x^{des_B(\sigma)}$. This proves (1.13).

To prove (1.14), we write $h_n(X-\overline{X})$ in terms of $e_\alpha(X-\overline{X})$, apply the homomorphism and multiply by $2^n n!$ to get the following.

$$\begin{aligned} 2^n n! \xi_B(h_n(X-\overline{X})) &= \sum_{\alpha \vdash n} (-1)^{n-l(\alpha)} 2^n n! B_{\alpha,(n)} \xi_B(e_\alpha(X-\overline{X})) \\ &= \sum_{\alpha \vdash n} (-1)^{n-l(\alpha)} 2^n n! B_{\alpha,(n)} \prod_{i=1}^{l(\alpha)} \frac{(1-x)^{\alpha_i-1} - x(1-x)^{\alpha_i-1}}{2^{\alpha_i}\alpha_i!} \\ &= \sum_{\alpha \vdash n} \sum_{T \in \mathcal{B}_{\alpha,(n)}} \binom{n}{\alpha_1, \cdots, \alpha_l} \\ &\times \prod_{i=1}^{l(\alpha)} \left((x-1)^{\alpha_i-1} - x(1-x)^{\alpha_i-1}\right). \end{aligned}$$

We again interpret this as a sum of signed, weighted objects $o \in \mathcal{O}_{Bhn-}$. The objects here are similar to those in the previous case, but the weights are slightly different. Again we have α-brick tabloids of shape (n), with the cells filled with the integers $1, 2, \ldots, n$ such that the integers decrease within each brick. Each brick is designated as regular or barred. Here, the weight of a cell c is defined by the following. If c is in a regular brick,

$$w(c) = \begin{cases} 1, & c \text{ is at the end of the brick} \\ -1 \text{ or } x, & \text{otherwise.} \end{cases}$$

This accounts for the $(x-1)^{\alpha_i-1}$ terms. If c is in a barred brick,

$$w(c) = \begin{cases} -x, & c \text{ is at the end of the brick} \\ -1 \text{ or } x, & \text{otherwise.} \end{cases}$$

This accounts for the $-x(1-x)^{\alpha_i-1}$ terms. The weight of an object o is defined by $w(o) = \prod_{c \in o} w(c)$. Then we can write

$$\sum_{\alpha \vdash n} \sum_{T \in \mathcal{B}_{\alpha,(n)}} \binom{n}{\alpha_1, \ldots, \alpha_{l(\alpha)}} \prod_{i=1}^{l(\alpha)} \left((x-1)^{\alpha_i-1} - x(1-x)^{\alpha_i-1} \right) = \sum_{o \in \mathcal{O}_{Bhn-}} w(o).$$

We will be able to perform two involutions on these objects. The first is similar to that in the previous case. Traverse the tableau from left to right and find the first occurrence of one of the following conditions, then perform the corresponding operation.

- If there is a decrease between the integer filling of the last cell c of a regular brick and that of the first element in an adjacent regular brick, join the bricks together and change the weight of c from 1 to -1.
- If there is a decrease between the integer filling of the last cell c of a barred brick and that of the first element in an adjacent barred brick, join the bricks together and change the weight of c from $-x$ to x.
- If there is a cell c in a regular brick with weight -1, cut the brick after c and change the weight of c from -1 to 1.
- If there is a cell c in a barred brick with weight x, cut the brick after c and change the weight of c from x to $-x$.

This is a sign-changing weight-preserving involution. Its fixed points have the following properties.

- The integer fillings decrease within bricks and increase between adjacent regular bricks and between adjacent barred bricks.

10	8	6	$\bar{9}$	$\bar{2}$	7	1	$\overline{12}$	$\bar{4}$	$\bar{3}$	$\overline{11}$	$\bar{5}$
x	x	1	-1	-x	x	1	-1	-1	-x	-1	-x

Figure 1.11: A fixed point of the first involution on $\mathcal{O}_{Bh_n-}$.

- In regular bricks, the last cell has weight 1 and all other cells have weight x.
- In barred bricks, the last cell has weight $-x$ and all other cells have weight -1.

An example of such a fixed point is given in Figure 1.11 .

We may now perform a second involution. Again, check from left to right in the tableau for the first occurrence of one of the following and perform the appropriate operation.

- If there is a barred brick of length more than one, separate the first cell c and make it into a separate regular brick of length one, changing the weight of c from -1 to 1.
- If there is a regular brick of length more than one, separate the first cell c and make it into a separate barred brick of length one, changing the weight of the cell c from x to $-x$.
- If there is a decrease between the integer filling of a regular brick which consists of a single cell c and that of the first cell of an adjacent barred brick, change the brick consisting of c into a barred brick, join the two bricks together, and change the weight of c from 1 to -1.
- If there is a decrease between the integer filling of a barred brick which consists of a single cell c and that of the first cell of an adjacent regular brick, change the brick consisting of c into a regular brick, join the two bricks together, and change the weight of c from $-x$ to x.

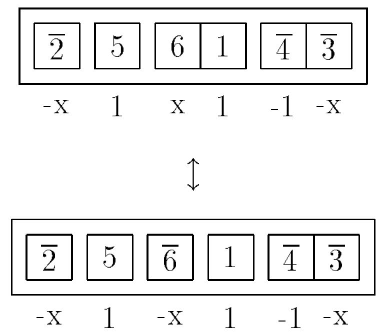

Figure 1.12: An example of the second involution on $\mathcal{O}_{Bh_n-}$.

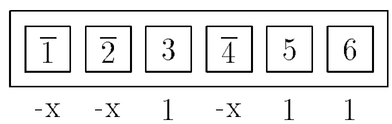

Figure 1.13: A fixed point of the second involution on $\mathcal{O}_{Bh_n-}$.

An example of this sign-changing weight-preserving involution is given in Figure 1.12 . This involution has fixed points with all bricks of length one such that the numbers increase between adjacent regular bricks, adjacent barred bricks, regular bricks adjacent to barred bricks, and barred bricks adjacent to regular bricks. That is, the numbers must be increasing throughout the tableau and there is only one way to fill it. Barred bricks are given weight $-x$ and regular bricks are given weight 1. Such a fixed point is shown in Figure 1.13 . Thus the only choice available is in choosing the weight of each cell. This is counted by $(1-x)^n$.

■

In the case where we consider h_λ rather than the special case h_n, we can perform involutions similar to those mentioned previously. We then obtain the following results. The proofs of these results can be found in [1].

Theorem 1.17. *Let $\lambda \vdash n$ and let ξ_B be defined as in Definition 1.15. Then*

$$2^n n! \xi_B(h_\lambda(X + \overline{X})) = \sum_{\sigma \in B_n} x^{des_{B,\lambda}(\sigma)},$$

and

$$2^n n! \xi_B(h_\lambda(X + \overline{X})) = \binom{n}{\lambda_1, \lambda_2, \ldots, \lambda_l}(1-x)^n,$$

where $des_{B,\lambda}(\sigma)$ is the number of B_n λ-descents of σ.

1.5.3 Λ_{B_n}-Power Symmetric Functions Under ξ_B

When we apply ξ_B to the power basis, we get the simple result that these count B_n-descedances over conjugacy classes. Here we will state the result without proof. The full proof can be found in [1].

Theorem 1.18. *If $\lambda \vdash n$ and ξ_B is the homomorphism defined in Definition 1.15, then*

$$\frac{2^n n!}{z_\lambda z_\mu} \xi_B(p_\lambda(x) p_\mu(\overline{x})) = \sum_{\sigma \in B_n(\lambda,\mu)} x^{de_B(\sigma)},$$

where $B_n(\lambda, \mu)$ is the conjugacy class of B_n indexed by the pair (λ, μ), and $de_B(\sigma)$ is the number of B_n-descedances of σ.

1.5.4 q-analogs for the B_n Case

As with the S_n case, we can introduce a q-analog of the homomorphism ξ_B.

Definition 1.19. Define the homomorphism $\overline{\xi}_B : \Lambda_B \to (Q[q])[x]$ on the elementary basis by

$$\overline{\xi}_B(e_k(X + \overline{X})) = \frac{q^{\binom{k}{2}} \left((1-x)^{k-1} + x(x-1)^{k-1}\right)}{2^k [k]!},$$

and

$$\overline{\xi}_B(e_k(X - \overline{X})) = \frac{q^{\binom{k}{2}} \left((1-x)^{k-1} - x(1-x)^{k-1}\right)}{2^k [k]!}$$

for $n \in \{1, 2, \ldots\}$ and by setting $\overline{\xi}_B(e_0(X+Y)) = \overline{\xi}_B(e_0(X-Y)) = 1$.

Application of this homomorphism to $h_n(X+\overline{X})$ and $h_n(X-\overline{X})$ gives the following results.

Theorem 1.20. *If $\overline{\xi}_B$ is the homomorphism defined by Definition 1.19, then*

$$2^n[n]!\overline{\xi}_B(h_n(X+\overline{X})) = \sum_{\sigma\in B_n} x^{des_B(\sigma)}q^{inv_B(\sigma)}$$

and

$$2^n[n]!\overline{\xi}_B(h_n(X-\overline{X})) = \sum_{\sigma\in \hat{B_n}} (-x)^{des_B(\sigma)}q^{inv_B(\sigma)}$$

where $\hat{B}_n = \{\sigma_1\sigma_2\cdots\sigma_n : \sigma_i \in \{i,\bar{i}\}\}$.

The proof of these statements uses involutions very similar to those used in the proof of Theorems 1.13 and 1.14. The one difference is the inclusion in the weight of a power q^{p_i} where p_i is the number of cells to the right of the cell in question whose integer contents is lower. In the case of $h_n(X-\overline{X})$, the second involution can not be performed.

Chapter 2

Lambda-ring Notation at Roots of Unity

We will write the irreducible characters of $C_k \S S_n$ using an extended version of λ-ring notation. In this chapter, we define this extension and prove some properties of it. Note that this is not the standard definition, which may be found in [5]. This definition allows particularly elegant combinatorial proofs of certain identities.

2.1 Definitions

Let $X = x_1 + x_2 + \cdots + x_l$ and $Y = y_1 + y_2 + \cdots + y_m$ be formal sums of alphabets. Define *λ-ring notation* (in its unextended form) on the power symmetric functions by

$$
\begin{aligned}
p_r(0) &= 0, \\
p_r(x) &= x^r, \\
p_r(X+Y) &= p_r(X) + p_r(Y), \\
p_r(-X) &= -p_r(X), \\
p_\mu(X) &= \prod_{i=1}^{l(\mu)} p_{\mu_i}(X).
\end{aligned}
$$

Extend this to other bases of the symmetric functions using the transition matrices between them and the power basis.

$$\begin{array}{rclrcl} h_r(X) &=& \sum_\mu \frac{1}{z_\mu} p_\mu(X); & h_\lambda(X) &=& \prod_{i=1}^{l(\lambda)} h_{\lambda_i}(X); \\ e_r(X) &=& \sum_\mu \frac{(-1)^{n-l(\mu)}}{z_\mu} p_\mu(X); & e_\lambda(X) &=& \prod_{i=1}^{l(\lambda)} e_{\lambda_i}(X); \\ s_\lambda(X) &=& \sum_\mu \frac{\chi_\mu^\lambda}{z_\mu} p_\mu(X), & & & \end{array}$$

where χ_μ^λ is the irreducible character of S_n indexed by λ evaluated at the conjugacy class indexed by μ. It is also possible to show that

$$p_\mu(X) = \sum_\lambda \chi_\mu^\lambda s_\lambda(X).$$

We extend the above definition to roots of unity by adding a single additional property. If $\epsilon = e^{\frac{2\pi i}{k}}$ for a positive integer k, then

$$p_r(\epsilon X) = \epsilon p_r(X).$$

2.2 Properties

Here we discuss properties of the extended λ-ring notation. The standard unextended λ-ring notation has the following properties. These also hold in the extended case.

Theorem 2.1.

$$s_\lambda(X+Y) = \sum_{\mu \subseteq \lambda} s_\mu(X) s_{\lambda/\mu}(Y), \tag{2.1}$$

$$s_\lambda(X-Y) = \sum_{\mu \subseteq \lambda} (-1)^{|\lambda/\mu|} s_\mu(X) s_{\lambda'/\mu'}(Y), \tag{2.2}$$

$$s_{\lambda/\mu}(-X) = (-1)^{|\lambda/\mu|} s_{\lambda'/\mu'}(X), \tag{2.3}$$

$$s_\lambda(XY) = \sum_{\mu,\nu} K_{\lambda,\mu,\nu} s_\mu(X) s_\nu(Y), \tag{2.4}$$

where

$$K_{\lambda,\mu,\nu} = \sum_\rho \frac{1}{z_\rho} \chi_\rho^\lambda \chi_\rho^\mu \chi_\rho^\nu$$

is the Kronecker coefficient.

Hooks: $\boxed{1}$ $\boxed{2}$ $\boxed{3\,|\,3}$ $\boxed{4\,|\,4}$ $\boxed{5\,|\,5\,|\,5}$

$$\begin{array}{|c|cccc}\hline 3 & & & \\ \hline 3 & \multicolumn{1}{c|}{5} & \multicolumn{1}{c|}{5} & \multicolumn{1}{c|}{5} \\ \hline 1 & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{4} & \multicolumn{1}{c|}{4} \\ \hline \end{array} \qquad \begin{array}{|c|cccc}\hline 4 & & & \\ \hline 4 & \multicolumn{1}{c|}{5} & \multicolumn{1}{c|}{5} & \multicolumn{1}{c|}{5} \\ \hline 1 & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{3} \\ \hline \end{array}$$

Figure 2.1: The rim-hook tableaux of shape (1,4,4) and type (1,1,2,2,3).

To prove this theorem, we need the following lemma.

Lemma 2.2. *If $\lambda, \nu \vdash n$ and α and β are partitions such that $\alpha + \beta = \nu$, then*

$$\chi_\nu^\lambda = \sum_{\mu \subseteq \lambda} \chi_\alpha^\mu \chi_\beta^{\lambda/\mu}.$$

Proof. (Sketch) Given a Ferrers' diagram, F_λ, of shape λ, a him hook of λ is a sequence of cells, h, along the northeast boundary of F_λ such that any two consecutive cells in h share an edge, and the removal of the cells of h from F_λ leaves the Ferrers' diagram of another partition. Given two partitions λ and μ, a *rim-hook tableau* of shape λ and type μ is a sequence of partitions

$$T = (= \lambda^{(0)} \subset \lambda^{(1)} \subset \cdots \lambda^{(k)} = \lambda) \tag{2.5}$$

such that for each $1 \leq i \leq k$, $\lambda^{(i)} - \lambda^{(i-1)}$ is a rim hook of size μ_i. Let $RH(\lambda, \mu)$ denote the set of all rim-hook tableaux of shape λ and type μ. Define the sign of a rim-hook h by

$$sgn(h) = (-1)^{r(h)-1}$$

where $r(h)$ is the number of rows occupied by h. Then the sign of a rim-hook tableau is the product of the signs of the hooks:

$$sgn(T) = \prod_{h \in T} sgn(H).$$

As an example, both of the rim-hook tableaux of shape (1,4,4) and type (1,1,2,2,3) are given in Figure 2.1.

Hooks: $\boxed{1}$ $\boxed{2\,2}$ $\boxed{3\,3\,3}$ $\boxed{4}$ $\boxed{5\,5}$

$$\begin{array}{|c|ccc}\hline 2 & \multicolumn{3}{c}{} \\ \hline 2 & \multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{4} & \multicolumn{1}{c|}{5} \\ \hline 1 & \multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{5} \\ \hline \end{array} \qquad \begin{array}{|c|ccc}\hline 2 & \multicolumn{3}{c}{} \\ \hline 2 & \multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{5} & \multicolumn{1}{c|}{5} \\ \hline 1 & \multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{4} \\ \hline \end{array} \qquad \begin{array}{|c|ccc}\hline 2 & \multicolumn{3}{c}{} \\ \hline 2 & \multicolumn{1}{|c|}{4} & \multicolumn{1}{c|}{5} & \multicolumn{1}{c|}{5} \\ \hline 1 & \multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{3} \\ \hline \end{array}$$

$$\begin{array}{|c|ccc}\hline 3 & \multicolumn{3}{c}{} \\ \hline 3 & \multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{4} & \multicolumn{1}{c|}{5} \\ \hline 1 & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{5} \\ \hline \end{array} \qquad \begin{array}{|c|ccc}\hline 3 & \multicolumn{3}{c}{} \\ \hline 3 & \multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{5} & \multicolumn{1}{c|}{5} \\ \hline 1 & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{4} \\ \hline \end{array} \qquad \begin{array}{|c|ccc}\hline 4 & \multicolumn{3}{c}{} \\ \hline 3 & \multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{5} \\ \hline 1 & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{5} \\ \hline \end{array}$$

Figure 2.2: The rim-hook tableaux of shape (1,4,4) and type (1,2,3) + (1,2).

The proof of the lemma depends on the fact (see [7]) that if χ^λ_ν is the irreducible character of S_n indexed by λ evaluated at the conjugacy class indexed by ν, then

$$\chi^\lambda_\mu = \sum_{T \in RH(\lambda,\nu)} sgn(T).$$

To see why the lemma holds, consider the example in Figure 2.1. Suppose that instead of filling λ with the hooks in the order given in the definition, we fill λ in another way. If $\alpha = (1,2,3)$ and $\beta = (1,2)$, then $\alpha + \beta = \mu$. Fill λ first by hooks of sizes 1, 2, 3, then by sizes 1,2. Figure 2.2 shows the result of this.

Classify these rim-hook tableaux by the partition, μ, that is formed by the hooks of type α. For each μ, filling λ the way that we did corresponds to filling μ with rim-hooks of type α and then filling λ/μ with rim-hooks of type β. This gives a product of the two characters: $\chi^\mu_\alpha \chi^{\lambda/\mu}_\beta$. Summing over all μ gives the result.

∎

We are now ready to prove Theorem 2.1.

Proof. (2.1) Begin by using the definition of lambda-ring notation for the Schur functions.

$$s_\lambda(X+Y) = \sum_{\nu \vdash n} \frac{\chi^\lambda_\nu}{z_\nu} p_\nu(X+Y).$$

Write ν as $(1^{\gamma_1}, 2^{\gamma_2}, \ldots, n^{\gamma_n})$ to obtain

$$\sum_{(1^{\gamma_1},\ldots,n^{\gamma_n})} \frac{\chi^{\lambda}_{(1^{\gamma_1},\ldots,n^{\gamma_n})}}{1^{\gamma_1}\cdots n^{\gamma_n}\gamma_1!\cdots\gamma_n!} \prod_{i=1}^{l(\nu)} (p_{\nu_i}(X) + p_{\nu_i}(Y))$$

$$= \sum_{(1^{\gamma_1},\ldots,n^{\gamma_n})} \frac{\chi^{\lambda}_{(1^{\gamma_1},\ldots,n^{\gamma_n})}}{1^{\gamma_1}\cdots n^{\gamma_n}\gamma_1!\cdots\gamma_n!} \prod_{i=1}^{n} (p_i(X) + p_i(Y))^{\gamma_i}.$$

The binomial theorem gives

$$\sum_{(1^{\gamma_1},\ldots,n^{\gamma_n})} \frac{\chi^{\lambda}_{(1^{\gamma_1},\ldots n^{\gamma_n})}}{1^{\gamma_1}\cdots n^{\gamma_n}\gamma_1!\cdots\gamma_n!} \prod_{i=1}^{n}\sum_{r_i=0}^{\gamma_i} \binom{\gamma_i}{r_i} p_i(X)^{r_i} p_i(Y)^{\gamma_i - r_i}.$$

Setting $\gamma_i = r_i + s_i$ gives

$$\sum_{1^{r_1+s_1}\ldots n^{r_n+s_n}} \chi^{\lambda}_{(1^{r_1+s_1},\cdots,n^{r_n+s_n})} \left(\frac{p_1(X)^{r_1}\cdots p_n(X)^{r_n}}{1^{r_1}\cdots n^{r_n} r_1!\cdots r_n!}\right)\left(\frac{p_1(Y)^{s_1}\cdots p_n(Y)^{s_n}}{1^{s_1}\cdots n^{s_n} s_1!\cdots s_n!}\right).$$

We now apply Lemma 2.2, with $\alpha = (1^{r_1}, \ldots, n^{r_n})$ and $\beta = (1^{s_1}, \ldots, n^{s_n})$.

$$\sum_{m=0}^{n}\sum_{\mu\vdash m}\left(\sum_{(1^{r_1},\ldots,n^{r_n})} \frac{\chi^{\mu}_{(1^{r_1},\ldots,n^{r_n})}}{1^{r_1}\cdots n^{r_n} r_1!\cdots r_n!} p_1(X)^{r_1}\cdots p_n(X)^{r_n}\right)$$

$$\times \left(\sum_{(1^{s_1},\ldots,n^{s_n})} \frac{\chi^{\lambda/\mu}_{(1^{s_1},\ldots,n^{s_n})}}{1^{s_1}\cdots n^{s_n} s_1!\cdots s_n!} p_1(Y)^{s_1}\cdots p_n(Y)^{s_n}\right) \quad (2.6)$$

$$= \sum_{m=0}^{n}\sum_{\mu\vdash m}\left(\sum_{\alpha\vdash m}\frac{\chi^{\mu}_{\alpha}}{z_\alpha}p_\alpha(X)\right)\left(\sum_{\beta\vdash n-m}\frac{\chi^{\lambda/\mu}_{\beta}}{z_\beta}p_\beta(Y)\right) = \sum_{m=0}^{n}\sum_{\mu\vdash m} s_\mu(X)s_{\lambda/\mu}(Y).$$

Since the Schur function $s_{\lambda/\mu}(Y)$ is zero if we do not have $\mu \subseteq \lambda$, this is equal to $\sum_{\mu\subseteq\lambda} s_\mu(X)s_{\lambda/\mu}(Y)$, completing the proof.

(2.2) The proof of this identity is extremely similar to the previous proof. Follow the same steps, using $-Y$ instead of Y. Then at (2.6) we have instead in the Y portion

$$\sum_{(1^{s_1},\ldots,n^{s_n})} \frac{\chi^{\lambda/\mu}_{(1^{s_1},\ldots,n^{s_n})}}{1^{s_1}\cdots n^{s_n} s_1!\cdots s_n!} p_1(-Y)^{s_1}\cdots p_n(-Y)^{s_n}$$

$$= (-1)^{n-m} \sum_{(1^{s_1},\ldots,n^{s_n})} \frac{(-1)^{n-m+l(1^{s_1},\ldots,n^{s_n})}\chi^{\lambda/\mu}_{(1^{s_1},\ldots,n^{s_n})}}{1^{s_1}\cdots n^{s_n} s_1!\cdots s_n!} p_1(Y)^{s_1}\cdots p_n(Y)^{s_n}.$$

Then since if $|\lambda| = n$ and $|\beta| = m$, $\chi_\beta^{\lambda'/\mu'} = (-1)^{n-m+l(\beta)}\chi_\beta^{\lambda/\mu}$, this becomes

$$(-1)^{n-m} \sum_{(1^{s_1},\ldots,n^{s_n})} \frac{\chi^{\lambda'/\mu'}_{(1^{s_1},\ldots,n^{s_n})}}{1^{s_1}\cdots n^{s_n}s_1!\cdots s_n!} p_1(Y)^{s_1}\cdots p_n(Y)^{s_n} = (-1)^{n-m}s_{\lambda'/\mu'}(Y).$$

Putting this into the full identity gives the result.

(2.3) This is an immediate consequence of the proof of (2.2).

(2.4) We use the definition of λ-ring notation for Schur functions to write

$$s_\lambda(XY) = \sum_\rho \frac{\chi_\rho^\lambda}{z_\rho} p_\rho(XY) = \sum_\rho \frac{\chi_\rho^\lambda}{z_\rho} p_\rho(X)p_\rho(Y).$$

Writing the p_ρ's in terms of Schur functions gives

$$\sum_\rho \frac{\chi_\rho^\lambda}{z_\rho}\left(\sum_\mu \chi_\rho^\mu s_\mu(X)\right)\left(\sum_\nu \chi_\rho^\nu s_\nu(Y)\right) = \sum_{\mu,\nu} s_\mu(X)s_\nu(Y)\sum_\rho \frac{1}{z_\rho}\chi_\rho^\lambda\chi_\rho^\mu\chi_\rho^\nu.$$

This completes the proof. ■

We have the following corollary of Theorem 2.1.

Corollary 2.3.

$$h_n(X+Y) = \sum_{m=0}^{n} h_m(X)h_{n-m}(Y), \tag{2.7}$$

$$h_n(XY) = \sum_\mu s_\mu(X)s_\mu(Y). \tag{2.8}$$

Proof. For the first identity,

$$\begin{aligned} h_n(X+Y) &= s_{(n)}(X+Y) \\ &= \sum_{\mu\subseteq(n)} s_\mu(X)s_{(n)/\mu}(Y) \\ &= \sum_{m=0}^{n} s_m(X)s_{n-m}(Y). \end{aligned}$$

For the second identity,

$$\begin{aligned} h_n(XY) &= s_{(n)}(XY) \\ &= \sum_{\mu,\nu,\rho} \frac{1}{z_\rho}\chi_\rho^{(n)}\chi_\rho^\mu\chi_\rho^\nu s_\mu(X)s_\nu(Y) \\ &= \sum_\mu s_\mu(X)\sum_\rho \frac{\chi_\rho^\mu}{z_\rho}\sum_\nu \chi_\rho^\nu s_\nu(Y) \\ &= \sum_\mu s_\mu(X)\sum_\rho \frac{\chi_\rho^\mu}{z_\rho}p_\rho(Y) \\ &= \sum_\mu s_\mu(X)s_\mu(Y) \end{aligned}$$

■

Some properties similar to those in Theorem 2.1 hold in the extended lambda ring notation. We will not prove these here as the proofs are nearly identical to those above.

Theorem 2.4. *If $\epsilon = e^{\frac{2\pi i}{k}}$, then*

$$s_\lambda(\epsilon^a X + \epsilon^b Y) = \sum_{\mu\subseteq\lambda} s_\mu(\epsilon^a X)s_{\lambda/\mu}(\epsilon^b Y),$$

for any integers a and b, and

$$\begin{aligned} s_\lambda(X\cdot\epsilon Y) &= s_\lambda(\epsilon X\cdot Y) \\ &= \sum_{\rho,\mu,\lambda} \epsilon^{l(\rho)}\frac{1}{z_\rho}\chi_\rho^\lambda\chi_\rho^\mu\chi_\rho^\nu s_\mu(X)s_\nu(Y) \\ &= \sum_{\mu,nu} K_{\lambda,\mu,\nu}s_\mu(X)s_\nu(\epsilon Y) = \sum_{\mu,nu} K_{\lambda,\mu,\nu}s_\mu(\epsilon X)s_\nu(Y). \end{aligned}$$

We will use a special case of the following Corollary in our determination of the irreducible characters of $C_k\S S_n$.

Corollary 2.5. *Let $\epsilon = e^{\frac{2\pi i}{k}}$, and let $a_1, a_2, \ldots, a_k$ be natural numbers. Then*

$$s_\lambda(\epsilon^{a_1}X^{(1)} + \cdots + \epsilon^{a_k}X^{(k)}) = \sum_{\mu^{(1)}\subseteq\mu^{(2)}\subseteq\cdots\mu^{(k-1)}\subseteq\lambda} s_{\mu^{(1)}}(\epsilon^{a_1}X^{(1)})s_{\mu^{(2)}/\mu^{(1)}}(\epsilon^{a_2}X^{(2)})\cdots s_{\lambda/\mu^{(k-1)}}(\epsilon^{a_k}X^{(k)}).$$

Chapter 3

The Representation Theory of $C_k \S S_n$

The main goal of this text is to explore the permutation enumeration of the wreath product $C_k \S S_n$, the group of signed permutations where there are k signs, $1, \epsilon, \epsilon^2, \ldots, \epsilon^{k-1}$, where $\epsilon = e^{\frac{2\pi i}{k}}$. In order to do this, however, we must know something of the representation theory of this group. In this chapter, we give a detailed presentation of the representation theory of $C_k \S S_n$ including the irreducible characters of $C_k \S S_n$ and their relationship to a space of symmetric functions.

3.1 Descriptions of $C_k \S S_n$

We begin by describing the group $C_k \S S_n$ in two ways. First, we can think of it as a Coxeter-like group, defined by generators and relations. There are n generators, $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}, \tau$, which satisfy the following relations:

$$
\begin{aligned}
\sigma_i^2 &= 1, \quad i = 1, 2, \ldots, n-1, \\
\tau^k &= 1, \\
(\sigma_i \sigma_j)^2 &= 1, \quad |i - j| > 1, \\
(\sigma_i \sigma_{i+1})^3 &= 1, \quad i = 1, 2, \ldots, n-2, \\
(\tau \sigma_{n-1})^{2k} &= 1.
\end{aligned}
$$

In fact, the generators σ_i are the transpositions $(i, i+1)$ which generate the symmetric group. The other generator is $\tau = (\epsilon n)$, that is, it maps n to ϵ times itself.

We can also write an element $\sigma \in C_k \S S_n$ in two line notation. For example, we could have

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 3 & \epsilon^2 6 & \epsilon^2 7 & 10 & \epsilon 5 & \epsilon^2 2 & \epsilon 1 & 9 & \epsilon^2 8 & 4 \end{pmatrix} \in C_3 \S S_n.$$

We can then write this in one-line form:

$$\sigma = \begin{array}{cccccccccc} 3 & \epsilon^2 6 & \epsilon^2 7 & 10 & \epsilon 5 & \epsilon^2 2 & \epsilon 1 & 9 & \epsilon^2 8 & 4 \end{array}.$$

We can also write the element in cyclic notation as

$$\sigma = (\epsilon 1, 3, \epsilon^2 7)(\epsilon^2 2, \epsilon^2 6)(\epsilon 5)(\epsilon^2 8, 9). \tag{3.1}$$

Note that when determining what a number is mapped to, one ignores the sign on that number and then considers only the sign on the next number in the cycle. Thus, in this example, we ignore the sign of ϵ on the 1 and note that then 1 maps to 3 since the sign on 3 is 1.

3.2 Conjugacy Classes of $C_k \S S_n$

In this section we will describe the conjugacy classes of $C_k \S S_n$. To do this, consider a single cycle γ. Conjugation by a generator σ_i does not change the structure of the cycle or which signs occur in the cycle. Conjugation by the generator τ does not change the structure of the cycle and for cycles of length at least 2, changes one sign by ϵ^{k-1} and one sign by ϵ, thus preserving the product of the signs within the cycle. Moreover, we can obtain any desired sign pattern by multiplying by appropriate products of σ_is and τs in the following way. If $\gamma = (\epsilon^{a_1} i_1, \epsilon^{a_2} i_2, \ldots, \epsilon^{a_m} i_m)$, we can conjugate by an element σ of S_n and the result is

$$\sigma(\epsilon^{a_1} i_1, \epsilon^{a_2} i_2, \ldots, \epsilon^{a_m} i_m)\sigma^{-1} = (\epsilon^{a_1} \sigma(i_1), \epsilon^{a_2} \sigma(i_2), \ldots, \epsilon^{a_m} \sigma(i_m)),$$

thus we can change the base set to whatever we choose. We can obtain the desired sign pattern in the following way. To increase the sign on i_j by l, conjugate by $(\epsilon^l i_j) = \sigma_{i_j}\sigma_{i_j+1}\cdots\sigma_{n-1}\tau^l\sigma_{n-1}\cdots\sigma_{i_j+1}\sigma_{i_j}$, giving

$$\begin{aligned}(\epsilon^l i_j)(\epsilon^{a_1}i_1,\ldots,\epsilon^{a_j}i_j,\ldots,\epsilon^{a_m}i_m)(\epsilon^{k-l}i_j) &= \\ &(\epsilon^{a_1}i_1,\ldots,\epsilon^{a_j+l}i_j,\ldots,\epsilon^{a_{j+1}+k-l}i_{j+1},\ldots,\epsilon^{a_m}i_m).\end{aligned}$$

One can then adjust all the signs one at a time as necessary.

If the product of all of the signs in a cycle is 1, then we say the cycle is a *1-cycle*. Similarly, if the product of the signs is ϵ^i, the cycle is called an ϵ^i-*cycle*. For example, in (3.1), $(\epsilon 1, 3, \epsilon^2 7)$ is a 1-cycle, $(\epsilon^2 2, \epsilon^2 6)$ and $(\epsilon 5)$ are ϵ-cycles, and $(\epsilon^2 8, 9)$ is an ϵ^2-cycle. Thus an ϵ^i-cycle remains so under conjugation by any element of $C_k \S S_n$. Let $(\lambda^{(1)},\ldots,\lambda^{(k)}) \vdash n$ denote a k-tuple of partitions such that $|\lambda^{(1)}|+\cdots+|\lambda^{(k)}| = n$. Then by the above argument we have the following lemma.

Lemma 3.1. *For some $(\lambda^{(1)},\ldots,\lambda^{(k)}) \vdash n$, let $C_{(\lambda^{(1)},\ldots,\lambda^{(k)})} = \{\sigma \in C_k \S S_n :$ the ϵ^i-cycles have lengths $\lambda^{(i)}_1,\ldots,\lambda^{(i)}_{l(\lambda^{(i)})}$ for $i = 1,\ldots,k\}$. Then the set of conjugacy classes of $C_k \S S_n$ is $\{C_{(\lambda^{(1)},\ldots,\lambda^{(k)})}\}_{(\lambda^{(1)},\ldots,\lambda^{(k)})\vdash n}$.*

The example in (3.1) then belongs to the conjugacy class $C_{((2,1),(2),(3))}$. We can also determine the size of each conjugacy class.

Lemma 3.2. *The conjugacy class $C_{(\lambda^{(1)},\ldots,\lambda^{(k)})}$ has order*

$$|C_{(\lambda^{(1)},\ldots,\lambda^{(k)})}| = \frac{k^n n!}{k^{l(\lambda^{(1)})+\cdots+l(\lambda^{(k)})} z_{\lambda^{(1)}}\cdots z_{\lambda^{(k)}}}.$$

Proof. Suppose that for each i, $|\lambda^{(i)}| = m_i$. Then we can choose the elements for each of the cycles in $\binom{n}{m_1,\ldots,m_k}$ ways. For each i, we choose the ϵ^i-cycles in $\frac{m_i!}{z_{\lambda^{(i)}}}$ ways. In each ϵ^i-cycle, one of the signs must be chosen so the product of the signs is ϵ^i; the other signs are arbitrary. We choose the sign pattern for the j^{th} ϵ^i-cycle in $k^{\lambda^{(i)}_j-1}$ ways. Putting all of this together, the total number of elements of $C_{(\lambda^{(1)},\ldots,\lambda^{(k)})}$ is

$$\binom{n}{m_1,\ldots,m_k}\frac{m_1!}{z_{\lambda^{(1)}}}\cdots\frac{m_k!}{z_{\lambda^{(k)}}}k^{n-l(\lambda 0)-\cdots-l(\lambda^{(k)})} = \frac{k^n n!}{k^{l(\lambda^{(1)})+\cdots+l(\lambda^{(k)})} z_{\lambda^{(1)}}\cdots z_{\lambda^{(k)}}}.$$

This completes the proof. ■

3.3 The Characteristic Map and Inner Products

Here we introduce an analog of the Frobenius Characteristic for $C_k \S S_n$, which preserves an inner product on the class functions of $C_k \S S_n$.

Let $1_{(\lambda^{(1)},\dots,\lambda^{(k)})}$, defined by

$$1_{(\lambda^{(1)},\dots,\lambda^{(k)})}(\sigma) = \begin{cases} 1, & \sigma \in C_{(\lambda^{(1)},\dots,\lambda^{(k)})}, \\ 0, & \text{otherwise}, \end{cases}$$

be the indicator function on the conjugacy class $C_{(\lambda^{(1)},\dots,\lambda^{(k)})}$. Then the collection $\{1_{(\lambda^{(1)},\dots,\lambda^{(k)})}\}_{(\lambda^{(1)},\dots,\lambda^{(k)})\vdash n}$ forms a basis for $C(C_k \S S_n)$, the class functions on $C_k \S S_n$.

Define the characteristic map

$$ch : C(C_k \S S_n) \longrightarrow \bigoplus_{m_1+\cdots+m_k=n} \Lambda_{m_1}(X^{(1)}) \otimes \cdots \otimes \Lambda_{m_k}(X^{(k)})$$

by

$$1_{(\lambda^{(1)},\dots,\lambda^{(k)})} \mapsto \frac{1}{z_{\lambda^{(1)}} \cdots z_{\lambda^{(k)}}} p_{\lambda^{(1)}}(X^{(1)}) \cdots p_{\lambda^{(k)}}(X^{(k)}), \tag{3.2}$$

where $X^{(i)} = (x_1^{(i)}, x_2^{(i)}, \dots)$ is a set of variables. We will denote the space

$$\bigoplus_{m_1+\cdots+m_k=n} \Lambda_{m_1}(X^{(1)}) \otimes \cdots \otimes \Lambda_{m_k}(X^{(k)}) = \Lambda_{W_{k,n}}(X^{(1)}, \dots, X^{(k)}).$$

We denote

$$\Lambda_{W_k}(X^{(1)}, \dots, X^{(k)}) = \bigoplus_{n\geq 0} \Lambda_{W_{k,n}}(X^{(1)}, \dots, X^{(k)}). \tag{3.3}$$

Now we will define inner products on the class functions of $C_k \S S_n$ and Λ_{W_k} such that the characteristic map is preserved. The usual inner product on class functions of a group G is

$$< \chi, \psi >_G = \frac{1}{|G|} \sum_{g\in G} \chi(g)\overline{\psi(g)}.$$

For $C_k \S S_n$, this becomes

$$\begin{aligned}
\left\langle 1_{(\lambda^{(1)},\ldots,\lambda^{(k)})}, 1_{(\mu^{(1)},\ldots,\mu^{(k)})} \right\rangle_{W_{k,n}} &= \frac{1}{k^n n!} \sum_{g \in C_k \S S_n} 1_{(\lambda^{(1)},\ldots,\lambda^{(k)})}(g) \overline{1_{(\mu^{(1)},\ldots,\mu^{(k)})}(g)} \\
&= \frac{1}{k^n n!} \sum_{g \in C_k \S S_n} 1_{(\lambda^{(1)},\ldots,\lambda^{(k)})}(g) \delta_{\lambda^{(1)},\mu^{(1)}} \cdots \delta_{\lambda^{(k)},\mu^{(k)}} \\
&= \frac{1}{k^n n!} |C_{(\lambda^{(1)},\ldots,\lambda^{(k)})}| \delta_{\lambda^{(1)},\mu^{(1)}} \cdots \delta_{\lambda^{(k)},\mu^{(k)}} \\
&= \frac{\delta_{\lambda^{(1)},\mu^{(1)}} \cdots \delta_{\lambda^{(k)},\mu^{(k)}}}{k^{l(\lambda^{(1)})+\cdots+l(\lambda^{(k)})} z_{\lambda^{(1)}} \cdots z_{\lambda^{(k)}}}
\end{aligned}$$

where $\delta_{\lambda,\mu} = 1$ if $\lambda = \mu$ or 0 otherwise.

We would like to define an inner product on Λ_{W_k} so that the characteristic map preserves the above inner product. Thus we use (3.2) to define $<, >_*$.

$$\left\langle \frac{p_{\lambda^{(1)}}(X^{(1)}) \cdots p_{\lambda^{(k)}}(X^{(k)})}{z_{\lambda^{(1)}} \cdots z_{\lambda^{(k)}}}, \frac{p_{\mu^{(1)}}(X^{(1)}) \cdots p_{\mu^{(k)}}(X^{(k)})}{z_{\mu^{(1)}} \cdots z_{\mu^{(k)}}} \right\rangle_* = \frac{\delta_{\lambda^{(1)},\mu^{(1)}} \cdots \delta_{\lambda^{(k)},\mu^{(k)}}}{k^{l(\lambda^{(1)})+\ldots+l(\lambda^{(k)})} z_{\lambda^{(1)}} \cdots z_{\lambda^{(k)}}},$$

that is,

$$\left\langle p_{\lambda^{(1)}}(X^{(1)}) \cdots p_{\lambda^{(k)}}(X^{(k)}), p_{\mu^{(1)}}(X^{(1)}) \cdots p_{\mu^{(k)}}(X^{(k)}) \right\rangle_* = \frac{z_{\lambda^{(1)}} \cdots z_{\lambda^{(k)}} \delta_{\lambda^{(1)},\mu^{(1)}} \cdots \delta_{\lambda^{(k)},\mu^{(k)}}}{k^{l(\lambda^{(1)})+\cdots+l(\lambda^{(k)})}}.$$

This defines a scalar product on Λ_{W_k} since $\{p_{\lambda^{(1)}}(X^{(1)}) \cdots p_{\lambda^{(k)}}(X^{(k)})\}$ is a basis of Λ_{W_k}.

3.4 Dual Bases

Here we discuss what it means for two bases of $\Lambda_{W_{k,n}}$ to be dual with respect to the inner product $<, >_*$ defined in the previous section. In later chapters, this will be used to determine the irreducible characters of $C_k \S S_n$.

Fix some standard order on k-tuples of partitions of n. Think of a basis $\{a_{\lambda^{(1)}} \cdots a_{\lambda^{(k)}}\}$ as a row vector $\langle a_{\lambda^{(1)}} \cdots a_{\lambda^{(k)}} \rangle$ with indices ranging over all k-tuples

$(\lambda^{(1)}, \ldots, \lambda^{(k)}) \vdash n$. From the previous section,

$$\left\langle \frac{p_{\lambda^{(1)}}(X^{(1)}) \cdots p_{\lambda^{(k)}}(X^{(k)})}{\sqrt{\frac{z_{\lambda^{(1)}} \cdots z_{\lambda^{(k)}}}{k^{l(\lambda^{(1)})+\cdots+l(\lambda^{(k)})}}}}, \frac{p_{\mu^{(1)}}(X^{(1)}) \cdots p_{\mu^{(k)}}(X^{(k)})}{\sqrt{\frac{z_{\mu^{(1)}} \cdots z_{\mu^{(k)}}}{k^{l(\mu^{(1)})+\cdots+l(\mu^{(k)})}}}} \right\rangle_* = \delta_{\lambda^{(1)},\mu^{(1)}} \cdots \delta_{\lambda^{(k)},\mu^{(k)}},$$

which means that the basis

$$\left\langle \frac{p_{\lambda^{(1)}}(X^{(1)}) \cdots p_{\lambda^{(k)}}(X^{(k)})}{\sqrt{\frac{z_{\lambda^{(1)}} \cdots z_{\lambda^{(k)}}}{k^{l(\lambda^{(1)})+\cdots+l(\lambda^{(k)})}}}} \right\rangle_{(\lambda^{(1)}, \ldots, \lambda^{(k)}) \vdash n}$$

is self-dual with respect to $<,>_*$.

We now want to determine a criterion for determining if two bases are dual. The following theorem will point the way to the general case, which follows. Let $\Omega^{2n}(X^{(1)}, \ldots, X^{(k)}, Y^{(1)}, \ldots, Y^{(k)})$ denote the sum of the terms of degree $2n$ in

$$\prod_{i=1}^{k} \prod_{r,s} \frac{1}{(1 - X_r^{(i)} Y_s^{(i)})^k}.$$

We then have the following theorem.

Theorem 3.3.

$$\sum_{(\lambda^{(1)}, \ldots, \lambda^{(k)}) \vdash n} \frac{p_{\lambda^{(1)}}(X^{(1)}) \cdots p_{\lambda^{(k)}}(X^{(k)})}{\sqrt{\frac{z_{\lambda^{(1)}} \cdots z_{\lambda^{(k)}}}{k^{l(\lambda^{(1)})+\cdots+l(\lambda^{(k)})}}}} \frac{p_{\lambda^{(1)}}(Y^{(1)}) \cdots p_{\lambda^{(k)}}(Y^{(k)})}{\sqrt{\frac{z_{\lambda^{(1)}} \cdots z_{\lambda^{(k)}}}{k^{l(\lambda^{(1)})+\cdots+l(\lambda^{(k)})}}}}$$
$$= \Omega^{2n}(X^{(1)}, \ldots, X^{(k)}, Y^{(1)}, \ldots, Y^{(k)}).$$

Proof. We rewrite the left hand side in the following way.

$$\sum_{(\lambda^{(1)}, \ldots, \lambda^{(k)}) \vdash n} \frac{p_{\lambda^{(1)}}(X^{(1)}) \cdots p_{\lambda^{(k)}}(X^{(k)})}{\sqrt{\frac{z_{\lambda^{(1)}} \cdots z_{\lambda^{(k)}}}{k^{l(\lambda^{(1)})+\cdots+l(\lambda^{(k)})}}}} \frac{p_{\lambda^{(1)}}(Y^{(1)}) \cdots p_{\lambda^{(k)}}(Y^{(k)})}{\sqrt{\frac{z_{\lambda^{(1)}} \cdots z_{\lambda^{(k)}}}{k^{l(\lambda^{(1)})+\cdots+l(\lambda^{(k)})}}}}$$
$$= \sum_{(\lambda^{(1)}, \ldots, \lambda^{(k)}) \vdash n} \frac{k^{l(\lambda^{(1)})+\cdots+l(\lambda^{(k)})}}{z_{\lambda^{(1)}} \cdots z_{\lambda^{(k)}}} p_{\lambda^{(1)}}(X^{(1)}) \cdots p_{\lambda^{(k)}}(X^{(k)}) p_{\lambda^{(1)}}(Y^{(1)}) \cdots p_{\lambda^{(k)}}(Y^{(k)})$$
$$= \sum_{m_1+\cdots+m_k=n} \prod_{i=1}^{k} \left(\sum_{\lambda^{(i)} \vdash m_i} \frac{k^{l(\lambda^{(i)})}}{z_{\lambda^{(i)}}} p_{\lambda^{(i)}}(X^{(i)}) p_{\lambda^{(i)}}(Y^{(i)}) \right).$$

Now if $\Omega^{2m_i}(X^{(i)}, Y^{(i)})$ is the sum of the terms of degree $2m_i$ in $\prod_{r,s} \frac{1}{(1-X_r^{(i)}Y_s^{(i)})^k}$, we will show that

$$\sum_{\lambda^{(i)} \vdash m_i} \frac{k^{l(\lambda^{(i)})}}{z_{\lambda^{(i)}}} p_{\lambda^{(i)}}(X^{(i)}) p_{\lambda^{(i)}}(Y^{(i)}) = \Omega^{2m_i}(X^{(i)}, Y^{(i)}),$$

which implies the theorem. We repeatedly rewrite the product as follows.

$$\begin{aligned}
\prod_{r,s} \frac{1}{(1-X_r^{(i)}Y_s^{(i)})^k} &= exp\left(log\left(\prod_{r,s} \frac{1}{(1-X_r^{(i)}Y_s^{(i)})^k}\right)\right) \\
&= exp\left(\sum_{r,s} k \; log\left(\frac{1}{1-X_r^{(i)}Y_s^{(i)}}\right)\right) \\
&= exp\left(\sum_{r,s} k \sum_{l \geq 1} \frac{(X_r^{(i)}Y_s^{(i)})^l}{l}\right) \\
&= exp\left(\sum_{l \geq 1} \frac{k}{l} p_l(X^{(i)}) p_l(Y^{(i)})\right) \\
&= 1 + \sum_{a \geq 1} \frac{1}{a!} \left(\sum_{l \geq 1} \frac{k}{l} p_l(X^{(i)}) p_l(Y^{(i)})\right)^a .
\end{aligned}$$

Because we only care about the terms of degree $2m_i$, we can write this again as

$$\sum_{a=1}^{m_i} \frac{1}{a!} \left(\sum_{l=1}^{m_i} \frac{k}{l} p_l(X^{(i)}) p_l(Y^{(i)})\right)^a .$$

Now take the terms of degree $2m_i$ to obtain

$$\begin{aligned}
\sum_{a=1}^{m_i} \frac{1}{a!} \sum_{b_1+2b_2+\cdots+m_i b_{m_i}=m_i} & \frac{k^{b_1}(p_1(X^{(i)})p_1(Y^{(i)}))^{b_1}}{1^{b_1} b_1!} \cdots \frac{k^{b_{m_i}}(p_{m_i}(X^{(i)})p_{m_i}(Y^{(i)}))^{b_{m_i}}}{1^{b_{m_i}} b_{m_i}!} \\
&= \sum_{\lambda^{(i)} \vdash m_i} \frac{k^{l(\lambda^{(i)})}}{z_{\lambda^{(i)}}} p_{\lambda^{(i)}}(X^{(i)}) p_{\lambda^{(i)}}(Y^{(i)}).
\end{aligned}$$

This completes the proof. ∎

We now generalize the previous result into a criterion for the duality of any two bases. This is expressed in the following theorem.

Theorem 3.4. *Let* $\{R_{\lambda^{(1)}}(X^{(1)})\cdots R_{\lambda^{(k)}}(X^{(k)})\}$ *and* $\{Q_{\lambda^{(1)}}(X^{(1)})\cdots Q_{\lambda^{(k)}}(X^{(k)})\}$ *be bases of* $\Lambda_{W_{k,n}}(X^{(1)},\ldots,X^{(k)})$. *Under the inner product* $<,>_*$, *these bases are dual if and only if*

$$\sum_{(\lambda^{(1)},\ldots,\lambda^{(k)})\vdash n} R_{\lambda^{(1)}}(X^{(1)})\cdots R_{\lambda^{(k)}}(X^{(k)})Q_{\lambda^{(1)}}(Y^{(1)})\cdots Q_{\lambda^{(k)}}(Y^{(k)})$$
$$= \Omega^{2n}(X^{(1)},\ldots,X^{(k)},Y^{(1)},\ldots,Y^{(k)}). \quad (3.4)$$

Proof. Let

$$\vec{p} = \left\langle \frac{p_{\lambda^{(1)}}(X^{(1)})\cdots p_{\lambda^{(k)}}(X^{(k)})}{\sqrt{\frac{z_{\lambda^{(1)}}\cdots z_{\lambda^{(k)}}}{k^{l(\lambda^{(1)})+\cdots+l(\lambda^{(k)})}}}} \right\rangle_{(\lambda^{(1)},\ldots,\lambda^{(k)})\vdash n},$$
$$\vec{R} = \left\langle R_{\lambda^{(1)}}(X^{(1)})\cdots R_{\lambda^{(k)}}(X^{(k)})\right\rangle_{(\lambda^{(1)},\ldots,\lambda^{(k)})\vdash n} = \vec{p}\cdot A,$$

and

$$\vec{Q} = \left\langle Q_{\lambda^{(1)}}(X^{(1)})\cdots Q_{\lambda^{(k)}}(X^{(k)})\right\rangle_{(\lambda^{(1)},\ldots,\lambda^{(k)})\vdash n} = \vec{p}\cdot B.$$

The proof depends entirely on linear algebra, and not on Ω^{2n} itself. It proceeds by showing that the bases are dual if and only if $A^TB = I$, and then that (3.4) holds if and only if $A^TB = I$.

The bases are dual if and only if

$$\vec{R}^T \odot \vec{Q} = ||\left\langle R_{\lambda^{(1)}}(X^{(1)})\cdots R_{\lambda^{(k)}}(X^{(k)}), Q_{\lambda^{(1)*}}(X^{(1)})\cdots Q_{\lambda^{(k)*}}(X^{(k)})\right\rangle_* || = I.$$

On the other hand,

$$\vec{R}^T \odot \vec{Q} = (\vec{p}A)^T \odot \vec{p}B = A^T\vec{p}^T \odot \vec{p}B,$$

and since $\vec{p}$ is a self-dual basis we have $\vec{p}^T\vec{p} = I$. Thus,

$$\vec{R}^T \odot \vec{Q} = A^TB.$$

Thus the bases are dual iff and only if $A^TB = AB^T = I$.

Now we can write the left hand side of (3.4) as

$$\sum_{(\lambda^{(1)},\ldots,\lambda^{(k)})\vdash n} R_{\lambda^{(1)}}(X^{(1)})\cdots R_{\lambda^{(k)}}(X^{(k)})Q_{\lambda^{(1)}}(Y^{(1)})\cdots Q_{\lambda^{(k)}}(Y^{(k)})$$
$$= \vec{R}\cdot\vec{Q}^T = \vec{p}A\cdot(\vec{p}B)^T = \vec{p}A\cdot B^T\vec{p}^T.$$

We have from the previous theorem that $\vec{p} \cdot \vec{p}^T = \Omega^{2n}$, which means that (3.4) holds if and only if

$$\vec{p}A \cdot B^T \vec{p}^T = \vec{p} \cdot \vec{p}^T.$$

This holds if and only if $(AB^T)_{(\lambda^{(1)},\ldots,\lambda^{(k)}),(\mu^{(1)},\ldots,\mu^{(k)})} = \delta_{(\lambda^{(1)},\ldots,\lambda^{(k)}),(\mu^{(1)},\ldots,\mu^{(k)})}$, that is, if $AB^T = I$.

We now have that the bases are dual if and only if $A^T B = I$ which is true if and only if (3.4) holds, which proves the theorem. ■

We can use the criterion given in Theorem 3.4 to show that two Schur function bases are dual. We will use this in the proof of the irreducible characters of $C_k \S S_n$.

Theorem 3.5. *Let $\epsilon = e^{\frac{2\pi i}{k}}$ and let $(\lambda^{(1)}, \ldots, \lambda^{(k)})$ extend over all k-tuples of partitions. Then the bases*

$$\{\prod_{i=1}^{k} s_{\lambda^{(i)}}(\epsilon^{1 \cdot i} X^{(1)} + \cdots + \epsilon^{k \cdot i} X^{(k)})\} \text{ and } \{\prod_{i=1}^{k} s_{\lambda^{(i)}}(\epsilon^{(k-1)i} X^{(1)} + \cdots + \epsilon^{(k-k)i} X^{(k)})\}$$

of Λ_{W_k} are dual with respect to $<,>_*$.

Proof. We proceed by showing that the the criterion in Theorem 3.4 is met in this case. Thus we consider the following sum.

$$\sum_{(\lambda^{(1)},\ldots,\lambda^{(k)})} \prod_{i-1}^{k} s_{\lambda^{(i)}}(\epsilon^{1i} X^{(1)} + \cdots + \epsilon^{ki} X^{(k)}) s_{\lambda^{(i)}}(\epsilon^{(k-1)i} Y^{(1)} + \cdots + \epsilon^{(k-k)i} Y^{(k)})$$
$$= \prod_{i=1}^{k} \left(\sum_{\lambda^{(i)}} s_{\lambda^{(i)}}(\epsilon^{1i} X^{(1)} + \cdots + \epsilon^{ki} X^{(k)}) s_{\lambda^{(i)}}(\epsilon^{(k-1)i} Y^{(1)} + \cdots + \epsilon^{(k-k)i} Y^{(k)}) \right).$$

Now apply expression (2.8) from Corollary 2.3 for

$$\prod_{i=1}^{k}\left(\sum_{a_i\geq 0} h_{a_i}\left((\epsilon^{1i}X^{(1)}+\cdots\epsilon^{ki}X^{(k)})(\epsilon^{(k-1)i}Y^{(1)}+\cdots+\epsilon^{(k-k)i}Y^{(k)})\right)\right)$$
$$=\prod_{i=1}^{k}\left(\sum_{a_i\geq 0} h_{a_i}\left(\sum_{p,q=1}^{k}\epsilon^{pi}X^{(p)}\epsilon^{(k-q)i}Y^{(q)}\right)\right)$$
$$=\sum_{a_1,\ldots,a_k\geq 0}\prod_{i=1}^{k} h_{a_i}\left(\sum_{p,q=1}^{k}\epsilon^{pi+(k-q)i}X^{(p)}Y^{(q)}\right)$$
$$=\sum_{m\geq 0} h_m\left(\sum_{i=1}^{k}\sum_{p,q=1}^{k}(\epsilon^{p+k-q})^{i}X^{(p)}Y^{(q)}\right)$$

where the last equality follows from expression (2.7) of Corollary 2.3, which allows us to express a sum of products of homogeneous functions over different alphabets as a single homogeneous function over the sum of the original alphabets. We then the sum to obtain

$$\sum_{m\geq 0} h_m\left(\sum_{p,q=1}^{k}X^{(p)}Y^{(q)}\sum_{i=1}^{k}(\epsilon^{p-q})^{i}\right)=\sum_{m\geq 0} h_m\left(\sum_{p=1}^{k}kX^{(p)}Y^{(p)}\right).$$

Again applying (2.7) gives

$$\sum_{a_1,\ldots,a_k\geq 0}\prod_{i=1}^{k} h_{a_i}(X^{(1)}Y^{(1)}+\cdots+X^{(k)}Y^{(k)})$$
$$=\left(\sum_{a\geq 0} h_a(X^{(1)}Y^{(1)}+\cdots+X^{(k)}Y^{(k)})\right)^{k}$$
$$=\left(\prod_{p=1}^{k}\prod_{r,s}\frac{1}{1-X_r^{(p)}Y_s^{(p)}}\right)^{k}.$$

The Theorem then follows immediately from this and Theorem 3.4. ■

3.5 Induction Products

The following theorem will be useful in the proof of the irreducible characters of $C_k\S S_n$.

Theorem 3.6. *Let $\phi_1, \dots, \phi_k$ be characters of $C_k \S S_{m_1}, \dots, C_k \S S_{m_k}$, respectively, with $m_1 + \cdots + m_k = n$. Then*

$$ch(\phi_1 \times \cdots \times \phi_k \uparrow^{C_k \S S_n}) = ch(\phi_1) \cdots ch(\phi_k),$$

where $\phi_1 \times \cdots \times \phi_k \uparrow^{C_k \S S_n}$ is the character obtained by inducing the product of characters to $C_k \S S_n$.

Proof. For a character χ, let $\chi(\lambda^{(1)}, \dots, \lambda^{(k)})$ be the value of the character when evaluated on the conjugacy class of $C_k \S S_n$ indexed by $(\lambda^{(1)}, \dots, \lambda^{(k)})$. Then we have

$$\begin{aligned}
ch(\chi) &= \sum_{(\lambda^{(1)}, \dots, \lambda^{(k)}) \vdash n} \chi(\lambda^{(1)}, \dots, \lambda^{(k)}) \frac{p_{\lambda^{(1)}}(X^{(1)}) \cdots p_{\lambda^{(k)}}(X^{(k)})}{z_{\lambda^{(1)}} \cdots z_{\lambda^{(k)}}} \\
&= \frac{1}{k^n n!} \sum_{(\lambda^{(1)}, \dots, \lambda^{(k)}) \vdash n} \frac{k^{l(\lambda^{(1)}) + \cdots + l(\lambda^{(k)})} k^n n!}{k^{l(\lambda^{(1)}) + \cdots + l(\lambda^{(k)})}} \chi(\lambda^{(1)}, \dots, \lambda^{(k)}) \frac{p_{\lambda^{(1)}}(X^{(1)}) \cdots p_{\lambda^{(k)}}(X^{(k)})}{z_{\lambda^{(1)}} \cdots z_{\lambda^{(k)}}} \\
&= \frac{1}{k^n n!} \sum_{\omega \in C_k \S S_n} k^{l(\lambda^{(1)}) + \cdots + l(\lambda^{(k)})} \chi(\lambda^{(1)}, \dots, \lambda^{(k)}) p_{\lambda^{(1)}}(X^{(1)}) \cdots p_{\lambda^{(k)}}(X^{(k)}) \\
&= < \chi, \psi_n >_{C_k \S S_n},
\end{aligned}$$

where for $\omega \in C_k \S S_n$ with cycle structure $(\lambda^{(1)}, \dots, \lambda^{(k)})$,

$$\psi_n(\omega) = k^{l(\lambda^{(1)}) + \cdots + l(\lambda^{(k)})} p_{\lambda^{(1)}}(X^{(1)}) \cdots p_{\lambda^{(k)}}(X^{(k)}).$$

A version of Frobenius Reciprocity holds using ψ_n. We prove this first, then use it to complete the proof.

$$\begin{aligned}
&\left\langle \phi_1 \times \cdots \times \phi_k \uparrow^{C_k\S S_n}, \psi_n \right\rangle_{C_k\S S_n} \\
&\qquad = \frac{1}{k^n n!} \sum_{g\in C_k\S S_n} \phi_1 \times \cdots \times \phi_k \uparrow^{C_k\S S_n} (g)\overline{\psi_n(g)} \\
&\qquad = \frac{1}{k^n n!} \sum_{g\in C_k\S S_n} \frac{1}{k^{m_1} m_1! \cdots k^{m_k} m_k!} \sum_{x\in C_k\S S_n} \phi_1 \times \cdots \times \phi_k(x^{-1}gx)\overline{\psi_n(g)} \\
&\qquad = \frac{1}{k^n n!} \frac{1}{k^n m_1! \cdots m_k!} \sum_{x,y\in C_k\S S_n} \phi_1 \times \cdots \times \phi_k(y)\overline{\psi_n(xyx^{-1})} \\
&\qquad = \frac{1}{k^n n!} \frac{1}{k^n m_1! \cdots m_k!} \sum_{x,y\in C_k\S S_n} \phi_1 \times \cdots \times \phi_k(y)\overline{\psi_n(y)} \\
&\qquad = \frac{1}{k^n m_1! \cdots m_k!} \sum_{y\in C_k\S S_n} \phi_1 \times \cdots \times \phi_k(y)\overline{\psi_n(y)} \\
&\qquad = \frac{1}{k^n m_1! \cdots m_k!} \sum_{y\in C_k\S S_{m_1}\times\cdots\times C_k\S S_{m_k}} \phi_1 \times \cdots \times \phi_k(y)\overline{\psi_n(y)} \\
&\qquad = \left\langle \phi_1 \times \cdots \times \phi_k, \psi_n \downarrow_{C_k\S S_{m_1}\times\cdots\times C_k\S S_{m_k}} \right\rangle_{C_k\S S_{m_1}\times\cdots\times C_k\S S_{m_k}},
\end{aligned}$$

where $\psi_n \downarrow_{C_k\S S_{m_1}\times\cdots\times C_k\S S_{m_k}}$ denotes the restriction of ψ_n to $C_k\S S_{m_1} \times \cdots \times C_k\S S_{m_k}$. Using this, we have

$$\begin{aligned}
ch(\phi_1 \times \cdots \times \phi_k \uparrow^{C_k\S S_n}) &= \left\langle \phi_1 \times \cdots \times \phi_k \uparrow^{C_k\S S_n}, \psi_n \right\rangle_{C_k\S S_n} \\
&= \left\langle \phi_1 \times \cdots \times \phi_k, \psi_n \downarrow_{C_k\S S_{m_1}\times\cdots\times C_k\S S_{m_k}} \right\rangle_{C_k\S S_{m_1}\times\cdots\times C_k\S S_{m_k}} \\
&= \prod_{i=1}^{k} \left(\frac{1}{|C_k\S S_{m_i}|} \sum_{\sigma_i\in C_k\S S_{m_i}} \phi_i(\sigma_i)\overline{\psi_{m_i}(\sigma_i)} \right) \\
&=< \phi_1, \psi_{m_1} >_{C_k\S S_{m_1}} \cdots < \phi_k, \psi_{m_k} >_{C_k\S S_{m_k}} \\
&= ch(\phi_1)\cdots ch(\phi_k)
\end{aligned}$$

This completes the proof. ■

Table 3.1: Linear characters of $C_k\S S_n$ applied at $C_{(\lambda^{(1)},\dots,\lambda^{(k)})}$.

$L(\sigma_i)$	$L(\tau)$	character	applied at $C_{(\lambda^{(1)},\dots,\lambda^{(k)})}$
1	1	1_n	1
1	ϵ^m	δ_{ϵ^m}	$(\epsilon^m)^{1l(\lambda^{(1)})+\cdots+kl(\lambda^{(1)})}$
-1	1	σ_n	$(-1)^{n-l(\lambda^{(1)})-\cdots-l(\lambda^{(k)})}$
-1	ϵ^m	$\sigma_n\delta_{\epsilon^m}$	$(-1)^{n-l(\lambda^{(1)})-\cdots-l(\lambda^{(k)})}(\epsilon^m)^{1l(\lambda^{(1)})+\cdots+kl(\lambda^{(1)})}$

Table 3.2: Images of the linear characters of $C_k\S S_n$ under ch.

character	image
1_n	$h_n(X^{(1)}+\cdots+X^{(k)})$
δ_{ϵ^m}	$h_n(\epsilon^{1m}X^{(1)}+\cdots+\epsilon^{km}X^{(k)})$
σ_n	$e_n(X^{(1)}+\cdots+X^{(k)})$
$\sigma_n\delta_{\epsilon^m}$	$e_n(\epsilon^{1m}X^{(1)}+\cdots+\epsilon^{km}X^{(k)}$

3.6 Linear Characters of $C_k\S S_n$

We can use the relations on the generators of $C_k\S S_n$ to calculate all of the one-dimensional characters of $C_k\S S_n$. Let L be a linear character of $C_k\S S_n$. Since $\sigma_i^2 = 1$, we have $L(\sigma_i^2) = L(1) = 1$. On the other hand, $L(\sigma_i^2) = L(\sigma_i)^2$ so we must have $L(\sigma_i) = \pm 1$. In addition, $(\sigma_i\sigma_{i+1})^3 = 1$ so $1 = L((\sigma_i\sigma_{i+1})^3) = L(\sigma_i)^3L(\sigma_{i+1})^3 = L(\sigma_i)L(\sigma_{i+1})$. This along with the fact that $\sigma_i, \sigma_{i+1} = \pm 1$ gives that $L(\sigma_i) = L(\sigma_{i+1})$. Thus we have $L(\sigma_1) = L(\sigma_2) = \cdots = L(\sigma_{n-1}) = \pm 1$. Also, since $\tau^k = 1$, $1 = L(1) = L(\tau^k) = L(\tau)^k$ so $L(\tau) \in \{1, \epsilon, \dots, \epsilon^{k-1}\}$.

Table 3.1 gives all of the linear characters of $C_k\S S_n$ when evaluated at the conjugacy class $C_{(\lambda^{(1)},\dots,\lambda^{(k)})}$. Note that here, m is taken to be in $\{1, \dots, k-1\}$.

Consider the images of the linear characters under the characteristic map. We have the following theorem.

Theorem 3.7. *The images of the linear characters of $C_k\S S_n$ under the characteristic map ch defined by (1.2) are as given in Table 3.2.*

Proof. We will give proofs of the images of 1_n and δ_{ϵ^m}. The proofs of the other two images are similar.

(Proof of $ch(1_n)$)

$$
\begin{aligned}
ch(1_n) &= ch\left(\sum_{(\lambda^{(1)},\dots,\lambda^{(k)})\vdash n} 1_{(\lambda^{(1)},\dots,\lambda^{(k)})}\right) \\
&= \sum_{(\lambda^{(1)},\dots,\lambda^{(k)})\vdash n} ch(1_{(\lambda^{(1)},\dots,\lambda^{(k)})}) \\
&= \sum_{(\lambda^{(1)},\dots,\lambda^{(k)})\vdash n} \frac{p_{\lambda^{(1)}}(X^{(1)})\cdots p_{\lambda^{(k)}}(X^{(k)})}{z_{\lambda^{(1)}}\cdots z_{\lambda^{(k)}}} \\
&= \sum_{a_1+\cdots+a_k=n} \left(\sum_{\lambda^{(1)}\vdash a_1} \frac{p_{\lambda^{(1)}}(X^{(1)})}{z_{\lambda^{(1)}}}\right)\cdots\left(\sum_{\lambda^{(k)}\vdash a_k} \frac{p_{\lambda^{(k)}}(X^{(k)})}{z_{\lambda^{(k)}}}\right) \\
&= \sum_{a_1+\cdots+a_k=n} h_{a_1}(X^{(1)})\cdots h_{a_k}(X^{(k)}) = h_n(X^{(1)}+\cdots+X^{(k)}).
\end{aligned}
$$

(Proof of $ch(\delta_{\epsilon^m})$)

$$
\begin{aligned}
ch(\delta_{\epsilon^m}) &= ch\left(\sum_{(\lambda^{(1)},\dots,\lambda^{(k)})\vdash n} (\epsilon^m)^{1l(\lambda^{(1)})+\cdots+kl(\lambda^{(k)})} 1_{(\lambda^{(1)},\dots,\lambda^{(k)})}\right) \\
&= \sum_{(\lambda^{(1)},\dots,\lambda^{(k)})\vdash n} (\epsilon^m)^{1l(\lambda^{(1)})+\cdots+kl(\lambda^{(k)})} ch(1_{(\lambda^{(1)},\dots,\lambda^{(k)})}) \\
&= \sum_{(\lambda^{(1)},\dots,\lambda^{(k)})\vdash n} (\epsilon^m)^{1l(\lambda^{(1)})+\cdots+kl(\lambda^{(k)})} \frac{p_{\lambda^{(1)}}(X^{(1)})\cdots p_{\lambda^{(k)}}(X^{(k)})}{z_{\lambda^{(1)}}\cdots z_{\lambda^{(k)}}} \\
&= \sum_{a_1+\cdots+a_k=n} \prod_{i=1}^{k} \left(\sum_{\lambda^{(i)}\vdash a_i} (\epsilon^m)^{il(\lambda^{(i)})} \frac{p_{\lambda^{(i)}}(X^{(i)})}{z_{\lambda^{(i)}}}\right) \\
&= \sum_{a_1+\cdots+a_k=n} \prod_{i=1}^{k} \left(\sum_{\lambda^{(i)}\vdash a_i} \frac{p_{\lambda^{(i)}}(\epsilon^{im}X^{(i)})}{z_{\lambda^{(i)}}}\right) \\
&= \sum_{a_1+\cdots+a_k=n} h_{a_1}(\epsilon^{1m}X^{(1)})\cdots h_{a_k}(\epsilon^{km}X^{(k)}) \\
&= h_n(\epsilon^{1m}X^{(1)}+\cdots+\epsilon^{km}X^{(k)}).
\end{aligned}
$$

$\blacksquare$

3.7 The Images of S_n-Characters Under ch

In this section we determine the images under the characteristics map of several characters of S_n viewed as characters of $C_k \S S_n$. We prove a number of lemmas that will be necessary for the proof of the irreducible characters of $C_k \S S_n$.

If χ^γ is an irreducible character of S_n, we may regard it as a character of $C_k \S S_n$ in the following way. For the generators σ_i, let $\chi^\gamma(\sigma_i)$ be defined in the same way as for $\sigma_i \in S_n$. In addition, set $\chi^\gamma(\tau)$ equal to the identity. We then have

$$\chi^\gamma_{(\lambda^{(1)},\dots,\lambda^{(k)})} = \chi^\gamma_{(\lambda^{(1)}\cup\cdots\cup\lambda^{(k)})}.$$

We then have the following lemma.

Lemma 3.8. *Let χ^γ be an irreducible character of S_n. Then*

$$ch(\chi^\gamma) = s_\gamma(X^{(1)} + \cdots + X^{(k)}).$$

Proof. We have that

$$ch(\chi^\gamma) = \sum_{(\lambda^{(1)},\dots,\lambda^{(k)})\vdash n} \chi^\gamma_{(\lambda^{(1)}\cup\cdots\cup\lambda^{(k)})} \frac{p_{\lambda^{(1)}}(X^{(1)})\cdots p_{\lambda^{(k)}}(X^{(k)})}{z_{\lambda^{(1)}}\cdots z_{\lambda^{(k)}}}.$$

Now for an alphabet W, consider the sum

$$\sum_\gamma ch(\chi^\gamma)s_\gamma(W) = \sum_\gamma \sum_{(\lambda^{(1)},\dots,\lambda^{(k)})\vdash n} \chi^\gamma_{(\lambda^{(1)}\cup\cdots\cup\lambda^{(k)})} \frac{p_{\lambda^{(1)}}(X^{(1)})\cdots p_{\lambda^{(k)}}(X^{(k)})}{z_{\lambda^{(1)}}\cdots z_{\lambda^{(k)}}} s_\gamma(W).$$

Recalling that $\sum_\gamma \chi^\gamma_\mu s_\gamma(Z) = p_\mu(Z)$, we can rewrite this as

$$\begin{aligned}
\sum_\gamma & ch(\chi^\gamma) s_\gamma(W) \\
&= \sum_{(\lambda^{(1)},\dots,\lambda^{(k)})\vdash n} \frac{p_{\lambda^{(1)}}(X^{(1)})\cdots p_{\lambda^{(k)}}(X^{(k)})}{z_{\lambda^{(1)}}\cdots z_{\lambda^{(k)}}} \sum_\gamma \chi^\gamma_{(\lambda^{(1)}\cup\cdots\cup\lambda^{(k)})} s_\gamma(W) \\
&= \sum_{(\lambda^{(1)},\dots,\lambda^{(k)})} \frac{p_{\lambda^{(1)}}(X^{(1)})\cdots p_{\lambda^{(k)}}(X^{(k)})}{z_{\lambda^{(1)}}\cdots z_{\lambda^{(k)}}} p_{(\lambda^{(1)}\cup\cdots\cup\lambda^{(k)})}(W) \\
&= \prod_{i=1}^k \left(\sum_{\lambda^{(i)}} \frac{1}{z_{\lambda^{(i)}}} p_{\lambda^{(i)}}(X^{(i)}) p_{\lambda^{(i)}}(W) \right) \\
&= \prod_{r,s} \frac{1}{1 - X_r^{(1)} W_s} \cdots \frac{1}{1 - X_r^{(k)} W_s} \\
&= \sum_\gamma s_\gamma(X^{(1)} + \cdots + X^{(k)}) s_\gamma(W).
\end{aligned}$$

Equating coefficients of $s_\gamma(W)$ gives the result. ∎

For $m = 1, 2, \dots, k$, define a homomorphism Δ_{ϵ^m} on $\Lambda_{W_{k,n}}$ by

$$\Delta_{\epsilon^m} : \Lambda_{W_{k,n}} \longrightarrow \Lambda_{W_{k,n}},$$

$$\Delta_{\epsilon^m} p_r(X^{(i)}) = p_r(\epsilon^{mi} X^{(i)}).$$

We then have

$$\Delta_{\epsilon^m} s_\lambda(X^{(i)}) = \Delta_{\epsilon^m} \sum_\mu \frac{\chi^\lambda_\mu}{z_\mu} p_\mu(X^{(i)}) = \sum_\mu \frac{\chi^\lambda_\mu}{z_\mu} p_\mu(\epsilon^{mi} X^{(i)}) = s_\lambda(\epsilon^{mi} X^{(i)}).$$

We now have the following lemma regarding the homomorphism Δ_{ϵ^m}.

Lemma 3.9. *Let χ be a character of $C_k \S S_n$ and let δ_{ϵ^m} be the linear character of $C_k \S S_n$ described in Tables 3.1 and 3.2. If $f = ch(\chi)$, then $\Delta_{\epsilon^m} f = ch(\delta_{\epsilon^m}\chi)$.*

Proof. By linearity, it is enough to show this for the indicator function $1_{(\lambda^{(1)},\dots,\lambda^{(k)})}$. By definition,

$$ch(1_{(\lambda^{(1)},\dots,\lambda^{(k)})}) = \frac{p_{\lambda^{(1)}}(X^{(1)})\cdots p_{\lambda^{(k)}}(X^{(k)})}{z_{\lambda^{(1)}}\cdots z_{\lambda^{(k)}}}.$$

Thus

$$\Delta_{\epsilon^m} ch(1_{(\lambda^{(1)},\dots,\lambda^{(k)})}) = \Delta_{\epsilon^m} \frac{p_{\lambda^{(1)}}(X^{(1)}) \cdots p_{\lambda^{(k)}}(X^{(k)})}{z_{\lambda^{(1)}} \cdots z_{\lambda^{(k)}}}$$
$$= \frac{p_{\lambda^{(1)}}(\epsilon^{1m} X^{(1)}) \cdots p_{\lambda^{(k)}}(\epsilon^{km} X^{(k)})}{z_{\lambda^{(1)}} \cdots z_{\lambda^{(k)}}}.$$

On the other hand,

$$ch(\delta_{\epsilon^m} 1_{(\lambda^{(1)},\dots,\lambda^{(k)})}) = (\epsilon^m)^{1l(\lambda^{(1)})+\cdots+kl(\lambda^{(k)})} \frac{p_{\lambda^{(1)}}(X^{(1)}) \cdots p_{\lambda^{(k)}}(X^{(k)})}{z_{\lambda^{(1)}} \cdots z_{\lambda^{(k)}}}$$
$$= \frac{p_{\lambda^{(1)}}(\epsilon^{1m} X^{(1)}) \cdots p_{\lambda^{(k)}}(\epsilon^{km} X^{(k)})}{z_{\lambda^{(1)}} \cdots z_{\lambda^{(k)}}}.$$

The combination of these two statements proves the lemma. ■

We now use the results of the previous two lemmas to determine the image of $\delta_{\epsilon^m}\chi^\lambda$ under ch.

Lemma 3.10. *Let χ^λ be an irreducible character of S_n and let δ_{ϵ^m} be the linear character of $C_k \S S_n$ described in Tables 3.1 and 3.2. Then*

$$ch(\delta_{\epsilon^m}\chi^\lambda) = s_\lambda(\epsilon^{1\cdot m} X^{(1)} + \cdots + \epsilon^{k\cdot m} X^{(k)}).$$

Proof. By Lemma 3.9,

$$ch(\delta_{\epsilon^m}\chi^\lambda) = \Delta_{\epsilon^m} ch(\chi^\lambda).$$

But Lemma 3.8 gives that

$$\Delta_{\epsilon^m} ch(\chi^\lambda) = \Delta_{\epsilon^m} s_\lambda(X^{(1)} + \cdots + X^{(k)}).$$

We then have

$$\begin{aligned} ch(\delta_{\epsilon^m}\chi^\lambda) &= \Delta_{\epsilon^m} s_\lambda(X^{(1)} + \cdots + X^{(k)}) \\ &= \Delta_{\epsilon^m} \sum_{\mu^{(1)}\subseteq\mu^{(2)}\subseteq\cdots\subseteq\mu^{(k-1)}\subseteq\lambda} s_{\mu^{(1)}}(X^{(1)}) s_{\mu^{(2)}/\mu^{(1)}}(X^{(2)}) \cdots s_{\lambda/\mu^{(k-1)}}(X^{(k)}) \\ &= \sum_{\mu^{(1)}\subseteq\mu^{(2)}\subseteq\cdots\subseteq\mu^{(k-1)}\subseteq\lambda} s_{\mu^{(1)}}(\epsilon^{1\cdot m} X^{(1)}) s_{\mu^{(2)}/\mu^{(1)}}(\epsilon^{2\cdot m} X^{(2)}) \cdots s_{\lambda/\mu^{(k-1)}}(\epsilon^{k\cdot m} X^{(k)}) \\ &= s_\lambda(\epsilon^{1\cdot m} X^{(1)} + \cdots + \epsilon^{k\cdot m} X^{(k)}), \end{aligned}$$

completing the proof. ■

3.8 The Irreducible Characters

We are now ready to characterize the irreducible characters of $C_k\S S_n$.

Theorem 3.11. *Let δ_{ϵ^m} be the linear character of $C_k\S S_n$ described in Tables 3.1 and 3.2. Then the irreducible characters of $C_k\S S_n$ are*

$$(\delta_{\epsilon^1}\chi^{\lambda^{(1)}} \times \cdots \times \delta_{\epsilon^k}\chi^{\lambda^{(k)}}) \uparrow^{C_k\S S_n},$$

their characteristics are

$$\prod_{i=1}^{k} s_{\lambda^{(i)}}(\epsilon^{1i}X^{(1)} + \cdots + \epsilon^{ki}X^{(k)}),$$

and their degrees are

$$\binom{n}{m_1, \ldots, m_k} f^{\lambda^{(1)}} \cdots f^{\lambda^{(k)}},$$

where the $\chi^{\lambda^{(i)}}$s are irreducible characters of symmetric groups and $f^{\lambda^{(i)}}$ is the number of standard tableaux of shape $\lambda^{(i)}$.

Proof. We will abuse notation by letting

$$\chi^{(\lambda^{(1)},\ldots,\lambda^{(k)})} = (\delta_{\epsilon^1}\chi^{\lambda^{(1)}} \times \cdots \times \delta_{\epsilon^k}\chi^{\lambda^{(k)}}) \uparrow^{C_k\S S_n},$$

$$s_{(\lambda^{(1)},\ldots,\lambda^{(k)})}(X^{(1)}, \ldots, X^{(k)}) = \prod_{i=1}^{k} s_{\lambda^{(i)}}(\epsilon^{1\cdot i}X^{(1)} + \cdots + \epsilon^{k\cdot i}X^{(k)}),$$

and

$$\overline{s_{(\lambda^{(1)},\ldots,\lambda^{(k)})}(X^{(1)}, \ldots, X^{(k)})} = \prod_{i=1}^{k} s_{\lambda^{(i)}}(\epsilon^{(k-1)i}X^{(1)} + \cdots + \epsilon^{(k-k)i}X^{(k)}).$$

(Characteristic) From Theorem 3.6 and Lemmas 3.8 and 3.10, it is clear that

$$ch(\chi^{(\lambda^{(1)},\ldots,\lambda^{(k)})}) = s_{(\lambda^{(1)},\ldots,\lambda^{(k)})}(X^{(1)}, \ldots, X^{(k)}).$$

(Irreducibility) We have that

$$\chi^{(\lambda^{(1)},\ldots,\lambda^{(k)})} = \sum_{(\alpha^{(1)},\ldots,\alpha^{(k)})} \chi^{(\lambda^{(1)},\ldots,\lambda^{(k)})}_{(\alpha^{(1)},\ldots,\alpha^{(k)})} 1_{(\alpha^{(1)},\ldots,\alpha^{(k)})}.$$

Then

$$ch(\chi^{(\lambda^{(1)},\dots,\lambda^{(k)})}) = s_{(\lambda^{(1)},\dots,\lambda^{(k)})}(X^{(1)},\dots,X^{(k)})$$
$$= \sum_{(\alpha^{(1)},\dots,\alpha^{(k)})} \chi^{(\lambda^{(1)},\dots,\lambda^{(k)})}_{(\alpha^{(1)},\dots,\alpha^{(k)})} \frac{p_{\alpha^{(1)}}(X^{(1)})\cdots p_{\alpha^{(k)}}(X^{(k)})}{z_{\alpha^{(1)}}\cdots z_{\alpha^{(k)}}}.$$

Similarly,

$$ch(\overline{\chi^{(\lambda^{(1)},\dots,\lambda^{(k)})}}) = \overline{s_{(\lambda^{(1)},\dots,\lambda^{(k)})}(X^{(1)},\dots,X^{(k)})}$$
$$= \sum_{(\alpha^{(1)},\dots,\alpha^{(k)})} \overline{\chi^{(\lambda^{(1)},\dots,\lambda^{(k)})}_{(\alpha^{(1)},\dots,\alpha^{(k)})}} \frac{p_{\alpha^{(1)}}(X^{(1)})\cdots p_{\alpha^{(k)}}(X^{(k)})}{z_{\alpha^{(1)}}\cdots z_{\alpha^{(k)}}}.$$

Thus by Theorem 3.5,

$$\prod_{p=1}^{k}\prod_{r,s}\left(\frac{1}{1-X_r^{(p)}Y_s^{(p)}}\right)^k$$
$$= \sum_{(\lambda^{(1)},\dots,\lambda^{(k)})} s_{(\lambda^{(1)},\dots,\lambda^{(k)})}(X^{(1)},\dots,X^{(k)})\overline{s_{(\lambda^{(1)},\dots,\lambda^{(k)})}(Y^{(1)},\dots,Y^{(k)})}$$
$$= \langle S\rangle \left\langle \overline{S}\right\rangle^T = \left\langle P\frac{\chi}{z_{\alpha^{(1)}}\cdots z_{\alpha^{(k)}}}\right\rangle \left\langle P\frac{\overline{\chi}}{z_{\alpha^{(1)}}\cdots z_{\alpha^{(k)}}}\right\rangle^T$$
$$= \left\langle \frac{P}{z_{\alpha^{(1)}}\cdots z_{\alpha^{(k)}}}\right\rangle \langle\chi\rangle \left\langle \frac{\overline{\chi}}{z_{\alpha^{(1)}}\cdots z_{\alpha^{(k)}}k^{l(\alpha^{(1)})+\cdots+l(\alpha^{(k)})}}\right\rangle^T \left\langle k^{l(\alpha^{(1)})+\cdots+l(\alpha^{(k)})}P\right\rangle^T.$$

On the other hand,

$$\prod_{p=1}^{k}\prod_{r,s}\left(\frac{1}{1-X_r^{(p)}Y_s^{(p)}}\right)^k$$
$$= \sum_{(\alpha^{(1)},\dots,\alpha^{(k)})} \frac{k^{l(\alpha^{(1)})+\cdots+l(\alpha^{(k)})}}{z_{\alpha^{(1)}}\cdots z_{\alpha^{(k)}}} p_{\alpha^{(1)}}(X^{(1)})\cdots p_{\alpha^{(k)}}(X^{(k)})p_{\alpha^{(1)}}(Y^{(1)})\cdots p_{\alpha^{(k)}}(Y^{(k)})$$
$$= \left\langle \frac{P}{z_{\alpha^{(1)}}\cdots z_{\alpha^{(k)}}}\right\rangle \left\langle k^{l(\alpha^{(1)})+\cdots+l(\alpha^{(k)})}P\right\rangle^T.$$

Since $\{p_{\lambda^{(1)}}(X^{(1)})\cdots p_{\lambda^{(k)}}(X^{(k)})\}$ and $\{s_{(\lambda^{(1)},\dots,\lambda^{(k)})}(X^{(1)},\dots,X^{(k)})\}$ are bases, we have

$$\langle\chi\rangle \left\langle \frac{\overline{\chi}}{z_{\alpha^{(1)}}\cdots z_{\alpha^{(k)}}k^{l(\alpha^{(1)})+\cdots+l(\alpha^{(k)})}}\right\rangle^T = I. \tag{3.5}$$

The left hand side of (3.5) is

$$\sum_{(\alpha^{(1)},\dots,\alpha^{(k)})} \chi^{(\lambda^{(1)},\dots,\lambda^{(k)})}_{\alpha^{(1)},\dots,\alpha^{(k)}} \overline{\chi^{(\mu^{(1)},\dots,\mu^{(k)})}_{\alpha^{(1)},\dots,\alpha^{(k)}}} \frac{1}{z_{\alpha^{(1)}}\cdots z_{\lambda^{(k)}} k^{(l(\alpha^{(1)})+\cdots+l(\alpha^{(k)})}}$$

$$= \frac{1}{k^n n!} \sum_{(\alpha^{(1)},\dots,\alpha^{(k)})} \chi^{(\lambda^{(1)},\dots,\lambda^{(k)})}_{\alpha^{(1)},\dots,\alpha^{(k)}} \overline{\chi^{(\mu^{(1)},\dots,\mu^{(k)})}_{\alpha^{(1)},\dots,\alpha^{(k)}}} |C_{(\alpha^{(1)},\dots,\alpha^{(k)})}|$$

$$= \frac{1}{k^n n!} \sum_{\sigma \in C_k \S S_n} \chi^{(\lambda^{(1)},\dots,\lambda^{(k)})}(\sigma) \overline{\chi^{(\mu^{(1)},\dots,\mu^{(k)})}(\sigma)}$$

$$= \left\langle \chi^{(\lambda^{(1)},\dots,\lambda^{(k)})}, \chi^{(\mu^{(1)},\dots,\mu^{(k)})} \right\rangle_{C_k \S S_n}.$$

Thus we have

$$\left\langle \chi^{(\lambda^{(1)},\dots,\lambda^{(k)})}, \chi^{(\mu^{(1)},\dots,\mu^{(k)})} \right\rangle_{C_k \S S_n} = \chi\left((\lambda^{(1)},\dots,\lambda^{(k)}) = (\mu^{(1)},\dots,\mu^{(k)}) \right).$$

$\{\chi^{(\lambda^{(1)},\dots,\lambda^{(k)})}\}$ thus has the right number of orthogonal characters and is therefore the set of irreducible characters of $C_k \S S_n$.

(Degree) Recall that if χ is a character, then $\chi(e)$ gives the degree of χ for the identity element e. We have

$$\left\langle \chi, 1_{(\emptyset,\dots,\emptyset,1^n)} \right\rangle_{C_k \S S_n} = \frac{1}{k^n n!} \sum_{\sigma \in C_k \S S_n} \chi(\sigma) 1_{(\emptyset,\dots,\emptyset,1^n)}(\sigma) = \frac{1}{k^n n!} \chi(e),$$

so

$$\chi(e) = k^n n! \left\langle \chi, 1_{(\emptyset,\dots,\emptyset,1^n)} \right\rangle_{C_k \S S_n}.$$

Thus we compute this inner product for $\chi = \chi^{(\lambda^{(1)},\dots,\lambda^{(k)})}$.

$$\begin{aligned}
\left\langle \chi^{(\lambda^{(1)},\dots,\lambda^{(k)})}, 1_{(\emptyset,\dots,\emptyset,1^n)} \right\rangle_{C_k \S S_n} &= \left\langle \prod_{i-1}^{k} s_{\lambda^{(i)}}(\epsilon^{1i} X^{(1)} + \cdots + \epsilon^{ki} X^{(k)}), \frac{p_{1^n}(X^{(k)})}{n!} \right\rangle_* \\
&= \left\langle \prod_{i=1}^{k} \sum_{\alpha^{(i,1)} \subseteq \alpha^{(i,2)} \subseteq \cdots \subseteq \alpha^{(i,k-1)} \subseteq \lambda} s_{\alpha^{(i,1)}}(\epsilon^{1i} X^{(1)}) s_{\alpha^{(i,2)}/\alpha^{(i,1)}}(\epsilon^{2i} X^{(2)}) \right. \\
&\qquad \left. \cdots s_{\lambda/\alpha^{(i,k-1)}}(\epsilon^{ki} X^{(k)}), \frac{p_{1^n}(X^{(k)})}{n!} \right\rangle_* \\
&= \left\langle \prod_{i=1}^{k} \sum_{\alpha^{(i,1)} \subseteq \alpha^{(i,2)} \subseteq \cdots \subseteq \alpha^{(i,k-1)} \subseteq \lambda} \prod_{j=1}^{k} \sum_{\beta^{(j)}} \chi^{\alpha^{(i,j)}}_{\beta^{(j)}} \frac{p_{\beta^{(j)}}(\epsilon^{ji} X^{(j)})}{z_{\beta^{(j)}}}, \frac{p_{1^n}(X^{(k)})}{n!} \right\rangle_* \\
&= \left\langle \prod_{i=1}^{k} \sum_{\beta^{(i)} \vdash |\lambda^{(i)}|} \chi^{\lambda^{(i)}}_{\beta^{(i)}} \frac{p_{\beta^{(i)}}(X^{(k)})}{z_{\beta^{(i)}}}, \frac{p_{1^n}(X^{(k)})}{n!} \right\rangle_* \\
&= \left\langle \prod_{i=1}^{k} \chi^{\lambda^{(i)}}_{1^{m_i}} \frac{p_{1^{m_i}}(X^{(k)})}{m_i!}, \frac{p_{1^n}(X^{(k)})}{n!} \right\rangle_* \\
&= \frac{f^{\lambda^{(1)}} \cdots f^{\lambda^{(k)}}}{m_1! \cdots m_k! n!} \left\langle p_{1^n}(X^{(k)}), p_{1^n}(X^{(k)}) \right\rangle_* \\
&= \frac{f^{\lambda^{(1)}} \cdots f^{\lambda^{(k)}}}{m_1! \cdots m_k! n!} \frac{n!}{k^n}.
\end{aligned}$$

We therefore have

$$\chi^{(\lambda^{(1)},\dots,\lambda^{(k)})}(e) = k^n n! \frac{f^{\lambda^{(1)}} \cdots f^{\lambda^{(k)}}}{m_1! \cdots m_k! n!} \frac{n!}{k^n} = \binom{n}{m_1, \dots, m_k} f^{\lambda^{(1)}} \cdots f^{\lambda^{(k)}},$$

the desired value. ■

Chapter 4

Transition Matrices Between Bases of the $C_3 \S S_n$-Symmetric Functions

In this chapter, we consider some of the bases of $\Lambda_{3,n}$, the space of symmetric functions associated with $C_3 \S S_n$ described in the previous chapter, and determine transition matrices between the bases.

Because of considerations of space, denote a basis $\{a_\lambda(X+Y+Z)b_\mu(X+\epsilon Y+\epsilon^2 Z)c_\nu(X+\epsilon^2 Y+\epsilon Z)\}$ with $(\lambda,\mu,\nu)\vdash n$ by $a_\lambda \tilde{b}_\mu \hat{c}_\nu$. Likewise, we denote the basis $p_\lambda(X)p_\mu(Y)p_\nu(Z)$ by $p_\lambda \tilde{p}_\mu \hat{p}_\nu$. Then a number of the bases of $\Lambda_{3,n}$ are given in Table 4.1.

We denote by $M(a\tilde{a}\hat{a}, b\tilde{b}\hat{b})$ the matrix that transforms the basis vector

Table 4.1: Some of the bases of $\Lambda_{3,n}$.

$e_\lambda \tilde{e}_\mu \hat{e}_\nu$	$h_\lambda \tilde{h}_\mu \hat{h}_\nu$	$m_\lambda \tilde{m}_\mu \hat{m}_\nu$	$f_\lambda \tilde{f}_\mu \hat{f}_\nu$	$s_\lambda \tilde{s}_\mu \hat{s}_\nu$	$p_\lambda \tilde{p}_\mu \hat{p}_\nu$
$e_\lambda \tilde{e}_\mu \hat{h}_\nu$	$e_\lambda \tilde{h}_\mu \hat{e}_\nu$	$h_\lambda \tilde{e}_\mu \hat{e}_\nu$	$e_\lambda \tilde{h}_\mu \hat{h}_\nu$	$h_\lambda \tilde{e}_\mu \hat{h}_\nu$	$h_\lambda \tilde{e}_\mu \hat{e}_\nu$
$m_\lambda \tilde{m}_\mu \hat{f}_\nu$	$m_\lambda \tilde{f}_\mu \hat{m}_\nu$	$f_\lambda \tilde{m}_\mu \hat{m}_\nu$	$m_\lambda \tilde{f}_\mu \hat{f}_\nu$	$f_\lambda \tilde{m}_\mu \hat{f}_\nu$	$f_\lambda \tilde{m}_\mu \hat{m}_\nu$

$< a_\lambda \tilde{a}_\mu \hat{a}_\nu >$ into the basis vector $< b_\lambda \tilde{b}_\mu \hat{b}_\nu >$. That is,

$$< b_\lambda \tilde{b}_\mu \hat{b}_\nu > = < a_\lambda \tilde{a}_\mu \hat{a}_\nu > M(a\tilde{a}\hat{a}, b\tilde{b}\hat{b}).$$

Then the $(\alpha, \beta, \gamma)(\lambda, \mu, \nu)$-entry of $M(a\tilde{a}\hat{a}, b\tilde{b}\hat{b})$ is defined by

$$b_\lambda \tilde{b}_\mu \hat{b}_\nu = \sum_{(\alpha,\beta,\gamma)\vdash n} a_\alpha \tilde{a}_\beta \hat{a}_\gamma M(a\tilde{a}\hat{a}, b\tilde{b}\hat{b})_{(\alpha,\beta,\gamma)(\lambda,\mu,\nu)}.$$

The transition matrices between all pairs of bases of $\Lambda_{3,n}$ that do not involve the basis $p_\lambda \tilde{p}_\mu \hat{p}_\nu$ consist of triples of transition matrices for the S_n case. Because of this and the fact that there are 306 transition matrices, we will only consider transition matrices involving the basis $p_\lambda \tilde{p}_\mu \hat{p}_\nu$. We will give the following transition matrices: $M(h\tilde{h}\hat{h}, p\tilde{p}\hat{p})$, $M(p\tilde{p}\hat{p}, h\tilde{h}\hat{h})$, $M(e\tilde{e}\hat{e}, p\tilde{p}\hat{p})$, $M(p\tilde{p}\hat{p}, e\tilde{e}\hat{e})$, $M(s\tilde{s}\hat{s}, p\tilde{p}\hat{p})$, $M(p\tilde{p}\hat{p}, s\tilde{s}\hat{s})$, $M(m\tilde{m}\hat{m}, p\tilde{p}\hat{p})$, $M(p\tilde{p}\hat{p}, m\tilde{m}\hat{m})$, $M(f\tilde{f}\hat{f}, p\tilde{p}\hat{p})$, and $M(p\tilde{p}\hat{p}, f\tilde{f}\hat{f})$. The proofs for the other transition matrices are similar to these.

4.1 $M(h\tilde{h}\hat{h}, p\tilde{p}\hat{p})$

Recall that a μ-brick tabloid of shape λ is a tabloid in the shape of λ filled with bricks of sizes $\mu_1, \mu_2, \ldots, \mu_{l(\mu)}$ such that each brick lies horizontally in a row. The set of all μ-brick tabloids of shape λ is denoted by $\mathcal{B}_{\mu,\lambda}$. We can weight these tabloids in the following way. The total weight is

$$w(B_{\mu,\lambda}) = \sum_{T \in \mathcal{B}_{\mu,\lambda}} w(T),$$

where the weight of a tabloid T is

$$w(T) = \prod_{b \in T} w_T(b),$$

and the weight of each brick in the tabloid is given by

$$w_T(b) = \begin{cases} |b|, & \text{if } b \text{ is at the end of a row;} \\ 1, & \text{otherwise.} \end{cases}$$

We begin with the following expression, which may be found in [7].

$$p_n = \sum_{\mu \vdash n} (-1)^{l(\mu)-1} w(B_{\mu,(n)}) h_\mu.$$

From this we obtain the following expressions.

$$p_n(X) + p_n(Y) + p_n(Z) = p_n(X+Y+Z) = \sum_{\alpha \vdash n} (-1)^{l(\alpha)-1} w(B_{\alpha,(n)}) h_\alpha(X+Y+Z), \quad (4.1)$$

$$p_n(X) + \epsilon p_n(Y) + \epsilon^2 p_n(Z) = p_n(X+\epsilon Y+\epsilon^2 Z) = \sum_{\beta \vdash n} (-1)^{l(\beta)-1} w(B_{\beta,(n)}) h_\beta(X+\epsilon Y+\epsilon^2 Z), \quad (4.2)$$

$$p_n(X) + \epsilon^2 p_n(Y) + \epsilon p_n(Z) = p_n(X+\epsilon^2 Y+\epsilon Z) = \sum_{\gamma \vdash n} (-1)^{l(\gamma)-1} w(B_{\gamma,(n)}) h_\gamma(X+\epsilon^2 Y+\epsilon Z). \quad (4.3)$$

If we sum (4.1) + (4.2) + (4.3), we get

$$3p_n(X) = \sum_{\alpha \vdash n} (-1)^{l(\alpha)-1} w(B_{\alpha,(n)}) h_\alpha(X+Y+Z) + \sum_{\beta \vdash n} (-1)^{l(\beta)-1} w(B_{\beta,(n)}) h_\beta(X+\epsilon Y+\epsilon^2 Z) + \sum_{\gamma \vdash n} (-1)^{l(\gamma)-1} w(B_{\gamma,(n)}) h_\gamma(X+\epsilon^2 Y+\epsilon Z). \quad (4.4)$$

Summing (4.1) + ϵ^2(4.2) + ϵ(4.3) gives

$$3p_n(Y) = \sum_{\alpha \vdash n} (-1)^{l(\alpha)-1} w(B_{\alpha,(n)}) h_\alpha(X+Y+Z) + \sum_{\beta \vdash n} (-1)^{l(\beta)-1} \epsilon^2 w(B_{\beta,(n)}) h_\beta(X+\epsilon Y+\epsilon^2 Z) + \sum_{\gamma \vdash n} (-1)^{l(\gamma)-1} \epsilon w(B_{\gamma,(n)}) h_\gamma(X+\epsilon^2 Y+\epsilon Z). \quad (4.5)$$

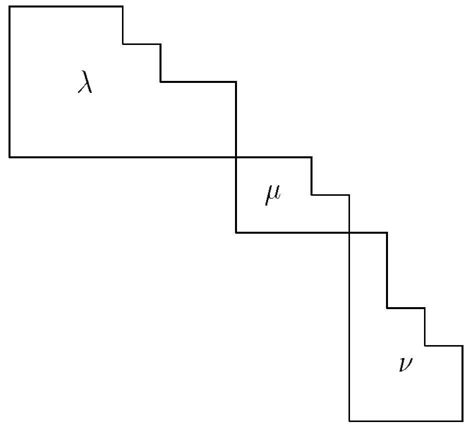

Figure 4.1: An illustration of $\lambda * \mu * \nu$.

Summing (4.1) + ϵ(4.2) + ϵ^2(4.3) gives

$$3p_n(Z) = \sum_{\alpha \vdash n} (-1)^{l(\alpha)-1} w(B_{\alpha,(n)}) h_\alpha(X+Y+Z)$$
$$+ \sum_{\beta \vdash n} (-1)^{l(\beta)-1} \epsilon w(B_{\beta,(n)}) h_\beta(X+\epsilon Y+\epsilon^2 Z)$$
$$+ \sum_{\gamma \vdash n} (-1)^{l(\gamma)-1} \epsilon^2 w(B_{\gamma,(n)}) h_\gamma(X+\epsilon^2 Y+\epsilon Z). \quad (4.6)$$

We now interpret products of the expressions (4.4), (4.5), and (4.6) combinatorially. If λ, μ, and ν are partitions, $\lambda * \mu * \nu$ is the diagram which results from consecutively placing the lower right corner of one partition at the upper left corner of the next. This is illustrated in Figure 4.1.

Let $\mathcal{F}^{\alpha,\beta,\gamma}_{\lambda*\mu*\nu}$ be the set of tabloids of shape $\lambda * \mu * \nu$ filled with α-bricks of sizes $\alpha_1, \alpha_2, \ldots, \alpha_{l(\alpha)}$, β-bricks of sizes $\beta_1, \beta_2, \ldots, \beta_{l(\beta)}$, and γ-bricks of sizes $\gamma_1, \gamma_2, \ldots, \gamma_{l(\gamma)}$, such that each row contains all bricks of the same type. An example of an element of $\mathcal{F}^{\alpha,\beta,\gamma}_{\lambda*\mu*\nu}$ with $\lambda = (3, 5^2)$, $\mu = (1, 4^2, 6)$, $\nu = (2, 4)$, $\alpha = (1^2, 2^4, 3)$, $\beta = (1, 2, 3^2)$, and $\gamma = (1^5, 2^2, 3)$ is given in Figure 4.2.

We use (4.4), (4.5), and (4.6) to write a product of power symmetric functions

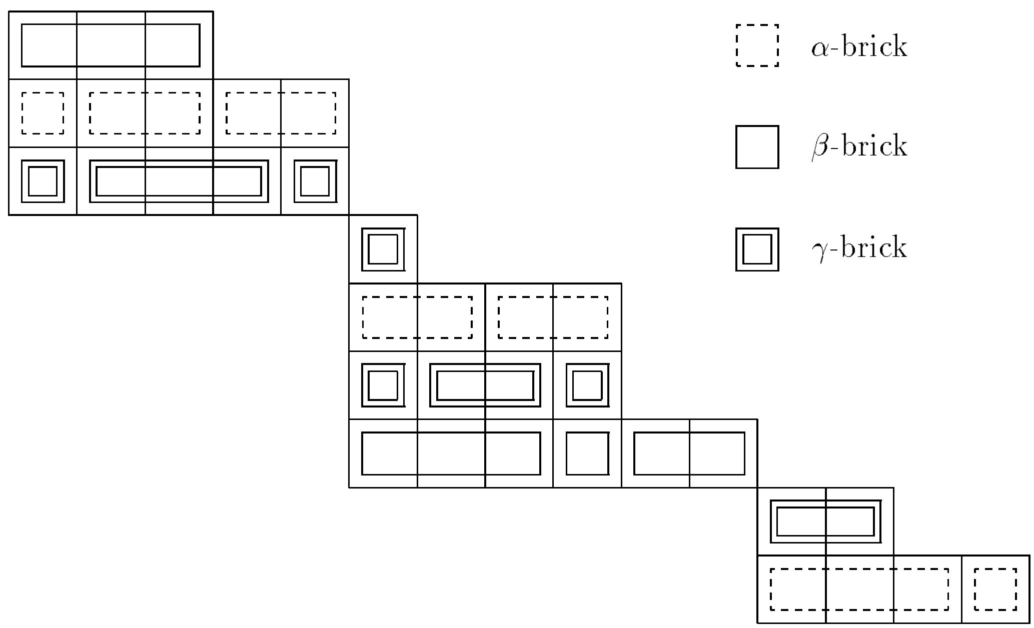

Figure 4.2: An example of an element of $\mathcal{F}^{\alpha,\beta,\gamma}_{\lambda*\mu*\nu}$.

in terms of a weight on elements of $\mathcal{F}^{\alpha,\beta,\gamma}_{\lambda*\mu*\nu}$.

$$3^{l(\lambda)+l(\mu)+l(\nu)} p_\lambda(X) p_\mu(Y) p_\nu(Z) \Big|_{h_\alpha \tilde{h}_\beta \hat{h}_\gamma} = \sum_{f \in \mathcal{F}^{\alpha,\beta,\gamma}_{\lambda*\mu*\nu}} W_1(f), \tag{4.7}$$

where $W_1(f)$ is defined by the product over all bricks b in f

$$W_1(f) = \prod_{b \in f} w_1(b),$$

and where $w_1(b)$ is defined according to the following cases.

- If b is an α-brick,

$$w_1(b) = \begin{cases} |b|, & b \text{ is at the end of a row,} \\ -1, & \text{otherwise.} \end{cases}$$

- If b is a β-brick,

$$w_1(b) = \begin{cases} |b|, & b \text{ is at the end of a row in } \lambda, \\ \epsilon^2|b|, & b \text{ is at the end of a row in } \mu, \\ \epsilon|b|, & b \text{ is at the end of a row in } \nu, \\ -1, & \text{otherwise.} \end{cases}$$

- If b is a γ-brick,

$$w_1(b) = \begin{cases} |b|, & b \text{ is at the end of a row in } \lambda, \\ \epsilon|b|, & b \text{ is at the end of a row in } \mu, \\ \epsilon^2|b|, & b \text{ is at the end of a row in } \nu, \\ -1, & \text{otherwise.} \end{cases}$$

It is helpful to rewrite this in terms of a more standard weight, defined by

$$w(f) = \prod_{b \in f} w(b),$$

where

$$w(b) = \begin{cases} |b|, & b \text{ is at the end of a row,} \\ 1, & \text{otherwise.} \end{cases}$$

Then (4.7) becomes

$$3^{l(\lambda)+l(\mu)+l(\nu)} p_\lambda(X) p_\mu(Y) p_\nu(Z) \Bigg|_{h_\alpha \tilde{h}_\beta \hat{h}_\gamma}$$
$$= \sum_{f \in \mathcal{F}^{\alpha,\beta,\gamma}_{\lambda*\mu*\nu}} (-1)^{l(\alpha)+l(\beta)+l(\gamma)-l(\lambda)-l(\mu)-l(\nu)} \epsilon^{2l^\beta(\mu)+l^\beta(\nu)+l^\gamma(\mu)+2l^\gamma(\nu)} w(f),$$

where $l^\beta(\mu)$ is the number of β-rows occurring in μ, and $l^\beta(\nu)$, $l^\gamma(\mu)$, and $l^\gamma(\nu)$ are defined similarly. This gives the following expression for the transition matrix.

$$M(h\tilde{h}\hat{h}, p\tilde{p}\hat{p})_{(\alpha,\beta,\gamma)(\lambda,\mu,\nu)}$$
$$= \sum_{f \in \mathcal{F}^{\alpha,\beta,\gamma}_{\lambda*\mu*\nu}} \frac{(-1)^{l(\alpha)+l(\beta)+l(\gamma)-l(\lambda)-l(\mu)-l(\nu)}}{3^{l(\lambda)+l(\mu)+l(\nu)}} \epsilon^{2l^\beta(\mu)+l^\beta(\nu)+l^\gamma(\mu)+2l^\gamma(\nu)} w(f).$$

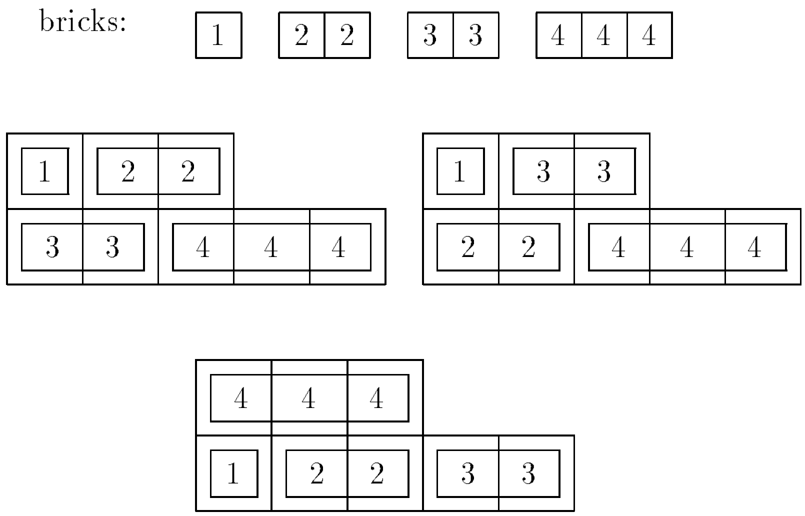

Figure 4.3: The (1,2,2,3)-brick tabloids of shape (3,5).

4.2 $M(p\tilde{p}\hat{p}, h\tilde{h}\hat{h})$

An *ordered μ-brick tabloid of shape λ* is similar to a μ-brick tabloid of shape λ in that it also consists of arrangements of bricks of lengths $\mu_1, \mu_2, \ldots, \mu_{l(\mu)}$ into the shape λ. In the ordered case, however, the bricks are labeled, with the smaller bricks getting the smaller labels. Then, when the bricks are placed in the tabloid, the labels on the bricks must increase left to right in each row. The number of ordered μ-brick tabloids of shape λ is denoted $OB_{\mu,\lambda}$. As an example, the $(1,2,2,3)$-brick tabloids of shape $(3,5)$ are given in Figure 4.3.

We begin with the following expression, which is given in [7].

$$h_\lambda(X+Y+Z) = \sum_{\phi\vdash|\lambda|} \frac{OB_{\phi,\lambda}}{z_\phi} p_\phi(X+Y+Z)$$
$$= \sum_{\phi\vdash|\lambda|} \frac{OB_{\phi,\lambda}}{z_\phi} \prod_{i=1}^{l(\phi)} \left(p_{\phi_i}(X) + p_{\phi_i}(Y) + p_{\phi_i}(Z)\right).$$

If we let $[n] = \{1, 2, \ldots, n\}$, and let $S+T$ be the union of two disjoint sets S and

T, we can rewrite the above expression as

$$\sum_{\phi \vdash |\lambda|} \frac{OB_{\phi,\lambda}}{z_\phi} \sum_{S+T+U=[l(\phi)]} \left(\prod_{i \in S} p_{\phi_i}(X)\right) \left(\prod_{i \in T} p_{\phi_i}(Y)\right) \left(\prod_{i \in U} p_{\phi_i}(Z)\right). \tag{4.8}$$

If $\phi = 1^{m_1}2^{m_2}\cdots n^{m_n}$, $\alpha = 1^{a_1}2^{a_2}\cdots n^{a_n}$, $\beta = 1^{b_1}2^{b_2}\cdots n^{b_n}$, and $\gamma = 1^{c_1}2^{c_2}\cdots n^{c_n}$, such that $0 \le a_i, b_i, c_i$ and $a_i + b_i + c_i = m_i$, then we write $\alpha + \beta + \gamma = \phi$. The number of ways α, β, and γ can be rearranged to form the partition ϕ is $C^{\phi}_{\alpha,\beta,\gamma}$ where

$$C^{\phi}_{\alpha,\beta,\gamma} = \binom{m_1}{a_1, b_1, c_1}\binom{m_2}{a_2, b_2, c_2}\cdots\binom{m_n}{a_n, b_n, c_n}.$$

We use this to rewrite (4.8) as

$$h_\lambda(X+Y+Z) = \sum_{\phi \vdash |\lambda|} \frac{OB_{\phi,\lambda}}{z_\phi} \sum_{\alpha+\beta+\gamma=\phi} C^{\phi}_{\alpha,\beta,\gamma} p_\alpha(X) p_\beta(Y) p_\gamma(Z).$$

Thus

$$h_\lambda(X+Y+Z)\Bigg|_{p_\alpha \tilde{p}_\beta \hat{p}_\gamma} = \frac{OB_{\alpha+\beta+\gamma,\lambda}}{z_{\alpha+\beta+\gamma}} \prod_{i=1}^{n} \binom{a_i+b_i+c_i}{a_i, b_i, c_i}$$

$$= OB_{\alpha+\beta+\gamma,\lambda} \prod_{i=1}^{n} \frac{1}{i^{a_i+b_i+c_i}(a_i+b_i+c_i)!} \frac{(a_i+b_i+c_i)!}{a_i! b_i! c_i!} = \frac{1}{z_\alpha z_\beta z_\gamma} OB_{\alpha_\beta+\gamma,\lambda}. \tag{4.9}$$

Similar arguments give the following expressions:

$$h_\mu(X+\epsilon Y+\epsilon^2 Z)\Bigg|_{p_\phi \tilde{p}_\psi \hat{p}_\pi} = \frac{\epsilon^{l(\psi)+2l(\pi)}}{z_\phi z_\psi z_\pi} OB_{\phi+\psi+\pi,\mu}, \tag{4.10}$$

$$h_\nu(X+\epsilon Y+\epsilon^2 Z)\Bigg|_{p_\delta \tilde{p}_\theta \hat{p}_\omega} = \frac{\epsilon^{2l(\theta)+l(\omega)}}{z_\delta z_\theta z_\omega} OB_{\delta+\theta+\omega,\nu}. \tag{4.11}$$

Combining (4.9), (4.10), and (4.11) gives

$$
\begin{aligned}
h_\lambda \tilde{h}_\mu \hat{h}_\nu = & \sum_{(\alpha,\beta,\gamma)\vdash|\lambda|} \frac{OB_{\alpha+\beta+\gamma,\lambda}}{z_\alpha z_\beta z_\gamma} p_\alpha \tilde{p}_\beta \hat{p}_\gamma \\
& \times \sum_{(\phi,\psi,\pi)\vdash|\mu|} \frac{\epsilon^{l(\psi)+2l(\pi)} OB_{\phi+\psi+\pi,\mu}}{z_\phi z_\psi z_\pi} p_\phi \tilde{p}_\psi \hat{p}_\pi \\
& \times \sum_{(\delta,\theta,\omega)\vdash|\nu|} \frac{\epsilon^{2l(\theta)+l(\omega)} OB_{\delta+\theta+\omega,\nu}}{z_\delta z_\theta z_\omega} p_\delta \tilde{p}_\theta \hat{p}_\omega \\
= & \sum_{\substack{(\alpha,\beta,\gamma)\vdash|\lambda| \\ (\phi,\psi,\pi)\vdash|\mu| \\ (\delta,\theta,\omega)\vdash|\nu|}} \frac{\epsilon^{l(\psi)+2l(\pi)+2l(\theta)+l(\omega)} OB_{\alpha+\beta+\gamma,\lambda} OB_{\phi+\psi+\pi,\mu} OB_{\delta+\theta+\omega,\nu}}{z_\alpha z_\beta z_\gamma z_\phi z_\psi z_\pi z_\delta z_\theta z_\omega} \\
& \times p_{\alpha+\phi+\delta} \tilde{p}_{\beta+\psi+\theta} \hat{p}_{\gamma+\pi+\omega}.
\end{aligned}
$$

We then have the transition matrix

$$
\begin{aligned}
M(p\tilde{p}\hat{p}, h\tilde{h}\hat{h})_{(\eta,\tau,\rho)(\lambda,\mu,\nu)} = & \left. h_\lambda \tilde{h}_\mu \hat{h}_\nu \right|_{p_\eta \tilde{p}_\tau \hat{p}_\rho} \\
= & \sum_{\substack{(\alpha,\beta,\gamma)\vdash|\lambda| \\ (\phi,\psi,\pi)\vdash|\mu| \\ (\delta,\theta,\omega)\vdash|\nu| \\ \alpha+\phi+\delta=\eta \\ \beta+\psi+\theta=\tau \\ \gamma+\pi+\omega=\rho}} \frac{\epsilon^{l(\psi)+2l(\pi)+2l(\theta)+l(\omega)} OB_{\alpha+\beta+\gamma,\lambda} OB_{\phi+\psi+\pi,\mu} OB_{\delta+\theta+\omega,\nu}}{z_\alpha z_\beta z_\gamma z_\phi z_\psi z_\pi z_\delta z_\theta z_\omega}.
\end{aligned}
$$

4.3 $M(e\tilde{e}\hat{e}, p\tilde{p}\hat{p})$

The transition matrix $M(e\tilde{e}\hat{e}, p\tilde{p}\hat{p})$ is very similar to $M(h\tilde{h}\hat{h}, p\tilde{p}\hat{p})$. The difference comes in the weight attached to elements of $\mathcal{F}^{\alpha,\beta,\gamma}_{\lambda*\mu*\nu}$. Again, we begin with an expression from [7].

$$p_n = \sum_\mu (-1)^{n-l(\mu)} w(B_{\mu,(n)}) e_\mu.$$

This gives the expressions

$$p_n(X) + p_n(Y) + p_n(Z) = p_n(X + Y + Z) = \sum_{\alpha \vdash n} (-1)^{n-l(\alpha)} w(B_{\alpha,(n)}) e_\alpha(X + Y + Z), \quad (4.12)$$

$$p_n(X) + \epsilon p_n(Y) + \epsilon^2 p_n(Z) = p_n(X + \epsilon Y + \epsilon^2 Z) = \sum_{\beta \vdash n} (-1)^{n-l(\beta)} w(B_{\beta,(n)}) e_\beta(X + \epsilon Y + \epsilon^2 Z), \quad (4.13)$$

$$p_n(X) + \epsilon^2 p_n(Y) + \epsilon p_n(Z) = p_n(X + \epsilon^2 Y + \epsilon Z) = \sum_{\gamma \vdash n} (-1)^{n-l(\gamma)} w(B_{\gamma,(n)}) e_\gamma(X + \epsilon^2 Y + \epsilon Z). \quad (4.14)$$

If we sum (4.12) + (4.13) + (4.14), we get

$$\begin{aligned} 3p_n(X) = \sum_{\alpha \vdash n} (-1)^{n-l(\alpha)} w(B_{\alpha,(n)}) e_\alpha(X + Y + Z) \\ + \sum_{\beta \vdash n} (-1)^{n-l(\beta)} w(B_{\beta,(n)}) e_\beta(X + \epsilon Y + \epsilon^2 Z) \\ + \sum_{\gamma \vdash n} (-1)^{n-l(\gamma)} w(B_{\gamma,(n)}) e_\gamma(X + \epsilon^2 Y + \epsilon Z). \quad (4.15) \end{aligned}$$

Summing (4.12) + ϵ^2(4.13) + ϵ(4.14) gives

$$\begin{aligned} 3p_n(Y) = \sum_{\alpha \vdash n} (-1)^{n-l(\alpha)} w(B_{\alpha,(n)}) e_\alpha(X + Y + Z) \\ + \sum_{\beta \vdash n} (-1)^{n-l(\beta)} \epsilon^2 w(B_{\beta,(n)}) e_\beta(X + \epsilon Y + \epsilon^2 Z) \\ + \sum_{\gamma \vdash n} (-1)^{n-l(\gamma)} \epsilon w(B_{\gamma,(n)}) e_\gamma(X + \epsilon^2 Y + \epsilon Z). \quad (4.16) \end{aligned}$$

Summing (4.12) + ϵ(4.13) + ϵ^2(4.14) gives

$$3p_n(Z) = \sum_{\alpha\vdash n}(-1)^{n-l(\alpha)}w(B_{\alpha,(n)})e_\alpha(X+Y+Z)$$
$$+\sum_{\beta\vdash n}(-1)^{n-l(\beta)}\epsilon w(B_{\beta,(n)})e_\beta(X+\epsilon Y+\epsilon^2 Z)$$
$$+\sum_{\gamma\vdash n}(-1)^{n-l(\gamma)}\epsilon^2 w(B_{\gamma,(n)})e_\gamma(X+\epsilon^2 Y+\epsilon Z). \quad (4.17)$$

We again use weights on elements of $\mathcal{F}^{\alpha,\beta,\gamma}_{\lambda*\mu*\nu}$ to interpret products of (4.15), (4.16), and (4.17). Here, if $(\lambda,\mu,\nu)\vdash n$, we have

$$3^{l(\lambda)+l(\mu)+l(\nu)}p_\lambda\tilde{p}_\mu\hat{p}_\nu\Big|_{e_\alpha\tilde{e}_\beta\hat{e}_\gamma} = (-1)^n\sum_{f\in\mathcal{F}^{\alpha,\beta,\gamma}_{\lambda*\mu*\nu}}W_2(f), \qquad (4.18)$$

where $W_2(f)$ is the product over all bricks b in f

$$W_2(f) = \prod_{b\in f} w_2(b),$$

and where $w_2(b)$ is defined by the following cases.

- If b is an α-brick,

$$w_2(b) = \begin{cases} -|b|, & b \text{ is at the end of a row,} \\ -1, & \text{otherwise.}\end{cases}$$

- If b is a β-brick,

$$w_2(b) = \begin{cases} -|b|, & b \text{ is at the end of a row in } \lambda, \\ -\epsilon^2|b|, & b \text{ is at the end of a row in } \mu, \\ -\epsilon|b|, & b \text{ is at the end of a row in } \nu, \\ -1, & \text{otherwise.}\end{cases}$$

- If b is a γ-brick,

$$w_2(b) = \begin{cases} -|b|, & b \text{ is at the end of a row in } \lambda, \\ -\epsilon|b|, & b \text{ is at the end of a row in } \mu, \\ -\epsilon^2|b|, & b \text{ is at the end of a row in } \nu, \\ -1, & \text{otherwise.} \end{cases}$$

We write this weight in terms of the standard weight to rewrite (4.18) as

$$3^{l(\lambda)+l(\mu)+l(\nu)} p_\lambda \tilde{p}_\mu \hat{p}_\nu \Big|_{e_\alpha \tilde{e}_\beta \hat{e}_\gamma} = \\ (-1)^n \sum_{f \in \mathcal{F}^{\alpha,\beta,\gamma}_{\lambda*\mu*\nu}} (-1)^{l(\alpha)+l(\beta)+l(\gamma)} \epsilon^{2l^\beta(\mu)+l^\beta(\nu)+l^\gamma(\mu)+2l^\gamma(\nu)} w(f). \quad (4.19)$$

Thus we have the transition matrix

$$M(e\tilde{e}\hat{e}, p\tilde{p}\hat{p})_{(\alpha,\beta,\gamma)(\lambda,\mu,\nu)} = \sum_{f \in \mathcal{F}^{\alpha,\beta,\gamma}_{\lambda*\mu*\nu}} \frac{(-1)^{n-l(\alpha)-l(\beta)-l(\gamma)} \epsilon^{2l^\beta(\mu)+l^\beta(\nu)+l^\gamma(\mu)+2l^\gamma(\nu)}}{3^{l(\lambda)+l(\mu)+l(\nu)}} w(f).$$

4.4 $M(p\tilde{p}\hat{p}, e\tilde{e}\hat{e})$

The transition matrix $M(p\tilde{p}\hat{p}, e\tilde{e}\hat{e})$ is very similar to $M(p\tilde{p}\hat{p}, h\tilde{h}\hat{h})$. The only difference is in the powers of -1. We start with the expression

$$e_\lambda = \sum_{\phi \vdash |\lambda|} (-1)^{|\lambda|-l(\phi)} \frac{OB_{\phi,\lambda}}{z_\phi} p_\phi,$$

which can be found in [7]. The argument is exactly the same as before, with the slight change of sign. This gives the following expressions.

$$e_\lambda(X+Y+Z) \Big|_{p_\alpha \tilde{p}_\beta \hat{p}_\gamma} = \frac{(-1)^{|\lambda|-l(\alpha)-l(\beta)-l(\gamma)}}{z_\alpha z_\beta z_\gamma} OB_{\alpha+\beta+\gamma,\lambda}, \quad (4.20)$$

$$e_\mu(X+\epsilon Y+\epsilon^2 Z) \Big|_{p_\phi \tilde{p}_\psi \hat{p}_\pi} = \frac{(-1)^{|\mu|-l(\phi)-l(\psi)-l(\pi)} \epsilon^{(l(\psi)+2l(\pi)}}{z_\phi z_\psi z_\pi} OB_{\phi+\psi+\pi,\mu}, \quad (4.21)$$

$$e_\nu(X+\epsilon^2 Y+\epsilon Z)\Bigg|_{p_\delta \tilde{p}_\theta \hat{p}_\omega} = \frac{(-1)^{|\nu|-l(\delta)-l(\theta)-l(\omega)}\epsilon^{(2l(\theta)+l(\omega))}}{z_\delta z_\theta z_\omega} OB_{\delta+\theta+\omega,\nu}. \quad (4.22)$$

If we combine the expressions (4.20), (4.21), and (4.22), and simplify them as in the previous case, we get the following transition matrix.

$$M(p\tilde{p}\hat{p}, e\tilde{e}\hat{e})_{(\eta,\tau,\rho)(\lambda,\mu,\nu)} = \sum_{\substack{(\alpha,\beta,\gamma)\vdash|\lambda| \\ (\phi,\psi,\pi)\vdash|\mu| \\ (\delta,\theta,\omega)\vdash|\nu| \\ \alpha+\phi+\delta=\eta \\ \beta+\psi+\theta=\tau \\ \gamma+\pi+\omega=\rho}} \frac{(-1)^{n-l(\alpha)-l(\beta)-l(\gamma)-l(\phi)-l(\psi)-l(\pi)-l(\delta)-l(\theta)-l(\omega)}\epsilon^{l(\psi)+2l(\pi)+2l(\theta)+l(\omega)}}{z_\alpha z_\beta z_\gamma z_\phi z_\psi z_\pi z_\delta z_\theta z_\omega}$$
$$\times OB_{\alpha+\beta+\gamma,\lambda} OB_{\phi+\psi+\pi,\mu} OB_{\delta+\theta+\omega,\nu}.$$

4.5 $M(s\tilde{s}\hat{s}, p\tilde{p}\hat{p})$ and $M(p\tilde{p}\hat{p}, s\tilde{s}\hat{s})$

We use the properties of dual bases and scalar products described in our study of the representation theory of $C_3 \S S_n$ to determine $M(s\tilde{s}\hat{s}, p\tilde{p}\hat{p})$. Begin by writing the element of the transition matrix as the scalar product below.

$$M(s\tilde{s}\hat{s}, p\tilde{p}\hat{p})_{(\alpha,\beta,\gamma)(\lambda,\mu,\nu)} = \langle p_\lambda \tilde{p}_\mu \hat{p}_\nu, s_\alpha \tilde{s}_\beta \hat{s}_\gamma \rangle_* = \left\langle 1_{\lambda,\mu,\nu} z_\lambda z_\mu z_\nu, \chi^{(\alpha,\beta,\gamma)} \right\rangle_{C_3 \S S_n}.$$

Recall that $\chi^{(\alpha,\beta,\gamma)}$ is the irreducible character of $C_3 \S S_n$ indexed by (α,β,γ). This scalar product is then equal to

$$\begin{aligned} M(s\tilde{s}\hat{s}, p\tilde{p}\hat{p})_{(\alpha,\beta,\gamma)(\lambda,\mu,\nu)} &= \frac{1}{3^n n!} \sum_{\sigma \in C_3 \S S_n} 1_{\lambda,\mu,\nu}(\sigma) z_\lambda z_\mu z_\nu \overline{\chi^{(\alpha,\beta,\gamma)}(\sigma)} \\ &= \frac{1}{3^n n!} |C_{\lambda,\mu,\nu}| z_\lambda z_\mu z_\nu \overline{\chi^{(\alpha,\beta,\gamma)}_{(\lambda,\mu,\nu)}(\sigma)} \\ &= \frac{1}{3^n n!} \frac{3^n n!}{3^{l(\lambda)+l(\mu)+l(\nu)} z_\lambda z_\mu z_\nu} z_\lambda z_\mu z_\nu \overline{\chi^{(\alpha,\beta,\gamma)}_{(\lambda,\mu,\nu)}(\sigma)} \\ &= \frac{\overline{\chi^{(\alpha,\beta,\gamma)}_{(\lambda,\mu,\nu)}(\sigma)}}{3^{l(\lambda)+l(\mu)+l(\nu)}}. \end{aligned}$$

We use the fact that $s_\lambda \tilde{s}_\mu \hat{s}_\nu$ and $p_\lambda \tilde{p}_\mu \hat{p}_\nu$ are self-dual bases to determine $M(p\tilde{p}\hat{p}, s\tilde{s}\hat{s})$. This gives

$$\begin{aligned} M(p\tilde{p}\hat{p}, s\tilde{s}\hat{s})_{(\alpha,\beta,\gamma)(\lambda,\mu,\nu)} &= M(s\tilde{s}\hat{s}, p\tilde{p}\hat{p})^T_{(\alpha,\beta,\gamma)(\lambda,\mu,\nu)} \cdot \frac{3^{l(\alpha)+l(\beta)+l(\gamma)}}{z_\alpha z_\beta z_\gamma} \\ &= \frac{\overline{\chi^{(\lambda,\mu,\nu)}_{(\alpha,\beta,\gamma)}}}{3^{l(\alpha)+l(\beta)+l(\gamma)}} \cdot \frac{3^{l(\alpha)+l(\beta)+l(\gamma)}}{z_\alpha z_\beta z_\gamma} = \frac{\overline{\chi^{(\lambda,\mu,\nu)}_{(\alpha,\beta,\gamma)}}}{z_\alpha z_\beta z_\gamma} \end{aligned}$$

We now give an analog of the Murnagham-Nakayama rule to interpret the character $\overline{\chi^{(\alpha,\beta,\gamma)}}$ combinatorially. Above, we showed that

$$3^{l(\lambda)+l(\mu)+l(\nu)} p_\lambda \tilde{p}_\mu \hat{p}_\nu \Big|_{s_\alpha \tilde{s}_\beta \hat{s}_\gamma} = \overline{\chi^{(\alpha,\beta,\gamma)}_{(\lambda,\mu,\nu)}}.$$

We now interpret this coefficient in terms of rim hook tabloids.

Start with the following relation (see [6]).

$$p_n s_\alpha = \sum_{\alpha \subseteq \rho} (-1)^{r(\rho/\alpha)-1} s_\rho,$$

where the sum is over all ρ such that ρ/α is a rim hook of length n, and $r(\rho/\alpha)$ is the number of rows occupied by the rim hook. We then have the following identities.

$$p_n(X+Y+Z) s_\alpha \tilde{s}_\beta \hat{s}_\gamma = \sum_{\alpha \subseteq \rho} (-1)^{r(\rho/\alpha)-1} s_\rho \tilde{s}_\beta \hat{s}_\gamma, \tag{4.23}$$

$$p_n(X+\epsilon Y+\epsilon^2 Z) s_\alpha \tilde{s}_\beta \hat{s}_\gamma = \sum_{\beta \subseteq \rho} (-1)^{r(\rho/\beta)-1} s_\alpha \tilde{s}_\rho \hat{s}_\gamma, \tag{4.24}$$

$$p_n(X+\epsilon^2 Y+\epsilon Z) s_\alpha \tilde{s}_\beta \hat{s}_\gamma = \sum_{\gamma \subseteq \rho} (-1)^{r(\rho/\alpha)-1} s_\alpha \tilde{s}_\beta \hat{s}_\rho. \tag{4.25}$$

Then, summing (4.23)+(4.24)+(4.25), we obtain

$$\begin{aligned} 3p_n(X) s_\alpha \tilde{s}_\beta \hat{s}_\gamma = &\sum_{\alpha \subseteq \rho} (-1)^{r(\rho/\alpha)-1} s_\rho \tilde{s}_\beta \hat{s}_\gamma \\ &+ \sum_{\beta \subseteq \rho} (-1)^{r(\rho/\beta)-1} s_\alpha \tilde{s}_\rho \hat{s}_\gamma + \sum_{\gamma \subseteq \rho} (-1)^{r(\rho/\alpha)-1} s_\alpha \tilde{s}_\beta \hat{s}_\rho. \end{aligned} \tag{4.26}$$

Summing (4.23)+ϵ^2(4.24)+ϵ(4.25), we obtain

$$3p_n(Y)s_\alpha\tilde{s}_\beta\hat{s}_\gamma = \sum_{\alpha\subseteq\rho}(-1)^{r(\rho/\alpha)-1}s_\rho\tilde{s}_\beta\hat{s}_\gamma + \sum_{\beta\subseteq\rho}\epsilon^2(-1)^{r(\rho/\beta)-1}s_\alpha\tilde{s}_\rho\hat{s}_\gamma + \sum_{\gamma\subseteq\rho}\epsilon(-1)^{r(\rho/\alpha)-1}s_\alpha\tilde{s}_\beta\hat{s}_\rho. \quad (4.27)$$

Summing (4.23)+ϵ(4.24)+ϵ^2(4.25), we obtain

$$3p_n(Z)s_\alpha\tilde{s}_\beta\hat{s}_\gamma = \sum_{\alpha\subseteq\rho}(-1)^{r(\rho/\alpha)-1}s_\rho\tilde{s}_\beta\hat{s}_\gamma + \sum_{\beta\subseteq\rho}\epsilon(-1)^{r(\rho/\beta)-1}s_\alpha\tilde{s}_\rho\hat{s}_\gamma + \sum_{\gamma\subseteq\rho}\epsilon^2(-1)^{r(\rho/\alpha)-1}s_\alpha\tilde{s}_\beta\hat{s}_\rho. \quad (4.28)$$

The expressions (4.26), (4.27), and (4.28) give rules to express the coefficient of $s_\alpha\tilde{s}_\beta\hat{s}_\gamma$ as a sum of weights of a row brick tabloid of shape $\alpha*\beta*\gamma$ with hooks of lengths $\lambda_1,\ldots,\lambda_{l(\lambda)},\mu_1,\ldots,\mu_{l(\mu)},\nu_1,\ldots,\nu_{l(\nu)}$. We have $\tilde{w}(T)=\prod_{h\in T}\tilde{w}(h)$ where $\tilde{w}(h)$ is defined as follows, depending on which part of the shape $\alpha*\beta*\gamma$ the hook appears in. Note that $r(h)$ denotes the number of rows that a hook h occupies.

- If h lies in α, then $\tilde{w}(h)=(-1)^{r(h)-1}$.
- If h lies in β, then

$$\tilde{w}(h)=\begin{cases}(-1)^{r(h)-1}, & \text{h corresponds to a } \lambda_i,\\ \epsilon^2(-1)^{r(h)-1}, & \text{h corresponds to a } \mu_i,\\ \epsilon(-1)^{r(h)-1}, & \text{h corresponds to a } \nu_i.\end{cases}$$

- If h lies in γ, then

$$\tilde{w}(h)=\begin{cases}(-1)^{r(h)-1}, & \text{h corresponds to a } \lambda_i,\\ \epsilon(-1)^{r(h)-1}, & \text{h corresponds to a } \mu_i,\\ \epsilon^2(-1)^{r(h)-1}, & \text{h corresponds to a } \nu_i.\end{cases}$$

In all cases the weight has a factor of $(-1)^{r(h)-1}$, the usual sign for a rim hook. Thus we can write $\tilde{w}(T)$ as

$$\tilde{w}(T) = sgn(T)\epsilon^{2h^\mu(\beta)+h^\mu(\gamma)+h^\nu(\beta)+2h^\nu(\gamma)},$$

where $h^\mu(\beta)$ denotes the number of hooks appearing in β that correspond to a μ_i, and so on. This then gives the following theorem.

Theorem 4.1. *Let $(\alpha,\beta,\gamma) \vdash n$ and $(\lambda,\mu,\nu) \vdash n$. Then*

$$\overline{\chi^{(\alpha,\beta,\gamma)}_{(\lambda,\mu,\nu)}} = \sum_{T \in RH^{\alpha*\beta*\gamma}_{(\lambda,\mu,\nu)}} \epsilon^{2h^\mu(\beta)+h^\mu(\gamma)+h^\nu(\beta)+2h^\nu(\gamma)} sgn(T),$$

where $RH^{\alpha*\beta*\gamma}_{(\lambda,\mu,\nu)}$ is the set of rim hook tabloids of shape $\alpha * \beta * \gamma$ filled with hooks of lengths $\lambda_1, \ldots, \lambda_{l(\lambda)}, \mu_1, \ldots, \mu_{l(\mu)}, \nu_1, \ldots, \nu_{l(\nu)}$.

4.6 $M(p\tilde{p}\hat{p}, m\tilde{m}\hat{m})$ and $M(m\tilde{m}\hat{m}, p\tilde{p}\hat{p})$

We use the fact that $m_\lambda \tilde{m}_\mu \hat{m}_\nu$ and $h_\lambda \tilde{h}_\mu \hat{h}_\nu$ are dual bases. To find $M(p\tilde{p}\hat{p}, m\tilde{m}\hat{m})$, begin with the relationship

$$\langle p_\lambda \tilde{p}_\mu \hat{p}_\nu \rangle = \left\langle h_\alpha \tilde{h}_\beta \hat{h}_\gamma \right\rangle M(h\tilde{h}\hat{h}, p\tilde{p}\hat{p}).$$

Taking duals gives

$$\langle m_\lambda \tilde{m}_\mu \hat{m}_\nu \rangle = \left\langle \frac{3^{l(\alpha)+l(\beta)+l(\gamma)}}{z_\alpha z_\beta z_\gamma} p_\alpha \tilde{p}_\beta \hat{p}_\gamma \right\rangle M(h\tilde{h}\hat{h}, p\tilde{p}\hat{p})^T.$$

Thus we have

$$\begin{aligned} M(p\tilde{p}\hat{p}, m\tilde{m}\hat{m})_{(\alpha,\beta,\gamma)(\lambda,\mu,\nu)} &= \frac{3^{l(\alpha)+l(\beta)+l(\gamma)}}{z_\alpha z_\beta z_\gamma} M(h\tilde{h}\hat{h}, p\tilde{p}\hat{p})_{(\lambda,\mu,\nu)(\alpha,\beta,\gamma)} \\ &= \sum_{f \in \mathcal{F}^{\lambda,\mu,\nu}_{\alpha*\beta*\gamma}} \frac{(-1)^{l(\alpha)+l(\beta)+l(\gamma)-l(\lambda)-l(\mu)-l(\nu)} \epsilon^{2l^\mu(\beta)+l^\nu(\beta)+l^\mu(\gamma)+2l^\nu(\gamma)}}{z_\alpha z_\beta z_\gamma} w(f). \quad (4.29) \end{aligned}$$

Note that here our objects are diagrams of shape $\alpha * \beta * \gamma$ filled with bricks of type α, μ, and ν such that each row contains all λ-, μ-, or γ-bricks.

Similarly, to find $M(m\tilde{m}\hat{m}, p\tilde{p}\hat{p})$, begin with

$$\left\langle h_\lambda \tilde{h}_\mu \hat{h}_\nu \right\rangle = \langle p_\alpha \tilde{p}_\beta \hat{p}_\gamma \rangle \, M(p\tilde{p}\hat{p}, h\tilde{h}\hat{h}).$$

Taking duals gives

$$\left\langle \frac{3^{l(\lambda)+l(\mu)+l(\nu)}}{z_\lambda z_\mu z_\nu} p_\lambda \tilde{p}_\mu \hat{p}_\nu \right\rangle = \langle m_\alpha \tilde{m}_\beta \hat{m}_\gamma \rangle \, M(p\tilde{p}\hat{p}, h\tilde{h}\hat{h})^T.$$

Thus we have

$$\begin{aligned} M(m\tilde{m}\hat{m}, p\tilde{p}\hat{p})_{(\eta,\tau,\rho)(\lambda,\mu,\nu)} &= \frac{z_\lambda z_\mu z_\nu}{3^{l(\lambda)+l(\mu)+l(\nu)}} M(p\tilde{p}\hat{p}, h\tilde{h}\hat{h})_{(\lambda,\mu,\nu)(\eta,\tau,\rho)} \\ = \sum_{\substack{(\alpha,\beta,\gamma)\vdash|\eta| \\ (\phi,\psi,\pi)\vdash|\tau| \\ (\delta,\theta,\omega)\vdash|\rho| \\ \alpha+\phi+\delta=\lambda \\ \beta+\psi+\theta=\mu \\ \gamma+\pi+\omega=\nu}} & \frac{\epsilon^{l(\psi)+2l(\pi)+2l(\theta)+l(\omega)} z_\lambda z_\mu z_\nu}{3^{l(\lambda)+l(\mu)+l(\nu)} z_\alpha z_\beta z_\gamma z_\phi z_\psi z_\pi z_\delta z_\theta z_\omega} OB_{\alpha+\beta+\gamma,\eta} OB_{\phi+\psi+\pi,\tau} OB_{\delta+\theta+\omega,\rho}. \end{aligned}$$

4.7 $M(p\tilde{p}\hat{p}, f\tilde{f}\hat{f})$ and $M(f\tilde{f}\hat{f}, p\tilde{p}\hat{p})$

As in the previous section, we use dual bases. Here, we use that $f_\lambda \tilde{f}_\mu \hat{f}_\nu$ and $e_\lambda \tilde{e}_\mu \hat{e}_\nu$ are dual. To find $M(p\tilde{p}\hat{p}, f\tilde{f}\hat{f})$, we begin with the relationship

$$\langle p_\lambda \tilde{p}_\mu \hat{p}_\nu \rangle = \langle e_\alpha \tilde{e}_\beta \hat{e}_\gamma \rangle \, M(e\tilde{e}\hat{e}, p\tilde{p}\hat{p}).$$

Taking duals, we have

$$\left\langle f_\lambda \tilde{f}_\mu \hat{f}_\nu \right\rangle = \left\langle \frac{3^{l(\alpha)+l(\beta)+l(\gamma)}}{z_\alpha z_\beta z_\gamma} p_\alpha \tilde{p}_\beta \hat{p}_\gamma \right\rangle M(e\tilde{e}\hat{e}, p\tilde{p}\hat{p})^T.$$

Thus we have

$$\begin{aligned} M(p\tilde{p}\hat{p}, f\tilde{f}\hat{f})_{(\alpha,\beta,\gamma)(\lambda,\mu,\nu)} &= \frac{3^{l(\lambda)+l(\beta)+l(\gamma)}}{z_\alpha z_\beta z_\gamma} M(e\tilde{e}\hat{e}, p\tilde{p}\hat{p})_{(\lambda,\mu,\nu)(\alpha,\beta,\gamma)} \\ &= \sum_{f\in\mathcal{F}^{\lambda,\mu,\nu}_{\alpha*\beta*\gamma}} \frac{(-1)^{n-l(\lambda)-l(\mu)-l(\nu)} \epsilon^{2l^\mu(\beta)+l^\nu(\beta)+l^\mu(\gamma)+2l^\nu(\gamma)}}{z_\alpha z_\beta z_\gamma} w(f). \end{aligned}$$

To find $M(f\tilde{f}\hat{f}, p\tilde{p}\hat{p})$, we begin with

$$\langle e_\lambda \tilde{e}_\mu \hat{e}_\nu \rangle = \langle p_\alpha \tilde{p}_\beta \hat{p}_\gamma \rangle \, M(p\tilde{p}\hat{p}, e\tilde{e}\hat{e}).$$

Taking duals gives

$$\left\langle \frac{3^{l(\lambda)+l(\mu)+l(\nu)}}{z_\lambda z_\mu z_\nu} p_\lambda \tilde{p}_\mu \hat{p}_\nu \right\rangle M(p\tilde{p}\hat{p}, e\tilde{e}\hat{e})^T.$$

Thus,

$$\begin{aligned} M(f\tilde{f}\hat{f}, p\tilde{p}\hat{p})_{(\eta,\tau,\rho)(\lambda,\mu,\nu)} &= \frac{z_\lambda z_\mu z_\nu}{3^{l(\lambda)+l(\mu)+l(\nu)}} M(p\tilde{p}\hat{p}, e\tilde{e}\hat{e})_{(\lambda,\mu,\nu)(\eta,\tau,\rho)} \\ = \sum_{\substack{(\alpha,\beta,\gamma)\vdash|\eta| \\ (\phi,\psi,\pi)\vdash|\tau| \\ (\delta,\theta,\omega)\vdash|\rho| \\ \alpha+\phi+\delta=\lambda \\ \beta+\psi+\theta=\mu \\ \gamma+\pi+\omega=\nu}} & \frac{(-1)^{n-l(\alpha)-l(\beta)-l(\gamma)-l(\phi)-l(\psi)-l(\pi)-l(\delta)-l(\theta)-l(\omega)} \epsilon^{l(\psi)+2l(\pi)+2l(\theta)+l(\omega)} z_\lambda z_\mu z_\nu}{3^{l(\lambda)+l(\mu)+l(\nu)} z_\alpha z_\beta z_\gamma z_\phi z_\psi z_\pi z_\delta z_\theta z_\omega} \\ & \times OB_{\alpha+\beta+\gamma,\eta} OB_{\phi+\psi+\pi,\tau} OB_{\delta+\theta+\omega,\rho}. \end{aligned}$$

Chapter 5

The Permutation Enumeration of $C_3 \S S_n$

In this chapter, we extend the ideas of Beck and Remmel's proofs regarding the permuation enumeration of S_n and B_n to $C_3 \S S_n$. We use the combinatorics and representation theory to define an appropriate analog, ξ_W (with W for wreath product), of ξ, and to prove similar results for the image of this analog applied to various bases of a certain space of symmetric functions. We conclude the chapter by indicating how the proofs can be extended to arbitrary wreath products $C_k \S S_n$.

It is important to note that in the case of $C_3 \S S_n$, we have much more choice than in the S_n and B_n cases. In B_n, there is a natural ordering on the elements that make up B_n elements which is natural when considering it as a Coxeter group. Since $C_3 \S S_n$ is not a Coxeter group, there is no longer a geometric interpretation with which to determine an ordering on the letters that make up elements of $C_3 \S S_n$. There are a lot of ways that we can define these orderings, which in turn lead to different definitions of statistics on these elements, and to different definitions of the analog of ξ. For example, we can have elements with certain signs be ordered in the reverse order, as are the barred elements in B_n. We can choose to give this reverse ordering to none, one, or two of the signs. We choose here to use the ordering where no sign has the reverse ordering because it gives results which are most easily generalizable. Note that in doing so, however, we sacrifice some information that we could otherwise have gained. The results obtained using a different ordering, that where one sign has the reverse ordering, including the appropriate definitions of statistics, are stated without proof in Appendix A.

5.1 Preliminaries

We begin by giving some definitions that we will need in this chapter.

Define a partial ordering Ω on the letters $1, 2, \ldots, n, \overline{1}, \overline{2}, \ldots, \overline{n}, \overline{\overline{1}}, \overline{\overline{2}}, \ldots, \overline{\overline{n}}$ by

$$1 <_\Omega 2 <_\Omega \cdots <_\Omega n,$$

$$\overline{1} <_\Omega \overline{2} <_\Omega \cdots <_\Omega \overline{n},$$

$$\overline{\overline{1}} <_\Omega \overline{\overline{2}} <_\Omega \cdots <_\Omega \overline{\overline{n}}.$$

Define a second partial ordering , which equates those letters with the same underlying letter.

$$1 \equiv \overline{1} \equiv \overline{\overline{1}} <_\Gamma 2 \equiv \overline{2} \equiv \overline{\overline{2}} <_\Gamma \cdots <_\Gamma n \equiv \overline{n} \equiv \overline{\overline{n}}.$$

Recall that for an element $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in C_3\S S_n$, the sign of a letter of the element is denoted by $\epsilon(\sigma_i)$. The sign of the element itself is just the product of the signs of the letters, $\prod_{i=1}^n \epsilon(\sigma_i)$.

Given all this, we define a number of statistics on elements of $C_3\S S_n$. Define the *number of* $C_3\S S_n$*-descents* of an element σ to be

$$\begin{aligned} des_W(\sigma) = |\{i : 1 \le i \le n-1, \sigma_i >_\Omega \sigma_{i+1}\}| = \\ |\{i : 1 \le i \le n-1, \epsilon(\sigma_i) = \epsilon(\sigma_{i+1}), \sigma_i > \sigma_{i+1}\}|. \end{aligned}$$

For example, if $\sigma = \overline{862}74\overline{\overline{31}}5$, then $des_W(\sigma) = |\{1, 2, 4, 6\}| = 4$. Given a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ of n, we define the number of λ-*descents*, $des_{W,\lambda}(\sigma)$ in the following way. Write $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ in one-line notation and break it into pieces of lengths $\lambda_1, \lambda_2, \ldots, \lambda_l$. Then count only the $C_3\S S_n$-descents that occur with both i and $i+1$ in the same piece. For example, if $\sigma = \overline{862}74\overline{\overline{31}}5$, we break σ into pieces $[\overline{8}][\overline{62}7][4\overline{\overline{31}}5]$, and $des_{W,\lambda}(\sigma) = |\{2,6\}| = 2$. We define the *number of* $C_3 \S S_n$*-inversions* of σ by

$$inv_W(\sigma) = |\{(i,j) : 1 \leq i < j \leq n, \sigma_i >_\Gamma \sigma_j\}|.$$

For example, if $\sigma = \overline{862}74\overline{\overline{31}}5$, then $inv_W(\sigma) = 7+5+1+4+2+1+0 = 20$. The *number of* $C_3 \S S_n$*-descedances* of the element σ is defined on the cycles of σ. Write σ in cycle notation as

$$\sigma = (\sigma_{1_1}, \sigma_{1_2}, \ldots, \sigma_{1_{l_1}})(\sigma_{2_1}, \sigma_{2_2}, \ldots, \sigma_{2_{l_2}}) \cdots (\sigma_{k_1}, \sigma_{k_2}, \ldots, \sigma_{k_{l_k}}).$$

Then the number of $C_3 \S S_n$-descedances of σ, denoted $de_W(\sigma)$, is given by

$$\begin{aligned} de_W(\sigma) = \sum_{i=1}^{k} (|\{j : 1 \leq j \leq l_i - 1, \epsilon(\sigma_{i_j}) = \epsilon(\sigma_{i_{j+1}}), \sigma_{i_j} > \sigma_{i_{j+1}}\}| \\ + \chi(\sigma_{i_{l_i}} > \sigma_{i_1}) \chi(\epsilon(\sigma_{i_{l_i}}) = \epsilon(\sigma_{i_1}))). \end{aligned}$$

For example, if $\sigma = (\overline{1}, 8, 6, \overline{\overline{5}}, \overline{\overline{3}})(\overline{2}, 7, \overline{4})$, then $de_W(\sigma) = 3$.

We now define an analog of ξ.

Definition 5.1. If Λ_{W_3} is the space of symmetric functions defined in (3.3), the homomorphism $\xi_{W_3} : \Lambda_W \longrightarrow \mathbf{Q}[x]$ is defined on the elementary basis by

$$\begin{aligned} \xi_W(e_n(X+Y+Z)) &= \frac{(1-x)^{n-1} + (1-x)^{n-1} + (1-x)^{n-1}}{3^n n!}, \\ \xi_W(e_n(X+\epsilon Y+\epsilon^2 Z)) &= \frac{(1-x)^{n-1} + \epsilon(\epsilon - \epsilon x)^{n-1} + \epsilon^2(\epsilon^2 - \epsilon^2 x)^{n-1}}{3^n n!}, \\ \xi_W(e_n(X+\epsilon^2 Y+\epsilon Z)) &= \frac{(1-x)^{n-1} + \epsilon^2(\epsilon^2 - \epsilon^2 x)^{n-1} + \epsilon(\epsilon - \epsilon x)^{n-1}}{3^n n!} \end{aligned}$$

for $n \in \{1, 2, \ldots\}$ and by setting $\xi_W(e_0(X+Y+Z)) = \xi_W(e_0(X+\epsilon Y+\epsilon^2 Z)) = \xi_W(e_0(X+\epsilon^2 Y+\epsilon Z)) = 1$.

Note that we do not write these expressions in the most concise manner possible. This is because this manner of writing them suggests the combinatorial proofs that will come later. We will now consider what the results are when ξ_W is applied to the homogeneous, power, and Schur bases of Λ_{W_3}.

5.2 ξ_W Applied to the $\Lambda_{W_{3,n}}$-Homogeneous Symmetric Functions

The transition matrices between $h_\lambda(X+Y+Z)h_\mu(X+\epsilon Y+\epsilon^2 Z)h_\nu(X+\epsilon^2 Y+\epsilon Z)$ and $e_\alpha(X+Y+Z)e_\beta(X+\epsilon Y+\epsilon^2 Z)e_\gamma(X+\epsilon^2 Y+\epsilon Z)$ are just triples of matrices from the S_n case. That is, we can express $h_\lambda(X+Y+Z)$ in terms of $e_\alpha(X+Y+Z)$, $h_\mu(X+\epsilon Y+\epsilon^2 Z)$ in terms of $e_\beta(X+\epsilon Y+\epsilon^2 Z)$, and $h_\nu(X+\epsilon^2 Y+\epsilon Z)$ in terms of $e_\gamma(X+\epsilon^2 Y+\epsilon Z)$. Because of this, we treat each of these cases separately.

5.2.1 ξ_W Applied to $h_n(X+Y+Z)$

If we apply ξ_W to $h_n(X+Y+Z)$, we achieve the following result.

Theorem 5.2. *Let ξ_W be the homomorphism defined in Definition 5.1. Then*

$$3^n n!\xi_W(h_n(X+Y+Z)) = \sum_{\sigma\in C_3\S S_n} x^{des_W(\sigma)},$$

where $des_W(\sigma)$ is the number of $C_3\S S_n$-descents.

Proof. We begin with the following expression, which can be found in [7].

$$h_n(X+Y+Z) = \sum_{\mu\vdash n}(-1)^{n-l(\mu)}B_{\mu,(n)}e_\mu(X+Y+Z).$$

We multiply by $3^n n!$ and apply ξ_W to get

$$\begin{aligned}
3^n n!\xi_W(h_n(X+Y+Z)) &= \sum_{\mu\vdash n} 3^n n!(-1)^{n-l(\mu)}B_{\mu,(n)}\xi_W(e_\mu(X+Y+Z)) \\
&= \sum_{\mu\vdash n} 3^n n!(-1)^{n-l(\mu)}B_{\mu,(n)}\prod_{i=1}^{l(\mu)}\frac{(1-x)^{\mu_i-1}+(1-x)^{\mu_i-1}+(1-x)^{\mu_i-1}}{3^{\mu_i}\mu_i!} \\
&= \sum_{\mu\vdash n}\sum_{T\in\mathcal{B}_{\mu,(n)}}\binom{n}{\mu_1,\ldots,\mu_{l(\mu)}}\prod_{i=1}^{l(\mu)}\left((x-1)^{\mu_i-1}+(x-1)^{\mu_i-1}+(x-1)^{\mu_i-1}\right).
\end{aligned}$$

We interpret this as a sum of signed, weighted objects $o\in\mathcal{O}_{3h_na}$. For a given partition μ, and a μ-brick tabloid of shape (n), the multinomial coefficient fills

13	8	2	$\overline{\overline{10}}$	$\overline{\overline{7}}$	5	1	$\overline{12}$	$\overline{9}$	$\overline{6}$	$\overline{11}$	$\overline{4}$	$\overline{3}$
x	-1	1	-1	1	x	1	x	-1	1	-1	x	1

Figure 5.1: An example of an object in $\mathcal{O}_{3h_na}$.

each brick with a decreasing sequence of integers such that exactly the integers $1, 2, \ldots, n$ are used. Each brick is designated as *regular*, *barred*, or *double barred.* Each cell c is given a weight according to the following rule.

$$w(c) = \begin{cases} 1, & c \text{ is at the end of a brick,} \\ -1 \text{ or } x, & \text{otherwise.} \end{cases}$$

This accounts for the $(x-1)^{\mu_i - 1}$ terms. Define the weight of an object o by $\prod_{c \in o} w(c)$. Then we can write

$$\sum_{\mu \vdash n} \sum_{T \in \mathcal{B}_{\mu,(n)}} \binom{n}{\mu_1, \ldots, \mu_{l(\mu)}} \prod_{i=1}^{l(\mu)} \left((x-1)^{\mu_i - 1} + (x-1)^{\mu_i - 1} + (x-1)^{\mu_i - 1}\right) = \sum_{o \in \mathcal{O}_{3h_na}} w(o).$$

An example of such an object is given in Figure 5.1.

We now perform a sign-changing, weight-preserving involution on these objects. Proceed from left to right through the tabloid until one of the following occurs, then perform the appropriate operation.

- If there is a cell c with weight -1, split the brick after c and change the weight of c from -1 to $+1$.

- If there is a decrease from the integer filling of the last cell of one brick to that of the first cell of the next brick, and the two bricks are of the same type, join the two bricks together and change the weight of c from $+1$ to -1.

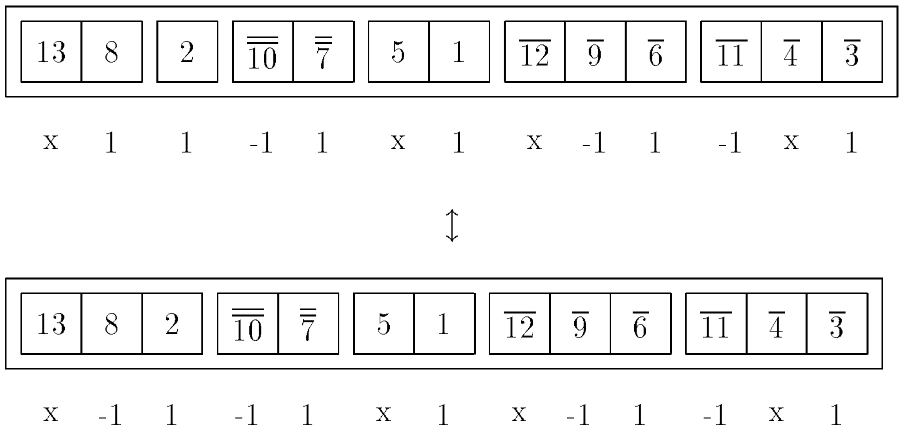

Figure 5.2: An example of the involution on $\mathcal{O}_{3h_na}$.

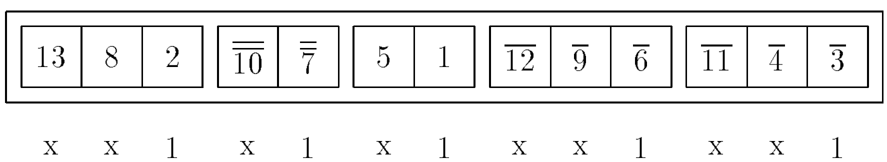

Figure 5.3: An example of a fixed point of the involution on $\mathcal{O}_{3h_na}$.

An example of the involution is given in Figure 5.2.

The fixed points of the involution are signed and weighted μ-brick tabloids of shape (n) filled with integers which have the following properties:

- The integer fillings decrease within each brick, and increase between consecutive bricks of the same type.
- A cell is weighted by 1 if it occurs at the end of a brick, or by x otherwise.

An example of a fixed point of the involution is given in Figure 5.3.

Interpret the integer fillings of the brick, read left to right, as an element of $C_3 \S S_n$, with elements in regular, barred, and double barred bricks corresponding to regular, barred, and double barred elements, respectively. Then each $C_3 \S S_n$-descent is weighted by x and all other transitions are weighted by 1, giving the result. ∎

5.2.2 ξ_W Applied to $h_n(X+\epsilon Y+\epsilon^2 Z)$ and $h_n(X+\epsilon^2 Y+\epsilon Z)$

When ξ_W is applied to the bases $h_n(X+\epsilon Y+\epsilon^2 Z)$ and $h_n(X+\epsilon^2 Y+\epsilon Z)$, the result is the following.

Theorem 5.3. *Let ξ_W be the homomorphism defined in Definition 5.1. Then*

$$3^n n!\xi_W(h_n(X+\epsilon Y+\epsilon^2 Z)) = \sum_{\sigma\in C_3\S S_n} \epsilon(\sigma)x^{des_W(\sigma)} \tag{5.1}$$

$$3^n n!\xi_W(h_n(X+\epsilon^2 Y+\epsilon Z)) = \sum_{\sigma\in C_3\S S_n} \overline{\epsilon(\sigma)}x^{des_W(\sigma)}, \tag{5.2}$$

where $des_W(\sigma)$ is the number of $C_k\S S_n$-descents of σ.

Proof. We begin by outlining the proof of (5.1). Following the same steps as in the proof of Theorem 5.2, we come to the identity

$$3^n n!\xi_W(h_n(X+\epsilon Y+\epsilon^2 Z)) =$$
$$\sum_{\mu\vdash n}\sum_{T\in\mathcal{B}_{\mu,(n)}} \binom{n}{\mu_1,\ldots,\mu_{l(\mu)}} \prod_{i=1}^{l(\mu)} \left((x-1)^{\mu_i-1}+\epsilon(\epsilon x-\epsilon)^{\mu_i-1}+\epsilon^2(\epsilon^2 x-\epsilon^2)^{\mu_i-1}\right).$$

We interpret this as a sum of signed, weighted objects $o\in\mathcal{O}_{3h_nb}$. We again have μ-brick tabloids of shape (n) filled with the integers $1,2,\ldots,n$ such that the integers decrease within each brick, and each brick is designated as regular, barred, or double barred. The difference comes with the weight on each cell. Here, the weight on a cell c depends on what kind of brick it lies in. The weights are given by the following.

- If c is in a regular brick,

$$w(c) = \begin{cases} 1, & c \text{ is at the end of a brick,} \\ -1 \text{ or } x, & \text{otherwise.} \end{cases}$$

  This accounts for the $(x-1)^{\mu_i-1}$ terms.

13	8	2	$\overline{\overline{10}}$	$\overline{\overline{7}}$	5	1	$\overline{12}$	$\overline{9}$	$\overline{6}$	$\overline{11}$	$\overline{4}$	$\overline{3}$
x	-1	1	$-\epsilon^2$	ϵ^2	x	1	ϵx	$-\epsilon$	ϵ	$-\epsilon$	ϵx	ϵ

Figure 5.4: An example of an object in $\mathcal{O}_{3h_nb}$.

- If c is in a barred brick,
$$w(c) = \begin{cases} \epsilon, & c \text{ is at the end of a brick,} \\ -\epsilon \text{ or } \epsilon x, & \text{otherwise.} \end{cases}$$
This accounts for the $\epsilon(\epsilon x - \epsilon)^{\mu_i - 1}$ terms.

- If c is in a double barred brick,
$$w(c) = \begin{cases} \epsilon^2, & c \text{ is at the end of a brick,} \\ -\epsilon^2 \text{ or } \epsilon^2 x, & \text{otherwise.} \end{cases}$$
This accounts for the $\epsilon^2(\epsilon^2 x - \epsilon^2)^{\mu_i - 1}$ terms.

Define the weight of an object o by $w(o) = \prod_{c \in o} w(c)$. Then we can write

$$\sum_{\mu \vdash n} \sum_{T \in \mathcal{B}_{\mu,(n)}} \binom{n}{\mu_1, \ldots, \mu_{l(\mu)}} \prod_{i=1}^{l(\mu)} \left((x-1)^{\mu_i - 1} + \epsilon(\epsilon x - \epsilon)^{\mu_i - 1} + \epsilon^2(\epsilon^2 x - \epsilon^2)^{\mu_i - 1}\right)$$
$$= \sum_{o \in \mathcal{O}_{3h_nb}} w(o).$$

An example of these objects is given in Figure 5.4.

We perform a similar involution to that in the proof of Theorem 5.2. Traverse the tabloid from left to right. At the first occurrence of one of the following, perform the corresponding operation.

- If a cell c has weight -1, $-\epsilon$, or $-\epsilon^2$, split the brick after c and change the weight of c from -1, $-\epsilon$, or $-\epsilon^2$ to $+1$, $+\epsilon$, or $+\epsilon^2$.

- If there is a decrease between the integer filling of the last cell of a brick and that of the first cell of the next brick and both bricks are of the same type, join the two bricks and change the weight of c from $+1$, $+\epsilon$, or $+\epsilon^2$ to -1, $-\epsilon$, of $-\epsilon^2$.

The fixed points then are μ-brick tabloids of shape (n), filled with the integers $1, 2, \ldots, n$ such that the integer fillings decrease within bricks and increase between consecutive bricks of the same type. Each brick is designated as regular, barred, or double barred. The weights on the cells are as follows. In a regular brick, the cell at the end has weight 1 while the others have weight x. In a barred brick, the cell at the end has weight ϵ while the others have weight ϵx. In a double barred brick, the cell at the end has weight ϵ^2 while the other cells have weight $\epsilon^2 x$. If we again consider the filling of an object as an element of $C_3 \S S_n$, the x weights appear precisely in the cells with $C_3 \S S_n$-descents. Each cell in a barred brick contributes ϵ and each cell in a double barred brick contributes ϵ^2. This contribution of signs corresponds to the sign of the element, $\epsilon(\sigma)$. Thus we have the result.

The proof of (5.2) is nearly the same. The only difference is in the weights placed on each cell in our interpretation of the sum. The weight on a cell is given by the following.

- If c is in a regular brick,

$$w(c) = \begin{cases} 1, & c \text{ is at the end of a brick,} \\ -1 \text{ or } x, & \text{otherwise.} \end{cases}$$

- If c is in a barred brick,

$$w(c) = \begin{cases} \epsilon,^2 & c \text{ is at the end of a brick,} \\ -\epsilon^2 \text{ or } \epsilon^2 x, & \text{otherwise.} \end{cases}$$

- If c is in a double barred brick,

$$w(c) = \begin{cases} \epsilon, & c \text{ is at the end of a brick,} \\ -\epsilon \text{ or } \epsilon x, & \text{otherwise.} \end{cases}$$

In this case, the fixed points are the same, except that barred cells contribute ϵ^2 and double barred cells contribute ϵ. This gives the complex conjugate of the sign of the underlying element of $C_3 \S S_n$. ■

5.2.3 ξ_W Applied to $h_\lambda(X+Y+Z)$, $h_\mu(X+\epsilon Y+\epsilon^2 Z)$, and $h_\nu(X+\epsilon^2 Y+\epsilon Z)$

If we apply ξ_W to the h_λ's rather than just the h_n's, we obtain generating functions for the statistic $des_{W,\lambda}(\sigma)$ on elements of $C_3 \S S_n$. The specific results follow.

Theorem 5.4. *Let ξ_W be the homomorphism defined in Definition 5.1. If λ is a partition of n, then*

$$3^n n! \xi_W(h_\lambda(X+Y+Z)) = \sum_{\sigma \in C_3 \S S_n} x^{des_{W,\lambda}(\sigma)}, \tag{5.3}$$

$$3^n n! \xi_W(h_\lambda(X+\epsilon Y+\epsilon^2 Z)) = \sum_{\sigma \in C_3 \S S_n} \epsilon(\sigma) x^{des_{W,\lambda}(\sigma)}, \tag{5.4}$$

$$3^n n! \xi_W(h_\lambda(X+\epsilon^2 Y+\epsilon Z)) = \sum_{\sigma \in C_3 \S S_n} \overline{\epsilon(\sigma)} x^{des_{W,\lambda}(\sigma)}, \tag{5.5}$$

where $des_{W,\lambda}(\sigma)$ is the number of $C_3 \S S_n$ λ-descents of σ.

Proof. We begin by outlining the proof of (5.3). The proofs of (5.4) and (5.5) are very similar. The only differences are the same as the differences between the proof of Theorem 5.2 and the proofs of (5.1) and (5.2).

By the same steps as before we obtain the expression

$$3^n n! \xi_W(h_\lambda(X+Y+Z)) =$$
$$\sum_{\mu \vdash n} \sum_{T \in \mathcal{B}_{\mu,\lambda}} \binom{n}{\mu_1, \ldots, \mu_{l(\mu)}} \prod_{i=1}^{l(\mu)} \left((x-1)^{\mu_i - 1} + (x-1)^{\mu_i - 1} + (x-1)^{\mu_i - 1}\right).$$

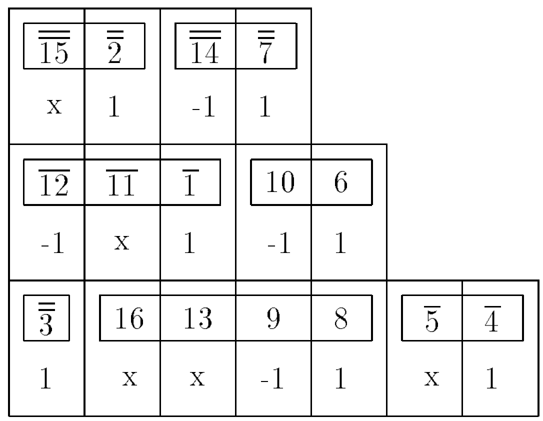

Figure 5.5: An example of an object in $\mathcal{O}_{3h_\lambda}$.

We again interpret this expression as a sum of signed, weighted objects $o \in \mathcal{O}_{3h_\lambda}$. Now we have μ-brick tabloids of shape λ, rather than one-row shapes. Each brick is again designated as regular, barred, or double barred. The multinomial coefficient fills each cell with the integers $1, 2, \ldots, n$ such that the numbers decrease within each brick. Each cell is given the same weight as in the proof of Theorem 5.2: 1 if it is at the end of a brick and either -1 or x otherwise. The weight of an object o is defined by $w(o) = \prod_{c \in o} w(c)$. Then we can write

$$\sum_{\mu \vdash n} \sum_{T \in \mathcal{B}_{\mu,\lambda}} \binom{n}{\mu_1, \ldots, \mu_{l(\mu)}} \prod_{i=1}^{l(\mu)} \left((x-1)^{\mu_i - 1} + (x-1)^{\mu_i - 1} + (x-1)^{\mu_i - 1}\right)$$
$$= \sum_{o \in \mathcal{O}_{3h_\lambda}} w(o).$$

An example of these objects is given in Figure 5.5.

We perform a sign-changing, weight-preserving involution on these objects. Traverse each row from left to right, considering the rows from top to bottom. At the first occurrence of one of the following conditions, perform the appropriate operation.

- If a cell c has weight -1, split the brick after c and change the weight of c from -1 to $+1$.

- If there is a decrease from the integer filling of the last cell c in a brick to that of the first cell of the next brick and both bricks are of the same type and lie in the same row, join the bricks together and change the weight of c from -1 to $+1$.

The fixed points of the involution are μ-brick tabloids of shape λ filled with the numbers $1, 2, \ldots, n$, such that the following properties hold.

- The integers decrease within each brick.
- The integers increase between consecutive bricks of the same type within a row.
- The last cell in each brick is weighted by 1. All other cells are weighted by x.

Consider the integer fillings, read left to right in rows and reading rows from top to bottom, to be an element of $C_3 \S S_n$. The cells that correspond to $C_3 \S S_n$-descents within each row are counted by an x, but we have no idea what happens between rows. This is the same as counting the statistic $des_{W,\lambda}(\sigma)$, which counts descents within pieces of sizes $\lambda_1, \lambda_2, \ldots, \lambda_{l(\lambda)}$, but says nothing about descents that might occur between the pieces. This proves (5.3). (5.4) and (5.5) follow by similar arguments. ∎

5.2.4 q-analogs for the Homogeneous Basis

Here we define a q-analog of ξ_W and determine its image on the homogeneous basis of Λ_{W_3}.

Definition 5.5. Define the homomorphism $\overline{\xi_W} : \Lambda_{W_3} \longrightarrow (\mathbf{Q}[q])[x]$ on the elementary basis by

$$\begin{aligned}
\overline{\xi_W}(e_n(X+Y+Z)) &= \frac{q^{\binom{n}{2}}\left((1-x)^{n-1}+(1-x)^{n-1}+(1-x)^{n-1}\right)}{3^n[n]!}, \\
\overline{\xi_W}(e_n(X+\epsilon Y+\epsilon^2 Z)) &= \frac{q^{\binom{n}{2}}\left((1-x)^{n-1}+\epsilon(\epsilon-\epsilon x)^{n-1}+\epsilon^2(\epsilon^2-\epsilon^2 x)^{n-1}\right)}{3^n[n]!}, \\
\overline{\xi_W}(e_n(X+\epsilon^2 Y+\epsilon Z)) &= \frac{q^{\binom{n}{2}}\left((1-x)^{n-1}+\epsilon^2(\epsilon^2-\epsilon^2 x)^{n-1}+\epsilon(\epsilon-\epsilon x)^{n-1}\right)}{3^n[n]!}
\end{aligned}$$

for $n \in \{1,2,\dots\}$ and by setting $\overline{\xi}_W(e_0(X+Y+Z)) = \overline{\xi}_W(e_0(X+\epsilon Y+\epsilon^2 Z)) = \overline{\xi}_W(e_0(X+\epsilon^2 Y+\epsilon Z)) = 1$.

Given this definition, if we apply $\overline{\xi_W}$ to $h_n(X+Y+Z)$, we get the following result.

Theorem 5.6. *Let $\overline{\xi}_W$ be the homomorphism defined in Definition 5.5. Then*

$$3^n[n]!\overline{\xi_W}(h_n(X+Y+Z)) = \sum_{\sigma\in C_3\S S_n} x^{des_W(\sigma)} q^{inv_W(\sigma)}, \tag{5.6}$$

$$3^n[n]!\overline{\xi_W}(h_n(X+\epsilon Y+\epsilon^2 Z)) = \sum_{\sigma\in C_3\S S_n} \epsilon(\sigma) x^{des_W(\sigma)} q^{inv_W(\sigma)}, \tag{5.7}$$

$$3^n[n]!\overline{\xi_W}(h_n(X+\epsilon^2 Y+\epsilon Z)) = \sum_{\sigma\in C_3\S S_n} \overline{\epsilon(\sigma)} x^{des_W(\sigma)} q^{inv_W(\sigma)}, \tag{5.8}$$

where $des_W(\sigma)$ is the number of $C_3\S S_n$-descents, and $inv_W(\sigma)$ is the number of $C_3\S S_n$-inversions of σ.

Proof. We will prove (5.6). The proofs of (5.7) and (5.8) are combinations of this proof with the proofs of (5.1) and (5.2).

We begin with the transition matrix from the S_n case

$$h_n = \sum_{\mu\vdash n} (-1)^{n-l(\mu)} B_{\mu,(n)} e_n.$$

Apply $\overline{\xi_W}$ and multiply by $3^n[n]!$ to get

$$\begin{aligned}
3^n[n]!\overline{\xi_W}(h_n(X+Y+Z)) &= \sum_{\mu\vdash n} 3^n[n]!(-1)^{n-l(\mu)}B_{\mu,(n)}\overline{\xi_W}(e_n(X+Y+Z)) \\
&= \sum_{\mu\vdash n} 3^n[n]!(-1)^{n-l(\mu)}B_{\mu,(n)}\prod_{i=1}^{l(\mu)}\frac{q^{\binom{\mu_i}{2}}\left((1-x)^{\mu_i-1}+(1-x)^{\mu_i-1}+(1-x)^{\mu_i-1}\right)}{3^{\mu_i}[\mu_i]!} \\
&= \sum_{\mu\vdash n}\sum_{T\in\mathcal{B}_{\mu,(n)}}\begin{bmatrix} n \\ \mu_1,\ldots,\mu_{l(\mu)}\end{bmatrix}\prod_{i=1}^{l(\mu)} q^{\binom{\mu_i}{2}}\left((1-x)^{\mu_i-1}+(1-x)^{\mu_i-1}+(1-x)^{\mu_i-1}\right).
\end{aligned}$$

We interpret this as a sum of signed, weighted objects $o \in \mathcal{O}_{3h_n q}$. The objects are similar to those in $\mathcal{O}_{3h_n}$. Again, we have μ-brick tabloids of shape (n). Each brick is designated as regular, barred, or double barred. The cells are filled with the integers $1, 2, \ldots, n$ in the following way. For a given tabloid, let $B_1, B_2, \ldots, B_l$ be the bricks that occur in order from left to right in the μ-brick tabloid. Let $b_i = |B_i|$ so $b_1, b_2, \ldots, b_l$ is a rearrangement of $\mu_1, \mu_2, \ldots, \mu_l$. We associate b_i i's with B_i and consider rearrangements in $\mathcal{R}(1^{b_1}, 2^{b_2}, \ldots, l^{b_l})$. For each rearrangement $r \in \mathcal{R}(1^{b_1}, 2^{b_2}, \ldots, l^{b_l})$, we create a permutation $\sigma(r)$ in the following way. Number, from right to left, first the 1's, then the 2's, and so on. We then take the inverse permutation $\sigma^{-1}(\sigma)$. An example of this process is given in Table 5.1. By the way we constructed $\sigma^{-1}(r)$, we have decreasing sequences of lengths $b_1, b_2, \ldots, b_l$, which then fit into the bricks $B_1, B_2, \ldots, B_l$. By Theorem 1.1,

$$\begin{bmatrix} n \\ \mu_1,\ldots,\mu_l\end{bmatrix} = \sum_{r\in\mathcal{R}(1^{b_1},2^{b_2},\ldots,l^{b_l})} q^{inv(r)}.$$

By the construction of $\sigma(r)$, we have

$$inv(\sigma^{-1}(r)) = inv(\sigma(r)) = inv(r) + \binom{b_1}{2} + \binom{b_2}{2} + \cdots + \binom{b_l}{2}.$$

We now have μ-brick tabloids of shape (n) filled with the integers $1, 2, \ldots, n$ such that the integers decrease within each brick. We give each cell an x-weight in the same way as the objects in $\mathcal{O}_{3h_n}$. If a cell is at the end of a brick, it gets a weight of 1. The other cells have weights of either -1 or x. Here, each cell is also

Table 5.1: Constructing the permutation $\sigma^{-1}(r)$.

r	1	3	2	1	3	3	1	2	1	3	3
$\sigma(r)$	4	11	6	3	10	9	2	5	1	8	7
$\sigma^{-1}(r)$	9	7	4	1	8	3	11	10	6	5	2

13	8	2	$\overline{\overline{10}}$	$\overline{\overline{7}}$	5	1	$\overline{12}$	$\overline{9}$	$\overline{6}$	$\overline{11}$	$\overline{4}$	$\overline{3}$
x	-1	1	-1	1	x	1	x	-1	1	-1	x	1
q^{12}	q^7	q^1	q^7	q^5	q^3	q^0	q^5	q^3	q^2	q^2	q^1	q^0

Figure 5.6: An example of an object in $\mathcal{O}_{3h_nq}$.

given a q-weight. If c is a cell filled with the integer i, the q-weight is q^p, where p is the number of integers that appear to the right of c in the tabloid, and which are smaller than i. As before, each brick is also designated as regular, barred, or double barred. An example of these objects appears in Figure 5.6.

We perform the same involution as we performed on the objects of $\mathcal{O}_{3h_n}$. Note that the q-weight does not change when this involution is performed. Thus the fixed points have the following characteristics.

- The integer fillings decrease within bricks, and increase between consecutive bricks of the same type.
- The x-weight of each cell is 1 if the cell is at the end of a brick and x otherwise.
- The q-weight is q^p where p is the number of cells to the right of the cell with smaller integer filling.

We again consider the filling as an element of $C_k \S S_n$. From the above characteristics we see that the x-weight counts precisely the $C_3\S S_n$-descents of the $C_3\S S_n$-element filling the tabloid. Meanwhile, the q-weight counts the number of inversions of that element.

■

If we combine the proofs of Theorem 5.4 and Theorem 5.6, we obtain the following corollary.

Corollary 5.7. *Let $\overline{\xi}_W$ be the homomorphism defined in Definition 5.5. If λ is a partition of n, then*

$$3^n[n]!\overline{\xi}_W(h_\lambda(X+Y+Z)) = \sum_{\sigma \in C_3 \S S_n} x^{des_{W,\lambda}(\sigma)} q^{inv_W(\sigma)},$$

$$3^n[n]!\overline{\xi}_W(h_\lambda(X+\epsilon Y+\epsilon^2 Z)) = \sum_{\sigma \in C_3 \S S_n} \epsilon(\sigma) x^{des_{W,\lambda}(\sigma)} q^{inv_W(\sigma)},$$

and

$$3^n[n]!\overline{\xi}_W(h_\lambda(X+\epsilon^2 Y+\epsilon Z)) = \sum_{\sigma \in C_3 \S S_n} \overline{\epsilon(\sigma)} x^{des_{W,\lambda}(\sigma)} q^{inv_W(\sigma)},$$

where $des_{W,\lambda}(\sigma)$ is the number of $C_3 \S S_n$ λ-inversions of σ and $q^{inv_W(\sigma)}$ is the number of $C_3 \S S_n$-inversions of σ.

5.3 Λ_{W_3}-Power Symmetric Functions Under ξ_W

Here we determine the image of the power basis, $p_\lambda(X)p_\mu(Y)p_\nu(Z)$, under the homomorphism ξ_W, defined in Definition 5.1, using the transition matrix developed Chapter 4 and the combinatorial ideas from the previous section. We then determine the q-analog of the image of $p_\lambda(X)p_\mu(Y)p_\nu(Z)$, using the homomorphism $\overline{\xi}_W$, defined in Definition 5.5.

5.3.1 ξ_W Applied to $p_\lambda(X)p_\mu(Y)p_\nu(Z)$

Here, we will prove the following theorem regarding the image of $p_\lambda(X)p_\mu(Y)p_\nu(Z)$ under ξ_W.

Theorem 5.8. *Let ξ_w be the homomorphism defined in Definition 5.1. If $(\lambda,\mu,\nu)\vdash n$, then*

$$\frac{3^n n!}{z_\lambda z_\mu z_\nu}\xi_W(p_\lambda(X)p_\mu(Y)p_\nu(Z)) = \sum_{\sigma\in C_{(\lambda,\mu,\nu)}} x^{de_W(\sigma)},$$

where $C_{(\lambda,\mu,\nu)}$ is the conjugacy class of $C_3\S S_n$ indexed by (λ,μ,ν), and $de_W(\sigma)$ is the number of $C_3\S S_n$-descedances of σ.

Proof. We begin with the expression (4.19) developed in the previous chapter.

$$3^{l(\lambda)+l(\mu)+l(\nu)}p_\lambda(x)p_\mu(Y)p_\nu(Z)\Bigg|_{e_\alpha\tilde{e}_\beta\hat{e}_\gamma} = \\ (-1)^n \sum_{f\in\mathcal{F}^{\alpha,\beta,\gamma}_{\lambda*\mu*\nu}} (-1)^{l(\alpha)+l(\beta)+l(\gamma)}\epsilon^{2l^\beta(\mu)+l^\beta(\nu)+l^\gamma(\mu)+2l^\gamma(\nu)}w(f).$$

Multiplying by the size of the conjugacy class indexed by (λ,μ,ν) and applying ξ_W gives

$$\begin{aligned}\frac{3^n n!}{z_\lambda z_\mu z_\nu}\xi_W(p_\lambda(X)p_\mu(Y)p_\nu(Z)) &= \sum_{(\alpha,\beta,\gamma)\vdash n}\sum_{f\in\mathcal{F}^{\alpha,\beta,\gamma}_{\lambda*\mu*\nu}} w(f)\frac{3^n n!}{3^{l(\lambda)+l(\mu)+l(\nu)}z_\lambda z_\mu z_\nu} \\ &\times(-1)^{n-l(\alpha)-l(\beta)-l(\gamma)}\epsilon^{2l^\beta(\mu)+l^\beta(\nu)+l^\gamma(\mu)+2l^\gamma(\nu)} \\ &\times\prod_{i=1}^{l(\alpha)}\left(\frac{(1-x)^{\alpha_i-1}+(1-x)^{\alpha_i-1}+(1-x)^{\alpha_i-1}}{3^{\alpha_i}\alpha_i!}\right) \\ &\times\prod_{i=1}^{l(\beta)}\left(\frac{(1-x)^{\beta_i-1}+\epsilon(\epsilon-\epsilon x)^{\beta_i-1}+\epsilon^2(\epsilon^2-\epsilon^2 x)^{\beta_i-1}}{3^{\beta_i}\beta_i!}\right) \\ &\times\prod_{i=1}^{l(\gamma)}\left(\frac{(1-x)^{\gamma_i-1}+\epsilon^2(\epsilon^2-\epsilon^2 x)^{\gamma_i-1}+\epsilon(\epsilon-\epsilon x)^{\gamma_i-1}}{3^{\gamma_i}\gamma_i!}\right)\end{aligned}$$

$$
\begin{aligned}
= \sum_{(\alpha,\beta,\gamma)\vdash n} \sum_{f\in\mathcal{F}^{\alpha,\beta,\gamma}_{\lambda*\mu*\nu}} & \frac{w(f)}{3^{l(\lambda)+l(\mu)+l(\nu)} z_\lambda z_\mu z_\nu} \\
& \times \epsilon^{2l^\beta(\mu)+l^\beta(\nu)+l^\gamma(\mu)+2l^\gamma(\nu)} \binom{n}{\alpha_1,\dots,\alpha_{l(\alpha)},\beta_1,\dots,\beta_{l(\beta)},\gamma_1,\dots,\gamma_{l(\gamma)}} \\
& \times \prod_{i=1}^{l(\alpha)} \left((x-1)^{\alpha_i-1} + (x-1)^{\alpha_i-1} + (x-1)^{\alpha_i-1}\right) \\
& \times \prod_{i=1}^{l(\beta)} \left((x-1)^{\beta_i-1} + \epsilon(\epsilon x-\epsilon)^{\beta_i-1} + \epsilon^2(\epsilon^2 x-\epsilon^2)^{\beta_i-1}\right) \\
& \times \prod_{i=1}^{l(\gamma)} \left((x-1)^{\gamma_i-1} + \epsilon^2(\epsilon^2 x-\epsilon^2)^{\gamma_i-1} + \epsilon(\epsilon x-\epsilon)^{\gamma_i-1}\right).
\end{aligned}
$$

We will adopt some new notation for the binomial coefficient which will allow the cancellation of $w(f)$. For a given $f \in \mathcal{F}^{\alpha,\beta,\gamma}_{\lambda*\mu*\nu}$, let $\alpha_{\phi(1)},\dots,\alpha_{\phi(j)}$, $\beta_{\psi(1)},\dots,\beta_{\psi(k)}$, $\gamma_{\pi(1)},\dots,\gamma_{\pi(l)}$, with $j+k+l = l(\lambda)+l(\mu)+l(\nu)$ denote the lengths of the α-, β-, and γ-bricks appearing at the end of rows in f. Then we express the multinomial coefficient as

$$
\begin{aligned}
&\binom{n}{\alpha_1,\dots,\alpha_{l(\alpha)},\beta_1,\dots,\beta_{l(\beta)},\gamma_1,\dots,\gamma_{l(\gamma)}} = \\
&\qquad \frac{n(n-1)\cdots(n-l(\lambda)-l(\mu)-l(\nu)+1)}{\alpha_{\phi(1)}\cdots\alpha_{\phi(j)}\beta_{\psi(1)}\cdots\beta_{\psi(k)}\gamma_{\pi(1)}\cdots\gamma_{\pi(l)}} \binom{n-l(\lambda)-l(\mu)-l(\nu)}{\tilde{\alpha}(f)\tilde{\beta}(f)\tilde{\gamma}(f)},
\end{aligned}
$$

where $\tilde{\alpha}(f)$ denotes $\hat{\alpha}_1,\hat{\alpha}_2,\dots,\hat{\alpha}_{l(\alpha)}$ with

$$
\hat{\alpha}_j = \begin{cases} \alpha_j - 1, & \text{the } \alpha_j \text{ brick is at the end of a row in } f, \\ \alpha_j, & \text{otherwise.} \end{cases}
$$

The notation for $\tilde{\beta}(f)$ and $\tilde{\gamma}(f)$ is similar. Using this notation, we have the following.

$$\frac{3^n n!}{z_\lambda z_\mu z_\nu}\xi_W(p_\lambda(X)p_\mu(Y)p_\nu(Z)) = \sum_{(\alpha,\beta,\gamma)\vdash n}\sum_{f\in\mathcal{F}^{\alpha,\beta,\gamma}_{\lambda*\mu*\nu}} \frac{w(f)}{3^{l(\lambda)+l(\mu)+l(\nu)}z_\lambda z_\mu z_\nu}$$
$$\times\ \epsilon^{2l^\beta(\mu)+l^\beta(\nu)+l^\gamma(\mu)+2l^\gamma(\nu)}$$
$$\times \frac{n(n-1)\cdots(n-l(\lambda)-l(\mu)-l(\nu)+1)}{\alpha_{\phi(1)}\cdots\alpha_{\phi(j)}\beta_{\psi(1)}\cdots\beta_{\psi(k)}\gamma_{\pi(1)}\cdots\gamma_{\pi(l)}}\binom{n-l(\lambda)-l(\mu)-l(\nu)}{\tilde{\alpha}(f)\tilde{\beta}(f)\tilde{\gamma}(f)}$$
$$\times\prod_{i=1}^{l(\alpha)}\left((x-1)^{\alpha_i-1}+(x-1)^{\alpha_i-1}+(x-1)^{\alpha_i-1}\right)$$
$$\times\prod_{i=1}^{l(\beta)}\left((x-1)^{\beta_i-1}+\epsilon(\epsilon x-\epsilon)^{\beta_i-1}+\epsilon^2(\epsilon^2x-\epsilon^2)^{\beta_i-1}\right)$$
$$\times\prod_{i=1}^{l(\gamma)}\left((x-1)^{\gamma_i-1}+\epsilon^2(\epsilon^2x-\epsilon^2)^{\gamma_i-1}+\epsilon(\epsilon x-\epsilon)^{\gamma_i-1}\right)$$

We now may cancel $w(f)$, the product of the sizes of the bricks appearing at the end of a row in f.

$$\frac{3^n n!}{z_\lambda z_\mu z_\nu}\xi_W(p_\lambda(X)p_\mu(Y)p_\nu(Z))$$
$$= \sum_{(\alpha,\beta,\gamma)\vdash n}\sum_{f\in\mathcal{F}^{\alpha,\beta,\gamma}_{\lambda*\mu*\nu}} \frac{n(n-1)\cdots(n-l(\lambda)-l(\mu)-l(\nu)+1)}{3^{l(\lambda)+l(\mu)+l(\nu)}z_\lambda z_\mu z_\nu}$$
$$\times\ \epsilon^{2l^\beta(\mu)+l^\beta(\nu)+l^\gamma(\mu)+2l^\gamma(\nu)}\binom{n-l(\lambda)-l(\mu)-l(\nu)}{\tilde{\alpha}(f)\tilde{\beta}(f)\tilde{\gamma}(f)}$$
$$\times\prod_{i=1}^{l(\alpha)}\left((x-1)^{\alpha_i-1}+(x-1)^{\alpha_i-1}+(x-1)^{\alpha_i-1}\right)$$
$$\times\prod_{i=1}^{l(\beta)}\left((x-1)^{\beta_i-1}+\epsilon(\epsilon x-\epsilon)^{\beta_i-1}+\epsilon^2(\epsilon^2x-\epsilon^2)^{\beta_i-1}\right)$$
$$\times\prod_{i=1}^{l(\gamma)}\left((x-1)^{\gamma_i-1}+\epsilon^2(\epsilon^2x-\epsilon^2)^{\gamma_i-1}+\epsilon(\epsilon x-\epsilon)^{\gamma_i-1}\right). \quad (5.9)$$

We interpret the above expression as a sum of signed, weighted objects $o\in\mathcal{O}_{3p}$. For some $(\alpha,\beta,\gamma)\vdash n$, we have an element of $\mathcal{F}^{\alpha,\beta,\gamma}_{\lambda*\mu*\nu}$. Designate each brick as

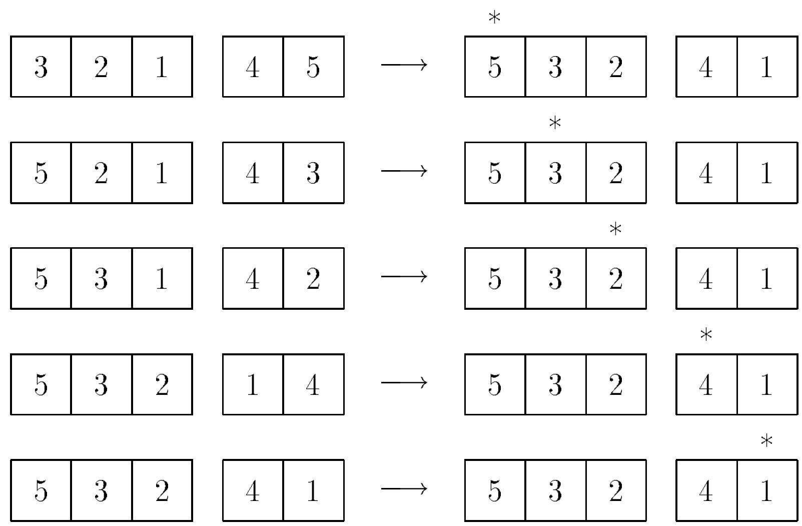

Figure 5.7: A row of length 5 has 5 possible origins.

regular, barred, or double barred. Fill the object with the integers $1, 2, \ldots, n$ in the following way. The term $n(n-1)\cdots(n-l(\lambda)-l(\mu)+l(\nu)+1)$ fills the last cell of the last brick of each row. The multinomial coefficient $\binom{n-l(\lambda)-l(\mu)-l(\nu)}{\tilde{\alpha}(f)\tilde{\beta}(f)\tilde{\gamma}(f)}$ fills all of the remaining cells with the integers not yet used in such a way that the numbers decrease within each brick, with the possible exception of the last cell in the row. To rectify this, for each row, we find the smallest integer, s_i, appearing in that row and move it to the last cell in the row. We move the number that was originally in the last cell, a_i, to the brick previously occupied by s_i and rearrange the numbers in that brick so they are decreasing. Place a star over the cell now occupied by a_i. For a row of length l, there are l ways to obtain a given filling. For a simple example of this, see Figure 5.7. We would like to ignore the marked cells and consider only the fillings where the smallest number is at the end of each row, so we must divide by the length of each row. If $\lambda = (1^{a_1}2^{a_2}\cdots n^{a_n})$, $\mu = (1^{b_1}2^{b_2}\cdots n^{b_n})$, and $\gamma = (1^{c_1}2^{c_2}\cdots n^{c_n})$, we must divide by $1^{a_1+b_1+c_1}2^{a_2+b_2+c_2}\cdots n^{a_n+b_n+c_n}$. Further, we would like to ignore the order of the rows within each of λ, μ, and ν, so we divide by $a_1!\cdots a_n!b_1!\cdots b_n!c_1\cdots c_n!$. Thus we have divided by $z_\lambda z_\mu z_\nu$.

Table 5.2: Weights of cells in the objects.

	end of:	regular cells row	regular cells brick	regular cells none	barred cells row	barred cells brick	barred cells none	double barred cells row	double barred cells brick	double barred cells none
λ	α	1	1	-1, x	1	1	-1, x	1	1	-1, x
	β	1	1	-1, x	ϵ	ϵ	$-\epsilon$, ϵx	ϵ^2	ϵ^2	$-\epsilon$, ϵ^2x
	γ	1	1	-1, x	ϵ^2	ϵ^2	$-\epsilon^2$, ϵ^2x	ϵ	ϵ	$-\epsilon$, ϵx
μ	α	1	1	-1, x	1	1	-1, x	1	1	-1, x
	β	ϵ^2	1	-1, x	1	ϵ	$-\epsilon$, ϵx	ϵ	ϵ^2	$-\epsilon$, ϵ^2x
	γ	ϵ	1	-1, x	1	ϵ^2	$-\epsilon^2$, ϵ^2x	ϵ^2	ϵ	$-\epsilon$, ϵx
ν	α	1	1	-1, x	1	1	-1, x	1	1	-1, x
	β	ϵ	1	-1, x	ϵ^2	ϵ	$-\epsilon$, ϵx	1	ϵ^2	$-\epsilon^2$, ϵ^2x
	γ	ϵ^2	1	-1, x	ϵ	ϵ^2	$-\epsilon^2$, ϵ^2x	1	ϵ	$-\epsilon$, ϵx

The weights placed on each cell depend on both whether the cell is in an α-, β-, or γ-brick, and on whether it is in a regular, barred, or double barred brick. They are defined as follows. A summary is given in Table 5.2, and an example of the objects is given in Figure 5.8.

(α) If the cell c is in an α-brick, the weight is given by

$$w(c) = \begin{cases} 1, & c \text{ is at the end of a brick,} \\ -1 \text{ or } x, & \text{otherwise.} \end{cases}$$

This holds no matter what type of brick c is in and accounts for the $(x-1)^{\alpha_i - 1}$ terms.

(β) If c lies in a β-brick, the weight is given by the following.

- If c is in a regular brick,

$$w(c) = \begin{cases} \epsilon^2, & c \text{ is at the end of the last brick in a row of } \mu, \\ \epsilon, & c \text{ is at the end of the last brick in a row of } \nu, \\ 1, & c \text{ is at the end of a brick but not a row,} \\ -1 \text{ or } x, & \text{otherwise.} \end{cases}$$

This accounts for the $\epsilon^{2l^{\beta}(\mu)}$, $\epsilon^{l^{\beta}(\nu)}$, and $(x-1)^{\beta_i - 1}$ terms.

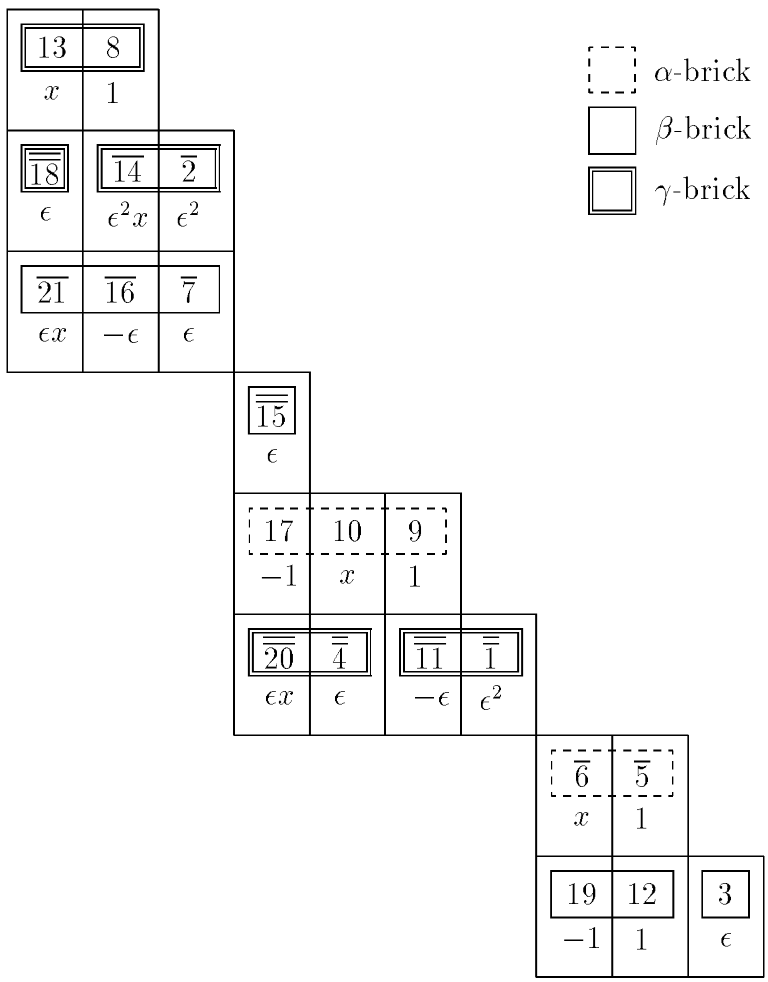

Figure 5.8: An example of an object in $\mathcal{O}_{3p}$.

- If c is in a barred brick,

$$w(c) = \begin{cases} 1, & c \text{ is at the end of the last brick in a row of } \mu, \\ \epsilon^2, & c \text{ is at the end of the last brick in a row of } \nu, \\ \epsilon, & c \text{ is at the end of a brick but not a row}, \\ -\epsilon \text{ or } \epsilon x, & \text{otherwise.} \end{cases}$$

This accounts for the $\epsilon^{2l^{\beta}(\mu)}$, $\epsilon^{l^{\beta}(\nu)}$, and $\epsilon(\epsilon x - \epsilon)^{\beta_i - 1}$ terms.

- If c is in a double barred brick,

$$w(c) = \begin{cases} \epsilon, & c \text{ is at the end of the last brick in a row of } \mu, \\ 1, & c \text{ is at the end of the last brick in a row of } \nu, \\ \epsilon^2, & c \text{ is at the end of a brick but not a row}, \\ -\epsilon^2 \text{ or } \epsilon^2 x, & \text{otherwise.} \end{cases}$$

This accounts for the $\epsilon^{2l^{\beta}(\mu)}$, $\epsilon^{l^{\beta}(\nu)}$, and $\epsilon^2(\epsilon^2 x - \epsilon^2)^{\beta_i - 1}$ terms.

(γ) If c lies in a γ-brick, the weight is given by the following.

- If c is in a regular brick,

$$w(c) = \begin{cases} \epsilon, & c \text{ is at the end of the last brick in a row of } \mu, \\ \epsilon^2, & c \text{ is at the end of the last brick in a row of } \nu, \\ 1, & c \text{ is at the end of a brick but not a row}, \\ -1 \text{ or } x, & \text{otherwise.} \end{cases}$$

This accounts for the $\epsilon^{l^{\gamma}(\mu)}$, $\epsilon^{2l^{\gamma}(\nu)}$, and $(x-1)^{\gamma_i - 1}$ terms.

- If c is in a barred brick,

$$w(c) = \begin{cases} 1, & c \text{ is at the end of the last brick in a row of } \mu, \\ \epsilon, & c \text{ is at the end of the last brick in a row of } \nu, \\ \epsilon^2, & c \text{ is at the end of a brick but not a row}, \\ -\epsilon^2 \text{ or } \epsilon^2 x, & \text{otherwise.} \end{cases}$$

This accounts for the $\epsilon^{l^{\gamma}(\mu)}$, $\epsilon^{2l^{\gamma}(\nu)}$, and $\epsilon^2(\epsilon^2 x - \epsilon^2)^{\gamma_i - 1}$ terms.

- If c is in a double barred brick,

$$w(c) = \begin{cases} \epsilon^2, & c \text{ is at the end of the last brick in a row of } \mu, \\ 1, & c \text{ is at the end of the last brick in a row of } \nu, \\ \epsilon, & c \text{ is at the end of a brick but not a row,} \\ -\epsilon \text{ or } \epsilon x, & \text{otherwise.} \end{cases}$$

This accounts for the $\epsilon^{l^{\gamma}(\mu)}$, $\epsilon^{2l^{\gamma}(\nu)}$, and $\epsilon(\epsilon x - \epsilon)^{\gamma_i - 1}$ terms.

Define the weight of an object o by $w(o) = \prod_{c \in o} w(c)$. Then we can write the expression in (5.9) as $\sum_{o \in \mathcal{O}_{3p}} w(o)$.

We will now perform an involution on the objects. Proceed by traversing, one at a time, the rows of the diagram, considering the rows from top to bottom, until we find one of the following conditions, then perform the appropriate operation.

- If there is a cell c with weight -1, $-\epsilon$, or $-\epsilon^2$, divide the brick after c, and change the weight of c from -1, $-\epsilon$, or $-\epsilon^2$ to $+1$, $+\epsilon$, or $+\epsilon^2$.

- If there is a decrease from the integer filling of the last cell of one brick to that of the first cell of the next, and both bricks are of the same type and in the same row, join together to two bricks and change the weight of c from $+1$, $+\epsilon$, or $+\epsilon^2$ to -1, $-\epsilon$, or $-\epsilon^2$.

An example of the involution is given in Figure 5.9.

The fixed points of the involution are elements of $\mathcal{F}^{\alpha,\beta,\gamma}_{\lambda*\mu*\nu}$ filled with the integers $1, 2, \ldots, n$ with the smallest number in each row appearing at the end. The integer fillings decrease within bricks, and increase between consecutive bricks of the same type in the same row. Each brick is designated as regular, barred, or double barred. The cells are weighted in the following way, and depend on the type of row and the type of brick they lie in. The weights are also summarized in Table 5.3, and an example of a fixed point is given in Figure 5.10.

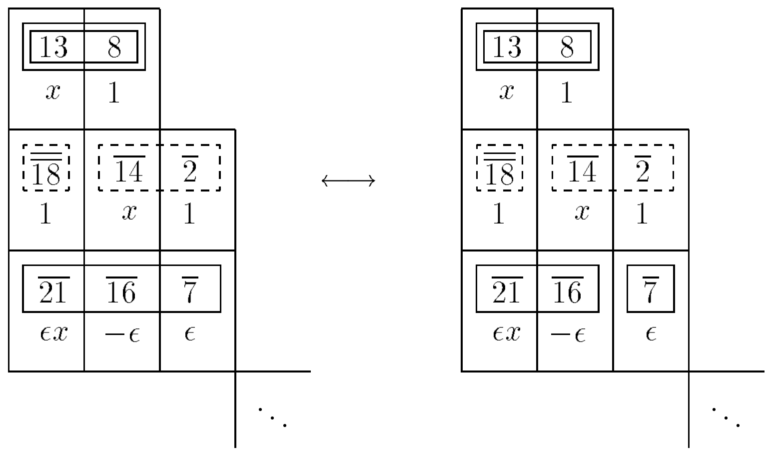

Figure 5.9: An example of the involution on $\mathcal{O}_{3p}$.

Table 5.3: Weights of cells in fixed points of the involution.

		regular cells			barred cells			double barred cells		
	end of:	row	brick	none	row	brick	none	row	brick	none
λ	α	1	1	x	1	1	x	1	1	x
	β	1	1	x	ϵ	ϵ	ϵx	ϵ^2	ϵ^2	ϵ^2x
	γ	1	1	x	ϵ^2	ϵ^2	ϵ^2x	ϵ	ϵ	ϵx
μ	α	1	1	x	1	1	x	1	1	x
	β	ϵ^2	1	x	1	ϵ	ϵx	ϵ	ϵ^2	ϵ^2x
	γ	ϵ	1	x	1	ϵ^2	ϵ^2x	ϵ^2	ϵ	ϵx
ν	α	1	1	x	1	1	x	1	1	x
	β	ϵ	1	x	ϵ^2	ϵ	ϵx	1	ϵ^2	ϵ^2x
	γ	ϵ^2	1	x	ϵ	ϵ^2	ϵ^2x	1	ϵ	ϵx

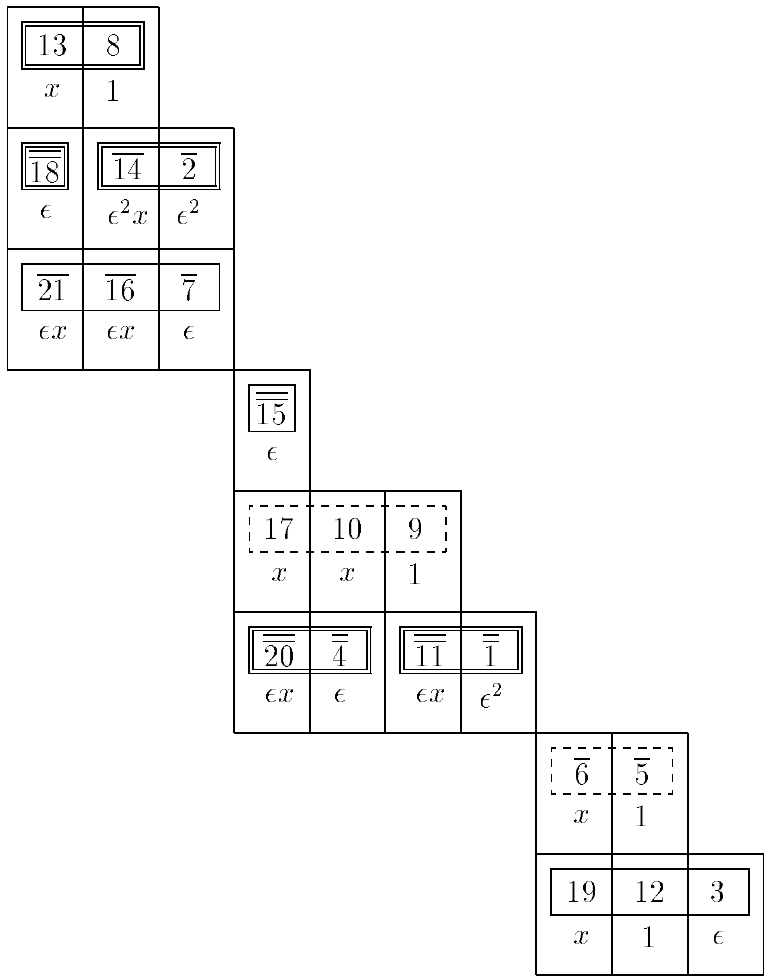

Figure 5.10: An example of a fixed point of the involution on $\mathcal{O}_{3p}$.

(α) If the cell c is in an α-brick, its weight is given by

$$w(c) = \begin{cases} 1, & c \text{ is at the end of a brick,} \\ x, & \text{otherwise.} \end{cases}$$

(β) If the cell is in a β-brick, its weight is given by the following.

- If c is in a regular brick, its weight is given by

$$w(c) = \begin{cases} \epsilon^2, & c \text{ is at the end of the last brick in a row of } \mu, \\ \epsilon, & c \text{ is at the end of the last brick in a row of } \nu, \\ 1, & c \text{ is at the end of a brick but not a row,} \\ x, & \text{otherwise.} \end{cases}$$

- If c is in a barred brick, its weight is given by

$$w(c) = \begin{cases} 1, & c \text{ is at the end of the last brick in a row of } \mu, \\ \epsilon^2, & c \text{ is at the end of the last brick in a row of } \nu, \\ \epsilon, & c \text{ is at the end of a brick but not a row,} \\ \epsilon x, & \text{otherwise.} \end{cases}$$

- If c is in a double barred brick, its weight is given by

$$w(c) = \begin{cases} \epsilon, & c \text{ is at the end of the last brick in a row of } \mu, \\ 1, & c \text{ is at the end of the last brick in a row of } \nu, \\ \epsilon^2, & c \text{ is at the end of a brick but not a row,} \\ \epsilon^2 x, & \text{otherwise.} \end{cases}$$

(γ) If the cell is in a γ-brick, its weight is given by the following.

- If c is in a regular brick, its weight is given by

$$w(c) = \begin{cases} \epsilon, & c \text{ is at the end of the last brick in a row of } \mu, \\ \epsilon^2, & c \text{ is at the end of the last brick in a row of } \nu, \\ 1, & c \text{ is at the end of a brick but not a row,} \\ x, & \text{otherwise.} \end{cases}$$

- If c is in a barred brick, its weight is given by

$$w(c) = \begin{cases} 1, & c \text{ is at the end of the last brick in a row of } \mu, \\ \epsilon, & c \text{ is at the end of the last brick in a row of } \nu, \\ \epsilon^2, & c \text{ is at the end of a brick but not a row,} \\ \epsilon^2 x, & \text{otherwise.} \end{cases}$$

- If c is in a double barred brick, its weight is given by

$$w(c) = \begin{cases} \epsilon^2, & c \text{ is at the end of the last brick in a row of } \mu, \\ 1, & c \text{ is at the end of the last brick in a row of } \nu, \\ \epsilon, & c \text{ is at the end of a brick but not a row,} \\ \epsilon x, & \text{otherwise.} \end{cases}$$

Define a *trivolution* to be a map from a set to itself, $f : S \to S$, with the property that for all $s \in S$, $f(f(f(s))) = s$. We perform the following trivolution on the remaining objects. Change each α-row to a β-row, change each β-row to a γ-row, and change each γ-row to an α-row, maintaining the type of each brick, but with the appropriate changes of weights on each cell. An example is given in Figures 5.11, 5.12, and 5.13.

By considering the weights in Table 5.3, we can see what happens to the weight of any given row throughout the trivolution. This depends, however, on whether the row lies in λ, μ, or ν. For this analysis, say that in all cases we begin with an

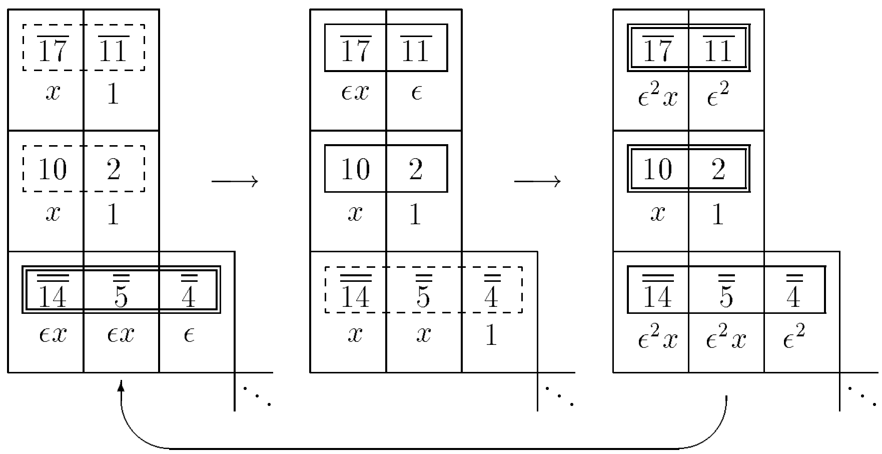

Figure 5.11: An example of the trivolution in λ.

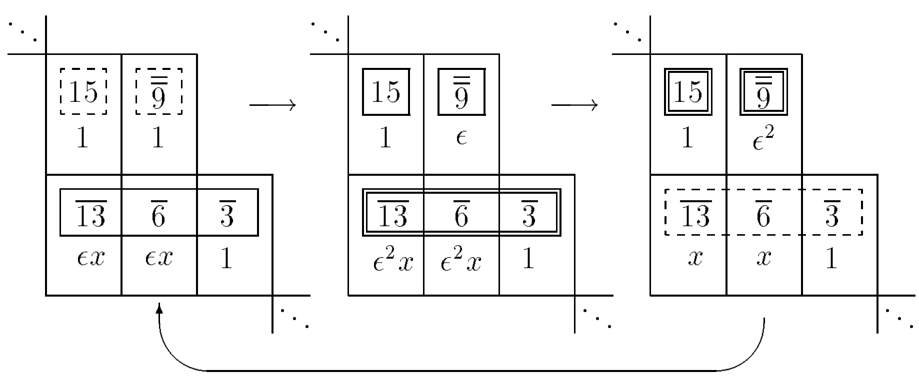

Figure 5.12: An example of the trivolution in μ.

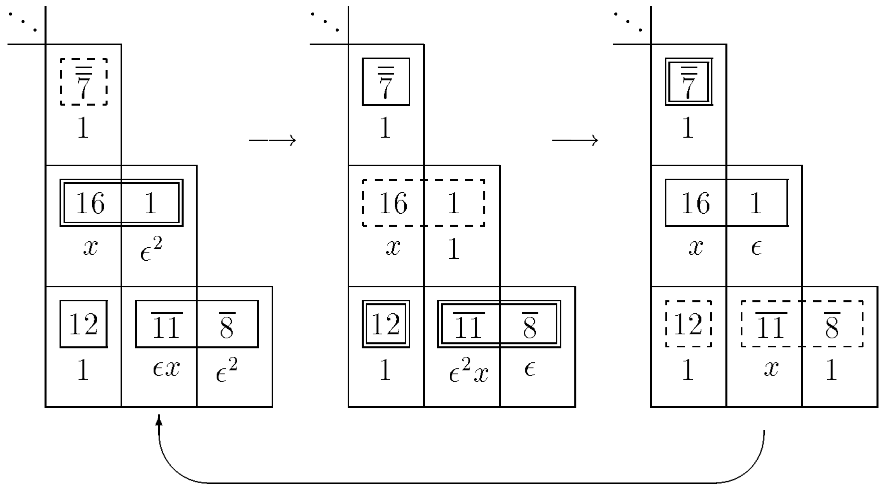

Figure 5.13: An example of the trivolution in ν.

α-row of weight w. Let b be the number of barred cells in the row, and let d be the number of double barred cells. We determine what changes occur in this weight, depending on where the row lies. We use this to determine the fixed points of the trivolution, an example of which appears in Figure 5.14.

(λ) If the row lies in λ, The first application of the trivolution changes the sign on each barred cell by ϵ and on each double barred cell by ϵ^2. When the involution is applied again, the weights on the barred cells are now ϵ^2 times their original weights, and the double barred cells are ϵ times their original weights. Thus the weights of the row as α-, β-, and γ-rows are w, $\epsilon^{b+2d}w$, and $\epsilon^{2b+d}q$, respectively. Summing over the objects, we have $w + \epsilon^{b+2d}w + \epsilon^{2b+d}w$. If $b + 2d \equiv 0 \pmod 3$, then this sum is 3. Otherwise, the sum is 0. Thus we are only left with those rows that have $b + 2d \equiv 0 \pmod 3$. In this case, we might as well consider each row as an α-row, so we divide by $3^{l(\lambda)}$ to disregard the α-, β-, and γ-rows.

(μ) If the row lies in μ, the first application of the trivolution changes the sign on each barred cell by ϵ and on each double barred cell by ϵ^2, with the exception of the cell at the end of the row. If the cell at the end of the row is a regular cell, its sign is ϵ^2 times the original weight; if it is a barred cell, its sign doesn't change;

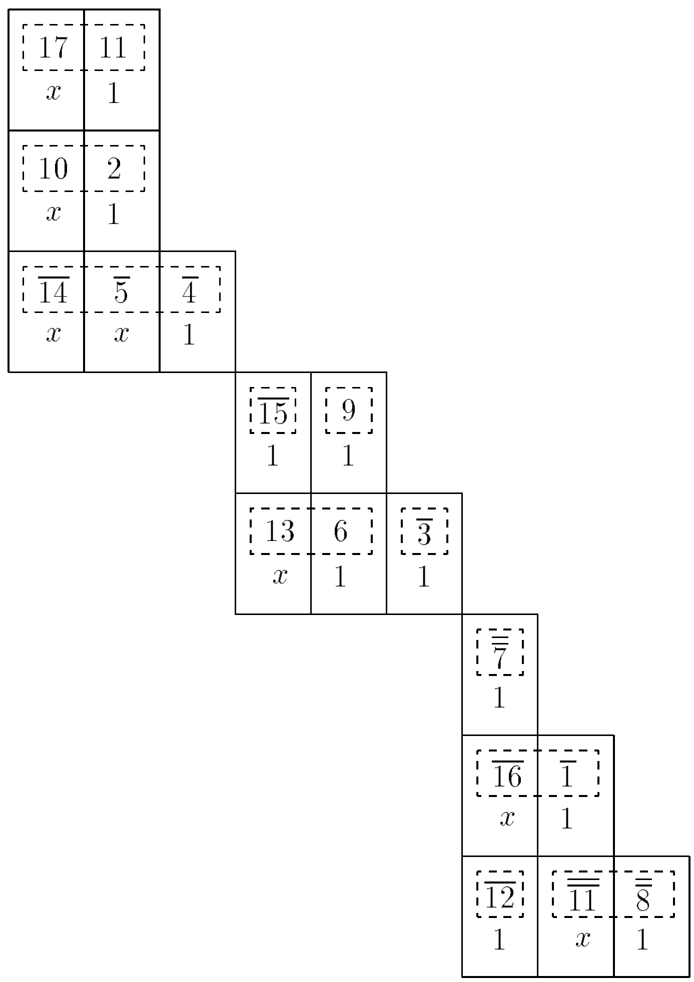

Figure 5.14: An example of a fixed point of the trivolution on $\mathcal{O}_{3p}$.

if it is double barred, it only increases by ϵ. When the trivolution is applied again, the weights on the barred cells are now ϵ^2 times their original weights, and the double barred cells are ϵ times their original weights, again with the exception of the last cell. If the last cell is regular, its weight is now ϵ times its original weight; if it is barred, the weight again remains the same; if it is double barred, the weight is ϵ^2 times its original weight. Thus the weights of the row as α-, β-, and γ-rows are w, $\epsilon^{b+2d+2\chi(\text{end cell reg})-\chi(\text{end cell barred})-\chi(\text{end cell double})}w = \epsilon^{b+2d+2}w$, and $\epsilon^{2b+d+\chi(\text{end cell reg})-2\chi(\text{end cell barred})+\chi(\text{end cell double})}w = \epsilon^{2b+d+1}w$, respectively, where $\chi(\text{statement})$ is 1 if the statement is true and 0 if it is false. Summing over the objects, we have $w + \epsilon^{b+2d+2}w + \epsilon^{2b+d+1}w$. If $b + 2d \equiv 1 \pmod 3$, then this sum is 3. Otherwise, the sum is 0. Thus we are only left with those rows that have $b + 2d \equiv 1 \pmod 3$. In this case, we might as well consider each row as an α-row, so we divide by $3^{l(\mu)}$ to disregard the α-, β-, and γ-rows.

(ν) If the row lies in ν, the first application of the trivolution changes the sign on each barred cell by ϵ and on each double barred cell by ϵ^2, with the exception of the cell at the end of the row. If the cell at the end of the row is a regular cell, its sign is ϵ times the original weight; if it is a barred cell, its sign is ϵ^2 times its original weight; if it is double barred, its weight doesn't change. When the trivolution is applied again, the weights on the barred cells are now ϵ^2 times their original weights, and the double barred cells are ϵ times their original weights, again with the exception of the last cell. If the last cell is regular, its weight is now ϵ^2 times its original weight; if it is barred, the weight is ϵ times its original weight; if it is double barred, the weight again does not change. Thus the weights of the row as α-, β-, and γ-rows are w, $\epsilon^{b+2d+\chi(\text{end cell reg})+\chi(\text{end cell barred})-2\chi(\text{end cell double})}w = \epsilon^{b+2d+1}w$, and $\epsilon^{2b+d+2\chi(\text{end cell reg})-\chi(\text{end cell barred})-\chi(\text{end cell double})}w = \epsilon^{2b+d+2}w$, respectively. Summing over the objects, we have $w + \epsilon^{b+2d+1}w + \epsilon^{2b+d+2}w$. If $b + 2d \equiv 2 \pmod 3$, then this sum is 3. Otherwise, the sum is 0. Thus we are only left with those rows that have $b + 2d \equiv 2 \pmod 3$. In this case, we might as well consider each row as an α-row, so we divide by $3^{l(\nu)}$ to disregard the α-, β-, and γ-rows.

We now interpret each row of a fixed point as a cycle in an element of $C_3 \S S_n$,

$$\begin{array}{|c|c|c|}\hline \overline{40} & \overline{18} & \overline{8} \\ \hline \multicolumn{1}{c}{\mathrm{x}} & \multicolumn{1}{c}{\mathrm{x}} & \multicolumn{1}{c}{1} \end{array} \quad \begin{array}{|c|c|}\hline \overline{22} & \overline{6} \\ \hline \multicolumn{1}{c}{\mathrm{x}} & \multicolumn{1}{c}{1} \end{array} \quad \begin{array}{|c|c|c|c|}\hline 35 & 12 & 11 & 5 \\ \hline \multicolumn{1}{c}{\mathrm{x}} & \multicolumn{1}{c}{\mathrm{x}} & \multicolumn{1}{c}{\mathrm{x}} & \multicolumn{1}{c}{1} \end{array} \quad \begin{array}{|c|c|}\hline \overline{\overline{10}} & \overline{\overline{2}} \\ \hline \multicolumn{1}{c}{\mathrm{x}} & \multicolumn{1}{c}{1} \end{array}$$

$$\longrightarrow \qquad (\overline{40}, \overline{18}, \overline{8}, \overline{22}, \overline{6}, 35, 12, 11, 5, \overline{\overline{10}}, \overline{\overline{2}})$$

Figure 5.15: An example of the interpretation of a row as a cycle.

with $\overline{k}$ and $\overline{\overline{k}}$ corresponding to ϵk and $\epsilon^2 k$, respectively. See Figure 5.15 as an example. Within each cycle, decreases between letters of the same type are weighted by x, while all decreases between letters of different types and increases are weighted by 1. This corresponds to the definition of $C_3\S S_n$-descedances. Note that in λ, we have cycles with $b+2d \equiv 0 \pmod 3$, so the sign of the cycle is $\epsilon^{b+2d} = 1$. In μ, we have cycles with $b+2d \equiv 1 \pmod 3$, so the cycles have sign ϵ. In ν, $b+2d \equiv 2 \pmod 3$, so the cycles have sign ϵ^2. Thus the element of $C_3\S S_n$ corresponding to each object remaining after the involution and trivolution belongs to the conjugacy class indexed by (λ, μ, ν), and its weight is $x^{de_W(\sigma)}$. ■

5.3.2 q-analogs for the Power Bases

Our ultimate goal here is to determine the image of $3^n[n]!p_\lambda(X)p_\mu(Y)p_\nu(Z)$ under the homomorphism $\overline{\xi}_W$, given in Definition 5.5. To do this, we will first determine the images of $3^k[k]!p_k(X)$, $3^k[k]!p_k(Y)$, and $3^k[k]!p_k(Z)$. We then obtain the desired result as a corollary.

We begin by proving the following theorem.

Theorem 5.9. *Let $\overline{\xi}_W$ be the homomorphism defined in Definition 5.5. Then*

$$3^k[k]!\overline{\xi}_W(p_k(X)) = \sum_{\sigma \in C_3\S S_k^{(1)}} x^{des_W(\sigma)+1-t(\sigma)} q^{inv_W(\sigma)} (x^{t(\sigma)} - (x-1)^{t(\sigma)}),$$

where $C_3\S S_k^{(1)}$ is the set $\{\sigma \in C_3\S S_n : \epsilon(\sigma) = 1\}$, and $t(\sigma)$ is the length of the cofinal strictly decreasing sequence of integers of the same type of elements in the *one-line notation of σ.*

Proof. We begin with the following expression, a specialization of (4.19).

$$3p_k(X)p_\emptyset(Y)p_\emptyset(Z) = \sum_{(\alpha,\beta,\gamma)\vdash k} \sum_{f\in\mathcal{F}^{\alpha,\beta,\gamma}_{(k)*\emptyset*\emptyset}} (-1)^{k-l(\alpha)-l(\beta)-l(\gamma)} w(f)$$
$$\times e_\alpha(X+Y+Z)e_\beta(X+\epsilon Y+\epsilon^2 Z)e_\gamma(X+\epsilon^2 Y+\epsilon Z).$$

We multiply this by $3^k[k]!$ and apply $\overline{\xi}_W$ to get the following.

$$3^{k+1}[k]!\overline{\xi}_W(p_k(X)) = \sum_{(\alpha,\beta,\gamma)\vdash k} \sum_{f\in\mathcal{F}^{\alpha,\beta,\gamma}_{(k)*\emptyset*\emptyset}} (-1)^{k-l(\alpha)-l(\beta)-l(\gamma)} 3^k[k]! w(f)$$
$$\times \prod_{i=1}^{l(\alpha)} \left(\frac{q^{\binom{\alpha_i}{2}}\left((1-x)^{\alpha_i-1}+(1-x)^{\alpha_i-1}+(1-x)^{\alpha_i-1}\right)}{3^{\alpha_i}[\alpha_i]!} \right)$$
$$\times \prod_{i=1}^{l(\beta)} \left(\frac{q^{\binom{\beta_i}{2}}\left((1-x)^{\beta_i-1}+\epsilon(\epsilon-\epsilon x)^{\beta_i-1}+\epsilon^2(\epsilon^2-\epsilon^2 x)^{\beta_i-1}\right)}{3^{\beta_i}[\beta_i]!} \right)$$
$$\times \prod_{i=1}^{l(\gamma)} \left(\frac{q^{\binom{\gamma_i}{2}}\left((1-x)^{\gamma_i-1}+\epsilon^2(\epsilon^2-\epsilon^2 x)^{\gamma_i-1}+\epsilon(\epsilon-\epsilon x)^{\gamma_i-1}\right)}{3^{\gamma_i}[\gamma_i]!} \right).$$

$$= \sum_{(\alpha,\beta,\gamma)\vdash k} \sum_{f\in\mathcal{F}^{\alpha,\beta,\gamma}_{(k)*\emptyset*\emptyset}} w(f) \begin{bmatrix} k \\ \alpha_1,\dots,\alpha_{l(\alpha)},\beta_1,\dots,\beta_{l(\beta)},\gamma_1,\dots,\gamma_{l(\gamma)} \end{bmatrix}$$
$$\times \prod_{i=1}^{l(\alpha)} q^{\binom{\alpha_i}{2}}\left((x-1)^{\alpha_i-1}+(x-1)^{\alpha_i-1}+(x-1)^{\alpha_i-1}\right)$$
$$\times \prod_{i=1}^{l(\beta)} q^{\binom{\beta_i}{2}}\left((x-1)^{\beta_i-1}+\epsilon(\epsilon x-\epsilon)^{\beta_i-1}+\epsilon^2(\epsilon^2 x-\epsilon^2)^{\beta_i-1}\right)$$
$$\times \prod_{i=1}^{l(\gamma)} q^{\binom{\gamma_i}{2}}\left((x-1)^{\gamma_i-1}+\epsilon^2(\epsilon^2 x-\epsilon^2)^{\gamma_i-1}+\epsilon(\epsilon x-\epsilon)^{\gamma_i-1}\right).$$

We interpret this as a sum of signed, weighted objects $o\in\mathcal{O}_{3pqx}$. These objects are clearly of shape $(k)*\emptyset*\emptyset$, filled with bricks such that the single row in λ is filled with all α-, β-, or γ-bricks. We consider these separately, and divide the above sum over objects in $\mathcal{F}^{\alpha,\beta,\gamma}_{(k)*\emptyset*\emptyset}$ into three sums over $\mathcal{B}_{\alpha,(k)}$, $\mathcal{B}_{\beta,(k)}$, and $\mathcal{B}_{\gamma,(k)}$.

$$
\begin{aligned}
&= \sum_{\alpha \vdash k} \sum_{f \in \mathcal{B}_{\alpha,(k)}} w(f) \begin{bmatrix} k \\ \alpha_1, \ldots, \alpha_{l(\alpha)} \end{bmatrix} \\
&\qquad \times \prod_{i=1}^{l(\alpha)} q^{\binom{\alpha_i}{2}} \left((x-1)^{\alpha_i - 1} + (x-1)^{\alpha_i - 1} + (x-1)^{\alpha_i - 1} \right) \\
&+ \sum_{\beta \vdash k} \sum_{f \in \mathcal{B}_{\beta,(k)}} w(f) \begin{bmatrix} k \\ \beta_1, \ldots, \beta_{l(\beta)} \end{bmatrix} \\
&\qquad \times \prod_{i=1}^{l(\beta)} q^{\binom{\beta_i}{2}} \left((x-1)^{\beta_i - 1} + \epsilon(\epsilon x - \epsilon)^{\beta_i - 1} + \epsilon^2(\epsilon^2 x - \epsilon^2)^{\beta_i - 1} \right) \qquad (5.10) \\
&+ \sum_{\gamma \vdash k} \sum_{f \in \mathcal{B}_{\gamma,(k)}} w(f) \begin{bmatrix} k \\ \gamma_1, \ldots, \gamma_{l(\gamma)} \end{bmatrix} \\
&\qquad \times \prod_{i=1}^{l(\gamma)} q^{\binom{\gamma_i}{2}} \left((x-1)^{\gamma_i - 1} + \epsilon^2(\epsilon^2 x - \epsilon^2)^{\gamma_i - 1} + \epsilon(\epsilon x - \epsilon)^{\gamma_i - 1} \right).
\end{aligned}
$$

We consider the three cases separately.

First, assume that the bricks in the single row are α-bricks. We will prove the following lemma.

Lemma 5.10.

$$
\begin{aligned}
&\sum_{\alpha \vdash k} \sum_{f \in \mathcal{B}_{\alpha,(k)}} w(f) \begin{bmatrix} k \\ \alpha_1, \ldots, \alpha_{l(\alpha)} \end{bmatrix} \\
&\qquad \times \prod_{i=1}^{l(\alpha)} q^{\binom{\alpha_i}{2}} \left((x-1)^{\alpha_i - 1} + (x-1)^{\alpha_i - 1} + (x-1)^{\alpha_i - 1} \right) \\
&\qquad\qquad = \sum_{\sigma \in C_3 \S S_k} x^{des_W(\sigma) + 1 - t(\sigma)} q^{inv_W(\sigma)} \left(x^{t(\sigma)} - (x-1)^{t(\sigma)} \right),
\end{aligned}
$$

where $t(\sigma)$ is the length of the cofinal strictly decreasing sequence of elements of the same type in the one-line notation of σ.

Proof. We have an α-brick tabloid of shape (k). Designate each brick as regular, barred, or double barred. Fill it with the integers $1, 2, \ldots, n$ in the following way. For a given tabloid, let $B_1, B_2, \ldots, B_l$ be the bricks that occur in order from left to right in the α-brick tabloid. Let $b_i = |B_i|$ so $b_1, b_2, \ldots, b_l$ is a rearrangement of $\alpha_1, \alpha_2, \ldots, \alpha_l$. We associate b_i i's with B_i and consider rearrangements in $\mathcal{R}(1^{b_1}, 2^{b_2}, \ldots, l^{b_l})$. For each rearrangement $r \in \mathcal{R}(1^{b_1}, 2^{b_2}, \ldots, l^{b_l})$, we create a permutation $\sigma(r)$ in the following way. Number, from right to left, first the 1's, then the 2's, and so on. We then take the inverse permutation $\sigma^{-1}(\sigma)$. An example of this process is given in Table 5.1. By the way we constructed $\sigma^{-1}(r)$, we have decreasing sequences of lengths $b_1, b_2, \ldots, b_l$, which then fit into the bricks $B_1, B_2, \ldots, B_l$. By Theorem 1.1,

$$\begin{bmatrix} n \\ \alpha_1, \ldots, \alpha_l \end{bmatrix} = \sum_{r \in \mathcal{R}(1^{b_1}, 2^{b_2}, \ldots, l^{b_l})} q^{inv(r)}.$$

By the construction of $\sigma(r)$, we have

$$inv(\sigma^{-1}(r)) = inv(\sigma(r)) = inv(r) + \binom{b_1}{2} + \binom{b_2}{2} + \cdots + \binom{b_l}{2}.$$

We now have an α-brick tabloid of shape (k), filled with integers such that they decrease within the bricks. We associate to each cell c a q-weight, $w_q(c) = q^{p(c)}$, where $p(c)$ is the number of cells to the right of c filled with a smaller number. By the above argument, these will count the $C_3 \S S_n$-inversions of the filling. As the involution we will perform does not change the q-weight, we will ignore it for now. We also associate to each cell an x-weight. This is given by

$$w_x(c) = \begin{cases} 1, & c \text{ is at the end of a brick,} \\ -1 \text{ or } x, & \text{otherwise.} \end{cases}$$

The last brick in the row is also weighted by its length. The weight of an object o is then defined as

$$w(o) = \left(\prod_{c \in o} w_x(c) w_q(c) \right) \left(\prod_{b \in o} |b|^{\chi(b \text{ is at the end of a row})} \right).$$

We can then write

$$\sum_{\alpha \vdash k} \sum_{f \in \mathcal{B}_{\alpha,(k)}} w(f) \begin{bmatrix} k \\ \alpha_1, \ldots, \alpha_{l(\alpha)} \end{bmatrix} \times \prod_{i=1}^{l(\alpha)} q^{\binom{\alpha_i}{2}} \left((x-1)^{\alpha_i - 1} + (x-1)^{\alpha_i - 1} + (x-1)^{\alpha_i - 1}\right) = \sum_{o \in \mathcal{O}_{3pqx}} w(o).$$

We now perform the following involution on the objects. Note that changing the size of the last brick in the row will change the weight of the object, since the last brick is weighted by its size. Thus in the following involution, we do not change anything about the last brick. Proceed left to right through the tabloid until the first occurrence of one of the following and perform the appropriate operation, *unless* the first occurrence occurs in the last brick.

- If there is a cell c with weight -1, divide the brick after c and change the weight of c from -1 to $+1$.
- If there is a decrease between the integer filling of the last cell c of one brick and that of the first cell of the next and both bricks are of the same type, join the two bricks together and change the weight of c from $+1$ to -1.

The fixed points of the involution are α-brick tabloids of shape (k), filled with the integers $1, 2, \ldots, n$. In addition, each cell is designated as regular, barred, or double barred. The fixed points have the following properties.

- The integer fillings decrease within bricks.
- The integer fillings increase between consecutive bricks of the same type, with the possible exception of a decrease between the second to the last brick and the last brick.
- The last brick has weight 1 in the last cell, and either x or -1 in the other cells. In addition, it is weighted by its length.

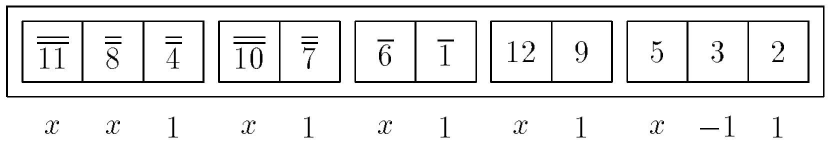

Figure 5.16: An example of a fixed point of the involution on $\mathcal{O}_{3pqx}$.

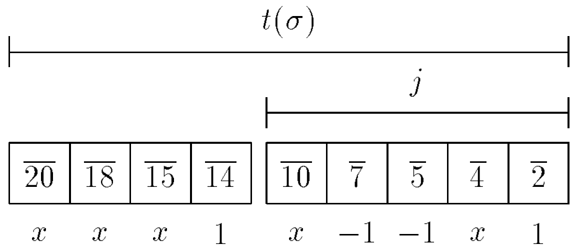

Figure 5.17: An example of $t(\sigma)$ and j.

- The other bricks have weight 1 in the last cell, and x in the other cells.
- Each cell has a q-weight as described above.

An example of a fixed point is given in Figure 5.16.

Let $t(\sigma)$ be the length of the cofinal strictly decreasing sequence of integers in bricks of the same type. Note that this might correspond to either just the last brick, or the last two bricks. Let j be the length of the last brick. See Figure 5.17 for an example of what this might look like. We want to count all of the descents of the object. We pull these out, which means we must divide each of the weights, except for the weight of the last cell, by x. These adjusted weights are shown in Figure 5.18.

We can now rewrite the sum for the α-brick case as

$$\sum_{\sigma \in C_3 \S S_k} x^{des_W(\sigma)} q^{inv_W(\sigma)} \left(\frac{1}{x} \sum_{j=1}^{t(\sigma)-1} \left((1 - \frac{1}{x})^{j-1} j \right) + t(\sigma)(1 - \frac{1}{x})^{t(\sigma)-1} \right)$$

by splitting the possibilities into the case in which the second to last brick is part

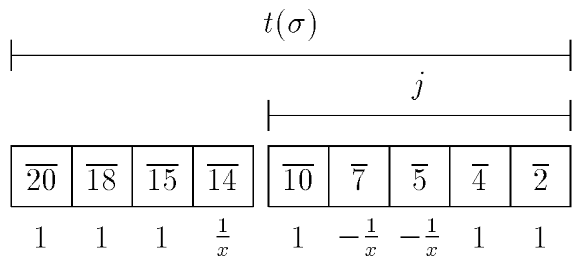

Figure 5.18: An example of the adjusted weights.

of the cofinal decreasing sequence, and the case when it is not. In both cases, we have pulled out the terms $x^{des_W(\sigma)}q^{inv_W(\sigma)}$, so we need only determine the adjusted weights. The first sum is over all possible lengths of the last brick, assuming that the second to last brick is part of the cofinal decreasing sequence. The $\frac{1}{x}$ comes from the last cell of the second to last brick. Within the sum, the j is from the weight of the last brick, and the term $(1-\frac{1}{x})^{j-1}$ gives all the possible weights of the other cells of the last brick. The last term corresponds to the case where the cofinal decreasing sequence occurs only within the last brick. In this case, the weight of the last brick is $j = t(\sigma)$. The last cell of the brick must have weight 1, but the term $(1-\frac{1}{x})^{t(\sigma)-1}$ counts all of the possible weights of the other cells in the brick. We can then rewrite this expression in the following ways to get the desired formula.

$$
\begin{aligned}
&= \sum_{\sigma \in C_3 \S S_k} x^{des_W(\sigma)} q^{inv_W(\sigma)} \left(x \sum_{j=1}^{t(\sigma)-1} \left((1-\frac{1}{x})^{j-1} \frac{j}{x^2} \right) + t(\sigma)(1-\frac{1}{x})^{t(\sigma)-1} \right) \\
&= \sum_{\sigma \in C_3 \S S_k} x^{des_W(\sigma)} q^{inv_W(\sigma)} \left(x \frac{d}{dx} \left(\sum_{j=1}^{t(\sigma)-1} (1-\frac{1}{x})^{j} \right) + t(\sigma)(1-\frac{1}{x})^{t(\sigma)-1} \right) \\
&= \sum_{\sigma \in C_3 \S S_k} x^{des_W(\sigma)} q^{inv_W(\sigma)} \left(x \frac{d}{dx} \left(\frac{(1-\frac{1}{x})^{t(\sigma)}-1}{(1-\frac{1}{x})-1} \right) + t(\sigma)(1-\frac{1}{x})^{t(\sigma)-1} \right) \\
&= \sum_{\sigma \in C_3 \S S_k} x^{des_W(\sigma)} q^{inv_W(\sigma)} \left(x \frac{d}{dx} x \left(1-(1-\frac{1}{x})^{t(\sigma)} \right) + t(\sigma)(1-\frac{1}{x})^{t(\sigma)-1} \right) \\
&= \sum_{\sigma \in C_3 \S S_k} x^{des_W(\sigma)} q^{inv_W(\sigma)} \left(x - x(1-\frac{1}{x})^{t(\sigma)} \right) \\
&= \sum_{\sigma \in C_3 \S S_k} x^{des_W(\sigma)+1-t(\sigma)} q^{inv_W(\sigma)} \left(x^{t(\sigma)} - (x-1)^{t(\sigma)} \right).
\end{aligned}
$$

This proves the lemma for the case that the tabloid is filled with α-bricks. ■

We now consider the case where the tabloid is filled with β-bricks, and prove the following lemma.

Lemma 5.11.

$$
\begin{aligned}
&\sum_{\beta \vdash k} \sum_{f \in \mathcal{B}_{\beta,(k)}} w(f) \begin{bmatrix} k \\ \beta_1, \ldots, \beta_{l(\beta)} \end{bmatrix} \\
&\qquad \times \prod_{i=1}^{l(\beta)} q^{\binom{\beta_i}{2}} \left((x-1)^{\alpha_i - 1} + \epsilon(\epsilon x - \epsilon)^{\beta_i - 1} + \epsilon^2(\epsilon^2 x - \epsilon^2)^{\beta_i - 1} \right) \\
&\qquad\qquad = \sum_{\sigma \in C_3 \S S_k} \epsilon^{b(\sigma)+2d(\sigma)} x^{des_W(\sigma)+1-t(\sigma)} q^{inv_W(\sigma)} \left(x^{t(\sigma)} - (x-1)^{t(\sigma)} \right),
\end{aligned}
$$

where $t(\sigma)$ is the length of the cofinal strictly decreasing sequence of elements of the same type in the one-line notation of σ, and $b(\sigma)$ and $d(\sigma)$ are the numbers of barred and double barred elements of σ, respectively.

Proof. The proof is nearly the same as the proof of Lemma 5.12. We again have tabloids filled with the integers $1, 2, \ldots, n$ using the same process as above, and

q-weights that count the number of inversions. Again, each brick is designated as regular, barred, or double barred. The only difference is in the x-weights placed on the cells in the β-brick tabloid. Here the x-weights are given by the following.

$$w(c) = \begin{cases} 1, & c \text{ is at the end of a regular brick,} \\ \epsilon, & c \text{ is at the end of a barred brick,} \\ \epsilon^2, & c \text{ is at the end of a double barred brick,} \\ -1 \text{ or } x, & c \text{ is elsewhere in a regular brick,} \\ -\epsilon \text{ or } \epsilon x, & c \text{ is elsewhere in a barred brick,} \\ -\epsilon^2 \text{ or } \epsilon^x, & c \text{ is elsewhere in a double barred brick.} \end{cases}$$

The weights are the same as in the proof of the previous lemma, except that each barred brick has an extra ϵ and each double barred brick has an extra ϵ^2. Thus we can pull out a product $\epsilon^{b(\sigma)+2d(\sigma)}$, where $b(\sigma)$ and $d(\sigma)$ are the number of barred and double barred elements in the $C_3 \S S_k$-element formed by the filling of the tabloid. Once that is done, we perform the same involution and the same simplification of the sum, but each term in the sum now is multiplied by $\epsilon^{b(\sigma)+2d(\sigma)}$. ∎

We finally consider the case where the tabloid is filled with γ-bricks. We have the following lemma.

Lemma 5.12.

$$\sum_{\gamma \vdash k} \sum_{f \in \mathcal{B}_{\gamma,(k)}} w(f) \begin{bmatrix} k \\ \gamma_1, \ldots, \gamma_{l(\gamma)} \end{bmatrix} \times \prod_{i=1}^{l(\gamma)} q^{\binom{\gamma_i}{2}} \left((x-1)^{\gamma_i - 1} + \epsilon^2(\epsilon^2 x - \epsilon^2)^{\gamma_i - 1} + \epsilon(\epsilon x - \epsilon)^{\gamma_i - 1} \right)$$
$$= \sum_{\sigma \in C_3 \S S_k} \epsilon^{2b(\sigma)+d(\sigma)} x^{des_W(\sigma)+1-t(\sigma)} q^{inv_W(\sigma)} \left(x^{t(\sigma)} - (x-1)^{t(\sigma)} \right),$$

where $t(\sigma)$ is the length of the cofinal strictly decreasing sequence of elements of the same type in the one-line notation of σ, and $b(\sigma)$ and $d(\sigma)$ are the numbers of barred and double barred elements of σ, respectively.

Proof. This proof is again nearly the same as the proofs of Lemmas 5.10 and 5.11. The objects are the same, with the only difference being the x-weights placed on the cells. These are given by the following.

$$w(c) = \begin{cases} 1, & c \text{ is at the end of a regular brick,} \\ \epsilon^2, & c \text{ is at the end of a barred brick,} \\ \epsilon, & c \text{ is at the end of a double barred brick,} \\ -1 \text{ or } x, & c \text{ is elsewhere in a regular brick,} \\ -\epsilon^2 \text{ or } \epsilon^2 x, & c \text{ is elsewhere in a barred brick,} \\ -\epsilon \text{ or } \epsilon x, & c \text{ is elsewhere in a double barred brick.} \end{cases}$$

This corresponds to each barred cell having an extra ϵ^2 and each double barred cell having an extra ϵ. We can then perform the same involution and simplifications with each term in the sum multiplied by a factor $\epsilon^{2b(\sigma)+d(\sigma)}$. ■

We are now ready to complete the proof of Theorem 5.9. We combine the results of Lemmas 5.10, 5.11, and 5.12 with (5.10) to obtain the following.

$$\begin{aligned} 3^{k+1}[k]!\overline{\xi}_W(p_k(X)) &= \sum_{\sigma \in C_3 \S S_k} x^{des_W(\sigma)+1-t(\sigma)} q^{inv_W(\sigma)} \left(x^{t(\sigma)} - (x-1)^{t(\sigma)}\right) \\ &+ \sum_{\sigma \in C_3 \S S_k} \epsilon^{b(\sigma)+2d(\sigma)} x^{des_W(\sigma)+1-t(\sigma)} q^{inv_W(\sigma)} \left(x^{t(\sigma)} - (x-1)^{t(\sigma)}\right) \\ &+ \sum_{\sigma \in C_3 \S S_k} \epsilon^{2b(\sigma)+d(\sigma)} x^{des_W(\sigma)+1-t(\sigma)} q^{inv_W(\sigma)} \left(x^{t(\sigma)} - (x-1)^{t(\sigma)}\right) = \\ \sum_{\sigma \in C_3 \S S_k} &\left(1 + \epsilon^{b(\sigma)+2d(\sigma)} + \epsilon^{2b(\sigma)+d(\sigma)}\right) x^{des_W(\sigma)+1-t(\sigma)} q^{inv_W(\sigma)} \left(x^{t(\sigma)} - (x-1)^{t(\sigma)}\right). \end{aligned}$$

Note that

$$1 + \epsilon^{b(\sigma)+2d(\sigma)} + \epsilon^{2b(\sigma)+d(\sigma)} = \begin{cases} 0, & b(\sigma) + 2d(\sigma) \equiv 1, 2 \pmod 3, \\ 3, & b(\sigma) + 2d(\sigma) \equiv 0 \pmod 3. \end{cases}$$

Thus the sum is actually over all elements σ of $C_3 \S S_k$ such that $b(\sigma) + 2d(\sigma) \equiv 0 \pmod 3$, that is, such that $\epsilon(\sigma) = 1$. If we let $C_3 \S S_k^{(1)}$ be the set of all such elements, we have

$$3^{k+1}[k]!\overline{\xi}_W(p_k(X)) = \sum_{\sigma\in C_3\S S_k^{(1)}} 3x^{des_W(\sigma)+1-t(\sigma)}q^{inv_W(\sigma)}\left(x^{t(\sigma)}-(x-1)^{t(\sigma)}\right).$$

This completes the proof of Theorem 5.9. ■

Our next step is to determine the image of $3^k[k]!p_k(Y)$ under $\overline{\xi}_W$. This is given by the following theorem.

Theorem 5.13. *Let $\overline{\xi}_W$ be the homomorphism defined in Definition 5.5. Then*

$$3^k[k]!\overline{\xi}_W(p_k(Y)) = \sum_{\sigma\in C_3\S S_k^{(\epsilon)}} x^{des_W(\sigma)+1-t(\sigma)}q^{inv_W(\sigma)}\left(x^{t(\sigma)}-(x-1)^{t(\sigma)}\right),$$

where $C_3\S S_k^{(\epsilon)}$ is the set $\{\sigma\in C_3\S S_k : \epsilon(\sigma)=\epsilon\}$, and $t(\sigma)$ is the length of the cofinal strictly decreasing sequence of elements of the same type in the one-line notation of σ.

Proof. We again begin with a specialization of (4.19).

$$3p_\emptyset(X)p_k(Y)p_\emptyset(Z) = \sum_{(\alpha,\beta,\gamma)\vdash k}\sum_{f\in\mathcal{F}^{\alpha,\beta,\gamma}_{\emptyset*(k)*\emptyset}} (-1)^{k-l(\alpha)-l(\beta)-l(\gamma)}\epsilon^{2l^\beta(\mu)+l^\gamma(\mu)}w(f)$$
$$\times\, e_\alpha(X+Y+Z)e_\beta(X+\epsilon Y+\epsilon^2 Z)e_\gamma(X+\epsilon^2 Y+\epsilon Z).$$

We multiply this by $3^k[k]!$, apply $\overline{\xi}_W$, and simplify in the same way as in the proof to Theorem 5.9 to get the following.

$$
\begin{aligned}
&3^{k+1}[k]!\overline{\xi}_W(p_k(Y)) = \\
&\sum_{(\alpha,\beta,\gamma)\vdash k}\ \sum_{f\in\mathcal{F}^{\alpha,\beta,\gamma}_{\emptyset*(k)*\emptyset}} \epsilon^{2l^{\beta}(\mu)+l^{\gamma}(\mu)} w(f) \begin{bmatrix} k \\ \alpha_1,\ldots,\alpha_{l(\alpha)},\beta_1,\ldots,\beta_{l(\beta)},\gamma_1,\ldots,\gamma_{l(\gamma)} \end{bmatrix} \\
&\qquad \times \prod_{i=1}^{l(\alpha)} q^{\binom{\alpha_i}{2}} \left((x-1)^{\alpha_i-1}+(x-1)^{\alpha_i-1}+(x-1)^{\alpha_i-1}\right) \\
&\qquad \times \prod_{i=1}^{l(\beta)} q^{\binom{\beta_i}{2}} \left((x-1)^{\beta_i-1}+\epsilon(\epsilon x-\epsilon)^{\beta_i-1}+\epsilon^2(\epsilon^2 x-\epsilon^2)^{\beta_i-1}\right) \\
&\qquad \times \prod_{i=1}^{l(\gamma)} q^{\binom{\gamma_i}{2}} \left((x-1)^{\gamma_i-1}+\epsilon^2(\epsilon^2 x-\epsilon^2)^{\gamma_i-1}+\epsilon(\epsilon x-\epsilon)^{\gamma_i-1}\right).
\end{aligned}
$$

We interpret this as a sum of signed, weighted objects of shape $\emptyset*(k)*\emptyset$, filled with bricks such that the single row in μ is filled with all α-, β-, or γ-bricks. We can consider these cases separately, and divide the above sum into three sums, as in the proof of Theorem 5.9. Note that here we must account for the term $\epsilon^{2l^{\beta}(\mu)+l^{\gamma}(\mu)}$ by multiplying by ϵ^2 the sum corresponding to filling the row with β-bricks, and multiplying by ϵ the sum corresponding to filling the row with γ-bricks. The above sum is then equal to

$$
\begin{aligned}
&= \sum_{\alpha\vdash k}\ \sum_{f\in\mathcal{B}_{\alpha,(k)}} w(f) \begin{bmatrix} k \\ \alpha_1,\ldots,\alpha_{l(\alpha)} \end{bmatrix} \\
&\qquad\qquad \times \prod_{i=1}^{l(\alpha)} q^{\binom{\alpha_i}{2}} \left((x-1)^{\alpha_i-1}+(x-1)^{\alpha_i-1}+(x-1)^{\alpha_i-1}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ \epsilon^2 \sum_{\beta\vdash k}\ \sum_{f\in\mathcal{B}_{\beta,(k)}} w(f) \begin{bmatrix} k \\ \beta_1,\ldots,\beta_{l(\beta)} \end{bmatrix} \\
&\qquad\qquad \times \prod_{i=1}^{l(\beta)} q^{\binom{\beta_i}{2}} \left((x-1)^{\beta_i-1}+\epsilon(\epsilon x-\epsilon)^{\beta_i-1}+\epsilon^2(\epsilon^2 x-\epsilon^2)^{\beta_i-1}\right)
\end{aligned}
$$

$$+\,\epsilon \sum_{\gamma \vdash k} \sum_{f \in \mathcal{B}_{\gamma,(k)}} w(f) \begin{bmatrix} k \\ \gamma_1, \ldots, \gamma_{l(\gamma)} \end{bmatrix} \times \prod_{i=1}^{l(\gamma)} q^{\binom{\gamma_i}{2}} \left((x-1)^{\gamma_i - 1} + \epsilon^2(\epsilon^2 x - \epsilon^2)^{\gamma_i - 1} + \epsilon(\epsilon x - \epsilon)^{\gamma_i - 1} \right).$$

These sums are the same as those evaluated in the proof of Theorem 5.9, so we apply Lemmas 5.10, 5.11, and 5.12 to obtain the following.

$$\begin{aligned} 3^{k+1}[k]!\overline{\xi}_W(p_k(Y)) &= \sum_{\sigma \in C_3 \S S_k} x^{des_W(\sigma)+1-t(\sigma)} q^{inv_W(\sigma)} \left(x^{t(\sigma)} - (x-1)^{t(\sigma)} \right) \\ &+ \sum_{\sigma \in C_3 \S S_k} \epsilon^{b(\sigma)+2d(\sigma)+2} x^{des_W(\sigma)+1-t(\sigma)} q^{inv_W(\sigma)} \left(x^{t(\sigma)} - (x-1)^{t(\sigma)} \right) \\ &+ \sum_{\sigma \in C_3 \S S_k} \epsilon^{2b(\sigma)+d(\sigma)+1} x^{des_W(\sigma)+1-t(\sigma)} q^{inv_W(\sigma)} \left(x^{t(\sigma)} - (x-1)^{t(\sigma)} \right) = \\ &\sum_{\sigma \in C_3 \S S_k} \left(1 + \epsilon^{b(\sigma)+2d(\sigma)+2} + \epsilon^{2b(\sigma)+d(\sigma)+1} \right) \\ &\times x^{des_W(\sigma)+1-t(\sigma)} q^{inv_W(\sigma)} \left(x^{t(\sigma)} - (x-1)^{t(\sigma)} \right). \end{aligned}$$

Note that

$$1 + \epsilon^{b(\sigma)+2d(\sigma)+2} + \epsilon^{2b(\sigma)+d(\sigma)+1} = \begin{cases} 0, & b(\sigma) + 2d(\sigma) \equiv 0, 2 \pmod 3, \\ 3, & b(\sigma) + 2d(\sigma) \equiv 1 \pmod 3. \end{cases}$$

Thus the sum is actually over all elements σ of $C_3 \S S_k$ such that $b(\sigma) + 2d(\sigma) \equiv 1 \pmod 3$, that is, such that $\epsilon(\sigma) = \epsilon$. If we let $C_3 \S S_k^{(\epsilon)}$ be the set of all such elements, we have

$$3^{k+1}[k]!\overline{\xi}_W(p_k(Y)) = \sum_{\sigma \in C_3 \S S_k^{(\epsilon)}} 3x^{des_W(\sigma)+1-t(\sigma)} q^{inv_W(\sigma)} \left(x^{t(\sigma)} - (x-1)^{t(\sigma)} \right).$$

This completes the proof of Theorem 5.13. ■

Next, we determine the image of $3^k[k]!p_k(Z)$ under $\overline{\xi}_W$. This is given by the following theorem.

Theorem 5.14. *Let $\overline{\xi}_W$ be the homomorphism defined in Definition 5.5. Then*

$$3^k[k]!\overline{\xi}_W(p_k(Z)) = \sum_{\sigma\in C_3\S S_k^{(\epsilon^2)}} x^{des_W(\sigma)+1-t(\sigma)} q^{inv_W(\sigma)} \left(x^{t(\sigma)} - (x-1)^{t(\sigma)}\right),$$

where $C_3\S S_k^{(\epsilon^2)}$ is the set $\{\sigma \in C_3\S S_k : \epsilon(\sigma) = \epsilon^2\}$, and $t(\sigma)$ is the length of the cofinal strictly decreasing sequence of elements of the same type in the one-line notation of σ.

Proof. We again begin with a specialization of (4.19).

$$\begin{aligned}3p_\emptyset(X)p_\emptyset(Y)p_k(Z) = \sum_{(\alpha,\beta,\gamma)\vdash k}\sum_{f\in\mathcal{F}^{\alpha,\beta,\gamma}_{\emptyset*\emptyset*(k)}} &(-1)^{k-l(\alpha)-l(\beta)-l(\gamma)}\epsilon^{l^\beta(\nu)+2l^\gamma(\nu)}w(f)\\ &\times e_\alpha(X+Y+Z)e_\beta(X+\epsilon Y+\epsilon^2 Z)e_\gamma(X+\epsilon^2 Y+\epsilon Z).\end{aligned}$$

We multiply this by $3^k[k]!$, apply $\overline{\xi}_W$, and simplify in the same way as in the proof to Theorem 5.9 to get the following.

$$\begin{aligned}&3^{k+1}[k]!\overline{\xi}_W(p_k(Z)) =\\ &\sum_{(\alpha,\beta,\gamma)\vdash k}\sum_{f\in\mathcal{F}^{\alpha,\beta,\gamma}_{\emptyset*\emptyset*(k)}} \epsilon^{l^\beta(\nu)+2l^\gamma(\nu)}w(f)\begin{bmatrix} k \\ \alpha_1,\dots,\alpha_{l(\gamma)},\beta_1,\dots,\beta_{l(\beta)},\gamma_1,\dots,\gamma_{l(\gamma)}\end{bmatrix}\\ &\times\prod_{i=1}^{l(\alpha)} q^{\binom{\alpha_i}{2}}\left((x-1)^{\alpha_i-1}+(x-1)^{\alpha_i-1}+(x-1)^{\alpha_i-1}\right)\\ &\times\prod_{i=1}^{l(\beta)} q^{\binom{\beta_i}{2}}\left((x-1)^{\beta_i-1}+\epsilon(\epsilon x-\epsilon)^{\beta_i-1}+\epsilon^2(\epsilon^2 x-\epsilon^2)^{\beta_i-1}\right)\\ &\times\prod_{i=1}^{l(\gamma)} q^{\binom{\gamma_i}{2}}\left((x-1)^{\gamma_i-1}+\epsilon^2(\epsilon^2 x-\epsilon^2)^{\gamma_i-1}+\epsilon(\epsilon x-\epsilon)^{\gamma_i-1}\right).\end{aligned}$$

We interpret this as a sum of signed, weighted objects of shape $\emptyset*\emptyset*(k)$, filled with bricks such that the single row in ν is filled with all α-, β-, or γ-bricks. We can consider these cases separately, and divide the above sum into three sums, as in the proof of Theorem 5.9. Note that here we must account for the term $\epsilon^{l^\beta(\nu)+2l^\gamma(\nu)}$

by multiplying by ϵ the sum corresponding to filling the row with β-bricks, and multiplying by ϵ^2 the sum corresponding to filling the row with γ-bricks. The above sum is then equal to

$$= \sum_{\alpha \vdash k} \sum_{f \in \mathcal{B}_{\alpha,(k)}} w(f) \begin{bmatrix} k \\ \alpha_1, \ldots, \alpha_{l(\alpha)} \end{bmatrix} \times \prod_{i=1}^{l(\alpha)} q^{\binom{\alpha_i}{2}} \left((x-1)^{\alpha_i - 1} + (x-1)^{\alpha_i - 1} + (x-1)^{\alpha_i - 1}\right)$$

$$+ \epsilon \sum_{\beta \vdash k} \sum_{f \in \mathcal{B}_{\beta,(k)}} w(f) \begin{bmatrix} k \\ \beta_1, \ldots, \beta_{l(\beta)} \end{bmatrix} \times \prod_{i=1}^{l(\beta)} q^{\binom{\beta_i}{2}} \left((x-1)^{\beta_i - 1} + \epsilon(\epsilon x - \epsilon)^{\beta_i - 1} + \epsilon^2(\epsilon^2 x - \epsilon^2)^{\beta_i - 1}\right)$$

$$+ \epsilon^2 \sum_{\gamma \vdash k} \sum_{f \in \mathcal{B}_{\gamma,(k)}} w(f) \begin{bmatrix} k \\ \gamma_1, \ldots, \gamma_{l(\gamma)} \end{bmatrix} \times \prod_{i=1}^{l(\gamma)} q^{\binom{\gamma_i}{2}} \left((x-1)^{\gamma_i - 1} + \epsilon^2(\epsilon^2 x - \epsilon^2)^{\gamma_i - 1} + \epsilon(\epsilon x - \epsilon)^{\gamma_i - 1}\right).$$

These sums are the same as those evaluated in the proof of Theorem 5.9, so we apply Lemmas 5.10, 5.11, and 5.12 to obtain the following.

$$\begin{aligned}
3^{k+1}[k]!\overline{\xi}_W(p_k(Z)) &= \sum_{\sigma \in C_3 \S S_k} x^{des_W(\sigma)+1-t(\sigma)} q^{inv_W(\sigma)} \left(x^{t(\sigma)} - (x-1)^{t(\sigma)}\right) \\
&\quad + \sum_{\sigma \in C_3 \S S_k} \epsilon^{b(\sigma)+2d(\sigma)+1} x^{des_W(\sigma)+1-t(\sigma)} q^{inv_W(\sigma)} \left(x^{t(\sigma)} - (x-1)^{t(\sigma)}\right) \\
&\quad + \sum_{\sigma \in C_3 \S S_k} \epsilon^{2b(\sigma)+d(\sigma)+2} x^{des_W(\sigma)+1-t(\sigma)} q^{inv_W(\sigma)} \left(x^{t(\sigma)} - (x-1)^{t(\sigma)}\right) \\
&= \sum_{\sigma \in C_3 \S S_k} \left(1 + \epsilon^{b(\sigma)+2d(\sigma)+1} + \epsilon^{2b(\sigma)+d(\sigma)+2}\right) \\
&\qquad \times x^{des_W(\sigma)+1-t(\sigma)} q^{inv_W(\sigma)} \left(x^{t(\sigma)} - (x-1)^{t(\sigma)}\right).
\end{aligned}$$

Note that

$$1+\epsilon^{b(\sigma)+2d(\sigma)+1}+\epsilon^{2b(\sigma)+d(\sigma)+2}=\begin{cases}0, & b(\sigma)+2d(\sigma)\equiv 0,1 \pmod 3,\\ 3, & b(\sigma)+2d(\sigma)\equiv 2 \pmod 3.\end{cases}$$

Thus the sum is actually over all elements σ of $C_3\S S_k$ such that $b(\sigma)+2d(\sigma)\equiv 2 \pmod 3$, that is, such that $\epsilon(\sigma)=\epsilon^2$. If we let $C_3\S S_k^{(\epsilon^2)}$ be the set of all such elements, we have

$$3^{k+1}[k]!\overline{\xi}_W(p_k(X))=\sum_{\sigma\in C_3\S S_k^{(\epsilon^2)}} 3x^{des_W(\sigma)+1-t(\sigma)}q^{inv_W(\sigma)}\left(x^{t(\sigma)}-(x-1)^{t(\sigma)}\right).$$

This completes the proof of Theorem 5.14. ■

If $(\lambda,\mu,\nu)\vdash n$, let $\overline{C_3\S S_n}(\lambda,\mu,\nu)$ be the set of all $\sigma=\sigma_1\sigma_2\cdots\sigma_n$ such that if σ is broken up into segments of lengths $\lambda_1,\lambda_2,\ldots,\lambda_{l(\lambda)}$, $\mu_1,\mu_2,\ldots,\mu_{l(\mu)}$, $\nu_1,\nu_2,\ldots,\nu_{l(\nu)}$, in that order, then the each segment corresponding to a part of λ has total sign 1, each segment corresponding to a part of μ has total sign ϵ, and each segment corresponding to a part of ν has total sign ϵ^2. We then have the following corollary of Theorems 5.9, 5.13, and 5.14.

Corollary 5.15. *Let $\overline{\xi}_W$ be the homomorphism defined in Definition 5.5. If $k+l+m=n$, then*

$$\begin{aligned}&3^n[n]!\overline{\xi}_W(p_k(X)p_l(Y)p_m(Z))=\\&\quad\sum_{\sigma\in\overline{C_3\S S_n}(k,l,m)} x^{des_{W,(k,l,m)}(\sigma)}q^{inv_W(\sigma)}\times x^{1-t(\sigma)}\left(x^{t(\sigma)}-(x-1)^{t(\sigma)}\right)\\&\qquad x^{1-u(\sigma)}\left(x^{u(\sigma)}-(x-1)^{u(\sigma)}\right)x^{1-v(\sigma)}\left(x^{v(\sigma)}-(x-1)^{v(\sigma)}\right),\end{aligned}$$

where $t(\sigma)$ is the length of the last strictly decreasing sequence of elements of the same type in the first k elements of the one-line notation of σ, $u(\sigma)$ is the length of the cofinal strictly decreasing sequence of elements of the same type in the $(k+1)^{st}$ through $(k+l)^{th}$ elements of σ, and $v(\sigma)$ is the length of the last strictly decreasing sequence of elements of the same type in the last m elements of σ.

Note that in the above theorem, $des_{W,(k,l,m)}$ is $des_{W,\rho}$ where ρ is the nondecreasing rearrangement of (k,l,m). The proof follows those of Theorems 5.9, 5.13, and 5.14. When creating a permutation such that decreasing sequences will fill the bricks, integers are associated with the bricks from right to left in rows, considering the rows in order from top to bottom of the diagram.

This easily generalizes to the following corollary, in which the partition $(\lambda \cup \mu \cup \nu)$ is defined to be the nondecreasing rearrangement of
$(\lambda_1, \ldots, \lambda_{l(\lambda)}, \mu_1, \ldots, \mu_{l(\mu)}, \nu_1, \ldots, \nu_{l(\nu)})$.

Corollary 5.16. *Let $\overline{\xi}_W$ be the homomorphism defined in Definition 5.5. If $(\lambda, \mu, \nu) \vdash n$, then*

$$3^n[n]!\overline{\xi}_W(p_\lambda(X)p_\mu(Y)p_\nu(Z)) =$$
$$\sum_{\sigma \in \overline{C_3 \S S_n}(\lambda,\mu,\nu)} x^{des_{W,(\lambda\cup\mu\cup\nu)}(\sigma)} q^{inv_W(\sigma)} \prod_{i=1}^{l(\lambda)} \left(x^{1-t_i(\sigma)}\left(x^{t_i(\sigma)} - (x-1)^{t_i(\sigma)}\right)\right)$$
$$\times \prod_{i=1}^{l(\lambda)} \left(x^{1-u_i(\sigma)}\left(x^{u_i(\sigma)} - (x-1)^{u_i(\sigma)}\right)\right) \times \prod_{i=1}^{l(\lambda)} \left(x^{1-v_i(\sigma)}\left(x^{v_i(\sigma)} - (x-1)^{v_i(\sigma)}\right)\right),$$

where $t_i(\sigma)$, $u_i(\sigma)$, and $v_i(\sigma)$ denote the length of the cofinal decreasing sequence of elements of the same type in the segment of σ corresponding to λ_i, μ_i, ν_i, respectively.

5.4 ξ_W and $\overline{\xi}_W$ Applied to $s_\lambda(X+Y+Z)$, $s_\lambda(X+\epsilon Y+\epsilon^2 Z)$, and $s_\lambda(X+\epsilon^2 Y+\epsilon Z)$

Here we consider the images of the Schur basis of Λ_{W_3} under the homomorphism ξ_W. In order to give the results for the Schur basis, we need some notation and definitions. Given a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$, let $D(\lambda)$ be the length of the Durfee square, that is, the largest square that will fit inside the digram of shape λ. We also define the following partitions.

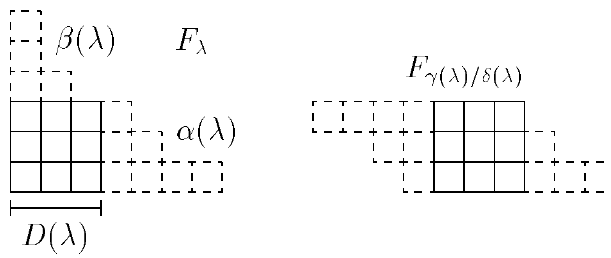

Figure 5.19: An example of $D(\lambda)$, $\alpha(\lambda)$, $\beta(\lambda)$, and $\gamma(\lambda)/\delta(\lambda)$.

- $\alpha(\lambda) = (\alpha_1, \ldots, \alpha_{D(\lambda)})$ where $\alpha_i = \lambda_{l-D(\lambda)+i} - D(\lambda)$, for $i = 1, \ldots, D(\lambda)$.
- $\beta(\lambda) = (\beta_1, \ldots, \beta_{D(\lambda)})$ where $\beta_i = \lambda'_{l-D(\lambda)+i} - D(\lambda)$, for $i = 1, \ldots, D(\lambda)$.
- $\gamma(\lambda) = (\alpha_{D(\lambda)} - \alpha_{D(\lambda)-1}, \alpha_{D(\lambda)} - \alpha_{D(\lambda)-2}, \ldots, \alpha_{D(\lambda)} - \alpha_1)$.
- $\delta(\lambda) = (\alpha_{D(\lambda)} + D(\lambda))^{D(\lambda)} + \beta(\lambda)$.

An example of $D(\lambda)$, $\alpha(\lambda)$, $\beta(\lambda)$, and the shape $\gamma(\lambda)/\delta(\lambda)$ which will occur in the following is given in Figure 5.19.

We now define the concept of special rim hook tabloids. Consider a Ferrers' diagram F_λ of shape λ. Recall that a rim hook is a sequence of cells in F_λ, such that any two consecutive cells share an edge, and removal of the cells from the diagram results in another Ferrers' diagram. A rim hook tabloid of shape λ, as defined in (2.5), is a sequence of rim hooks that together form the shape λ. A *special rim hook* of λ is a rim hook of λ if one of its cells lies in the first column of λ. A *special rim hook tabloid* of shape λ and type μ is a rim hook tabloid of shape λ and type μ such that all of the rim hooks are special rim hooks. The sign of a special rim hook tabloid is defined by $sgn(T) = \prod_{h \in T} sgn(h)$, where the sign of a hook is $sgn(h) = (-1)^{r(h)-1}$, and $r(h)$ is the number of hooks occupied by the hook h. If $SRH_{\mu,\lambda}$ is the set of all special rim hook tabloids of shape λ and type μ, define

$$K^{-1}_{\mu,\lambda} = \sum_{T \in SRH_{\mu,\lambda}} sgn(T).$$

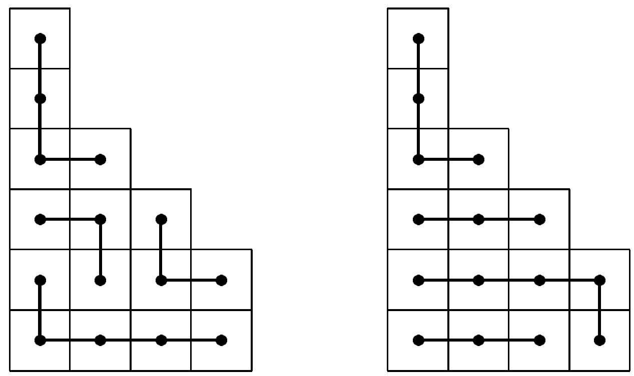

Figure 5.20: An example of a rim hook tabloid and a special rim hook tabloid.

An example of a rim hook tabloid and a special rim hook tabloid, both of shape $(1,1,2,3,4,4)$ and type $(3,3,4,5)$, are given in Figure 5.20.

We now have the necessary notation to state the image of the basis $s_\lambda(X+Y+Z)s_\mu(X+\epsilon Y+\epsilon^2 Z)s_\nu(X+\epsilon^2 Y+\epsilon Z)$ under ξ_W. We consider each term separately.

Theorem 5.17. *Let ξ_W be the homomorphism defined in Definition 5.1. If $\lambda \vdash n$, then*

$$3^n n!\xi_W(s_\lambda(X+Y+Z)) = \sum_{m=D(\lambda)}^{D(\lambda)+\alpha_{D(\lambda)}} 3^m(1-x)^{n-m}C_{m,\lambda'}, \tag{5.11}$$

$$3^n n!\xi_W(s_\lambda(X+\epsilon Y+\epsilon^2 Z)) = \sum_{m=D(\lambda)}^{D(\lambda)+\alpha_{D(\lambda)}} 3^m(1-x)^{n-m}C^{(3)}_{m,\lambda'}, \tag{5.12}$$

and

$$3^n n!\xi_W(s_\lambda(X+\epsilon^2 Y+\epsilon Z)) = \sum_{m=D(\lambda)}^{D(\lambda)+\alpha_{D(\lambda)}} 3^m(1-x)^{n-m}C^{(3)}_{m,\lambda'}, \tag{5.13}$$

where

$$C_{m,\lambda'} = \sum_{\substack{\rho\vdash n\\ l(\rho)=m}} \binom{n}{\rho_1,\ldots,\rho_m} K^{-1}_{\rho,\lambda'},$$

$$C^{(3)}_{m,\lambda'} = \sum_{\substack{\rho\vdash n\\ l(\rho)=m\\ 3|\rho_i\forall i}} \binom{n}{\rho_1,\ldots,\rho_m} K^{-1}_{\rho,\lambda'},$$

and $\alpha_{D(\lambda)}$ is the largest part of the partition $\alpha(\lambda)$ defined above.

Proof. To prove (5.11), begin with the following identity, which is given in [7].

$$s_\lambda = \sum_{\rho\vdash n} K^{-1}_{\rho,\lambda'} e_\rho.$$

Multiply by $3^n n!$ and apply ξ_W to this to obtain

$$\begin{aligned}
&3^n n!\xi_W(s_\lambda(X+Y+Z))\\
&\qquad= \sum_{\rho\vdash n} 3^n n! K^{-1}_{\rho,\lambda'} \prod_{i=1}^{l(\rho)} \frac{(1-x)^{\rho_i-1}+(1-x)^{\rho_i-1}+(1-x)^{\rho_i-1}}{3^{\rho_i}\rho_i!}\\
&= \sum_{\rho\vdash n} K^{-1}_{\rho,\lambda}\binom{n}{\rho_1,\ldots,\rho_{l(\rho)}} \prod_{i=1}^{l(\rho)} \left((1-x)^{\rho_i-1}+(1-x)^{\rho_i-1}+(1-x)^{\rho_i-1}\right)\\
&\qquad\qquad= \sum_{\rho\vdash n} K^{-1}_{\rho,\lambda'}\binom{n}{\rho_1,\ldots,\rho_{l(\rho)}} 3^{l(\rho)}(1-x)^{n-l(\rho)}.
\end{aligned}$$

For $K^{-1}_{\rho,\lambda'}$ to be nonzero, we need $D(\lambda)\le l(\rho)\le D(\lambda)+\alpha_{D(\lambda)}$. Thus we can rewrite the above as

$$\begin{aligned}
&3^n n!\xi_W(s_\lambda(X+Y+Z))\\
&\qquad= \sum_{m=D(\lambda)}^{D(\lambda)+\alpha_{D(\lambda)}} \sum_{\substack{\rho\vdash n\\ l(\rho)=m}} K^{-1}_{\rho,\lambda'}\binom{n}{\rho_1,\ldots,\rho_m} 3^m(1-x)^{n-m}\\
&\qquad\qquad= \sum_{m=D(\lambda)}^{D(\lambda)+\alpha_{D(\lambda)}} 3^m(1-x)^{n-m} C_{m,\lambda'}.
\end{aligned}$$

This completes the proof of (5.11). The proof of (5.12) and (5.13) is similar. Begin with the same identity, multiply by $3^n n!$, and apply ξ_W to obtain

$$3^n n!\xi_W(s_\lambda(X+\epsilon Y+\epsilon^2 Z)) = 3^n n!\xi_W(s_\lambda(X+\epsilon^2 Y+\epsilon Z))$$
$$= \sum_{\rho\vdash n} 3^n n! K^{-1}_{\rho,\lambda'} \prod_{i=1}^{l(\rho)} \frac{(1-x)^{\rho_i-1}+\epsilon(\epsilon-\epsilon x)^{\rho_i-1}+\epsilon^2(\epsilon^2-\epsilon^2 x)^{\rho_i-1}}{3^{\rho_i}\rho_i!}$$
$$= \sum_{\rho\vdash n} K^{-1}_{\rho,\lambda}\binom{n}{\rho_1,\ldots,\rho_{l(\rho)}} \prod_{i=1}^{l(\rho)} \left((1-x)^{\rho_i-1}+\epsilon(\epsilon-\epsilon x)^{\rho_i-1}+\epsilon^2(\epsilon^2-\epsilon^2 x)^{\rho_i-1}\right)$$
$$= \sum_{\rho\vdash n} K^{-1}_{\rho,\lambda'}\binom{n}{\rho_1,\ldots,\rho_{l(\rho)}}(1-x)^{n-l(\rho)} \prod_{i=1}^{l(\rho)} \left(1+\epsilon^{\rho_i}+\epsilon^{2\rho_i}\right).$$

We have that $1+\epsilon^{\rho_i}+\epsilon^{2\rho_i}=0$ unless $\rho_i \equiv 0 \pmod 3$, when it equals 3. As before, $K^{-1}_{\rho,\lambda'}=0$ unless $D(\lambda)\le l(\rho)\le D(\lambda)+\alpha_{D(\lambda)}$. Thus the above becomes

$$3^n n!\xi_W(s_\lambda(X+\epsilon Y+\epsilon^2 Z)) = 3^n n!\xi_W(s_\lambda(X+\epsilon^2 Y+\epsilon Z))$$
$$= \sum_{m=D(\lambda)}^{D(\lambda)+\alpha_{D(\lambda)}} \sum_{\substack{\rho\vdash n\\ l(\rho)=m\\ 3\mid\rho_i\forall i}} K^{-1}_{\rho,\lambda'}\binom{n}{\rho_1,\ldots,\rho_m} 3^m(1-x)^{n-m}$$
$$= \sum_{m=D(\lambda)}^{D(\lambda)+\alpha_{D(\lambda)}} 3^m(1-x)^{n-m} C^{(3)}_{m,\lambda'}.$$

■

5.4.1 Other Expressions for $C_{m,\lambda'}$ and $C^{(3)}_{m,\lambda'}$

We would like to find expressions for $C_{m,\lambda'}$ and $C^{(3)}_{m,\lambda'}$ that are easier to compute. This is possible for $C_{m,\lambda'}$. For $C^{(3)}_{m,\lambda'}$, the expression we obtain is not necessarily easier to compute, but it is interesting because it involves a new lattice condition. In order to define this condition properly, we will pursue the following for arbitrary $C_k\S S_n$, rather than the specific case with $k=3$. Now, we would like to find expressions for

$$C_{m,\lambda'} = \sum_{\substack{\mu\vdash n\\ l(\mu)=m}} \binom{n}{\mu_1,\ldots,\mu_m} K^{-1}_{\mu,\lambda'} \tag{5.14}$$

and

$$C^{(k)}_{m,\lambda'} = \sum_{\substack{\mu\vdash n\\ l(\mu)=m\\ k\mid\mu_i}} \binom{n}{\mu_1,\ldots,\mu_m} K^{-1}_{\mu,\lambda'}. \tag{5.15}$$

In order to do this, however, we will need a number of definitions and lemmas.

The first thing we will consider is the case where $m = D(\lambda)$. Brenti [3] proved and Beck [1] gave a combinatorial proof of the following identity.

Theorem 5.18.

$$C_{D(\lambda),\lambda'} = \sum_{\substack{\mu\vdash n\\ l(\mu)=D(\lambda)}} \binom{n}{\mu_1,\mu_2,\ldots,\mu_{D(\lambda)}} K^{-1}_{\mu,\lambda'} = (-1)^{|\alpha(\lambda)|} f^{\gamma(\lambda)/\delta(\lambda)},$$

where $F^{\gamma(\lambda)/\delta(\lambda)}$ is the number of standard tableaux of shape $\gamma(\lambda)/\delta(\lambda)$.

In order to examine $C^{(k)}_{m,\lambda'}$ for $m = D(\lambda)$, we need more definitions. Eğecioğlu and Remmel [7] introduced a sign-changing involution on the set of special rim hook tabloids of shape λ called *hook switching*. This maps a special rim hook tabloid T to another special rim hook tabloid T' as follows. If two hooks h_i and h_{i+1} begin in the adjacent rows i and $i+1$ in T, respectively, and end in rows s and t, then Eğecioğlu and Remmel show that there is only one other way to cover the cells occupied by h_i and h_{i+1} by two other special rim hooks h'_i and h'_{i+1} which begin in rows i and $i+1$, respectively. Figures 5.21 and 5.22 show the two cases that need to be considered, with the cells of the diagram represented by dots. In case (a), $t > s$ and h'_i and h'_{i+1} will end in rows $t-1$ and s, respectively. In case (b), $t \leq s$ and h'_i and h'_{i+1} end in rows t and $s+1$, respectively. Case (a) has a special case in which h_{i+1} has length 1. In this case, the switch consists of gluing together h_i and h_{i+1} to form h'_{i+1}, and h'_i is an empty hook. We will not need this

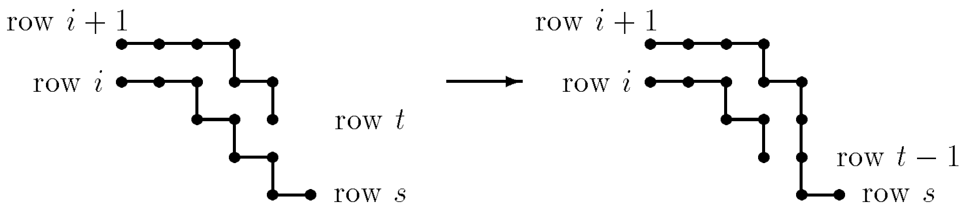

Figure 5.21: Case (a), $t > s$.

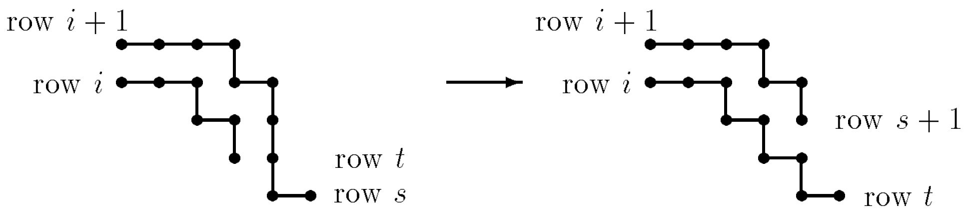

Figure 5.22: Case (b), $t \leq s$.

case in what follows, as our sum requires keeping the number of hooks constant. Note that in both cases,

$$sgn(h_i)sgn(h_{i+1}) = -sgn(h'_i)sgn(h'_{i+1}),$$

thus this is a sign-changing involution. Note also that the lengths of the new hooks are

$$|h'_i| = |h_{i+1}| - 1 \quad \text{and} \quad |h'_{i+1}| = |h_i| + 1.$$

If there are two hooks h_p and h_q that begin in rows p and q, respectively, such that h_j begins with a vertical segment consisting of $q-p$ cells in the first column, we say the hooks are adjacent, and they can be switched in the following way. We switch them in $2(q-p)-1$ steps. Let $h_{p+1}, h_{p+2}, \ldots, h_{q-1}$ be the empty hooks beginning in rows $p+1, p+2, \ldots, q-1$. Let $h_i^{(r)}$ be the hook beginning in row i after step r. In step r for $1 \leq r \leq q-p$, switch $h_{q-r-1}^{(r-1)}$ and $h_{q-r}^{(r-1)}$, that is, proceed from the top row to the bottom, switching adjacent hooks. At step r for $q-p+1 \leq r \leq 2(q-p)-1$, switch $h_{r-q+2p}^{(r-1)}$ and $h_{r-q+2p+1}^{(r-1)}$, that is, go back up the rows, switching adjacent hooks. At each step, if we have changed hooks $h_m^{(r-1)}$

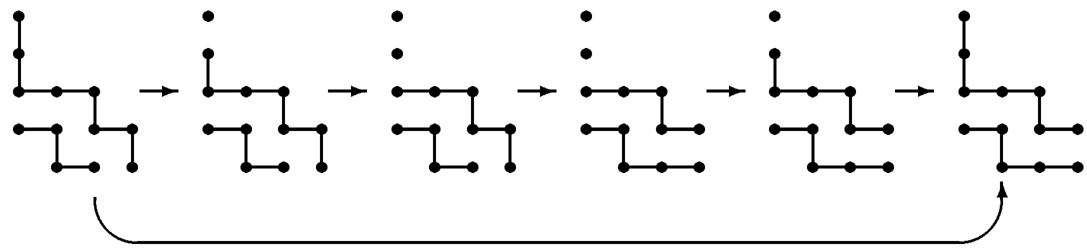

Figure 5.23: Switching adjacent hooks beginning in nonadjacent rows.

and $h_{m+1}^{(r-1)}$, we have lengths $|h_m^{(r)}| = |h_{m+1}^{(r-1)}| - 1$ and $|h_{m+1}^{(r)}| = |h_m^{(r-1)}| + 1$. Thus in the end,

$$|h'_p| = |h_q| - (q-p), \text{ and } |h'_q| = |h_p| + (q-p),$$

where $h'_p = h_p^{(2q-2p-1)}$ and $h'_q = h_q^{(2q-2p-1)}$ are the hooks remaining at the end of the process. In addition, since the sign changes at each step and there are an odd number of steps, we have $sgn(h'_p)sgn(h'_q) = -sgn(h_p)sgn(h_q)$. See Figure 5.23 for an example of this process.

One may also switch two hooks which are not adjacent in the following way. Note that by not adjacent we mean that there are nonempty hooks that begin in rows between the rows in which the two hooks we want to switch begin. Say we want to switch h_{i_1} and h_{i_m} such that there are $m-2$ hooks $h_{i_2}, h_{i_3}, \ldots, h_{i_{m-1}}$ between them. This can be done in $2m-3$ steps. Let the hooks after step j be denoted by $h_{i_1}^{(j)}, h_{i_2}^{(j)}, \ldots, j_{i_m}^{(j)}$. Then at step j for $1 \leq j \leq m-1$, switch $h_{i_j}^{(j-1)}$ and $h_{i_{j+1}}^{(j-1)}$. For $m \leq j \leq 2m-3$, switch $h_{i_{2m-1-j}}^{(j-1)}$ and $h_{i_{2m-2-j}}^{(j-1)}$. Since each switch of two adjacent hooks gives a sign change and there are an odd number of switches of adjacent hooks, we have

$$\prod_{j=1}^{m} sgn(h_{i_j}) = -\prod_{j=1}^{m} sgn(h_{i_j}^{(2m-3)}).$$

In addition, if h_{i_1} begins in row p and h_{i_m} begins in row q, we have

$$|h_{i_1}^{(2m-3)}| = |h_{i_m}| - (q-p) \quad \text{and} \quad |h_{i_m}^{(2m-3)}| = |h_{i_1}| + (q-p).$$

An example is given in Figure 5.24.

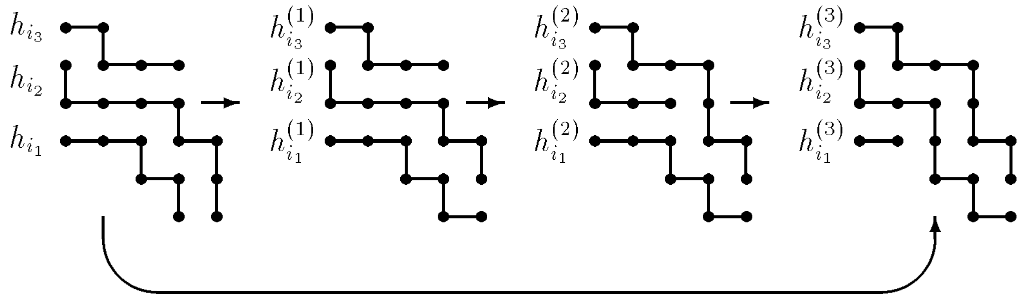

Figure 5.24: Switching nonadjacent hooks.

We say that two hooks are *k-switchable* if they begin in rows p and q such that k divides $q-p$, and the hook beginning in row q is nonempty. If two k-switchable hooks h_i and h_j are switched, then the new lengths differ from the old by multiples of k.

Using the definition of $K^{-1}_{\mu,\lambda'}$, the expression (5.15) for $m = D(\lambda)$ can be rewritten as

$$\sum_{\substack{\mu \vdash n \\ l(\mu)=D(\lambda) \\ k|\mu_i}} \binom{n}{\mu_1, \ldots, \mu_{D(\lambda)}} \sum_{T \in SRH_{\mu,\lambda'}} sgn(T),$$

where $SRH_{\mu,\lambda'}$ is the set of μ-brick tabloids of shape λ'. Thus we can interpret this as a sum of signed objects, $T \in \mathcal{O}_{ks}$. They are μ-brick tabloids of shape λ' for a given partition λ, such that the number of parts of μ is equal to the size of the Durfee square of λ and each part of μ is divisible by μ. In addition, the multinomial coefficient fills each tabloid with the integers $1, 2, \ldots, n$ in such a way that the integers increase along the hooks from top to bottom and from left to right.

We now label the hooks in an object T in the following way. The bottom-most hook is labeled h_1. Then $h_2, h_3, \ldots, h_{i_1}$ are the hooks, in order from the bottom of T to the top, such that the difference between the row in which they begin and the row in which h_1 begins is divisible by k. h_{i_1+1} is then the lowest hook not yet labeled. The hooks $h_{i_1+2}, h_{i_1+3}, \ldots, h_{i_2}$ are then the hooks, in order from bottom to top, such that the difference between the row in which they begin and the row

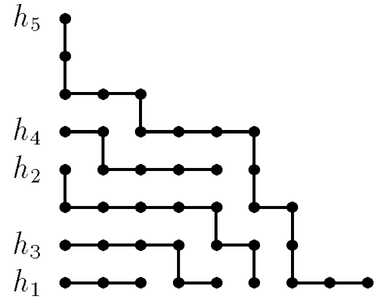

Figure 5.25: An example of the hook labeling for $k = 3$.

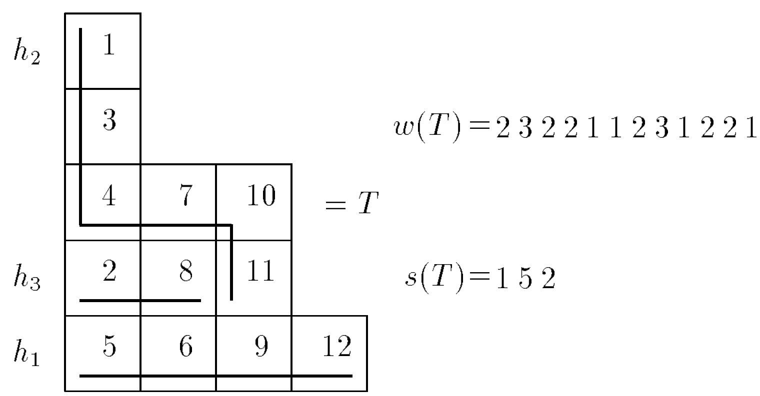

Figure 5.26: An example of an object in $\mathcal{O}_{ks}$ and its associated words.

in which h_{i_1+1} begins is divisible by k. Continue in this manner until all of the hooks are labeled. An example of this labeling system is given in Figure 5.25.

Now for each object T with its hooks labeled as above, we introduce two words associated with the object. The *hook word of* T, $w(T)$ is given by $w(T) = w_1 w_2 \cdots w_n \in \{1, 2, \ldots, D(\lambda)\}^n$ such that $w_i = j$ if and only if i lies in h_j in T. The *row word of* T is given by $s(T) = s_1 s_2 \cdots s_{D(\lambda)}$ such that h_i begins in row s_i in T. An example of an object in $\mathcal{O}_{ks}$ and its associated words is given in Figure 5.26.

We now perform an operation called *r-pairing* on the hook words of the objects. This was first introduced by Remmel and Shimozono [8]. This operation consists

$$w(T) \;=\; 3\ 2\ 2\ 2\ 3\ 1\ 1\ 2\ 2\ 2\ 1\ 3\ 2\ 2\ 2\ 1\ 3\ 1$$

$$d_r w(T) \;=\; 3\ 2\ 2\ 2\ 3\ 1\ 1\ 2\ 2\ 2\ 1\ 3\ 2\ 2\ 2\ 2\ 3\ 1$$

Figure 5.27: An example of r-pairing and d_r for $r = 1$.

of the following steps, ignoring any letter in $w(T)$ that is not an r or $r + 1$.

1. Pair any r and $r + 1$ that appear in that order with no r or $r + 1$ between them.

2. Remove the letters paired in Step 1 and repeat until no pairs remain.

The remaining letters form a subword of the form $(r+1)^p r^q$, called the *r-unpaired subword* of $w(T)$. If $w(T)$ has the r-unpaired subword $(r + 1)^p r^q$ and h_r and h_{r+1} begin in rows s_r and s_{r+1} in T, respectively, then define an operator d_r by declaring that in $d_r w(T)$ the r-unpaired subword of $w(T)$ is replaced by $(r + 1)^{q+s_{r+1}-s_r} r^{p+s_r-s_{r+1}}$. Note that d_r is only defined on $w(T)$ if $p \geq s_{r+1} - s_r$. An example of r-pairing and d_r is given in Figure 5.27. This is given for $r = 1$ and the word of r-unpaired letters are circled.

Proposition 5.19. *If d_r is defined on a hook word $w(T)$, then the r-unpaired subwords of $w(T)$ and $d_r w(T)$ occupy the same positions.*

Proof. It is enough to show that $w(T) = w_1 w_2 \cdots w_n$ and $d_r w(T) = v_1 v_2 \cdots v_n$ have the same r-pairs at Step 1. Suppose that we have a $w_i = r$ and $w_j = r + 1$ paired in $w(T)$ in Step 1, that is, there are no r's or $r+1$'s between them. If $v_i = r$ and $v_j = r + 1$ in $d_r w(T)$, they will still be paired and there is no problem. So consider what happens if one or both of them are different.

Case 1. $v_i = r$ and $v_j = r$. In this case only v_j has changed, so $w_i = r$ must have been paired and $w_j = r + 1$ must not have been paired in $w(T)$. But this contradicts the fact that there were no r's between w_i and w_j to pair with $w_j = r + 1$.

Case 2. $v_i = r+1$ and $v_j = r+1$. Here, $v_i \neq w_i$, so $w_j = r+1$ must have been paired in $w(T)$, but $w_i = r+1$ must not have been. This is again a contradiction since there were no r's between w_i and w_j to pair with w_j.

Case 3. $v_i = r+1$ and $v_j = r$. In this case, both w_i and w_j have changed, therefore both must have been unpaired in $w(T)$. Again this is a contradiction since there are no r's or $r+1$'s between them to prevent pairing w_i and w_j.

Thus the only possibility is that w_i and w_j are paired in both $w(T)$ and $d_r w(T)$, and the two words have the same r-pairs. ■

Let $o_i(r)$ be the number of occurrences of the letter r in the subword $w_1 w_2 \cdots w_i$ of $w(T) = w_1 w_2 \cdots w_n$. We then have the following definition.

Definition 5.20. An object T in $\mathcal{O}_{ks}$ is *hook k-lattice* if $o_i(r+1) < o_i(r) + s_{r+1} - s_r$ for all $1 \leq i \leq n$ and for all r such that k divides $s_{r+1} - s_r$.

Proposition 5.21. *An object T is hook k-lattice if and only if the r-unpaired subword of $w(T)$ is $(r+1)^p r^q$ with $p < s_{r+1} - s_r$ for all r such that k divides $s_{r+1} - s_r$.*

Proof. In the following proof, we will use the notation given above for the number of occurrences of the letter r in a subword of $w(T)$. In addition, let $u_i(r)$ be the number of letters r in the subword $w_1 w_2 \cdots w_i$ that are unpaired in $w(T)$. Fix a given r such that k divides $s_{r+1} - s_r$. For simplicity, set $a = s_{r+1} - s_r - 1$.

($\Longrightarrow$) Assume that $w(T)$ is hook k-lattice. We need to prove that $u_n(r+1) \leq a$. We do this by showing by induction on i that if $u_i(r+1) \leq a$, then $u_{i+1}(r+1) \leq a$. The case $i = 1$ is trivially true unless $k = 1$ and $a = 0$. Then for $w(T)$ to be 1-hook lattice, $w(T)$ cannot begin with $r+1$ anyway. Now assume that $u_i(r+1) \leq a$. If $w_{i+1} \neq r+1$ or if this is a strict inequality, $u_{i+1}(r+1) \leq a$ is trivially true, so assume $u_i(r+1) = a$ and $w_{i+1} = r+1$. If w_{i+1} is paired, then $u_{i+1}(r+1) = u_i(r+1) \leq a$. If w_{i+1} is not paired, we can assume that $u_{i+1}(r) = 0$, because if there were an unpaired r, it would be paired with w_{i+1}. Thus $o_{i+1}(r+1)$ is the sum of the number of unpaired $r+1$'s ($u_{i+1}(r+1)$) and the number of paired $r+1$'s ($o_{i+1}(r)$). We then have $u_{i+1}(r+1) = o_{i+1}(r+1) - o_{i+1}(r)$ but this is less than or equal to a by the lattice condition. Thus we must have that if $u_i(r+1) \leq a$, then $u_{i+1}(r+1) \leq a$, and ultimately, $u_n(r+1) \leq a$.

($\Longleftarrow$) Here we assume that $u_n(r+1) \leq a$ and show that for all i, $o_i(r+1) \leq o_i(r) + a$. We assume this is true for i and proceed by induction. Again, if $a \geq 1$, the base case $i = 1$ is trivial. If $k = 1$ and $a = 0$, then by assumption there are no unpaired $r+1$'s, so $o_1(r+1) = 0 \leq o_1(r) + a$. Now assume $o_i(r+1) \leq o_i(r) + a$. If $w_{i+1} \neq r, r+1$, then $o_{i+1}(r) = o_i(r)$ and $o_{i+1}(r+1) = o_i(r+1)$ so it is clear that $o_{i+1}(r+1) \leq o_{i+1}(r) + a$. If $w_{i+1} = r$, then we have $o_{i+1}(r+1) = o_i(r+1) \leq o_i(r) + a + 1 = o_{i+1}(r) + a$. That leaves us with the case that $w_{i+1} = r+1$ and thus $o_{i+1}(r+1) = o_i(r+1) + 1$ and $o_{i+1}(r) = o_i(r)$. If $o_i(r+1) < o_i(r) + a$, it is trivial that $o_{i+1}(r+1) \leq o_{i+1}(r) + a$, so consider the case in which $o_i(r+1) = o_i(r) + a$. Since $r+1$'s can only be paired with r's that come before them, at most $o_i(r)$ of the $r+1$'s in $w_1 w_2 \cdots w_i$ can be paired. Thus $u_i(r+1) \geq o_i(r+1) - o_i(r) = a$. But since we are assuming that $w(T)$ has at most a unpaired $r+1$'s, we must have $u_i(r+1) = a$ and $u_i(r) = 0$. Because of this, there are no r's to pair $w_{i+1} = r+1$ with, so we must have $u_{i+1}(r+1) = a+1$, but this contradicts our assumption. Hence if $w_{i+1} = r+1$, we must have $o_i(r+1) < o_i(r) + a$ and hence that $o_{i+1}(r+1) \leq o_{i+1}(r) + a$. ∎

Given all this, we may now state the following theorem.

Theorem 5.22. *Given a partition $\lambda \vdash n$,*

$$C^{(k)}_{D(\lambda),\lambda'} = \sum_{\substack{\mu \vdash n \\ l(\mu) = D(\lambda) \\ k \mid \mu_i}} \binom{n}{\mu_1, \ldots, \mu_{D(\lambda)}} K^{-1}_{\mu,\lambda'} = \sum_{\substack{\mu \vdash n \\ l(\mu) = D(\lambda) \\ k \mid \mu_i}} \sum_{T \in SRH^{(k)}_{\mu,\lambda'}} sgn(T),$$

where $SRH^{(k)}_{\mu,\lambda'}$ is the set of special rim hook tabloids of shape λ' and type μ which are hook k-lattice.

Proof. We perform the following involution θ on the objects in $\mathcal{O}_{ks}$. For a given object T, consider $w(T)$ from left to right and look for the first violation of the hook k-lattice condition.

- If there is no violation, then T is hook k-lattice and $\theta(T) = T$.
- If there is a violation, let $r+1$ be the first letter to violate the hook k-lattice condition. Then by Proposition 5.21, there are at least $s_{r+1} - s_r$ r-unpaired $r+1$'s in $w(T)$ and k divides $s_{r+1} - s_r$. Then set $\theta(T)$ to be the object with $s(\theta(T)) = s(T)$ and $w(\theta(T)) = d_r w(T)$.

First, we show that this is a well-defined involution. If T is hook k-lattice, this is trivial, so assume not. Then there is a least $r+1$ such that there are at leftmost $s_{r+1}-s_r$ r-unpaired $(r+1)$'s in $w(T)$ and k divides $s_{r+1}-s_r$. By the first condition, d_r is defined for $w(T)$. By Proposition 5.19, the r-unpaired subwords of $w(T)$ and $d_r w(T)$ occupy the same positions, so the first violation of the lattice condition in $d_r w(T)$ will be in the same position as that of $w(T)$. Moreover, if the r-unpaired subword of $w(T)$ is $(r+1)^p r^q$, then that of $d_r w(T)$ is $(r+1)^{q+s_{r+1}-s_r} r^{p+s_r-s_{r+1}}$, and then that of $d_r d_r w(T)$ is $(r+1)^{p+s_r-s_{r+1}-(s_{r+1}-s_r)} r^{q+s_{r+1}-s_r+(s_r-s_{r+1})} = (r+1)^p r^q$, showing that θ is an involution. That the involution is sign-changing can be seen in the following way. We have $s(\theta(T)) = s(T)$ so the hooks in $\theta(T)$ begin in the same rows as the hooks in T do. However, the lengths of the hooks change. Let $P(r)$ be the number of r-pairs in $w(T)$ and therefore in $d_r w(T)$. Then the length of the hook h_r in T is $|h_r| = P(r) + q$ but the length of the corresponding hook h'_r in $\theta(T)$ is $|h'_r| = P(r) + p + s_r - s_{r+1}$. Similarly, the lengths of the hooks h_{r+1} and h'_{r+1} are $|h_{r+1}| = P(r) + q$ and $|h'_{r+1}| = P(r) + p + s_{r+1} - s_r$. We thus have $|h'_r| = |h_{r+1}| - (s_{r+1} - s_r)$ and $|h'_{r+1}| = |h_r| + (s_{r+1} - s_r)$, the same change of lengths as in the switching of the hooks h_r and h_{r+1}. In addition, given the lengths of some hooks, the rows in which each of them must begin, and the cells they must cover, there is only one way to place the hooks. Thus $\theta(T)$ has the same rim hook configuration as the result of switching h_r and h_{r+1} in T, and therefore $sgn(\theta(T)) = -sgn(T)$. The fixed points of the involution are then those objects T which are hook k-lattice, completing the proof. ■

In order to state our expressions for $C_{m,\lambda'}$ and $C^{(k)}_{m,\lambda'}$, we need the language of $r$border rim hook tabloids and hook shifting.

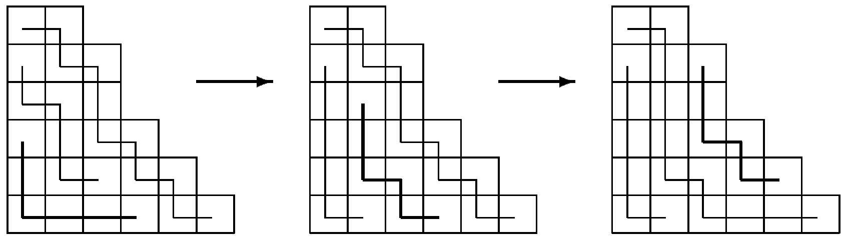

Figure 5.28: An example of hook shifting.

Hook Shifting, introduced by White [11], is a method of switching a hook from the inside of a tabloid to the northeastern border or from the border to the inside of the tabloid. If beginning from the inside, the hook is shifted upward and outward at a diagonal. Meanwhile, another hook is shifted down to occupy the cells that the shifted hook occupied. It is easiest to understand this procedure through an example, such as that given in Figure 5.28, where the bold hook is shifted to the outside.

An r-border rim hook tabloid of shape ν, $h_\nu = (h_1, h_2, \ldots, h_k)$, is defined in the following way. Place r hooks $h_1, h_2, \ldots, h_r$ into a Ferrers' diagram of shape ν such that

- h_1 is a rim hook of the Ferrers' diagram of shape ν, and for $1 < i \leq r$, h_i is a rim hook of the Ferrers' diagram of shape $\nu - (h_1, h_2, \ldots, h_{i-1})$, that is, the diagram of the shape of ν with the hooks $h_1, h_2, \ldots, h_{i-1}$ removed.

- h_i begins above h_j for $i < j$ the first cell of h_i is northwest of the first cell of h_j.

An example of a 4-border rim hook tabloid of shape $(1^2, 3^3, 4, 6^2)$ is given in Figure 5.29.

Let $|h_i|$ be the length of the hook h_i, that is the number of cells it occupies. Define the sign of a hook as for any rim hook, that is, $sgn(h_i) = (-1)^{r(h_i)-1}$ where $r(h_i)$ is the number of rows the hook occupies. Then the sign of the r-border rim hook tabloid is $sgn(H_\nu) = \prod_{i=1}^{r} sgn(h_i)$.

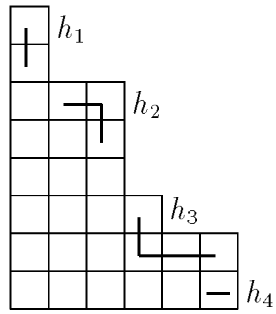

Figure 5.29: An example of a 4-border rim hook tabloid of shape $(1^2, 3^3, 4, 6^2)$.

Let $\mathcal{B}_\nu^r$ be the set of all r-border rim hook tabloids of shape ν. Let $\nu_{\overline{H}}$ be the shape $\nu - (h_1, h_2, \ldots, h_r)$. Then set $sh(H_\nu) = \nu/\nu_{\overline{H}}$. This is the shape formed by the cells in which the hooks lie. In the following, the shape of the r-border rim hook tabloid will correspond to the shape of $\alpha(\lambda)$ for some partition λ. Let $\lambda - sh(H_{\alpha(\lambda)})$ be the Ferrers' diagram of the shape which remains after the r-border rim hooks are removed from $\alpha(\lambda)$.

We are now ready to give our expressions for $C_{m,\lambda'}$ and $C_{m,\lambda'}^{(k)}$. The expression for $C_{m,\lambda'}$ is due to Beck and a proof can be found in [1].

Theorem 5.23. *Let λ be a partition of n, $D(\lambda) \leq m \leq D(\lambda) + \alpha_{D(\lambda)}$, and $r = m - D(\lambda)$. Then*

$$C_{m,\lambda'} = \sum_{H_{\alpha(\lambda)} = (h_1, \ldots, h_r) \in \mathcal{B}_\nu^r} sgn(H_{\alpha(\lambda)}) \binom{n}{|h_1|, \ldots, |h_r|} \times (-1)^{|\alpha(\lambda - sh(H_{\alpha(\lambda)}))|} f^{\gamma(\lambda - sh(H_{\alpha(\lambda)}))/\delta(\lambda - sh(H_{\alpha(\lambda)}))},$$

and

$$C_{m,\lambda'}^{(k)} = \sum_{\substack{H_{\alpha(\lambda)} = (h_1, \ldots, h_r) \in \mathcal{B}_\nu^r \\ k | h_i}} sgn(H_{\alpha(\lambda)}) \binom{n}{|h_1|, \ldots, |h_r|} \times \sum_{\substack{\omega \vdash |\nu| \\ l(\omega) = D(\nu) \\ k | \omega_i}} \sum_{T \in SRH_{\omega,\nu'}^{(k)}} sgn(T), \quad (5.16)$$

where $SRH^{(k)}_{\omega,\nu'}$ is the set of special rim hook tabloids of shape ν' and type ω which are hook k-lattice.

Proof. We will prove (5.16). To do this, we interpret the expressions in (5.15) and (5.16) as two sets of signed objects, then give a bijection between the two sets to show that the expressions are equal.

First, we can express $C^{(k)}_{m,\lambda'}$ as

$$C^{(k)}_{m,\lambda'} = \sum_{\substack{\mu \vdash n \\ l(\mu)=m \\ k|\mu_i}} \sum_{T \in SRH_{\mu,\lambda'}} \binom{n}{\mu_1, \ldots, \mu_m} sgn(T),$$

where $SRH_{\mu,\lambda'}$ is the set of special rim hook tabloids of shape λ' and type μ. We interpret the right hand side of the above equation as a sum of signed objects $o \in \mathcal{O}_{sA}$. These objects are special rim hook tabloids of shape λ' and type μ such that $l(\mu) = m$ and each part of μ is divisible by k. The binomial coefficient fills the cells of the objects with the integers $1, 2, \ldots, n$ such that each integer appears exactly once, and the integers increase along each hook, from top to bottom and from left to right. Each hook is given a sign $sgn(h) = (-1)^{r(h)-1}$ where $r(h)$ is the number of rows occupied by the hook h. The sign of an object o is then defined by $sgn(o) = \prod_{h \in o} sgn(h)$. Note that this is the same as the sign of a rim hook tabloid. Thus we can see that

$$\sum_{\substack{\mu \vdash n \\ l(\mu)=m \\ k|\mu_i}} \sum_{T \in SRH_{\mu,\lambda'}} \binom{n}{\mu_1, \ldots, \mu_m} sgn(T) = \sum_{o \in \mathcal{O}_{sA}} sgn(o).$$

We now consider the right hand side of the equation (5.16), letting ν be the shape $\lambda - H_{\alpha(\lambda)}$. By Theorem 5.22, this expression is equal to

$$\sum_{\substack{H_{\alpha(\lambda)}=(h_1,\ldots,h_r)\in\mathcal{B}^r_\nu \\ k|h_i}} sgn(H_{\alpha(\lambda)}) \binom{n}{|h_1|, \ldots, |h_r|}$$
$$\times \sum_{\substack{\omega \vdash |\nu| \\ l(\omega)=D(\nu) \\ k|\omega_i}} \sum_{T \in SRH_{\omega,\nu'}} \binom{|\nu|}{\omega_1, \ldots, \omega_{D(\nu)}} sgn(T).$$

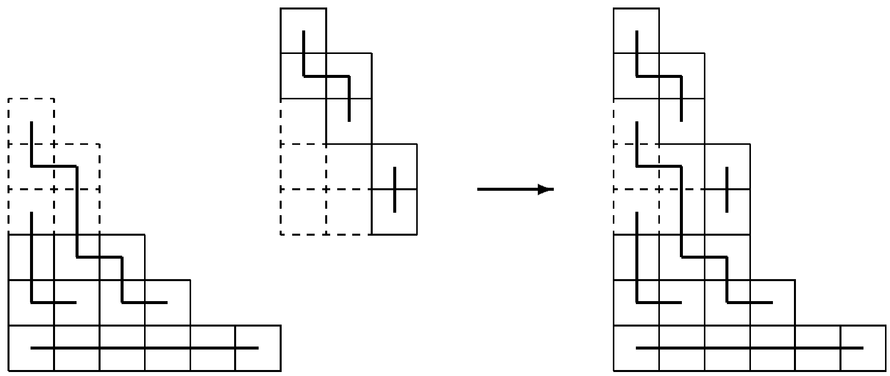

Figure 5.30: An example of an object in $\mathcal{O}_{sB}$.

We interpret this as a sum of signed objects $o \in \mathcal{O}_{sB}$. The objects are r-border rim hook tabloids $H_{\alpha(\lambda)}$ of shape $\alpha(\lambda)$ such that each hook has length divisible by k, and special rim hook tabloids of shape ν' such that the special rim hook tabloid is filled with $D(\nu) = D(\lambda)$ hooks each of which has length divisible by k. The shapes are joined into a single object as shown in Figure 5.30.

The multinomial coefficient $\binom{n}{|h_1|,\ldots,|h_r|}$ fills the the r-border rim hooks with integers from $1, 2, \ldots, n$ such that the integers increase along the hooks, it also leaves $n-|h_1|-|h_2|-\cdots-|h_r| = |\nu'|$ integers unused. The multinomial coeffiecient $\binom{|\nu|}{\omega_1,\ldots,\omega_{D(\nu)}}$ then fills the hooks of the special rim hook tabloid with the remaining integers. The sign of an object is defined by $sgn(o) = \prod_{h\in o} sgn(h)$ where the sign of a hook is $sgn(h) = (-1)^{r(h)}$ and $r(h)$ is the number of rows occupied by the hook h. Thus it is clear that we can write

$$\sum_{\substack{H_{\alpha(\lambda)}=(h_1,\ldots,h_r)\in\mathcal{B}^r_\nu \\ k|h_i}} sgn(H_{\alpha(\lambda)})\binom{n}{|h_1|,\ldots,|h_r|} \times \sum_{\substack{\omega\vdash|\nu| \\ l(\omega)=D(\nu) \\ k|\omega_i}} \sum_{T\in SRH_{\omega,\nu'}} \binom{|\nu|}{\omega_1,\ldots,\omega_{D(\nu)}} sgn(T) = \sum_{o\in\mathcal{O}_{sB}} sgn(o).$$

We now give a bijection between the two sets of signed objects $\mathcal{O}_{sA}$ and $\mathcal{O}_{sB}$. Consider $a \in \mathcal{O}_{sA}$. Draw a diagonal line from the lower right corner of $\alpha(\lambda)$

in the Ferrers diagram of λ', extending down to the left. Then there are exactly $r = l(\mu) - D(\lambda)$ hooks that lie entirely above the diagonal in a. To see this, note that exactly $l(\alpha(\lambda))$ hooks cross the diagonal and at most $D(\lambda) - l(\alpha(\lambda))$ hooks start in the first column below the diagonal. All of these r hooks can be shifted to the border of $\alpha(\lambda)$. We order these hooks from bottom to top according to the row in which the hooks start. The hook that starts in the topmost cell is denoted by h_1, the next lowest is h_2, and so on. First, we shift h_1 to the border of $\alpha(\lambda)$. Then, shift h_2 to the border of $\alpha(\lambda) - h_1$, and so on. Because we shift the hooks along diagonals, the relative positions of the hooks will not change. We clearly obtain an object in $\mathcal{O}_{sB}$.

Now, consider an object $b \in \mathcal{O}_{sB}$. Label the r-border rim hooks from top to bottom, such that h_1 is the topmost hook and h_r is the bottommost. First shift h_r down and to the left so that it begins in the first column. Then shift h_{r-1} in the same way, and so on. An example of the bijection is given in Figure 5.31. In [1], Beck shows that since the movement of the hooks is along diagonals, no border rim hook begins on the dame diagonal as one of the $D(\lambda) = D(\nu)$ special rim hooks in the tabloid of shape ν' Because of considerations of space, we do not include the proof here.

■

5.4.2 q-analogs for the Schur Basis

It is also possible to give an expression for the image of the Schur basis under $\overline{\xi}_W$. The results follow.

Theorem 5.24. *Let $\overline{\xi}_W$ be the homomorphism defined in Definition 5.5. If $\lambda \vdash n$, then*

$$3^n[n]!\overline{\xi}_W(s_\lambda(X+Y+Z)) = \sum_{\sigma \in C_3 \S S_n} q^{inv_W(\sigma)} r_{W,\lambda}(\sigma), \tag{5.17}$$

$$3^n[n]!\overline{\xi}_W(s_\lambda(X+\epsilon Y+\epsilon^2 Z)) = \sum_{\sigma \in C_3 \S S_n} \epsilon(\sigma) q^{inv_W(\sigma)} r_{W,\lambda}(\sigma), \tag{5.18}$$

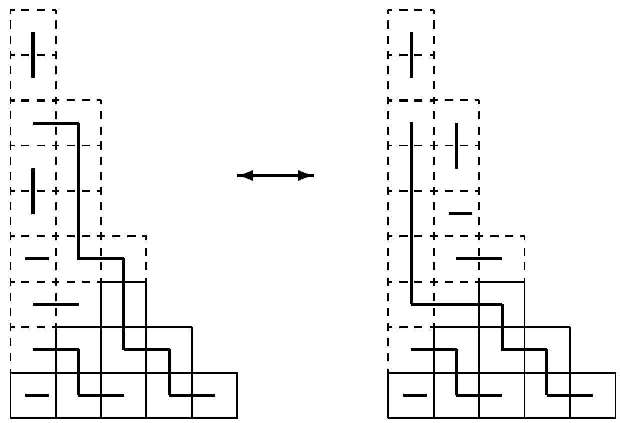

Figure 5.31: An example of the bijection between $\mathcal{O}_{sA}$ and $\mathcal{O}_{sB}$.

and

$$3^n[n]!\overline{\xi}_W(s_\lambda(X+\epsilon^2 Y+\epsilon Z)) = \sum_{\sigma\in C_3\S S_n} \epsilon(\sigma) q^{inv_W(\sigma)} r_{W,\lambda}(\sigma), \tag{5.19}$$

where

$$r_{W,\lambda}(\sigma) = \sum_{\mu\vdash n} K^{-1}_{\mu,\lambda} x^{des_{W,\mu}(\sigma)},$$

$inv_W(\sigma)$ is the number of $C_3\S S_n$-inversions of σ, and $des_{W,\mu}(\sigma)$ is the number of $C_3\S S_n$ μ-descents of σ.

Proof. Begin with the following expression, given in [7].

$$s_\lambda = \sum_{\mu\vdash n} K^{-1}_{\mu,\lambda} h_\mu.$$

To prove (5.17), multiply by $3^n[n]!$ and apply $\overline{\xi}_W$ to get

$$\begin{aligned} 3^n[n]!\overline{\xi}_W(s_\lambda(X+Y+Z)) &= \sum_{\mu\vdash n} K^{-1}_{\mu,\lambda}\overline{\xi}_W(h_\mu(X+Y+Z)) \\ &= \sum_{\mu\vdash n} K^{-1}_{\mu,\lambda} \sum_{\sigma\in C_3\S S_n} q^{inv_W(\sigma)} x^{des_{W,\mu}(\sigma)} \\ &= \sum_{\sigma\in C_3\S S_n} q^{inv_W(\sigma)} r_{W,\lambda}(\sigma). \end{aligned}$$

The proofs of (5.18) and (5.19) are similar. ■

5.5 ξ_W Applied to Other Bases of Λ_{W_3}

Here, for the sake of completeness, we determine the images of the monomial and forgotten bases by writing them in terms of the power basis and then applying ξ_W. For example, using (4.29) we get

$$\begin{aligned}
&3^n n! \xi_W(m_\lambda(X+Y+Z)m_\mu(X+\epsilon Y+\epsilon^2 Z)m_\nu(X+\epsilon^2 Y+\epsilon Z)) \\
&\quad = \sum_{(\alpha,\beta,\gamma)\vdash n} \sum_{f\in\mathcal{F}^{\lambda,\mu,\nu}_{\alpha*\beta*\gamma}} (-1)^{l(\alpha)+l(\beta)+l(\gamma)-l(\lambda)-l(\mu)-l(\nu)} \epsilon^{2l^\mu(\beta)+l^\nu(\beta)+l^\mu(\gamma)+2l^\nu(\gamma)} w(f) \\
&\qquad\qquad \times \frac{3^n n!}{z_\alpha z_\beta z_\gamma} \xi_W(p_\alpha(X)p_\beta(Y)p_\gamma(Z)).
\end{aligned}$$

A similar approach is taken with the forgotten basis. We then have the following theorem.

Theorem 5.25. *If $(\lambda,\mu,\nu)\vdash n$, then*

$$\begin{aligned}
&3^n n! \xi_W(m_\lambda(X+Y+Z)m_\mu(X+\epsilon Y+\epsilon^2 Z)m_\nu(X+\epsilon^2 Y+\epsilon Z)) = \\
&\quad = \sum_{(\alpha,\beta,\gamma)\vdash n} \sum_{f\in\mathcal{F}^{\lambda,\mu,\nu}_{\alpha*\beta*\gamma}} (-1)^{l(\alpha)+l(\beta)+l(\gamma)-l(\lambda)-l(\mu)-l(\nu)} \epsilon^{2l^\mu(\beta)+l^\nu(\beta)+l^\mu(\gamma)+2l^\nu(\gamma)} w(f) \\
&\qquad\qquad \times \sum_{\sigma\in C_{(\alpha,\beta,\gamma)}} x^{de_W(\sigma)}.
\end{aligned}$$

and

$$\begin{aligned}
&3^n n! \xi_W(f_\lambda(X+Y+Z)f_\mu(X+\epsilon Y+\epsilon^2 Z)f_\nu(X+\epsilon^2 Y+\epsilon Z)) = \\
&\sum_{(\alpha,\beta,\gamma)\vdash n} \sum_{f\in\mathcal{F}^{\lambda,\mu,\nu}_{\alpha*\beta*\gamma}} (-1)^{n-l(\lambda)-l(\mu)-l(\nu)} \epsilon^{2l^\mu(\beta)+l^\nu(\beta)+l^\mu(\gamma)+2l^\nu(\gamma)} w(f) \sum_{\sigma\in C_{(\alpha,\beta,\gamma)}} x^{de_W(\sigma)}.
\end{aligned}$$

5.6 The Permutation Enumeration of $C_k \S S_n$

Here we will indicate how the previous results for $C_3\S S_n$ can be extended to arbitrary wreath products $C_k\S S_n$. We begin by defining a number of statistics on elements of $C_k \S S_n$. If $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in C_k\S S_n$, the sign of the element σ_i is denoted $\epsilon(\sigma_i)$ and the sign of σ is the product $\epsilon(\sigma) = \prod_{i=1}^n \epsilon(\sigma_i)$. The number of *$C_k\S S_n$-descents* of σ is given by

$$des_{W_k}(\sigma) = |\{i : 1 \le i \le n-1, \epsilon(\sigma_1) = \epsilon(\sigma_{i+1}), \sigma_i > \sigma_{i+1}\}|.$$

Given a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$, the number of $C_k\S S_n$ *λ-descents*, denoted $des_{W_k,\lambda}(\sigma)$ is defined in the following way. Write σ in one-line notation and break it into segments of lengths $\lambda_1, \lambda_2, \ldots, \lambda_l$. Then count only the $C_3\S S_n$-descents which occur with i and $i+1$ in the same segment. The number of *$C_k\S S_n$-inversions* is given by

$$inv_{W_k}(\sigma) = |\{(i,j) : 1 \le i < j \le n, \sigma_i >_\Gamma \sigma_j\}|,$$

where , is the partial order

$$1 \equiv \epsilon 1 \equiv \cdots \equiv \epsilon^{k-1}1 <_\Gamma 2 \equiv \epsilon 2 \equiv \cdots \equiv \epsilon^{k-1}2 <_\Gamma \cdots <_\Gamma n \equiv \epsilon n \equiv \cdots \equiv \epsilon^{k-1}n.$$

The number of *$C_k\S S_n$-descedances* of the element σ is defined on the cycles of σ. Write σ in cycle notation as

$$\sigma = (\sigma_{1_1}, \sigma_{1_2}, \ldots, \sigma_{1_{l_1}})(\sigma_{2_1}, \sigma_{2_2}, \ldots, \sigma_{2_{l_2}})\cdots(\sigma_{m_1}, \sigma_{m_2}, \ldots, \sigma_{m_{l_m}}).$$

Then the number of $C_k\S S_n$-descadences of σ is given by

$$\begin{aligned} de_{W_k}(\sigma) = \sum_{i=1}^{k}(|\{j : 1 \le j \le l_i - 1, \epsilon(\sigma_{i_j}) = \epsilon(\sigma_{i_{j+1}}), \sigma_{i_j} >_\Gamma \sigma_{i_{j+1}}\}| \\ + \chi(\sigma_{i_{l_i}} >_\Gamma \sigma_{i_1})\chi(\epsilon(\sigma_{i_{l_i}}) = \epsilon(\sigma_{i_1}))), \end{aligned}$$

where $\chi(\text{statement})$ is 1 if the statement is true and 0 if it is false.

We also need definitions of analogs of ξ and $\overline{\xi}$ for $C_k\S S_n$. These are given below.

Definition 5.26. The homomorphism $\xi_{W_k} : \Lambda_{W_k} \longrightarrow \mathbf{Q}[\epsilon][x]$ is defined on the elementary basis by

$$\xi_{w_k}(e_n(\epsilon^{m\cdot 1}X^{(1)} + \cdots \epsilon^{m\cdot k}X^{(k)})) = \frac{\epsilon^{m\cdot 1}(\epsilon^{m\cdot 1} - \epsilon^{m\cdot 1}x)^{n-1} + \cdots + \epsilon^{m\cdot k}(\epsilon^{m\cdot k} - \epsilon^{m\cdot k}x)^{n-1}}{k^n n!}$$

for each $m = 1, 2, \ldots, k$.

Definition 5.27. The homomorphism $\overline{\xi}_{W_k} : \Lambda_{W_k} \longrightarrow \mathbf{Q}[q][\epsilon][x]$ is defined on the elementary basis by

$$\overline{\xi}_{W_k}(e_n(\epsilon^{m\cdot 1}X^{(1)} + \cdots \epsilon^{m\cdot k}X^{(k)})) = \frac{q^{\binom{n}{2}}\left(\epsilon^{m\cdot 1}(\epsilon^{m\cdot 1} - \epsilon^{m\cdot 1}x)^{n-1} + \cdots + \epsilon^{m\cdot k}(\epsilon^{m\cdot k} - \epsilon^{m\cdot k}x)^{n-1}\right)}{k^n[n]!}$$

for each $m = 1, 2, \ldots, k$.

5.6.1 Λ_{W_k}-Homogeneous Bases Under ξ_{W_k}

When ξ_{W_k} and $\overline{\xi}_{W_k}$ are applied to the homogeneous basis, the following results are obtained.

Theorem 5.28. *Let ξ_{W_k} and $\overline{\xi}_{W_k}$ be the homomorphisms defined in Definitions 5.26 and 5.27. Then for $m = 1, 2, \ldots, k$, we have*

$$k^n n! \xi_{W_k}(h_n(\epsilon^{m\cdot 1}X^{(1)} + \cdots + \epsilon^{m\cdot k}X^{(k)})) = \sum_{\sigma \in C_k \S S_n} \epsilon(\sigma)^m x^{des_{W_k}(\sigma)}, \tag{5.20}$$

$$k^n n! \xi_{W_k}(h_\lambda(\epsilon^{m\cdot 1}X^{(1)} + \cdots + \epsilon^{m\cdot k}X^{(k)})) = \sum_{\sigma \in C_k \S S_n} \epsilon(\sigma)^m x^{des_{W_k,\lambda}(\sigma)}, \tag{5.21}$$

$$k^n n! \overline{\xi}_{W_k}(h_n(\epsilon^{m\cdot 1}X^{(1)} + \cdots + \epsilon^{m\cdot k}X^{(k)})) = \sum_{\sigma \in C_k \S S_n} \epsilon(\sigma)^m x^{des_{W_k}(\sigma)} q^{inv_{W_k}(\sigma)}, \tag{5.22}$$

and

$$k^n n! \overline{\xi}_{W_k}(h_\lambda(\epsilon^{m\cdot 1}X^{(1)} + \cdots + \epsilon^{m\cdot k}X^{(k)})) = \sum_{\sigma \in C_k \S S_n} \epsilon(\sigma)^m x^{des_{W_k,\lambda}(\sigma)} q^{inv_{W_k}(\sigma)}, \tag{5.23}$$

where $des_{W_k}(\sigma)$, $des_{W_k,\lambda}(\sigma)$, and $inv_{W_k}(\sigma)$ are the number of $C_k \S S_n$-descents, the number of $C_k \S S_n$ λ-descents, and the number of $C_k \S S_n$ inversions of σ, respectively.

We will sketch the proof of (5.20). The variations in the proof necessary to prove (5.21), (5.22), and (5.23) are similar to the variations in the proof of Theorem 5.2 necessary to prove (5.1), (5.2), (5.3), (5.4), (5.5), and Theorem 5.6.

Proof. We begin with the relation

$$h_n = \sum_{\mu \vdash n} (-1)^{n-l(\mu)} B_{\mu,(n)} e_\mu.$$

Multiply by $k^n n!$ and apply ξ_{W_k} to obtain

$$k^n n! \xi_{W_k}(h_n(\epsilon^{m\cdot 1}X^{(1)} + \cdots + \epsilon^{m\cdot k}X^{(k)}))$$

$$= \sum_{\mu \vdash n} (-1)^{n-l(\mu)} B_{\mu,(n)} k^n n! \prod_{i=1}^{l(\mu)} \frac{\epsilon^{m\cdot 1}(\epsilon^{m\cdot 1} - \epsilon^{m\cdot 1}x)^{\mu_i - 1} + \cdots + \epsilon^{m\cdot k}(\epsilon^{m\cdot k} - \epsilon^{m\cdot k}x)^{\mu_i - 1}}{k^{\mu_i}\mu_i!}$$

$$= \sum_{\mu \vdash n} \sum_{T \in \mathcal{B}_{\mu,(n)}} \binom{n}{\mu_1, \ldots, \mu_{l(\mu)}}$$

$$\times \prod_{i=1}^{l(\mu)} \left(\epsilon^{m\cdot 1}(\epsilon^{m\cdot 1}x - \epsilon^{m\cdot 1})^{\mu_i - 1} + \cdots + \epsilon^{m\cdot k}(\epsilon^{m\cdot k}x - \epsilon^{m\cdot k})^{\mu_i - 1}\right). \quad (5.24)$$

We interpret this as a sum of signed, weighted objects $o \in \mathcal{O}_{kh_n}$ similar to those in the proof of Theorem 5.2. They are μ-brick tabloids of shape (n), filled with the integers $1, 2, \ldots, n$ such that each integer is used exactly once and the integers decrease within each brick. Each brick is designated as an i-brick for some $i = 1, 2, \ldots, k$. Each cell c is weighted in the following way.

$$w(c) = \begin{cases} \epsilon^{mi}, & c \text{ is at the end of an } i\text{-brick}, \\ -\epsilon^{mi} \text{ or } \epsilon^{mi}x, & c \text{ is elsewhere in an } i\text{-brick}. \end{cases}$$

Define the weight of an object o by $w(o) = \prod_{c \in o} w(c)$. Then we can write the expression in (5.24) as $\sum_{o \in \mathcal{O}_{kh_n}} w(o)$.

We perform an involution on the objects. Traverse the row from left to right. At the first occurrence of one of the following, perform the appropriate operation.

- If a cell c has weight $-\epsilon^{mi}$, split the brick after c and change the weight of c from $-\epsilon^{mi}$ to $+\epsilon^{mi}$.
- If there is a decrease between the integer filling of the last cell c of a brick and the first cell of the next brick and both are i-bricks for some i, join the two bricks together and change the weight of c from $+\epsilon^{mi}$ to $-\epsilon^{mi}$.

The fixed points of the involution are μ-brick tabloids of shape (n) filled with the integers $1, 2, \ldots, n$ such that the integers decrease within each brick and increase between consecutive bricks of the same type. Each brick is designated as an i-brick for some $i = 1, 2, \ldots, k$. Each cell is weighted by the following

$$w(c) = \begin{cases} \epsilon^{mi}, & c \text{ is at the end of an } i\text{-brick}, \\ \epsilon^{mi}x, & c \text{ is elsewhere in an } i\text{-brick}. \end{cases}$$

Reading left to right, we consider the filling to be an element of $C_k \S S_n$ with each cell in an i-brick corresponding to an element with sign ϵ^i, the descents of the element have an x-weight and cells that do not correspond to a descent do not have an x-weight. Thus the x-weights count the $C_k \S S_n$-descents. In addition, each cell in an i-brick has a sign of $\epsilon^{mi} = (\epsilon^i)^m$. The sign of the element is obtained by contributing ϵ^i for each i-element. Thus the sign counted here is the m^{th} power of the sign of the element. This completes the proof of (5.20). ■

5.6.2 Λ_{W_k}-Power Symmetric Functions Under ξ_{W_k}

When ξ_{W_k} is applied to the power basis, the following result is obtained.

Theorem 5.29. *Let ξ_{W_k} be the homomorphism defined in Definition 5.26. If $(\lambda^{(1)}, \ldots, \lambda^{(k)}) \vdash n$, then*

$$\frac{k^n n!}{z_{\lambda^{(1)}} \cdots z_{\lambda^{(k)}}} \xi_{W_k}(p_{\lambda^{(1)}}(X^{(1)}) \cdots p_{\lambda^{(k)}}(X^{(k)})) = \sum_{\sigma \in C_{(\lambda^{(1)}, \ldots, \lambda^{(1)})}} x^{de_{W_k}(\sigma)},$$

where $C_{(\lambda^{(1)}, \ldots, \lambda^{(k)})}$ is the conjugacy class of $C_k \S S_n$ indexed by $(\lambda^{(1)}, \ldots, \lambda^{(k)})$ and $de_{W_k}(\sigma)$ is the number of $C_k \S S_n$-descedances of σ.

In order to prove Theorem 5.29, we need the following lemma regarding the transition matrix between the basis $p_\lambda^{(1)}(X^{(1)}) \cdots p_\lambda^{(k)}(X^{(k)})$ and the basis $e_{\lambda^{(1)}}(\epsilon^{1 \cdot 1} X^{(1)} + \cdots + \epsilon^{1 \cdot k} X^{(k)}) \cdots e_{\lambda^{(k)}}(\epsilon^{k \cdot 1} X^{(1)} + \cdots + \epsilon^{k \cdot k} X^{(k)})$.

Lemma 5.30. *If* $(\lambda^{(1)},\dots,\lambda^{(k)}),(\alpha^{(1)},\dots,\alpha^{(k)})\vdash n$*, then*

$$p_{\lambda^{(1)}}(X^{(1)})\cdots p_{\lambda^{(k)}}(X^{(k)}) = \sum_{f\in\mathcal{F}^{\alpha^{(1)},\dots,\alpha^{(k)}}_{\lambda^{(1)}*\cdots*\lambda^{(k)}}} \frac{(-1)^{n-l(\alpha^{(1)})-\cdots-l(\alpha^{(k)})}}{k^{l(\lambda^{(1)})+\cdots+l(\lambda^{(k)})}}\epsilon^{\sum_{s,t=1}^{k}-stl^{\alpha^{(t)}}(\lambda^{(s)})}$$
$$\times\, w(f)e_{\alpha^{(1)}}(\epsilon^{1\cdot 1}X^{(1)}+\cdots+\epsilon^{1\cdot k}X^{(k)})\cdots e_{\alpha^{(k)}}(\epsilon^{k\cdot 1}X^{(1)}+\cdots+\epsilon^{k\cdot k}X^{(k)}),$$

where $l^{\alpha^{(t)}}(\lambda^{(s)})$ *is the number of* $\alpha^{(t)}$*-rows that appear in* $\lambda^{(s)}$*.*

Proof. We begin with the expression

$$p_n = \sum_{\mu\vdash n}(-1)^{n-l(\mu)}w(B_{\mu,(n)})e_\mu.$$

For $t=1,2,\dots,k$, we then have

$$\epsilon^{t\cdot 1}p_n(X^{(1)})+\cdots+\epsilon^{t\cdot k}p_n(X^{(k)}) = p_n(\epsilon^{t\cdot 1}X^{(1)}+\cdots+\epsilon^{t\cdot k}X^{(k)}) =$$
$$\sum_{\alpha^{(t)}\vdash n}(-1)^{n-l(\alpha^{(t)})}w(B_{\alpha^{(t)},(n)})e_{\alpha^{(t)}}(\epsilon^{t\cdot 1}X^{(1)}+\cdots+\epsilon^{t\cdot k}X^{(k)}).$$

Given this, for each $s=1,2,\dots,k$, we have

$$\begin{aligned} kp_n(x^{(s)}) &= \sum_{t=1}^{k}\epsilon^{-st}p_n(\epsilon^{t\cdot 1}X^{(1)}+\cdots+\epsilon^{t\cdot k}X^{(k)}) \\ &= \sum_{t=1}^{k}\sum_{\alpha^{(t)}\vdash n}\epsilon^{-st}(-1)^{n-l(\alpha^{(t)})}w(B_{\alpha^{(t)},(n)})e_{\alpha^{(t)}}(\epsilon^{t\cdot 1}X^{(1)}+\cdots+\epsilon^{t\cdot k}X^{(k)}). \end{aligned}$$

We interpret this as a sum of signed, weighted objects. We have diagrams of shape $\lambda^{(1)}*\cdots*\lambda^{(k)}$ such that each row is filled with all $\alpha^{(i)}$-bricks for some $i=1,2,\dots,k$. We use the above expression to weight the bricks of the tabloid. The coefficient of $e_{\alpha^{(1)}}(\epsilon^{1\cdot 1}X^{(1)}+\cdots+\epsilon^{1k}X^{(k)})\cdots e_{\alpha^{(k)}}(\epsilon^{k1}X^{(1)}+\cdots+\epsilon^{kk}X^{(k)})$ in $k^{l(\lambda^{(1)})+\cdots+l(\lambda^{(k)})}p_{\lambda^{(1)}}(X^{(1)})\cdots p_{\lambda^{(k)}}(X^{(k)})$ is given by

$$(-1)^n\sum_{f\in\mathcal{F}^{\alpha^{(1)},\dots,\alpha^{(k)}}_{\lambda^{(1)}*\cdots*\lambda^{(k)}}} W_k(f),$$

where

$$W_k(f) = \prod_{b \in f} w_k(b)$$

is the product of the weights of the bricks, and

$$w_k(b) = \begin{cases} -\epsilon^{-st}|b|, & b \text{ is at the end of an } \alpha^{(t)}\text{-row in } \lambda^{(s)}, \\ -1, & \text{otherwise.} \end{cases}$$

Writing the above coefficient in terms of the usual weight gives

$$(-1)^n \sum_{f \in \mathcal{F}^{\alpha^{(1)},\dots,\alpha^{(k)}}_{\lambda^{(1)} * \cdots * \lambda^{(k)}}} (-1)^{l(\alpha^{(1)})+\cdots+l(\alpha^{(k)})} \epsilon^{\sum_{s,t=1}^{k} -st l^{\alpha^{(t)}}(\lambda^{(s)})} w(f),$$

where $l^{\alpha^{(t)}}(\lambda^{(s)})$ is the number of $\alpha^{(t)}$-rows appearing in $\lambda^{(s)}$. This completes the proof of Lemma 5.30. ■

We are now ready to prove Theorem 5.29.

Proof. We multiply the expression given in Lemma 5.30 by $\frac{k^n n!}{z_{\lambda^{(1)}} \cdots z_{\lambda^{(k)}}}$ and apply ξ_{W_k} to obtain the following expression.

$$\frac{k^n n!}{z_{\lambda^{(1)}} \cdots z_{\lambda^{(k)}}} \xi_{W_k}(p_{\lambda^{(1)}}(X^{(1)}) \cdots p_{\lambda^{(k)}}(X^{(k)})) =$$

$$\sum_{(\alpha^{(1)},\dots,\alpha^{(k)}) \vdash n} \sum_{f \in \mathcal{F}^{\alpha^{(1)},\dots,\alpha^{(k)}}_{\lambda^{(1)} * \cdots * \lambda^{(k)}}} \frac{k^n n! (-1)^{n - l(\alpha^{(1)}) - \cdots - l(\alpha^{(k)})}}{k^{l(\lambda^{(1)})+\cdots+l(\lambda^{(k)})} z_{\lambda^{(1)}} \cdots z_{\lambda^{(k)}}} \epsilon^{\sum_{s,t=1}^{k} -st l^{\alpha^{(t)}}(\lambda^{(s)})} w(f)$$

$$\times \prod_{m=1}^{k} \prod_{i=1}^{l(\alpha^{(m)})} \frac{\epsilon^{m \cdot 1}(\epsilon^{m \cdot 1} - \epsilon^{m \cdot 1} x)^{\alpha_i^{(m)}-1} + \cdots + \epsilon^{m \cdot k}(\epsilon^{m \cdot k} - \epsilon^{m \cdot k} x)^{\alpha_i^{(m)}-1}}{k^{\alpha_i^{(m)}} \alpha_i^{(m)}!}.$$

This in turn is equal to

$$\sum_{(\alpha^{(1)},\dots,\alpha^{(k)}) \vdash n} \sum_{f \in \mathcal{F}^{\alpha^{(1)},\dots,\alpha^{(k)}}_{\lambda^{(1)} * \cdots * \lambda^{(k)}}} \frac{1}{k^{l(\lambda^{(1)})+\cdots+l(\lambda^{(k)})} z_{\lambda^{(1)}} \cdots z_{\lambda^{(k)}}}$$

$$\times \epsilon^{\sum_{s,t=1}^{k} -st l^{\alpha^{(t)}}(\lambda^{(s)})} w(f) \binom{n}{\alpha_1^{(1)}, \dots, \alpha_{l(\alpha^{(1)})}^{(1)}, \dots, \alpha_1^{(k)}, \dots, \alpha_{l(\alpha^{(k)})}^{(k)}}$$

$$\times \prod_{m=1}^{k} \prod_{i=1}^{l(\alpha^{(m)})} \left(\epsilon^{m \cdot 1}(\epsilon^{m \cdot 1} x - \epsilon^{m \cdot 1})^{\alpha_i^{(m)}-1} + \cdots + \epsilon^{m \cdot k}(\epsilon^{m \cdot k} x - \epsilon^{m \cdot k})^{\alpha_i^{(m)}-1} \right).$$

If $b_{m,i}$ is the brick in f corresponding to $\alpha_i^{(m)}$, define

$$\hat{\alpha}_i^{(m)} = \begin{cases} \alpha_i^{(m)} - 1, & b_{m,i} \text{ is at the end of a row in } f, \\ \alpha_i^{(m)}, & \text{otherwise.} \end{cases}$$

Then set $\tilde{\alpha}^{(m)}(f) = (\hat{\alpha}_1^{(m)}, \hat{\alpha}_2^{(m)}, \dots, \hat{\alpha}_{l(\alpha^{(m)})}^{(m)})$. We may then rewrite the above as

$$\sum_{(\alpha^{(1)},\dots,\alpha^{(k)})\vdash n} \sum_{f \in \mathcal{F}_{\lambda^{(1)}*\cdots*\lambda^{(k)}}^{\alpha^{(1)},\dots,\alpha^{(k)}}} \frac{1}{k^{l(\lambda^{(1)})+\cdots+l(\lambda^{(k)})} z_{\lambda^{(1)}} \cdots z_{\lambda^{(k)}}} \epsilon^{\sum_{s,t=1}^{k} -stl^{\alpha^{(t)}}(\lambda^{(s)})}$$
$$\times\, n(n-1)\cdots(n - l(\lambda^{(1)}) - \cdots - l(\lambda^{(k)}) + 1) \binom{n - l(\lambda^{(1)}) - \cdots - l(\lambda^{(k)})}{\tilde{\alpha}^{(1)}(f) \cdots \tilde{\alpha}^{(k)}(f)}$$
$$\times \prod_{m=1}^{k} \prod_{i=1}^{l(\alpha^{(m)})} \left(\epsilon^{m\cdot 1}(\epsilon^{m\cdot 1}x - \epsilon^{m\cdot 1})^{\alpha_i^{(m)}-1} + \cdots + \epsilon^{m\cdot k}(\epsilon^{m\cdot k}x - \epsilon^{m\cdot k})^{\alpha_i^{(m)}-1} \right).$$

We interpret the above expression as a sum of signed, weighted objects $o \in \mathcal{O}_{kp}$. They are elements of $\mathcal{F}_{\lambda^{(1)}*\cdots*\lambda^{(k)}}^{\alpha^{(1)},\dots,\alpha^{(k)}}$, filled with the integers $1, 2, \dots, n$, such that each integer is used exactly once, the numbers decrease within each brick, and the smallest number in each row appears at the end of the row. In addition, one cell in each row is distinguished. Each brick is designated as an i-brick for some $i = 1, 2, \dots, k$. Each cell is given a weight according to the following rule. If c is a cell in an i-brick in an $\alpha^{(m)}$-row, then

$$w(c) = \begin{cases} \epsilon^{m(i-s)}, & c \text{ is at the end of a row in } \lambda^{(s)}, \\ \epsilon^{mi}, & c \text{ is at the end of a brick but not a row,} \\ -\epsilon^{mi} \text{ or } \epsilon^{mi}x, & \text{otherwise.} \end{cases}$$

We wish to ignore the distinguished cell and the order of the rows with in each of $\lambda^{(1)}, \dots, \lambda^{(k)}$, so we divide by $z_{\lambda^{(1)}} \cdots z_{\lambda^{(k)}}$.

We perform the following involution on the objects. Traverse the diagram, considering first $\lambda^{(1)}$, then $\lambda^{(2)}$, and so on, and in each part, considering the rows from top to bottom and within each row, considering the cells from left to right. Look for the first occurrence of one of the following and perform the appropriate operation.

- If a cell c has weight $-\epsilon^{mi}$, divide the brick after c and change the weight of c from $-\epsilon^{mi}$ to $+\epsilon^{mi}$.

- If there is a decrease from the integer filling of the last cell c in a brick to that of the filling of the first cell in the next brick and both bricks are i-bricks for some i and lie in the same row, join the two bricks together and change the weight of c from $+\epsilon^{mi}$ to $-\epsilon^{mi}$

The involution has fixed points with the following properties. They are elements of $\mathcal{F}^{\alpha^{(1)},\dots,\alpha^{(k)}}_{\lambda^{(1)}*\dots*\lambda^{(k)}}$, filled with the integers $1, 2, \dots, n$ such that the integers decrease within each brick and increase between consecutive bricks of the same type, with the smallest number in each row appearing at the end. The weight of a cell c in an i-brick in an $\alpha^{(m)}$-row is given by

$$w(c) = \begin{cases} \epsilon^{m(i-s)}, & c \text{ is at the end of a row in } \lambda^{(s)}, \\ \epsilon^{mi}, & c \text{ is at the end of a brick but not a row}, \\ \epsilon^{mi}x, & \text{otherwise}. \end{cases}$$

Define a *k-volution* to be a function from a set S to itself such that for any $s \in S$, $f^k(s) = s$. We now perform a k-volution on the fixed points of the previous involution. For $m = 1, 2, \dots, k-1$, change each $\alpha^{(m)}$-row into an $\alpha^{(m+1)}$-row. In addition, change each $\alpha^{(k)}$-row into an $\alpha^{(1)}$-row. All of this is to be done with the appropriate changes of weight. Apply this k-volution k times.

Consider an α-row in one of the objects, and say that its weight is w. Let b_i be the number of cells that appear in i-bricks in the row. If we assume that the row lies in $\lambda^{(s)}$, then when the row is an $\alpha^{(m)}$-row, its weight can be written as

$$\epsilon^{-ms+\sum_{i=1}^{k} mib_i} w.$$

The sum of these weights is

$$w \left(\sum_{m=1}^{k} \epsilon^{m\left(-s+\sum_{i=1}^{k} ib_i\right)} \right).$$

If $\sum_{i=1}^{k} ib_i \equiv s \pmod{k}$, then this sum is kw. Otherwise, the sum is 0. Thus we are only left with rows in $\lambda^{(s)}$ with $\sum_{i=1}^{k} ib_i \equiv s \pmod{k}$. It no longer matters what type of row we have, since it can be considered as an $\alpha^{(k)}$-row, so we must divide by $k^{l(\lambda^{(1)})+\cdots+l(\lambda^{(k)})}$.

Our objects are now diagrams of shape $\lambda^{(1)} * \cdots * \lambda^{(k)}$, filled with bricks of lengths the parts of $\alpha^{(1)}, \ldots, \alpha^{(k)}$. Each brick is designated as an i-brick for some i. The cells are filled with the integers $1, 2, \ldots, n$ such that each integer is used exactly once, the smallest integer in each row appears at the end of the row, the integers decrease within each brick, and they increase between consecutive bricks of the same type in the same row. The cells are weighted according to the following rule.

$$w(c) = \begin{cases} 1, & c \text{ is at the end of a brick,} \\ x, & \text{otherwise.} \end{cases}$$

Interpret each row as a cycle in a $C_k \S S_n$ element, where if a is the filling of a cell in an i-brick, then it corresponds to $\epsilon^i a$ in the $C_k \S S_n$ cycle. Within each cycle, decreases between elements of the same type are weighted by x. All other transitions are weighted by 1. Note that there can never be a decrease from the last cell of the row to the first cell, since the last cell is filled with the smallest integer in the row. Thus the x-weight counts the decedences of the cycle. In $\lambda^{(s)}$, we have cycles with $\sum_{i=1}^{k} ib_i \equiv s \pmod{k}$, so the sign of the cycle is ϵ^s. This holds for each s, so the element consisting of the cycles formed by all of the rows belongs to the conjugacy class indexed by $(\lambda^{(1)}, \ldots, \lambda^{(k)})$. ■

Conclusion

We have extended the results of Brenti, Beck and Remmel to analogous results for the wreath products $C_k \S S_n$. There are, however, many questions remaining which are related to the problems discussed in this text and which might be solved using combinatorial methods. First of all, we would like to find a more satisfactory expression for the image of $s_\lambda(X + \epsilon Y + \epsilon^2 Z)$ and $s_\lambda(X + \epsilon^2 + \epsilon Z)$ under ξ_W, and we would like to determine if there are other uses for the involution we used to switch hooks while preserving restricions on their lengths. In addition, the methods used here could perhaps be extended to other groups, such as the alternating group, the dihedral group, or other Coxeter groups. If they can be extended, one must find the best way to define statistics on elements of the groups that may less resemble permutations than the elements of the wreath products studied here. There also seems to be a connection between the type of permutation enumeration studied here and some classical identities regarding permutation statistics which deserves further study. We hope to pursue the solutions to some of these problems in the future.

Appendix A

The Permutation Enumeration of $C_3 \S S_n$ for Another Choice of Ordering

Here we state without proof the definitions and results for an ordering on the letters that make up elements of $C_3 \S S_n$ which is different than that in Chapter 5.

We define a partial ordering, Θ', on the letters by the following.

$$\begin{matrix} 1 <_{\Theta'} 2 <_{\Theta'} \cdots <_{\Theta'} n \\ \overline{1} <_{\Theta'} \overline{2} <_{\Theta'} \cdots <_{\Theta'} \overline{n} \end{matrix} \;<_{\Theta'} \overline{\overline{n}} <_{\Theta'} \overline{\overline{n-1}} <_{\Theta'} \cdots <_{\Theta'} \overline{\overline{1}}.$$

We will also make use of the following partial order that was also used in Chapter 5.

$$1 \equiv \overline{1} \equiv \overline{\overline{1}} <_{\Gamma} 2 \equiv \overline{2} \equiv \overline{\overline{2}} <_{\Gamma} \cdots <_{\Gamma} n \equiv \overline{n} \equiv \overline{\overline{n}}.$$

We use these partial orders to define statistics on elements of $C_3 \S S_n$. The number of modified $C_3 \S S_n$-*descents* is given by

$$des_{W'}(\sigma) = |\{i : 1 \leq i \leq n-1, \sigma_i >_{\Theta'} \sigma_{i+1}\}| + \chi(\epsilon(\sigma_n) = \epsilon^2).$$

If an element $\sigma = \sigma_1 \sigma_2 \cdots \sigma_n$ is divided into segments of lengths $\lambda_1, \lambda_2, \ldots, \lambda_{l(\lambda)}$ for some $\lambda \vdash n$, then the number of modified $C_3 \S S_n$ λ-*descents* is the number of modified descents, $des_{W',\lambda}$, which occur with both i and $i+1$ in the same segment. The number of modified $C_3 \S S_n$-*inversions* is given by

$$inv_{W'}(\sigma) = inv_W(\sigma) = |\{(i,j) : 1 \leq i < j \leq n, \sigma_i >_\Gamma \sigma_j\}|.$$

If we write σ in cyclic form as

$\sigma = (\sigma_{1_1}, \sigma_{1_2}, \ldots, \sigma_{1_{l_1}})(\sigma_{2_1}, \sigma_{2_2}, \ldots, \sigma_{2_{l_2}}) \cdots (\sigma_{k_1}, \sigma_{k_2}, \ldots, \sigma_{k_{l_k}})$, then the number of modified $C_3 \S S_n$-*descedances* is given by

$$de_{W'}(\sigma) = \sum_{i=1}^{k} \Big(|\{j : 1 \leq j \leq l_i - 1, \sigma_{i_j} >_{\Theta'} \sigma_{i_{j+1}}\}| + \chi(\sigma_{i_{l_i}} >_{\Theta'} \sigma_{i_1}) \Big).$$

We define modified versions of ξ_W and $\overline{\xi}_W$ as follows.

Definition A.1. Define the homomorphism $\xi_{W'} : \Lambda_{W_3} \longrightarrow \mathbf{Q}[x]$ on the elementary basis by

$$\xi_{W'}(e_n(X+Y+Z)) = \frac{(1-x)^{n-1} + (1-x)^{n-1} + x(x-1)^{n-1}}{3^n n!},$$

$$\xi_{W'}(e_n(X+\epsilon Y+\epsilon^2 Z)) = \frac{(1-x)^{n-1} + \epsilon(\epsilon - \epsilon x)^{n-1} + \epsilon^2 x(\epsilon^2 x - \epsilon^2)^{n-1}}{3^n n!},$$

and

$$\xi_{W'}(e_n(X+\epsilon^2 Y+\epsilon Z)) = \frac{(1-x)^{n-1} + \epsilon^2(\epsilon^2 - \epsilon^2 x)^{n-1} + \epsilon x(\epsilon x - \epsilon)^{n-1}}{3^n n!}.$$

Definition A.2. Define the homomorphism $\overline{\xi}_{W'} : \Lambda_{W_3} \longrightarrow (\mathbf{Q}[q])[x]$ on the elementary basis by

$$\overline{\xi}_{W'}(e_n(X+Y+Z) = \frac{q^{\binom{n}{2}}\left((1-x)^{n-1} + (1-x)^{n-1} + x(x-1)^{n-1}\right)}{3^n n!},$$

$$\overline{\xi}_{W'}(e_n(X+\epsilon Y+\epsilon^2 Z) = \frac{q^{\binom{n}{2}}\left((1-x)^{n-1} + \epsilon(\epsilon - \epsilon x)^{n-1} + \epsilon^2 x(\epsilon^2 x - \epsilon^2)^{n-1}\right)}{3^n n!},$$

and

$$\overline{\xi}_{W'}(e_n(X+\epsilon^2 Y+\epsilon Z) = \frac{q^{\binom{n}{2}}\left((1-x)^{n-1} + \epsilon^2(\epsilon^2 - \epsilon^2 x)^{n-1} + \epsilon x(\epsilon x - \epsilon)^{n-1}\right)}{3^n n!}.$$

We are now ready to state the results for this modified version.

When $\xi_{W'}$ is applied to the homogeneous basis we have the following. Note that here, $\lambda \vdash n$.

$$3^n n! \xi_{W'}(h_n(X+Y+Z)) = \sum_{\sigma \in C_3 \S S_n} x^{des_{W'}(\sigma)}.$$

$$3^n n! \xi_{W'}(h_n(X+\epsilon Y+\epsilon^2 Z)) = \sum_{\sigma \in C_3 \S S_n} \epsilon(\sigma) x^{des_{W'}(\sigma)}.$$

$$3^n n! \xi_{W'}(h_n(X+\epsilon^2 Y+\epsilon Z)) = \sum_{\sigma \in C_3 \S S_n} \overline{\epsilon(\sigma)} x^{des_{W'}(\sigma)}.$$

$$3^n n! \xi_{W'}(h_\lambda(X+Y+Z)) = \sum_{\sigma \in C_3 \S S_n} x^{des_{W',\lambda}(\sigma)}.$$

$$3^n n! \xi_{W'}(h_\lambda(X+\epsilon Y+\epsilon^2 Z)) = \sum_{\sigma \in C_3 \S S_n} \epsilon(\sigma) x^{des_{W',\lambda}(\sigma)}.$$

$$3^n n! \xi_{W'}(h_\lambda(X+\epsilon^2 Y+\epsilon Z)) = \sum_{\sigma \in C_3 \S S_n} \overline{\epsilon(\sigma)} x^{des_{W',\lambda}(\sigma)}.$$

For the q-analogs, if we apply $\overline{\xi}_{W'}$ to the homogeneous basis, the results are as follows. Again, $\lambda \vdash n$.

$$3^n [n]! \overline{\xi}_{W'}(h_n(X+Y+Z)) = \sum_{\sigma \in C_3 \S S_n} x^{des_{W'}(\sigma)} q^{inv_{W'}(\sigma)}.$$

$$3^n [n]! \overline{\xi}_{W'}(h_n(X+\epsilon Y+\epsilon^2 Z)) = \sum_{\sigma \in C_3 \S S_n} \epsilon(\sigma) x^{des_{W'}(\sigma)} q^{inv_{W'}(\sigma)}.$$

$$3^n [n]! \overline{\xi}_{W'}(h_n(X+\epsilon^2 Y+\epsilon Z)) = \sum_{\sigma \in C_3 \S S_n} \overline{\epsilon(\sigma)} x^{des_{W'}(\sigma)} q^{inv_{W'}(\sigma)}.$$

$$3^n [n]! \overline{\xi}_{W'}(h_\lambda(X+Y+Z)) = \sum_{\sigma \in C_3 \S S_n} x^{des_{W',\lambda}(\sigma)} q^{inv_{W'}(\sigma)}.$$

$$3^n [n]! \overline{\xi}_{W'}(h_\lambda(X+\epsilon Y+\epsilon^2 Z)) = \sum_{\sigma \in C_3 \S S_n} \epsilon(\sigma) x^{des_{W',\lambda}(\sigma)} q^{inv_{W'}(\sigma)}.$$

$$3^n [n]! \overline{\xi}_{W'}(h_\lambda(X+\epsilon^2 Y+\epsilon Z)) = \sum_{\sigma \in C_3 \S S_n} \overline{\epsilon(\sigma)} x^{des_{W',\lambda}(\sigma)} q^{inv_{W'}(\sigma)}.$$

When $\xi_{W'}$ is applied to the power basis the result is

$$\frac{3^n n!}{z_\lambda z_\mu z_\nu}\xi_{W'}(p_\lambda(X)p_\mu(Y)p_\nu(Z)) = \sum_{\sigma \in C_{(\lambda,\mu,\nu)}} x^{de_{W'}(\sigma)}.$$

For the q-analog, when $\overline{\xi}_{W'}$ is applied to the power basis, the result is

$$\begin{aligned}
&3^n[n]!\overline{\xi}_{W'}(p_\lambda(X)p_\mu(Y)p_\nu(Z)) = \\
&\quad \sum_{\substack{\sigma\in\overline{C_3\S S_n}(\lambda,\mu,\nu)\\ \epsilon(\sigma_n)=1,\epsilon}} x^{des_{W',(\lambda\cup\mu\cup\nu)}(\sigma)} q^{inv_{W'}(\sigma)} \prod_{i=1}^{l(\lambda)} \left(x^{1-t_i(\sigma)}\left(x^{t_i(\sigma)}-(x-1)^{t_i(\sigma)}\right)\right) \\
&\quad \times \prod_{i=1}^{l(\mu)} \left(x^{1-u_i(\sigma)}\left(x^{u_i(\sigma)}-(x-1)^{u_i(\sigma)}\right)\right) \prod_{i=1}^{l(\nu)} \left(x^{1-v_i(\sigma)}\left(x^{v_i(\sigma)}-(x-1)^{v_i(\sigma)}\right)\right) \\
&+ \sum_{\substack{\sigma\in\overline{C_3\S S_n}(\lambda,\mu,\nu)\\ \epsilon(\sigma_n)=\epsilon^2}} x^{des_{W',(\lambda\cup\mu\cup\nu)}(\sigma)-1} q^{inv_{W'}(\sigma)} \prod_{i=1}^{l(\lambda)} \left(x+((2t_i(\sigma)+1)x-1)(1-x)^{t_i(\sigma)}\right) \\
&\qquad \times \prod_{i=1}^{l(\mu)} \left(x+((2u_i(\sigma)+1)x-1)(1-x)^{u_i(\sigma)}\right) \\
&\qquad\qquad \prod_{i=1}^{l(\nu)} \left(x+((2v_i(\sigma)+1)x-1)(1-x)^{v_i(\sigma)}\right),
\end{aligned}$$

where $\overline{C_3\S S_n}(\lambda,\mu,\nu)$ is the set of all $\sigma = \sigma_1,\sigma_2,\ldots,\sigma_n \in C_3\S S_n$ such that if σ is broken up into segments of lengths $\lambda_1,\lambda_2,\ldots,\lambda_{l(\lambda)}$, $\mu_1,\mu_2,\ldots,\mu_{l(\mu)}$, $\nu_1,\nu_2,\ldots,\nu_{l(\nu)}$, in that order, then each segment corresponding to a part of λ has total sign 1, each segment corresponding to a part of μ has total sign ϵ, and each segment corresponding to a part of ν has total sign ϵ^2, and where $t_i(\sigma)$, $u_i(\sigma)$, and $v_i(\sigma)$ denote the lengths of the cofinal decreasing sequence of elements of the same type in the segment of σ corresponding to λ_i, μ_i, and ν_i, respectively.

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