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Stability properties for q-multiplicities and branching formulas for representations of the classical groups

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The dissertation of Jeb Faulkner Willenbring is approved, and it is acceptable in quality and form for publication on microfilm:

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To Shari

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ABSTRACT OF THE DISSERTATION

Stability properties for q-multiplicities and branching formulas for representations of the classical groups

by

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By q-multiplicity we mean the generalization of a multiplicity formula for an irreducible representation in a graded space to a generating function for the multiplicity in the graded components. The q-multiplicity refines the (non graded) multiplicity formula. The main result of this thesis is a stable range in the space of harmonic polynomials associated to the $(GL(n), O(n))$ case of the Kostant-Rallis theorem. In the stable range the q-multiplicity is deduced from certain symmetric function identities and Littlewood's restriction rules. Chapter 3 gives an alternative proof of Littlewood's restriction rules from Howe duality and a classification of unitary highest weight modules due to Enright, Howe and Wallach.

For $n \geq 3$, the q-multiplicity for the spherical harmonics is given both for the $(GL(n), O(n))$ and the $(SL(n), SO(n))$ cases. The full q analog of the Kostant-Rallis theorem is described in detail for the symmetric pair $(SL(4), SO(4))$. The significance of this example is that it has implications in the study of entanglement of the mixed 2 qubit states in quantum computation. In chapter 2, a problem from classical invariant theory is addressed. Specifically, the complex orthogonal group acts on the $n \times n$ matrices by restricting the adjoint action of $GL(n, \mathbb{C})$. This gives us an action on the ring of complex valued polynomial functions on the matrices. A combinatorial description of the Hilbert series for the invariant polynomials under this action is given.

Chapter 1

Introduction

There are two parallel themes that are of importance in this thesis. The first is the interplay between combinatorics and the representation theory of the classical groups. The second comes from the fact that the structure of certain infinite dimensional representations of Lie algebras provides information about finite dimensional representation theory. In order to be more precise, let W be a completely reducible representation of a group G. For any irreducible representation V of G, we will call the largest number of copies of V in a direct sum that can be embedded as a subrepresentation of W. By Schur's lemma, this number is equal to the dimension of the space of linear maps between V and W which commute with the action of G. If W is a graded representation with the d^{th} graded component denoted W^d, the graded multiplicity of V in W is to be the formal power series,

$$\sum_{d \geq 0} \dim Hom_G(E, W^d)\, q^d$$

Here we take q to be an indeterminate. Observe that the multiplicity of a representation is the graded multiplicity evaluated at 1. One idea behind this formalization is that often the multiplicity of a representation has a purely combinatorial meaning and the graded multiplicity becomes a natural q analog of the (non-graded) multiplicity. This supplies us with an important interaction between combinatorics and representation theory.

We will now give several examples of this phenomenon. We first need some notation. Let E be a representation of G. Let $\mathcal{P}(E)$ be the representation of G consisting of the complex valued polynomial functions on E, under the usual action. Observe that we have a natural grading by degree, and let $\mathcal{P}^d(E)$ denote the degree d homogeneous polynomial functions on E. Lastly, by $\mathcal{P}(E)^G$ we will mean the G-invariant polynomial functions.

Consider the case when G is a semi-simple linear algebraic group with Lie algebra $\mathfrak{g}$. G acts by the adjoint representation on $\mathfrak{g}$. Let $\mathcal{I}(\mathfrak{g})$ be the ideal generated by the G invariant polynomials on $\mathfrak{g}$ which vanish at the origin. Set:

$$\mathcal{I}^d(\mathfrak{g}) = \mathcal{P}^d(\mathfrak{g}) \cap \mathcal{I}(\mathfrak{g})$$

Let $\mathcal{H}^d(\mathfrak{g})$ be the G invariant complement of $\mathcal{I}^d(\mathfrak{g})$ in the representation $\mathcal{P}^d(\mathfrak{g})$. Set:

$$\mathcal{H}(\mathfrak{g}) = \bigoplus_d \mathcal{H}^d(\mathfrak{g})$$

In [14], B. Kostant proved that $\mathcal{P}(\mathfrak{g})$ is a free module over the G-invariant polynomials, $\mathcal{P}(\mathfrak{g})^G$. That is to say that,

$$\mathcal{P}(\mathfrak{g}) = \mathcal{P}(\mathfrak{g})^G \otimes \mathcal{H}(\mathfrak{g})$$

Furthermore, Kostant's result also gives the multiplicity of all irreducible representations of G in $\mathcal{H}(\mathfrak{g})$. Indeed, if T is a maximal torus in G and V is an irreducible regular (in the sense of algebraic groups) representation of G then the multiplicity of an irreducible representation V in $\mathcal{H}(\mathfrak{g})$ is the dimension of the space of T-invariant vectors in V, denoted V^T. In other words,

$$\dim Hom_G(V, \mathcal{H}(\mathfrak{g})) = \dim V^T$$

The space $\mathcal{H}(\mathfrak{g})$ has a grading from $\mathcal{P}(\mathfrak{g})$. Nineteen years later in [9], Hesselink gave a graded multiplicity as well. Another proof of this graded multiplicity formula is also given in [20].

Another case with a similar flavor is as follows: let θ denote a regular involution on G with differential at the identity (also denoted by) θ. Let K be the subgroup of G consisting of the fixed points of θ. Denote the Lie algebra of K by $\mathfrak{k}$. We take $\mathfrak{p}$ to be the -1 eigenspace of θ.

Let $\mathcal{I}(\mathfrak{p})$ be the ideal generated by the K invariant polynomials on $\mathfrak{p}$ which vanish at the origin. Set:

$$\mathcal{I}^d(\mathfrak{p}) = \mathcal{P}^d(\mathfrak{p}) \cap \mathcal{I}(\mathfrak{p})$$

Let $\mathcal{H}^d(\mathfrak{p})$ be the K invariant complement of $\mathcal{I}^d(\mathfrak{p})$ in the representation $\mathcal{P}^d(\mathfrak{p})$. Set:

$$\mathcal{H}(\mathfrak{p}) = \bigoplus_d \mathcal{H}^d(\mathfrak{p})$$

In [13], B. Kostant and S. Rallis prove the following theorem.

Theorem 1 (Kostant, Rallis) *The space of polynomial functions on $\mathfrak{p}$ is a free module over the ring of K invariant polynomials. That is,*

$$\mathcal{P}(\mathfrak{p}) \cong \mathcal{P}(\mathfrak{p})^K \otimes \mathcal{H}(\mathfrak{p})$$

and furthermore, as a representation of K, $\mathcal{H}(\mathfrak{p})$ is equivalent to the representation algebraically induced from the trivial representation of a subgroup M to K. Where M is the centralizer in K of an Abelian subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ maximal among such algebras both contained in $\mathfrak{p}$ and containing only semisimple elements.

A proof of this theorem can be found in [8]. This thesis will address the special case when $G = GL(n, \mathbb{C})$, $\theta(g) = (g^{-1})^T$ and hence $K = O(n, \mathbb{C})$. Let D_n denote the set of diagonal $n \times n$ complex matrices. In this special case, one can show that $\mathfrak{a} = \mathfrak{p} \cap D_n$ and $M = K \cap D_n$.

The fact that $\mathcal{H}(\mathfrak{p})$ is an induced representation and Frobenius reciprocity determine the multiplicity of an irreducible representation of K in the space of harmonics. Indeed, let V^M denote the space of M-invariant vectors in V, where V is an irreducible representation of K. Then,

$$\dim Hom_K(V, \mathcal{H}(\mathfrak{p})) = \dim V^M$$

The question which will be addressed in the thesis concerns the distribution of the above multiplicity in the graded components of $\mathcal{H}(\mathfrak{p})$. This would provide a natural q-analog of the above multiplicity formula. More specifically, if V^λ is an irreducible representation of K with highest weight λ then we will find

$$m_\lambda(q) = \sum_{d \geq 0} \dim Hom_K(V^\lambda, \mathcal{H}^d)\, q^d$$

Here a multiplicity formula is also given, but in general the formula for the graded multiplicity is not yet known. In [20] an important family of examples is given where we do have such a graded multiplicity formula. An example of a case not completely understood is the pair $(SL(n,\mathbb{C}), SO(n,\mathbb{C}))$. The first non-trivial example of this case is when $n = 4$. [20] gives the graded multiplicity for this case, but the techniques employed were special to that example. Chapter 6 of this work is in part a recapitulation of [20] with alternative proofs of some of the theorems.

It is interesting that the $n = 4$ case is also important for the subject of quantum computation. In fact, results in this direction have applications to other problems. In classical invariant theory, they aid in the calculation of Hilbert series, which are graded multiplicity formulae for the trivial representation. In [20], the Hilbert series is given for the $SO(4,\mathbb{C})$-invariants in the polynomial functions on the 4×4 complex matrices under the action of conjugation. This material is also reprinted for the convenience of the reader in chapter 6. Another direction of research is to generalize this result for all n. Some progress has been made in this direction and is mentioned in chapter 2.

The purpose of chapter 5 is to address part of the problem of finding the graded multiplicity in the harmonic polynomials for the $(GL(n,\mathbb{C}), O(n,\mathbb{C}))$ case of the Kostant-Rallis theorem. Specifically, Chapter 5 addresses larger values of n by describing a certain stable range in which the graded components of the harmonic polynomials can be completely decomposed. The components in the stable range are precisely those of degree not greater than $\lfloor \frac{n}{2} \rfloor$. Thus a graded component of degree d is decomposed for all but finitely many n. The existence of the stable range is a consequence of a new interpretation of results of Enright, Howe and Wallach, (see [2]) and Howe's theory of dual pairs (see [10]). We will summarize this point of view shortly, but first consider the following consequence of this program in classical invariant theory.

Consider the action of the complex orthogonal group $O(n)$ on the $n \times n$ matrices, M_n, by restricting the adjoint action of $GL(n,\mathbb{C})$. This action gives us an action on the ring of complex valued polynomial functions on the $n \times n$ matrices, $\mathcal{P}(M_n)$. The polynomials of degree d, denoted $\mathcal{P}^d(M_n)$ form a finite dimensional representation of $O(n)$ and provide a graded module structure on $\mathcal{P}(M_n)$ as well as the subring of invariant polynomials, $\mathcal{P}(M_n)^{O(n)}$.

We can study this example by decomposing the $n \times n$ matrices into a direct sum of symmetric, SM_n and antisymmetric, AM_n subspaces. The dimension of $\mathcal{P}^d(M_n)^{O(n)}$ would follow from a complete graded decomposition of $\mathcal{P}(AM_n)$ and $\mathcal{P}(SM_n)$. In the former, the results of Hesselink (see [9]), imply such a graded decomposition, while in the latter a graded decomposition, which refines the non-graded decomposition is essentially the program just described. This is one reason that this case is the emphasis of this thesis.

From another point of view, it is shown in chapter 2 that for $0 \leq d \leq n$, $\dim \mathcal{P}^d(M_n)^{O(n)}$ is equal to the coefficient of q^d in,

$$\prod_{k \geq 1} \left(\frac{1}{1-q^k} \right)^{c_k}$$

Where c_k is the number of k vertex cyclic graphs with directed edges counted up to dihedral symmetry. The above formula gives a combinatorial interpretation of the Hilbert series for this ring.

An important observation for these types of questions is that often the representation we consider is the restriction of a representation of a larger group. This fact provides a connection with another problem. Given an irreducible representation of a group, how does it decompose if we restrict the action to a subgroup? Of particular importance are the representations of $GL(n) = GL(n, \mathbb{C})$. In [15] and [16], Littlewood gives a formula for the multiplicity of an irreducible representation of the group $O(n) = O(n, \mathbb{C})$ in certain irreducible representations of $GL(n)$. Results in chapter 3 give insight into when such formulae exist and what can be said in more generality. Littlewood's restriction formulae have received recent attention from another point of view in [7] and [6].

The problem of decomposing an irreducible, regular, $GL(n)$ representation into irreducible (regular) $O(n)$ representations by recasting the statement to one of infinite dimensional representation theory. Let $\{E^\mu\}_{\mu \in \mathcal{E}}$ and $\{F^\lambda\}_{\lambda \in \mathcal{F}}$ be representatives of the equivalence classes of irreducible regular representations of $O(n)$ and $GL(n)$ respectively. In [8] it is explained how to take these index sets to be non-negative integer partitions with at most n parts. For $\lambda \in \mathcal{F}$ and $\mu \in \mathcal{E}$, define the non-negative integer b^λ_μ to be the multiplicity of the representation E^μ in the representation F^λ restricted to $O(n)$.

$M_{n \times m}$ denotes the representation of $O(n)$ on the $n \times m$ matrices defined by matrix multiplication on the left. This fact in turn gives us a representation of $O(n)$ on

$\mathcal{P}(M^*_{n\times m})$. The algebra of operators commuting with the image of this representation is generated by a set of polynomial coefficient differential operators. These, under the usual bracket, span a Lie algebra isomorphic to the rank m symplectic Lie algebra, $sp(m) = sp(m,\mathbb{C})$.

The significance of studying the algebra of operators commuting with a representation is given by a theorem known as the "double commutant theorem" (see [8]). The double commutant theorem gives us a pairing of the irreducible $sp(m)$ representations, denoted E_μ, in $\mathcal{P}(M^*_{n\times m})$ with the irreducible regular representations of the group $O(n)$, occurring in $\mathcal{P}(M^*_{n\times m})$. The pairing in this case is part of Roger Howe's general duality theory (see [10]). For a recent treatment of these results see [8]. As an $O(n) \times sp(m)$ representation the space $\mathcal{P}(M^*_{n\times m})$ is a multiplicity free space. Indeed,

$$\mathcal{P}(M^*_{n\times m}) = \bigoplus_{\mu} E^\mu \otimes E_\mu \tag{1.1}$$

In this light, the isotypic component of an irreducible $O(n)$ representation has the structure of an $sp(m)$ representation. The irreducible $sp(m)$-representations in this space are (infinite dimensional) unitary highest weight representations. The interesting fact is that they have a structure which implies a new interpretation of the numbers b^λ_μ.

There is a rich literature on the structure of these representations, much of which can be traced back to [2]. A careful survey of this theory will give great insight into the restriction problem. One reason for this opinion is a consequence of the following line of reasoning. $GL(n)$ acts on $M_{n\times m}$ by left multiplication while $GL(m)$ acts by right multiplication. As a $GL(n) \times GL(m)$ representation, $\mathcal{P}(M^*_{n\times m})$ is a multiplicity free space decomposing as,

$$\mathcal{P}(M^*_{n\times m}) = \bigoplus_{\lambda} F^\lambda \otimes F^\lambda \tag{1.2}$$

In the above decomposition, F^λ is the irreducible representation of $GL(n,\mathbb{C})$ with highest weight λ. Observe that if we restrict the action of the first $GL(n)$ to the orthogonal group and write out the decomposition, we obtain,

$$\mathcal{P}(M^*_{n\times m}) = \bigoplus_{\lambda} \left(\bigoplus_{\mu} b^\lambda_\mu E^\mu \right) \otimes F^\lambda \tag{1.3}$$

Reordering the summands in the above leads us to a new decomposition of the space on the left.

$$\mathcal{P}(M^*_{n\times m}) = \bigoplus_{\mu} \left(E^{\mu} \otimes \bigoplus_{\lambda} b^{\lambda}_{\mu} F^{\lambda} \right) \tag{1.4}$$

$GL(m)$ is contained in the commutant of the algebra generated by the image of the $O(n)$ action, thus we have an action of $GL(m)$ on the modules E_{μ}. This deduction gives an expression for the $GL(m)$ decomposition of these $sp(m)$ representations, E_{μ} occurring in $\mathcal{P}(M^*_{n\times m})$. Indeed,

$$E_{\mu} = \bigoplus_{\lambda} b^{\lambda}_{\mu}\, F^{\lambda} \tag{1.5}$$

This interpretation of the modules E_{μ} allows for the development of the stable range of chapter 5 mentioned earlier. It is important to note the special case when $m = 1$, gives the classical example of decomposing the polynomials on the standard representation of $O(n)$ into spherical harmonics. More specifically, let the Laplacian be defined by,

$$\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial {x_i}^2}$$

We now define the degree d spherical harmonics,

$$H_d(\mathbb{C}^n) = \left\{ f \in P^d(\mathbb{C}^n) | \Delta(f) = 0 \right\}$$

We will abbreviate $H_d = H_d(\mathbb{C}^n)$.

The fact that the spherical harmonics are contained in the polynomials on the standard representation provides additional structure. In chapter 4 a theorem is proven which implies a graded multiplicity formula for the spherical harmonics in the space $\mathcal{H}(\mathfrak{p})$ for both the $(SL(n), SO(n))$ and the $(GL(n), O(n))$ case of the Kostant-Rallis theorem. In the former we have,

$$\sum_{k,d\geq 0} \dim Hom_{SO(n,\mathbb{C})}(H_k, \mathcal{H}^d(\mathfrak{p}))\, q^d\, t^k = \frac{1 + t^n q^{\binom{n}{2}}}{\prod_{1\leq i\leq n-1}(1 - q^i t^2)}.$$

While in the latter we have,

$$\sum_{k,d\geq 0} \dim Hom_{O(n,\mathbb{C})}(H_k, \mathcal{H}^d(\mathfrak{p}))\, q^d\, t^k = \frac{1}{\prod_{1\leq i\leq n-1}(1 - q^i t^2)}.$$

Chapter 2

Stable result

2.1 The Problem

Let's recall the standard notation. Let M_n denote the vector space of $n \times n$ matrices with entries from $\mathbb{C}$. Let GL_n the group of invertible matrices in M_n, and denote by O_n the subgroup of GL_n consisting of orthogonal matrices, that is,

$$O_n = \left\{ g \in GL_n | g^T g = I \right\}$$

Denote the ring of polynomial functions on M_n by $\mathcal{P}(M_n)$. O_n acts linearly on M_n by conjugation, which gives a linear group action on $\mathcal{P}(M_n)$ by, $g.f\,(X) = f(g^{-1}Xg)$ for $g \in O_n$, $X \in M_n$. Let $\mathcal{P}(M_n)^{O_n}$ be the ring of invariants under this action. This action does not change the degree of f so we have a grading,

$$\mathcal{P}(M_n) = \bigoplus_{d \geq 0} \mathcal{P}^d(M_n) \tag{2.1}$$

where we take $\mathcal{P}^d(M_n)$ to be the degree d homogeneous polynomials on M_n. This also gives us a grading on the invariants of the action. Our problem is to find an explicit form for the Hilbert series of invariants. Let,

$$H_n(q) = \sum_{d \geq 0} \dim \mathcal{P}^d(M_n)^{O_n} q^d \tag{2.2}$$

Where: $\mathcal{P}(M_n)^{O_n}$ denotes the space of invariant polynomials of homogeneous degree d. For notation we will also define,

$$H(q,t) = \sum_{n\geq 0} H_n(q)t^n \tag{2.3}$$

Expressing these formal power series in a simpler form is the subject of this work. Specifically, $H_n(q)$ is to be written as a rational function of the form,

$$H_n(q) = \frac{a_0 + a_1 q + a_2 q^2 + \cdots + a_r q^r}{\prod_{1\leq i\leq k}(1-q^{e_i})} \tag{2.4}$$

Where: k, r, a_i, and e_i, are non-negative integers depending only on n. In general, k is the Krull dimension of the ring of invariants. Or equivalently, the dimension of the variety defined by the invariant polynomials without constant term.

Note that for $n = 1, 2$, or 3 it is not hard to study the invariants directly because the representations of the orthogonal group in low rank are well understood.

$$H_1(q) = \frac{1}{1-q} \tag{2.5}$$

$$H_2(q) = \frac{1}{(1-q)(1-q^2)^2} \tag{2.6}$$

$$H_3(q) = \frac{1+q^6}{(1-q)(1-q^2)^2(1-q^3)^2(1-q^4)} \tag{2.7}$$

We refer the reader to [20] for the highly non-trivial and interesting case of $SO(4)$.

2.2 Littlewood-Richardson coefficients

Irreducible regular representations of the group GL_n (or for any connected reductive linear algebraic group for that matter) are indexed by highest weight vectors. The highest weight vectors for polynomial representations of GL_n are in one to one correspondence with non-negative integer partitions. For a detailed development see [8]. The irreducible regular representation of GL_n indexed by $\lambda = (\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_n)$ will be denoted by F^λ. The sum of the parts of a partition λ is denoted by $|\lambda|$, while the number of parts is $l(\lambda)$.

Definition 1 *Let F^μ and F^ν be irreducible GL_n representations with highest weights μ and ν respectfully. Define the coefficients $\{N^\lambda_{\mu\nu}\}$ by,*

$$F^\mu \otimes F^\nu = \bigoplus_\lambda N^\lambda_{\mu\nu} F^\lambda$$

The numbers $N^\lambda_{\mu,\nu}$ are called the Littlewood-Richardson coefficients. Although, these numbers are often defined as the structure constants for multiplication of the Schur basis of the ring of symmetric functions. We will discuss this point somewhat further in chapter 5. In the following we will derive yet another characterization in terms of representations of the symmetric group. This is a consequence of Schur-Weyl duality.

The irreducible representations of the symmetric group are in bijective correspondence with the partitions of m. A precise indexing of the representations of S_m is given by the *Young symmetrizers* (see [8], [4], [18], [17], [19], [11] etc.) So to each partition λ of m we can associate an irreducible representation, U_λ of S_m. The correspondence is implicitly stated in the following theorem:

Theorem 2 (Schur-Weyl Duality) *S_m acts on $\bigotimes^m \mathbb{C}^n$ by permutation of the tensor factors while GL_n acts on the same space diagonally. As a $GL_n \times S_m$ representation we have,*

$$\bigotimes^m \mathbb{C}^n = \bigoplus_{\substack{\lambda\,:\,|\lambda| = m \\ l(\lambda) \leq n}} F^\lambda \otimes U_\lambda \tag{2.8}$$

Where the U_λ are irreducible representations of S_m.

We now begin an argument establishing the interpretation of the Littlewood-Richardson coefficients in terms of the representations of the symmetric groups. First we set $m = m_1 + m_2$ for some non-negative integers m_1 and m_2 and apply theorem 2 to $\bigotimes^{m_1} \mathbb{C}^n$ and $\bigotimes^{m_2} \mathbb{C}^n$.

$$\bigotimes^{m_1+m_2} \mathbb{C}^n = \Big(\bigoplus_{\substack{\mu\,:\,|\mu| = m_1 \\ l(\mu) \leq n}} F^\mu \otimes U_\mu\Big) \otimes \Big(\bigoplus_{\substack{\nu\,:\,|\nu| = m_2 \\ l(\nu) \leq n}} F^\nu \otimes U_\nu\Big) \tag{2.9}$$

Next we expand the product and view $\bigotimes^{m_1+m_2} \mathbb{C}^n$ as a $GL_n \times GL_n \times S_{m_1} \times S_{m_2}$ representation.

$$\bigotimes^{m_1+m_2}\mathbb{C}^n = \bigoplus_{\substack{\mu,\nu \\ |\mu|=m_1 \\ |\nu|=m_2 \\ l(\mu),l(\nu)\leq n}} F^\mu \otimes F^\nu \otimes U_\mu \otimes U_\nu \tag{2.10}$$

Restrict the action of $GL_n \times GL_n$ to the diagonal GL_n and decompose into irreducible $GL_n \times S_{m_1} \times S_{m_2}$ representations. The isotypic component of the GL_n representation F^λ then has the structure of an $S_{m_1} \times S_{m_2}$ representation. That is,

$$\bigotimes^{m_1+m_2}\mathbb{C}^n = \bigoplus_{\substack{\mu,\nu,\lambda \\ |\mu|=m_1,|\nu|=m_2 \\ |\lambda|=m_1+m_2 \\ l(\lambda),l(\mu),l(\nu)\leq n}} N_{\mu\nu}^\lambda \left(U_\mu \otimes U_\nu\right) \otimes F^\lambda \tag{2.11}$$

Of course we also view $\bigotimes^{m_1+m_2}\mathbb{C}^n$ as an $GL_n \times S_{m_1+m_2}$ representation, and then the isotypic component of the GL_n representation F^λ has the structure of an (irreducible) $S_{m_1+m_2}$ representation.

$$\bigotimes^{m_1+m_2}\mathbb{C}^n = \bigoplus_{\substack{\lambda\,:\,|\lambda|=m_1+m_2 \\ l(\lambda)\leq n}} U_\lambda \otimes F^\lambda \tag{2.12}$$

We are lead to the following rule for restricting an irreducible $S_{m_1+m_2}$ representation to the subgroup $S_{m_1} \times S_{m_2}$.

$$\mathrm{Res}\begin{array}{c} S_{m_1+m_2} \\ S_{m_1}\times S_{m_2}\end{array} U_\lambda = \bigoplus_{\substack{\mu,\nu \\ |\mu|=m_1 \\ |\nu|=m_2}} N_{\mu\nu}^\lambda U_\mu \otimes U_\nu \tag{2.13}$$

By Frobenius reciprocity for finite groups we can restate the above in terms of induced representations as,

$$\mathrm{Ind}\begin{array}{c} S_{m_1+m_2} \\ S_{m_1}\times S_{m_2}\end{array} U_\mu \otimes U_\nu = \bigoplus_{\substack{\lambda \\ |\lambda|=m_1+m_2}} N_{\mu\nu}^\lambda U_\lambda \tag{2.14}$$

2.3 The symmetric pair (GL_n, O_n)

As an O_n representation, the conjugation action on M_n is equivalent to the diagonal action of O_n on the the space $\mathbb{C}^n \otimes \mathbb{C}^n$. This is a consequence of the fact that the standard representation of O_n is equivalent to its dual. In order to understand the space $\mathcal{P}(M_n)$ as a graded O_n representation we will investigate the restriction of the standard $GL_n \times GL_n$ action on $\mathbb{C}^n \otimes \mathbb{C}^n$ to the diagonal GL_n (that is, $\{(g, g) | g \in GL_n\}$), and then restricted to the group O_n. For this we will use the following special case of the Carton-Helgason theorem.

Theorem 3 *Let F^λ denote the irreducible finite dimensional representation of the group GL_n with highest weight λ. Then,*

$$\begin{aligned} \dim {(F^\lambda)}^{O_n} &= 1 \quad \textit{if } \lambda \textit{ has all even parts.} \\ &= 0 \quad \textit{otherwise} \end{aligned}$$

This theorem follows from the fact that (GL_n, O_n) is a symmetric pair, see [8], chapters 11 and 12. So the dimension of the O_n invariant space in $\mathcal{P}^d(M_n)$ can be computed from a complete GL_n decomposition. We begin this program by asserting the following result sometimes referred to as Cauchy's identity or the Chauchy-Littlewood identity (see [4]), but which is also an instance of Roger Howe's theory of dual pairs (see [10]).

Theorem 4 *The standard action of $GL_n \times GL_n$ on $\mathbb{C}^n \otimes \mathbb{C}^n$ defines an action on $\mathcal{P}^d(\mathbb{C}^n \otimes \mathbb{C}^n)$ by,*

$$(g, h) f\,(x \otimes y) = f(g^{-1}x \otimes h^{-1}y)$$

for $(g, h) \in GL_n \times GL_n$ and f a degree d homogeneous polynomial function on $\mathbb{C}^n \otimes \mathbb{C}^n$. Under this action the space of functions decomposes as,

$$\mathcal{P}^d(\mathbb{C}^n \otimes \mathbb{C}^n) = \bigoplus_{\substack{\lambda\,:\,l(\lambda) \le n \\ |\lambda| = d}} (F^\lambda)^* \otimes (F^\lambda)^*$$

We will now need to restrict the action from $GL_n \times GL_n$ to the diagonal GL_n. This is exactly the problem of finding the decomposition of a tensor product of an irreducible GL_n representation with itself.

Using theorems 3, and 4, we can write an expression for the Hilbert series defined in section 2.1 as follows,

$$\mathcal{P}(\mathbb{C}^n \otimes \mathbb{C}^n)^{O_n} = \bigoplus_{\mu:l(\mu)\leq n} (F^{\mu*} \otimes F^{\mu*})^{O_n} \tag{2.15}$$

$$= \bigoplus_{\mu:l(\mu)\leq n} (F^{\mu} \otimes F^{\mu})^{O_n} \tag{2.16}$$

$$= \bigoplus_{\mu,\nu:l(\mu),l(\nu)\leq n} N^{\nu}_{\mu\,\mu}(F^{\nu})^{O_n} \tag{2.17}$$

$$= \bigoplus_{\mu,\lambda:l(\mu),l(\nu)\leq n} N^{2\lambda}_{\mu\,\mu}(F^{2\lambda})^{O_n} \tag{2.18}$$

Note that 2λ means double all the parts of λ. Theorem 3 then implies,

$$H_n(q) = \sum_{\lambda,\mu:l(\lambda),l(\mu)\leq n} N^{2\lambda}_{\mu\mu} q^{|\lambda|} \left(= \sum_{\lambda,\mu:l(\lambda),l(\mu)\leq n} N^{2\lambda}_{\mu\mu} q^{|\mu|} \right) \tag{2.19}$$

The second equality is a consequence of the fact that if $N^{\lambda}_{\mu\nu} \neq 0$ then $|\mu| + |\nu| = |\lambda|$. It is interesting to point out that, if μ is a partition of n, and $N^{2\lambda}_{\mu\mu} \neq 0$ then λ must be a partition of n as well, hence λ can have at most n parts. This implies the following result.

Theorem 5 (Stability range for invariants) *For $d, n \geq 0$,*

$$\dim \mathcal{P}^d(M_n)^{O_n} \leq \sum_{\lambda,\mu:|\lambda|=|\mu|=d} N^{2\lambda}_{\mu\mu} \tag{2.20}$$

with equality holding if and only if $d \leq n$.

For $d > n$ the exact dimension is a subject of investigation. Consider the formal power series,

$$\widetilde{H}(q,t) = \sum_{\lambda,\mu} N^{2\lambda}_{\mu\mu} q^{|\lambda|} t^{l(\lambda)}$$

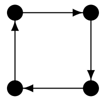

Figure 2.1: An element in $\mathcal{C}_4$

and for $m \geq 0$, set $d_m(t) = \widetilde{H}(q,t)|_{q^m}$ (which is a polynomial in the t with non-negative integer coefficients. Now observe,

$$H(q,t) = \sum_{\lambda,\mu,n:l(\lambda),l(\mu)\leq n} N_{\mu\mu}^{2\lambda} q^{|\lambda|} t^n \tag{2.21}$$

$$= \frac{1}{1-t} \sum_{\lambda,\mu} N_{\mu\mu}^{2\lambda} q^{|\lambda|} t^{l(\lambda)} \tag{2.22}$$

$$= \frac{\widetilde{H}(q,t)}{1-t}. \tag{2.23}$$

2.4 A Combinatorial Result.

Equation 2.19 defines a generating function which is a rational function. This rational function is what we wish to compute. Unfortunately, a closed form expression is not available yet, however, if we drop the length conditions on the partitions λ and μ something more can be said.

A *directed cycle* is a (unlabeled) cyclic graph with its edges oriented. That is, an unlabeled directed graph with cyclic underlying graph. Let $\mathcal{C}_k$ denote the set of k vertex directed cycles, and $\mathcal{D}_m$ denote the set of (not necessarily connected) directed k vertex graphs in which each component is a directed cycle. It is helpful to introduce some easy terminology. Here we shall call a directed edge an *arc*. The arrow end of an arc will be called the *head*, while the other end will be called the *tail*. Note that we will allow the one or two arc cases, so an arc joining a vertex to itself is a directed cycle, as well as the case of two arcs joining a pair of vertices. An example of an element in $\mathcal{C}_4$ is shown in figure 2.1, while an example of an element in $\mathcal{D}_{11}$ is shown in figure 2.2

The elements of $\mathcal{D}_m$ are multisubsets of $\cup C_i$. Set $d_m = |\mathcal{D}_m|$, and $c_k = |\mathcal{C}_k|$. It

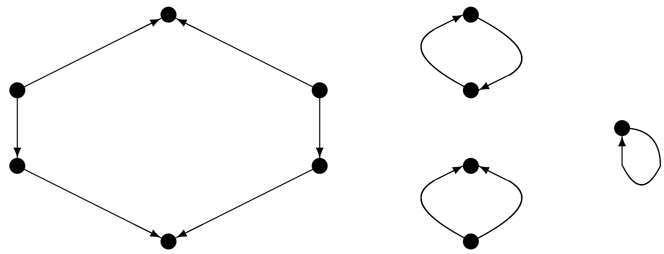

Figure 2.2: An example of an element in $\mathcal{D}_{11}$

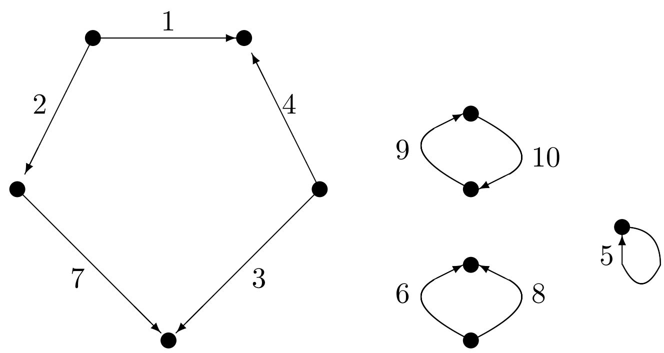

Figure 2.3: An example of an element in $\mathcal{LD}_{10}$

is instructive to check that,

$$\sum_{m\geq 0} d_m q^m = \prod_{k\geq 1}\left(\frac{1}{1-q^k}\right)^{c_k} \tag{2.24}$$

$\mathcal{LD}_m$ will denote the set of graphs from $\mathcal{D}_m$ with each arc labeled by an element of the set $\{1, 2, \ldots, m\}$ such that each label is used exactly once. An example of an element in $\mathcal{LD}_{10}$ is shown in figure 2.3.

Let I_r be the set of involutions on the set $\{1, 2, \ldots, r\}$, and let $\widetilde{I}_r$ be the subset of I_r consisting of involutions which do not have fixed points. Our strategy is to set up a bijective correspondence between $\mathcal{LD}_m$ and $\widetilde{I}_{2m}$, The bijective correspondence is

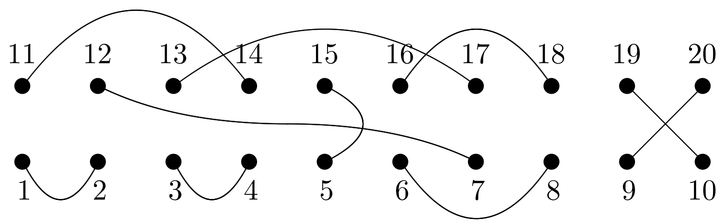

Figure 2.4: Arcs between the vertices paired by the involution.

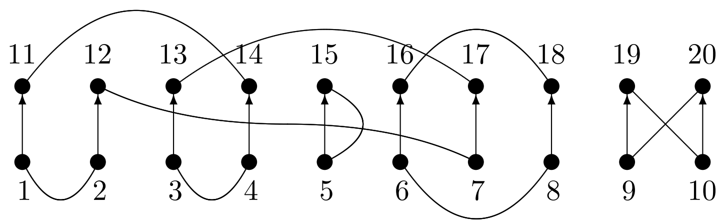

Figure 2.5: An arrow from vertex i to vertex i+n

proven shortly. First we illuminate the idea of the proof by an example. Consider the involution,

$$(1\ 2)(3\ 4)(5\ 15)(6\ 8)(7\ 12)(9\ 20)(10\ 19)(11\ 14)(13\ 17)(16\ 18)$$

(Here we are using the disjoint cycle representation.) Now we will describe how this involution corresponds to figure 2.3. Write the numbers from 1 to 10 in a row and above them write the numbers from 11 to 20. Next, connect each number with its image under the involution, as shown in figure 2.4.

In the diagram above draw and arrow from each number to the number directly above it, see figure 2.5.

Each number in the top row labels the head of an arc, while each number in the

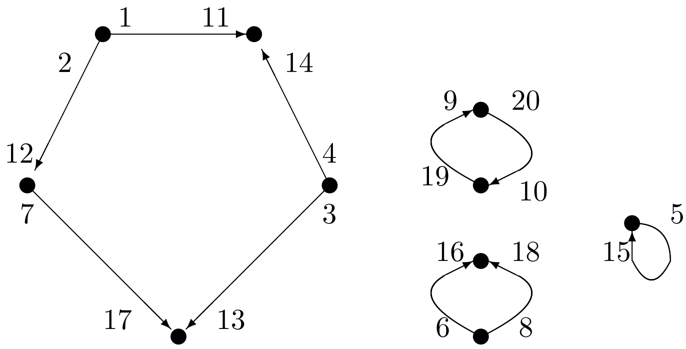

Figure 2.6: Collapsed arcs in figure 2.5

bottom row labels the tail of an arc. Next, identify the pairs of vertices in the diagram whenever a vertex is connected to another. The resulting picture will be an element of $\mathcal{D}_{10}$ which has each arc labeled by a pair $(i, i+10)$, where i labels the tail of the arc and $i+10$ labels the head of the arc. See figure 2.6.

Note that nothing is lost if one relabels the arc $(i, i+10)$ with just i. This completes the correspondence for this example. The following is a more precise description.

Lemma 1 *There exists a bijective map* $\Theta : \mathcal{LD}_m \longrightarrow \widetilde{I}_{2m}$

Proof:

Given $g \in \mathcal{LD}_m$ we will make an involution $\sigma \in \widetilde{I}_{2m}$ as follows, pick $k \in \{1, \ldots, 2m\}$, we will define the value of σ at k in two cases. The two cases are $1 \leq k \leq m$ and $m+1 \leq k \leq 2m$.

CASE 1: If $1 \leq k \leq m$ then find the arc labeled by k if the *tail* of arc k is attached to the *tail* of arc j then define, $\sigma(k) = j$. On the other hand, if the *tail* of arc k is attached to the *head* of arc j then define, $\sigma(k) = j + m$.

CASE 2: If $m+1 \leq k \leq 2m$ then find the arc labeled by $k - m$ if the *head* of arc $m - k$ is attached to the *tail* of arc j then define, $\sigma(k) = j$. On the other hand, if the *head* of arc $k - m$ is attached to the *head* of arc j then define, $\sigma(k) = j + m$.

It can be checked that σ is an involution in S_{2m}. We will now describe the inverse correspondence. Given an involution σ create an element of $\mathcal{LD}_m$ by starting with m non-joined arcs labeled with the numbers 1 through m. We will now use the values of $\sigma(k)$ and $\sigma(k+n)$ to identify the vertices. Identify the vertices at the head and tail of arc k and arc j according to the following four cases:

$\sigma(k) = j$	Identify the tail of arc k with the tail of arc j.
$\sigma(k) = j + m$	Identify the tail of arc k with the head of arc j.
$\sigma(k + m) = j$	Identify the head of arc k with the tail of arc j.
$\sigma(k + m) = j + m$	Identify the head of arc k with the head of arc j.

Q.E.D.

The following theorem establishes that the polynomials $d_m(t)$ defined in the last section are a t-analog of the numbers d_m.

Theorem 6

$$\sum_{\lambda,\mu} N^{2\lambda}_{\mu\mu} q^{|\mu|} = \prod_{k\geq 1} \left(\frac{1}{1-q^k}\right)^{c_k}$$

Proof:

For a partition λ of m, let U_λ denote the irreducible representation of S_m. Recall that the Littlewood-Richardson coefficients can be defined equivalently by,

$$\mathrm{Ind}\ {}^{S_{m_1+m_2}}_{S_{m_1}\times S_{m_2}} (U_\nu \otimes U_\mu) = \bigoplus_\lambda N^\lambda_{\mu\nu} U_\lambda \tag{2.25}$$

Let ΔS_m denote the diagonally embedded copy of S_m in $S_m \times S_m$. By relabeling the letters in the second copy of S_m we will embed $S_m \times S_m$ in S_{2m}. Hence, view both ΔS_m and $S_m \times S_m$ as subgroups of S_{2m}. The representation of S_{2m} induced from the trivial representation of ΔS_m can be decomposed into irreducible representations as follows:

$$\operatorname{Ind}_{\Delta S_m}^{S_{2m}} 1 = \operatorname{Ind}_{S_m \times S_m}^{S_{2m}} \left(\operatorname{Ind}_{\Delta S_m}^{S_m \times S_m} 1 \right) \tag{2.26}$$

$$= \operatorname{Ind}_{S_m \times S_m}^{S_{2m}} \left(\bigoplus_{\mu} U_\mu \otimes U_\mu \right) \tag{2.27}$$

$$= \bigoplus_{\mu} \operatorname{Ind}_{S_m \times S_m}^{S_{2m}} (U_\mu \otimes U_\mu) \tag{2.28}$$

$$= \bigoplus_{\lambda,\mu} N_{\mu\mu}^{\lambda} U_\lambda \tag{2.29}$$

Note that here 1 denotes the trivial representation. The above computation first uses the fact that induction of representations is transitive in equation 2.26. Then it is observed that representations of the symmetric group are self dual (this is because every permutation is conjugate to its inverse), so by Schur's lemma we obtain, equation 2.27. After noting that induction distributes across the direct sums, we use the (equivalent) definition of the Littlewood-Richardson coefficients in 2.25.

Let τ be the element of S_{2m} with disjoint cycle representation

$$(1\,2)(3\,4)\ldots(i\;i+1)\ldots(2m-1\;2m)$$

Define H_m to be the centralizer group of τ in S_{2m}. H_m is isomorphic to the Weyl group B_m (or C_m). An important fact explained in [17], chapter VII, section 2 (page 402) is,

$$\operatorname{Ind}_{H_m}^{S_{2m}} 1 = \bigoplus_{\lambda:|\lambda|=m} U_{2\lambda} \tag{2.30}$$

Equation 2.29 and equation 2.30 lead us to a formula which is a consequence of only Schur's Lemma,

$$\sum_{\lambda,\mu:|\lambda|=|\mu|=m} N^{2\lambda}_{\mu\mu} = \dim Hom_{S_{2m}}\left(\operatorname{Ind}_{H_m}^{S_{2m}} 1, \operatorname{Ind}_{\Delta S_m}^{S_{2m}} 1\right) \tag{2.31}$$

$$= \dim Hom_{\Delta S_m}\left(1, \operatorname{Res}_{\Delta S_m}^{S_{2m}} \operatorname{Ind}_{H_m}^{S_{2m}} 1\right) \tag{2.32}$$

$$= \dim \left(\operatorname{Res}_{\Delta S_m}^{S_{2m}} \operatorname{Ind}_{H_m}^{S_{2m}} 1\right)^{\Delta S_m} \tag{2.33}$$

The group ΔS_m acts on the left cosets of H_m in S_{2m} by restricting the left coset action of S_{2m}. By 2.33, it is clear that the orbits of this action are to be enumerated. As usual, let S_{2m}/H_m denote the left cosets of H_m in S_{2m}. ΔS_m acts on $\widetilde{I}_{2m}$ by conjugation and this action is equivalent to the left coset action of ΔS_m on S_{2m}/H_m, by the following correspondence:

$$\begin{array}{rlcl} \psi: & S_{2m}/H_m & \longrightarrow & \widetilde{I}_{2m} \\ & \sigma H_m & \longmapsto & \sigma\tau\sigma^{-1} \end{array}$$

A quick check will establish that the above bijection is defined. The remainder of the proof is to establish a bijective correspondence between the orbits $\widetilde{I}_{2m}/\Delta S_m$ and the set $\mathcal{D}_m$ defined above. Observe that two involutions σ_1 and σ_2 are in the same orbit under the action of ΔS_m, if and only if they correspond to two elements $g_1, g_2 \in \mathcal{LD}_m$ which are different labelings of the same *unlabeled* directed graph. **Q.E.D.**

Observe that the space $\bigotimes^2 \mathbb{C}^n = S^2(\mathbb{C}^n) \oplus \wedge^2(\mathbb{C}^n)$. So under the action of GL_n we have,

$$S(\bigotimes^2 \mathbb{C}^n) = S(S^2(\mathbb{C}^n)) \otimes S(\wedge^2(\mathbb{C}^n))$$

$$S\left(S^2(\mathbb{C}^n)\right) = \bigoplus_{\mu\,:\,l(\mu)\,\leq\, n} F^{2\mu} \tag{2.34}$$

$$S\left(\wedge^2(\mathbb{C}^n)\right) = \bigoplus_{\nu\,:\,l((2\nu)^c)\,\leq\, n} F^{(2\nu)^c} \tag{2.35}$$

Here we use the notation, λ^c to denote conjugation λ. Combining the last two results together we obtain the following decomposition as a representation of GL_n.

$$
\begin{aligned}
S\left(\bigotimes^2(\mathbb{C}^n)\right) &= \bigoplus_{\substack{\mu,\,\nu \\ l((2\nu)^c)\,\leq\, n \\ l(\mu)\,\leq\, n}} F^{2\mu}\otimes F^{(2\nu)^c} \\
&= \bigoplus_{\substack{\lambda,\,\mu,\,\nu \\ l((2\nu)^c)\,\leq\, n \\ l(\mu),\,\lambda\,\leq\, n}} N^{\lambda}_{2\mu,(2\nu)^c}\, F^{\lambda}
\end{aligned}
$$

Then, find the O_n invariants as before by taking one invariant for each even λ. This approach leads to the following identity:

$$
\sum_{\substack{\lambda,\,\mu \\ l(\mu),\,l(\lambda)\,\leq\, n}} N^{2\lambda}_{\mu\mu} q^{\lambda} = \sum_{\substack{\lambda,\,\mu,\,\nu \\ l((2\nu)^c)\,\leq\, n \\ l(\mu),\,l(\lambda)\,\leq\, n}} N^{2\lambda}_{2\mu,(2\nu)^c} q^{\lambda} \tag{2.36}
$$

Taking n to infinity gives another identity of formal power series.

Corollary 7

$$
\sum_{\lambda,\mu} N^{2\lambda}_{\mu\mu} q^{\lambda} = \sum_{\lambda,\mu,\nu} N^{2\lambda}_{2\mu,(2\nu)^c} q^{\lambda} = \prod_{k\geq 1}\left(\frac{1}{1-q^k}\right)^{c_k} \tag{2.37}
$$

It is interesting to note that there is a correspondence between elements of $\mathcal{C}_k$ and certain O_n invariant functions on M_n given by the following description. Let $C \in \mathcal{C}_k$. Choose arbitrarily an arc A_1 in C. Let $\{A_2, A_3, \ldots, A_k\}$ denote the sequence of arcs traversed clockwise from A_1. Let $f : M_n \to \mathbb{C}$ be defined by, $f(X) = Trace\ \left(\prod_{i=1}^{k} f_i(X)\right)$. Where:

$$
f_i = \begin{cases} X & \text{If } A_i \text{ is oriented clockwise.} \\ X^T & \text{If } A_i \text{ is oriented counterclockwise.} \end{cases}
$$

It is easy to see that these functions are O_n invariant. Observe that the definition is independent of the initial choice of A_1 because a cyclic permutation of the variables will not effect the value of the trace. Also, we could have interchanged the words clockwise and counterclockwise because the trace of a matrix is the same as the trace of its transpose.

2.5 An enumeration of $\mathcal{C}_k$ and $\mathcal{D}_m$

In this section we enumerate the set $\mathcal{C}_k$ by computing a generating function. A vertex in a directed cycle is called a *sink* vertex if the two arcs joined to it are pointing into the vertex. A vertex is called a *source* vertex if the two arcs joined to it are pointed away from the vertex. A vertex is called a *flow* vertex if one arc points into it and one arc points out of it. Next, let

$$\mathcal{C}_k^{(i)} = \{c \in \mathcal{C}_k | c \text{ has } i \text{ sinks and } i \text{ sources}\} \tag{2.38}$$

So $\mathcal{C}_k = \cup_i \mathcal{C}_k^{(i)}$. Observe that the source and sink vertices alternate with flow vertices scattered between them. For each directed cycle, construct a polygon whose corners correspond to the source vertices, with each side corresponding to a sink vertex. The flow vertices are then represented by ordered pairs of non-negative integers assigned to each side. In order to enumerate these, we will use Burnside's theorem by averaging the fix point set cardinalities of the action of the dihedral group on such n sided polygons. The generating function $g_n(x)$ in which x^k records the directed cycles with exactly n sources, n sinks and k flow vertices is,

$$g_n(x) = \frac{1}{2(1-x^2)^n} + \frac{1}{2n} \sum_{d|n} \frac{\phi(d)}{(1-x^d)^{\frac{2n}{d}}} \tag{2.39}$$

The generating function in which the coefficient of x^n is the number of directed cycles with n vertices is then,

$$C(x) = \sum_{k \geq 1} g_k(x) x^{2k} \tag{2.40}$$

$$= \sum_{n \geq 1} \left(\frac{x^{2n}}{2(1-x^2)^n} + \frac{x^{2n}}{2n} \sum_{d|n} \frac{\phi(d)}{(1-x^d)^{\frac{2n}{d}}} \right). \tag{2.41}$$

It is interesting to see that the initial segment of the sequence $\{c_k\}_{k \geq 1}$ is 1, 2, 2, 4, 4, 9, 10, 22, 30, 62, 94, 192, 316, 623, 1096, 2122, 3856, 7429, 13798, ...

An initial segment of the numbers $\mathcal{D}_m$, for $m = 0...19$ is, 1, 1, 3, 5, 12, 20, 44, 76, 157, 281, 559, 1021, 2005, 3721, 7237, 13631, 26433, 50297, 97543, 187129. In section

2.3, we defined a t-analog of these numbers. This is what is currently being investigated. The following table gives some initial data. The coefficient of $t^{(column)}q^{(row)}$ in $\widetilde{H}(q,t)$ is displayed for rows and columns from 0 to 9. Recall then that in the m^{th} row are the coefficients of $d_m(t)$ and the sum of the first n columns is the sequence of coefficients of the formal power series expansion of $H_n(q)$.

1	0	0	0	0	0	0	0	0	0	...
1	2	0	0	0	0	0	0	0	0	...
1	2	2	0	0	0	0	0	0	0	...
1	5	3	3	0	0	0	0	0	0	...
1	5	7	4	3	0	0	0	0	0	...
1	9	13	12	5	4	0	0	0	0	...
1	9	21	21	14	6	4	0	0	0	...
1	14	33	48	30	19	7	5	0	0	...
1	14	51	75	67	39	21	8	5	0	...
1	20	73	145	133	98	48	26	9	6	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

2.6 On the Hilbert Series of $\mathcal{P}(\mathfrak{g})^K$

2.6.1 The restricted adjoint representation

Let G denote a connected semi-simple linear algebraic group over $\mathbb{C}$ with Lie algebra $\mathfrak{g}$ and let θ denote a regular involution with differential (also denoted) $\theta : \mathfrak{g} \to \mathfrak{g}$. Let K be the set of fixed points of θ in G and let $\mathfrak{k}$ denote the Lie algebra of K. $\mathfrak{k}$ is a Lie subalgebra of $\mathfrak{g}$ and under the adjoint representation of K, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ where, $\mathfrak{p} = \{X \in \mathfrak{g} | \theta(X) = -X\}$.

Of course for the purpose of this work we are interested in the reductive case of the pair $(GL_n(\mathbb{C}), O_n(\mathbb{C})$, but for the time being we will take G to be the semisimple group $SL_n(\mathbb{C})$ and $K = SO_n(\mathbb{C})$. θ is inverse transpose, while its differential at the identity is negative transpose. $\mathfrak{g}$ can be taken as the the trace zero matrices, $\mathfrak{k}$ the skew symmetric matrices and $\mathfrak{p}$ the symmetric matrices in $\mathfrak{g}$.

Let V be a regular representation of K. As before we have a linear action of

K on the polynomial functions on V. Semisimplicity of G implies that K is reductive, and in the case of K reductive the ring $\mathcal{P}(V)^K$ is finitely generated, but not necessarily a polynomial ring. Also, we have that $\mathcal{P}(V)$ is a finitely generated module over $\mathcal{P}(V)^K$, but not necessarily a free module. Two deep results are the following.

Theorem 8 (Kostant) $\mathcal{P}(\mathfrak{g})$ *is free as a* $\mathcal{P}(\mathfrak{g})^G$ *-module*

Of course applying the theorem to $\mathfrak{k}$, we have that $\mathcal{P}(\mathfrak{k})$ is free as a $\mathcal{P}(\mathfrak{k})^K$ -module, but in addition we also have,

Theorem 9 (Kostant-Rallis) $\mathcal{P}(\mathfrak{p})$ *is free as a* $\mathcal{P}(\mathfrak{p})^K$ *-module.*

The Chevalley restriction theorem gives that $\mathcal{P}(\mathfrak{g})^G$, $\mathcal{P}(\mathfrak{k})^K$ and $\mathcal{P}(\mathfrak{p})^K$ are polynomial rings in which case the Hilbert series for each is of the form, $\frac{1}{\prod_{i=1}^{l}(1-q^{d_i})}$. Where the numbers d_i are the degrees the algebraically independent generators. Let the ideal generated by the K-invariant polynomial functions of positive degree be denoted by $\mathcal{I}_V$ and for each $d \geq 0$ set $\mathcal{I}_V^d = \mathcal{P}^d(V) \cap \mathcal{I}_V$ and let $\mathcal{H}_V^d$ be the K-invariant complement of $\mathcal{I}_V^d$ in $\mathcal{P}^d(V)$. Set $\mathcal{H}_V = \oplus_{d\geq 0}\mathcal{H}_V^d$. Another way to say theorems 8 and 9 is,

$$\mathcal{P}(\mathfrak{g}) = \mathcal{P}(\mathfrak{g})^G \otimes \mathcal{H}_{\mathfrak{g}}$$

$$\mathcal{P}(\mathfrak{k}) = \mathcal{P}(\mathfrak{k})^K \otimes \mathcal{H}_{\mathfrak{k}}$$

$$\mathcal{P}(\mathfrak{p}) = \mathcal{P}(\mathfrak{p})^K \otimes \mathcal{H}_{\mathfrak{p}}$$

The goal of this work is to understand the Hilbert series of the ring of invariants, $\mathcal{P}(\mathfrak{g})^K$. One approach is to view $\mathcal{P}(\mathfrak{g})$ as $\mathcal{P}(\mathfrak{k}) \otimes \mathcal{P}(\mathfrak{p})$ and then apply the above results to obtain,

$$\mathcal{P}(\mathfrak{g})^K = \mathcal{P}(\mathfrak{k})^K \otimes \mathcal{P}(\mathfrak{p})^K \otimes (\mathcal{H}_{\mathfrak{k}} \otimes \mathcal{H}_{\mathfrak{p}})^K$$

One way of carrying out this program might be to obtain a complete graded decomposition of the spaces $\mathcal{H}_{\mathfrak{p}}$ and $\mathcal{H}_{\mathfrak{k}}$ and then pair each representation of K with its dual to find the invariants. The graded decompositions of $\mathcal{H}_{\mathfrak{k}}$ is given in [9], but currently the graded decomposition of $\mathcal{H}_{\mathfrak{p}}$ is not known for general $(K, \mathfrak{p})$. See [20], for some important cases where such a graded decomposition is given. Because of this the program will be to first use the results in [9] to obtain a graded decomposition of $\mathcal{H}_{\mathfrak{g}}$ into irreducible representations and then use the Cartan and Helgason theorem for the multiplicities of K-invariants in irreducible G representations.

We will now recall a formula which gives the graded multiplicity of each irreducible representation of G in $\mathcal{H}_{\mathfrak{g}}$. Fix, T, a maximal (algebraic) torus of G and $\mathfrak{h}$ the corresponding Cartan subalgebra of $\mathfrak{g}$. Let $l = \dim T$. Let Φ denote the root system of T acting on G. Choose a system, Φ^+, of positive roots in Φ and let Δ be the simple roots in Φ^+. Let $W(G,T) = W$ be the Weyl group of T in G acting on $\mathfrak{h}$. If $\alpha \in \Phi$ then $s_\alpha \in W$ denotes the reflection about the hyperplane $\alpha = 0$ in $\mathfrak{h}$. Let $\rho = \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha$ (as usual). Next, we will define Lustig's q-analog of the Kostant partition function which is a function $\wp_q$ on $\mathfrak{h}^*$ taking values in the ring of polynomials in q with non-negative integer coefficients.

$$\frac{1}{\prod_{\alpha\in\Phi^+} 1 - qe^{-\alpha}} = \sum_{\xi\in\mathfrak{h}^*} \wp_q(\xi)e^{-\xi} \tag{2.42}$$

For any vector $\lambda \in \mathfrak{h}^*$ define the function,

$$\mathcal{L}_\lambda(q) = \sum_{s\in W} sgn(s)\wp_q(s(\lambda+\rho)-\rho).$$

Theorem 10 (Hesselink) *If μ is a dominant integral character of T we denote by F^μ the irreducible finite dimensional representation of (of the simply connected covering group of) G with highest weight μ. Then,*

$$\mathcal{L}_\mu(q) = \sum_d \dim Hom_G(F^\mu, \mathcal{H}^d_{\mathfrak{g}})\; q^d \tag{2.43}$$

Here we quote the Cartan, Helgason theorem which asserts that the dimension of the space of K-invariants in the restriction of an irreducible G-representation, F^μ is at most 1. Define

$$I(G,K) = \left\{\mu \in \mathcal{P}_+(\mathfrak{g}) \middle| \dim\,(F^\mu)^K = 1\right\}$$

With this in mind, we are lead to a (not so closed) formula for the Hilbert series of the invariants,

$$Hilb_q\,\mathcal{P}(\mathfrak{g})^K = \frac{\sum_{\lambda\in I(G,K)}\mathcal{L}_\lambda(q)}{\prod_{i=1}^{l}(1-q^{d_i})} \tag{2.44}$$

The above hints at an algorithm to compute the Hilbert series of $\mathcal{P}(\mathfrak{g})^K$. For the cases of (GL_n, O_n) and (GL_n, SO_n), I have implemented this algorithm.

2.7 O_n-stability of the tensor algebra

Define the formal series,

$$\mathcal{T}_n(q) = \sum_{m\geq 0} \dim\left(\bigotimes^m End(\mathbb{C}^n)\right)^{O_n} q^m$$

We will let the GL_n decomposition of $\bigotimes^m \mathbb{C}^n$ be defined by the numbers f_λ,

$$\bigotimes^m \mathbb{C}^n = \bigoplus_{\substack{\lambda \\ l(\lambda)\leq n \\ |\lambda| = m}} f_\lambda F^\lambda \tag{2.45}$$

Schur-Weyl duality implies that f_λ is the dimension of the irreducible S_m representation indexed by λ. Again, we will find the dimension of the O_n-invariants by theorem 3, and obtain,

$$\mathcal{T}_n(q) = \sum_{\lambda : l(\lambda)\leq n} f_{2\lambda} q^{|\lambda|} \tag{2.46}$$

Theorem 11 *For $m, n \geq 0$ we have,*

$$\dim\left(\bigotimes^m End(\mathbb{C}^n)\right)^{O_n} \leq \frac{(2m)!}{2^m m!} \tag{2.47}$$

with equality if and only if $m \leq n$

Proof:

$$\begin{aligned}
\dim\left(\bigotimes^m End(\mathbb{C}^n)\right)^{O_n} &= \dim\left(\bigotimes^m \left(\mathbb{C}^n \otimes (\mathbb{C}^n)^*\right)\right)^{O_n} \\
&= \dim\left(\bigotimes^m \left(\mathbb{C}^n \otimes \mathbb{C}^n\right)\right)^{O_n} \\
&= \dim\left(\bigotimes^{2m} \mathbb{C}^n\right)^{O_n}
\end{aligned}$$

From the theory of representations of the symmetric group, $f_{2\lambda}$ is the number of standard tableaux on the shape 2λ. The RSK algorithm gives a correspondence between pairs of standard tableaux (of the same shape) and permutations (see [4], chapter 4). The involutions are sent to the diagonal pairs via this correspondence (see [4], page 52), moreover the number of odd columns is the number of fixed points in the involution. We can conjugate all of these to change column parity conditions to row parity conditions.

$$\begin{aligned}
\dim\left(\bigotimes^m End(\mathbb{C}^n)\right)^{O_n} &= \sum_{\lambda:|\lambda|=m, l(\lambda)\leq n} f_{2\lambda} \\
&\leq \sum_{\lambda:|\lambda|=m} f_{2\lambda} \\
&= |\widetilde{I}_{2m}| \\
&= \frac{(2m)!}{2^m\, m!}
\end{aligned}$$

Observe that the above inequality is actually an equality if and only if $m \leq n$. **Q.E.D.**

It is important to note that the above proof gives an algorithm for computing the exact dimension, by using the hook length formula for $f_{2\lambda}$.

Chapter 3

Branching

3.1 Restriction of representations

We address the problem of decomposing an irreducible $GL(n) = GL(n, \mathbb{C})$ representation into irreducible representations of the orthogonal subgroup, $O(n) = O(n, \mathbb{C})$ or in the case $n = 2k$, the rank k symplectic group, $Sp(k) = Sp(k, \mathbb{C})$. In general this the problem of decomposing an irreducible representation of a group into irreducible representations for one of its subgroups is called *branching.* From the Weyl character formula, one can derive very general formulae in terms of partition functions, see [8] chapter 8. There are quite a few sources in the literature involving branching problems. Some of the most recent papers are [7], and [6]. Although the problem is address much earlier. In [15] and [16], Littlewood refers to the work of Schur and Frobenius. The physics literature also has a wide variety of treatments, see [3].

[5] has a section on branching rules. However, some of the hypotheses are not strong enough in certain theorems. For example, the rules for branching from $GL(n, \mathbb{C})$ to $O(n, \mathbb{C})$ in formula (25.37) on page 427 in [5] has the following counterexample: The square of the determinant is an irreducible $GL(n, \mathbb{C})$ representation which restricts to the trivial representation of $O(n, \mathbb{C})$. The stated formula indicates that the restricted representation should contain irreducible components which have dimension greater than 1. If one adds the requirement that the partition indexing the $GL(n, \mathbb{C})$ representation has at most $\lfloor \frac{n}{2} \rfloor$ parts then this example is removed. Littlewood indicates this assumption in [16]. This example indicates that some care must be given when attacking this problem.

In particular, one should expect to find formulas which work only in certain ranges. I will refer to such a range as a *stable* range.

The representations will in general be regular representations in the category of linear algebraic groups. This is to say that, the matrix coefficients are rational functions of the entries in $GL(n, \mathbb{C})$ where the denominators are powers of the determinant. Indeed, the ring of matrix coefficients is isomorphic to the ring of polynomials in n^2 variables localized at the determinant. In this category the irreducible representations are indexed by dominant weights. The over all structure of this chapter is to describe first and indexing of the irreducible representations of the groups involved (even the disconnected case of $O(n)$). Then we introduce Howe duality for the pairs $(GL(n), GL(m))$, $(O(n), sp(m))$ and $(Sp(k), so^*(2m))$. Using the Howe duality theorems will will prove the Littlewood restriction rules for the pairs $(GL(n), O(n))$ and $(GL(2k), Sp(k))$.

3.1.1 Irreducible Representations of $GL(n)$

For any integer partition $\lambda = (\lambda_1, \ldots, \lambda_n)$, let $F^\lambda_{(n)}$ be the irreducible regular representations of $GL(n)$ indexed in the usual way by highest weights, $\lambda_1 e_1 + \cdots + \lambda_n e_n$. Where the e_i denote the standard linear functionals on the diagonal Cartan subalgebra of the Lie algebra $gl_n(\mathbb{C})$. That is,

$$e_i : \begin{bmatrix} a_{11} & 0 & 0 & \ldots & 0 \\ 0 & a_{22} & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & a_{nn} \end{bmatrix} \mapsto a_{ii}$$

Note that if it is clear which group we are dealing with, we will usually omit the (n) subscript from our notation. If $\lambda_n \geq 0$ then we will say that F^λ is a polynomial representation. This terminology is motivated by the fact that the matrix coefficients are in fact polynomial functions in this case. Polynomial representations are not closed under duality, while regular representations are closed. We consider polynomial representations for a convenience of notation. Branching to the orthogonal or symplectic groups is essentially the same in either category because every regular representation of $GL(n)$ is a polynomial representation if we tensor with a sufficiently high positive integer power of the determinant.

3.1.2 Irreducible Representations of $Sp(k)$

$\{V^\mu\}_\mu$ will be representatives of the equivalence classes of irreducible regular representations of $Sp(k,\mathbb{C})$. As with the $GL(n,C)$ case we will index this by non-negative integer partitions in which each ν corresponds to an irreducible regular (or polynomials in this case) representation of $Sp(n)$ with highest weight $\mu_1 e_1 + \cdots + \mu_n e_k$.

3.1.3 Irreducible Representations of $O(n)$

$\{E^\mu\}_\mu$ will be representatives of the equivalence classes of irreducible regular representations of $O(n,\mathbb{C})$. We can again take the index set to be non-negative integer partitions with at most n parts as described in [8] chapter 10. We will repeat the indexing here. Let Q be the symmetric nondegenerate bilinear form defining the complex orthogonal group.

Let

$$C_{ij} : \bigotimes^m \mathbb{C}^n \to \bigotimes^{m-2} \mathbb{C}^n$$

Defined by:

$$C_{ij}(v_1 \otimes \cdots \otimes v_i \otimes \cdots \otimes v_j \otimes \cdots \otimes v_m) =$$
$$Q(v_i, v_j)\; v_1 \otimes \cdots \otimes v_{i-1} \otimes v_{i+1} \otimes \cdots \otimes v_{j-1} \otimes v_{j+1} \otimes \cdots \otimes v_m$$

Let

$$\mathcal{H}(\bigotimes^m \mathbb{C}^n) = \bigcap_{1 \le i < j \le m} \ker C_{ij}$$

For each λ with $|\lambda| = m$ and $l(\lambda) \le n$ we can choose a representative, F^λ for the irreducible $GL(n)$ representation with highest weight λ as a subspace of $\bigotimes^m \mathbb{C}^n$. This tensor space decomposes by Schur-Weyl duality as,

$$\bigotimes^m \mathbb{C}^n = \bigoplus_{\substack{\lambda : |\lambda| = m \\ l(\lambda) \le n}} F^\lambda \otimes U_\lambda$$

Where U_λ is an irreducible representation of the symmetric group on m letters, S_m. See chapter 2 for more comments on this decomposition.

Given a partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n \geq 0)$ with $|\lambda| = m$ and $l(\lambda) \leq n$, set $E^\lambda = F^\lambda \cap \mathcal{H}(\otimes^m \mathbb{C}^n)$. Denote by λ^c the conjugate partition of λ. That is to say the partition with its i part defined by $\lambda_i^c = \#\{j : \lambda_j \geq i\}$. If λ is such that $\lambda_1^c + \lambda_2^c > n$ then $E^\lambda = (0)$. In the case that $\lambda_1^c + \lambda_2^c \leq n$, E^λ will be an irreducible representation of $O(n, \mathbb{C})$. It is the case that all irreducible regular representations of $O(n, \mathbb{C})$ occur as invariant subspaces of tensor powers of the standard representation. As a consequence of this fact, every irreducible regular representation of $O(n)$ corresponds to a unique partition λ. The proofs of these facts can be found in [8], chapter 10. In particular, we have the decomposition of the harmonic tensors into irreducible $O(n) \times S_m$ representations,

$$\mathcal{H}(\bigotimes^m \mathbb{C}^n) = \bigoplus_{\substack{\lambda : |\lambda| = m \\ \lambda_1^c + \lambda_2^c \leq n}} E^\lambda \otimes U_\lambda$$

Note that because $O(n, \mathbb{C})$ is not a connected group we do not index representations by highest weights. Next we will describe how representations of $O(n)$ indexed in this way decomposes into irreducible representations for $SO(n)$ where we do describe the representations by highest weights.

Branching from $O(n, \mathbb{C})$ to $SO(n, \mathbb{C})$

In chapter 5 of [8] the irreducible $O(n)$ representations are obtained from the fact that $SO(n)$ is an index two subgroup of $O(n)$. This implies that the irreducible representations of $O(n)$ either remain irreducible when restricted to $SO(n)$ or decompose into two irreducible representations of $SO(n)$. We will now describe this in terms of the above parameters.

Proposition 12 ($(O(n), SO(n))$ branching, see [8]) *Let E^λ be an irreducible representation of $O(n, \mathbb{C})$ indexed by a non-negative integer partition, λ subject to the condition that $\lambda_1^c + \lambda_2^c \leq n$. (A partition satisfying this condition will be called admissible.) Set:*

$$V = Res_{SO(n,\mathbb{C})}^{O(n,\mathbb{C})} E^\lambda$$

Case 1: $n = 2r$ and $l(\lambda) = r$ then the representation V is reducible with two irreducible components with highest weights,

$$\lambda_1 e_1 + \cdots + \lambda_{r-1} e_{r-1} + \lambda_r e_r$$

and,

$$\lambda_1 e_1 + \cdots + \lambda_{r-1} e_{r-1} - \lambda_r e_r$$

Case 2: $l(\lambda) \neq r$ then for all n the representation V is irreducible with highest weight,

$$\lambda_1 e_1 + \cdots + \lambda_{r-1} e_{r-1} + \lambda_r e_r$$

Remark: The proof of this theorem is contained in chapter 5 of [8] together with the comments about the indexing of the $O(n)$ representations found in chapter 10. Let λ and μ be partitions such that,

- $\lambda_1 + \mu_1 = n$
- $\lambda_i = \mu_i$ for all i such that $2 \leq i \leq n$

A pair of partitions satisfying the above two conditions will be called associates.

Corollary 13 (Associate partition theorem, see [8] chapter 10)
As representations of $SO(n)$,

$$Res^{O(n)}_{SO(n)} E^{\lambda} \cong Res^{O(n)}_{SO(n)} E^{\mu}$$

if and only if λ and μ are associates.

3.1.4 Branching Coefficients

For a non-negative integer partitions λ and μ, define the non-negative integers b^{λ}_{μ} to be the multiplicity of E^{μ} in the representation F^{λ} restricted to the orthogonal group.

As $Sp(k) \subset GL(2k)$, we can view an irreducible representation of $GL(n)$ as a representation of $Sp(k)$. In the space manner, let c_μ^λ be the multiplicity of V^μ in F^λ restricted to $Sp(k)$.

By Schur's lemma we have:

$$b_\mu^\lambda = \dim Hom_{O(n)}\left(E^\mu,\ Res\,{}_{O(n)}^{GL(n)}\, F^\lambda\right)$$

$$c_\mu^\lambda = \dim Hom_{Sp(k)}\left(V^\mu,\ Res\,{}_{Sp(k)}^{GL(2k)}\, F^\lambda\right)$$

3.1.5 Littlewood's Restriction Rules

Given integer partitions, $\mu = (\mu_1 \geq \cdots \geq \mu_n)$, $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n)$, and $\nu = (\nu_1 \geq \cdots \geq \nu_n)$, define the classical *Littlewood-Richardson coefficients* $N_{\mu,\nu}^\lambda$ by,

$$N_{\mu,\nu}^\lambda = \dim Hom_{GL(n)}(F^\lambda, F^\mu \otimes F^\nu)$$

Littlewood's restriction formula for $O(n)$

If ν is a partition then set 2ν to be the partition obtained by doubling each part of ν.

Theorem 14 (See [16]) *If $r = \lfloor \frac{n}{2} \rfloor$, and $\lambda = (\lambda_1 \geq \cdots \geq \lambda_r \geq 0)$, then for $\mu = (\mu_1 \geq \cdots \geq \mu_r \geq 0)$,*

$$\dim Hom_{O(n)}(E^\mu, Res\,{}_{O(n)}^{GL(n)} F^\lambda) = \sum_{\nu : l(\nu) \leq r} N_{2\nu,\,\mu}^\lambda$$

Littlewood's restriction formula for $Sp(k)$

If ν is a non-negative integer partition then $(2\nu)^c$ is a non-negative integer partition with even columns in its Young diagram.

Littlewood's restriction formula for $Sp(k)$ asserts:

Theorem 15 (See [16]) *For* $\lambda = (\lambda_1 \geq \cdots \geq \lambda_k \geq 0)$*,*
then for $\mu = (\mu_1 \geq \cdots \geq \mu_k)$*,*

$$\dim Hom_{Sp(k)}(V^{\mu}, Res\,{}^{GL(2k)}_{Sp(k)} F^{\lambda}) = \sum_{\nu : l((2\nu)^c) \leq k} N^{\lambda}_{(2\nu)^c,\, \mu}$$

The goal of this work is to give a proof of the above formulae by using the structure theory of infinite dimensional highest weight modules.

3.2 Howe duality

3.2.1 Howe duality for the groups $(GL(n), GL(m))$

Theorem 16 (Howe duality A-A) *$GL(n)$ acts on $M_{n\times m}$ by left multiplication and $GL(m)$ acts by right multiplication. As a $GL(n)\times GL(m)$ representation, $\mathcal{P}(M^*_{n\times m})$ is multiplicity free space decomposing as,*

$$\mathcal{P}(M^*_{n\times m}) = \bigoplus_{\lambda : l(\lambda) \leq min(n,m)} F^{\lambda}_{(n)} \otimes F^{\lambda}_{(m)}$$

Corollary 17 *All irreducible polynomial representations of $GL(n)$ occur as invariant subspaces of the left multiplication representation on $\mathcal{P}(M^*_{n\times n})$.*

Corollary 18 *All irreducible regular representations of $O(n)$ occur as invariant subspaces of $\mathcal{P}(M^*_{n\times n})$*

Corollary 19 *All irreducible regular representations of $Sp(k)$ occur as invariant subspaces of $\mathcal{P}(M^*_{2k\times k})$*

Let $\mathcal{A}^L_{n,m}$ be the algebra of operators in $End(\mathcal{P}(M^*_{n\times m}))$ generated by the image of the representation of $GL(n)$.

Let $\mathcal{A}^R_{n,m}$ be the algebra of operators in $End(\mathcal{P}(M^*_{n\times m}))$ generated by the image of the representation of $GL(m)$.

Part of the Howe duality theorem (in full form) is the assertion that the commutant of the algebra $\mathcal{A}^L_{n,m}$ is the algebra $\mathcal{A}^R_{n,m}$ and that the commutant of the algebra $\mathcal{A}^R_{n,m}$ is the algebra $\mathcal{A}^L_{n,m}$.

3.2.2 Howe duality for $(O(n), sp(m))$

Consider the representation of $O(n)$ defined by matrix multiplication on $M_{n\times m}$ from the left. We then have a left (linear) action of $O(n)$ on $\mathcal{P}(M^*_{n\times m})$. For $1 \leq i \leq j \leq m$, the algebra of operators commuting with the algebra generated by the image of this representation has a generating set defined in terms of the following operators:

$$\begin{aligned} E_{ij} &= \sum_{s=1}^{n} x_{ki} \frac{\partial}{\partial x_{s,j}} \\ \Delta_{ij} &= \sum_{s=1}^{n} \frac{\partial^2}{\partial x_{s,i} \partial x_{s,j}} \\ M_{ij} &= \sum_{s=1}^{n} x_{s,i} x_{s,j} \end{aligned}$$

Let $\mathcal{C}_{n,m}$ be the algebra of operators in $End\,(\mathcal{P}(M_{n\times m}))$ commuting with the image of the left action of $O(n)$.

The above operators generate the algebra of linear operators commuting with the $O(n)$ action. Under the usual bracket, the vector space span of the operators $\{M_{ij},\, E_{ij} + \frac{n}{2}\delta_{ij},\, \Delta_{ij}\}$ has the structure a Lie algebra isomorphic to the rank m symplectic Lie algebra, $sp(m) = sp(m, \mathbb{C})$.

The double commutant theorem gives us a pairing of the irreducible $sp(m)$ modules, denoted E_μ in $\mathcal{P}(M^*_{n\times m})$ with the irreducible representation of the group $O(n)$, occurring in $\mathcal{P}(M^*_{n\times m})$. The pairing in this case is part of Roger Howe's general duality theory see [10]. For a recent treatment of these results see [8]. We quote the result here:

Theorem 20 *As an $O(n) \times sp(m)$ representation the space $\mathcal{P}(M^*_{n\times m})$ is a multiplicity free space. That is,*

$$\mathcal{P}(M^*_{n\times m}) = \bigoplus_{\mu : l(\mu) \leq min(\lfloor \frac{n}{2} \rfloor, m)} E^\mu \otimes E_\mu$$

In this light, the isotypic component of an irreducible $O(n)$ representation has the structure of an $sp(m)$ module. The irreducible $sp(m)$ modules are infinite dimensional highest weight modules.

The interesting fact is that the structure of these modules gives insight into the computation of the branching coefficients described in subsection 3.1.5. These facts will be discussed further in section 3.3.

3.2.3 Howe duality for $(Sp(k), so^*(2m))$

Consider the representation of $Sp(k)$ defined on $M_{2k\times m}$ by matrix multiplication on the left. We then have a locally finite dimensional representation of $Sp(k)$ on $\mathcal{P}(M^*_{k\times m})$. For $1 \leq i < j \leq m$, the algebra of operators commuting with the algebra generated by the image of this representation has a generating set defined in terms of the operators,

$$
\begin{aligned}
D_{ij} &= \sum_{s=1}^{k} \left(\frac{\partial^2}{\partial x_{s,i} \partial x_{(k+s),j}} - \frac{\partial^2}{\partial x_{s,j} \partial x_{(k+s),i}} \right) \\
S_{ij} &= \sum_{s=1}^{k} \left(x_{s,i} x_{(k+s),j} - x_{s,j} x_{(k+s),i} \right)
\end{aligned}
$$

And, recall that for all $1 \leq i, j \leq m$ we defined:

$$
E_{ij} = \sum_{s=1}^{2k} x_{s,i} \frac{\partial}{\partial x_{s,j}}
$$

Let $\mathcal{D}_{k,m}$ be the algebra of operators in $End\left(\mathcal{P}(M_{2k\times m})\right)$ commuting with the image of the left action of $Sp(k)$.

As with the $(O(n), sp(m))$ case, the above operators generate the algebra of operators commuting with the $Sp(k)$ action. Under the usual bracket, the vector space span of $\{M_{ij},\, \Delta_{ij}\}$ for $1 \leq i < j \leq n$ and $E_{ij} + k\delta_{ij}$ for all $1 \leq i, j \leq m$ has the structure a Lie algebra isomorphic to the Lie algebra $so(2m, \mathbb{C})$. This is a Lie algebra over $\mathbb{C}$, which is the complexification of the real Lie algebra $so^*(2m)$.

The double commutant theorem gives us a pairing of the irreducible $Sp(k)$ representations and irreducible $so(2m)$ modules that occur in $\mathcal{P}(M^*_{2k\times m})$. Denote the $so(2m)$ modules by V_μ. The pairing in this case is again part of Roger Howe's general duality theory.

Theorem 21 *As an $Sp(k) \times so(2m)$ representation the space $\mathcal{P}(M^*_{2k\times m})$ is a multiplicity free space. That is,*

$$\mathcal{P}(M^*_{2k\times m}) = \bigoplus_{\mu:l(\mu)\leq min(k,m)} V^\mu \otimes V_\mu$$

In this light, the isotypic component of an irreducible $Sp(k)$ representation has the structure of an $so(2m)$ module. The irreducible $so(m)$ modules are infinite dimensional highest weight modules.

3.2.4 A change in the order of summation.

Recall $\mathcal{A}_{n,m}$, the centralizer algebra of operators commuting with the left $GL(n)$ action on $\mathcal{P}(M^*_{n,m})$. From theorem 16 we have,

$$\mathcal{P}(M^*_{n\times m}) = \bigoplus_{\lambda:l(\lambda)\leq min(n,m)} F^\lambda_{(n)} \otimes F^\lambda_{(m)}$$

We will restrict the action of the $GL(n)$ in the above to the orthogonal group and write out the decomposition,

$$\mathcal{P}(M^*_{n\times m}) = \bigoplus_\lambda \left(\bigoplus_\mu b^\lambda_\mu E^\mu \right) \otimes F^\lambda$$

In the above sum, μ runs over non-negative integer partitions with at most $r = \lfloor \frac{n}{2} \rfloor$ parts. Upon reordering the summands in each of the decompositions we obtain,

$$\mathcal{P}(M^*_{n\times m}) = \bigoplus_\mu \left(E^\mu \otimes \bigoplus_\lambda b^\lambda_\mu F^\lambda \right)$$

The $sp(m)$ modules, E_μ are representations of the $O(n)$ centralizer algebra, $\mathcal{C}_{n,m}$ in $End(\mathcal{P}(M^*_{n\times m}))$. Observe that $\mathcal{A}^R_{n,m} \subset \mathcal{C}_{n,m}$ because $O(n) \subset GL(n)$. The image of $GL(m)$ is then contained in the algebra, $\mathcal{C}_{n,m}$. This implies that E_μ is a representation of the group $GL(m)$. We shall call this representation the *right* representation of $GL(m)$ on E_μ.

In the same way, consider the case where $n = 2k$ and restrict the action of the $GL(2k)$ to the symplectic group and write out the decomposition after reversing the order of summation.

$$\mathcal{P}(M^*_{2k\times m}) = \bigoplus_{\mu} \left(V^{\mu} \otimes \bigoplus_{\lambda} c^{\lambda}_{\mu} F^{\lambda} \right)$$

In the above, μ runs over all partitions with at most $min(k,m)$ non-zero parts, while λ runs over all partitions with at most $min(2k,m)$ non-zero parts. The $so(2m)$ modules, V_μ are representations of the $Sp(k)$ centralizer algebras, $\mathcal{D}_{k,m}$. Observe that $\mathcal{A}_{2k,m} \subset \mathcal{D}_{k,m}$ because $Sp(k) \subset GL(2k)$. Hence the image of $GL(m)$ is contained in the algebra, $\mathcal{D}_{n,m}$. This implies that V_μ is a representation of the group $GL(m)$. Again, we shall call this the right representation of $GL(m)$ on V_μ.

This gives an expression for the $GL(m)$ decomposition (under the right action) of the $sp(m)$-modules, E_μ as well as the $so(2m)$-modules, V_μ occurring in $\mathcal{P}(M^*_{n\times m})$.

Proposition 22 (**$(O(n), sp(k))$ case**) *As a $GL(m)$ representation under the right action, an $sp(m)$ representation E_μ occurring in the space $\mathcal{P}(M^*_{n,m})$ decomposes as,*

$$E_\mu = \bigoplus_{\lambda : l(\lambda) \leq min(n,m)} b^{\lambda}_{\mu} \, F^{\lambda}$$

Proposition 23 (**$(Sp(k), so^*(2m))$ case**) *As a $GL(m)$ representation under the right action, an $so(2m)$ representation V_μ occurring in the space $\mathcal{P}(M^*_{2k,m})$ decomposes as,*

$$V_\mu = \bigoplus_{\lambda : l(\lambda) \leq min(2k,m)} c^{\lambda}_{\mu} \, F^{\lambda}$$

3.3 Generalized Verma Modules

In this section we will show how the results of Enright, Howe and Wallach (see [2]) imply that in a certain stable range the modules E_μ and V_μ have a special structure. We will use these results to give another proof of the the Littlewood restriction rules. In the most general situation, let G be simple real Lie group with a subgroup K such that, (G, K) is a Hermitian symmetric pair. Let $\mathfrak{g}_0$ be the Lie algebra of G, while $\mathfrak{g}$ denotes the complexification of $\mathfrak{g}_0$. Let $\mathfrak{k}_0$ be the Lie algebra of K, and again let $\mathfrak{k}$ be the complexification of $\mathfrak{k}_0$. Let $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ be the Cartan decomposition. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the complexified Cartan decomposition. $\mathfrak{h}_0 \subset \mathfrak{k}_0$ will denote a Cartan subalgebra. Our assumptions imply that $\mathfrak{h}_0 \subset \mathfrak{g}_0$ and is a Cartan subalgebra for $\mathfrak{g}_0$ as well. $\mathfrak{h}$ will denote the complexification of $\mathfrak{h}_0$ as before. Let $\mathfrak{b}$ be a Borel subalgebra containing $\mathfrak{t}$ such that $\mathfrak{q} = \mathfrak{k} + \mathfrak{b}$ is a parabolic subalgebra of $\mathfrak{g}$. $\Phi = \Phi(\mathfrak{g}, \mathfrak{h})$ will denote the roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$.

$\Phi^+ = \Phi^+(\mathfrak{g}, \mathfrak{b})$ will denote the positive roots. Let Φ_c be the roots of $(\mathfrak{k}, \mathfrak{h})$. We call these roots, compact roots, while the remaining roots of $\mathfrak{g}$ will be denoted by Φ_n and referred to as noncompact roots. Set $\Phi_c^+ = \Phi^+ \cap \Phi_c$ and $\Phi_n^+ = \Phi^+ \cap \Phi_n$.

For a dominant weight $\lambda \in \mathfrak{h}$, let F^λ be a irreducible $\mathfrak{k}$ module. Now,

$$N(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{q})} F^\lambda$$

. We will call $N(\lambda)$ a generalized Verma module. Define:

$$\mathfrak{p}^+ = \sum_{\alpha \in \Phi_n^+} \mathfrak{g}_\alpha$$

$$\mathfrak{p}^- = \sum_{\alpha \in \Phi_n^+} \mathfrak{g}_{-\alpha}$$

Both $\mathfrak{p}^+$ and $\mathfrak{p}^-$ are $\mathfrak{k}$ invariant subspaces. As a $\mathfrak{k}$-module we have $\mathfrak{g} = \mathfrak{p}^- \oplus \mathfrak{k} \oplus \mathfrak{p}^+$. This implies that as a $\mathfrak{k}$-module,

$$N(\lambda) \cong S(p^-) \otimes F^\lambda$$

Let the irreducible quotient of $N(\lambda)$ be denoted by $L(\lambda)$. The goal of this section is to use a sufficient condition of when $N(\lambda) = L(\lambda)$ and use this fact to prove the Littlewood restriction rules.

For $\lambda \in \mathfrak{h}^*$, and $\alpha \in \Phi$ set

$$\lambda_\alpha = \frac{2\langle \lambda, \alpha \rangle}{(\alpha, \alpha)}$$

Let β be the unique maximal noncompact positive root of $\mathfrak{g}$. Let $\mathfrak{h}_1^*$ be the span of Φ_c. By the assumptions on (G, K), $\mathfrak{h}_1^*$ has codimension 1 in $\mathfrak{h}^*$. That is, $\mathfrak{k}$ has a one dimensional center. There is then a unique choice $\zeta \in \mathfrak{h}^*$ so that, ζ is orthogonal to Φ_c and $\zeta_\beta = 1$. Equivalently,

$$\langle \zeta, \beta \rangle = \frac{1}{2}(\beta, \beta)$$

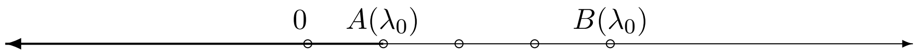

Figure 3.1: $\lambda_0 + z\zeta$ for $z \in \mathbb{R}$

The span of ζ is a complement to $\mathfrak{k}_1^*$. Each highest weight λ is a point on a line defined by $\lambda_0 + z\zeta$ where λ_0 is normalized so that:

$$\langle \lambda_0 + \rho, \beta \rangle = 0$$

λ is then determined by λ_0 and z by either of the two equivalent conditions,

$$\begin{aligned} \langle \lambda + \rho, \beta \rangle &= z \langle \zeta, \rho \rangle && (3.1) \\ &= \frac{z(\beta, \beta)}{2} && (3.2) \end{aligned}$$

Theorem 24 (Enright, Howe, Wallach [2]) *The set of real numbers z with $L(\lambda_0 + z\zeta)$ a unitarizable $\mathfrak{g}$-module is given in figure 3.1.*

- *The set includes the half line ending at $A(\lambda_0)$.*
- *The discrete series representations of G correspond to values $z < 0$*
- *The limit of the discrete series corresponds to $z = 0$.*
- *The smallest value of z with $N(\lambda_0 + z\zeta)$ reducible is $z = A(\lambda_0$. We call $A(\lambda_0)$ the first reduction point.*
- *In addition to the half line there are a number of equally spaced points in the set ending at $B(\lambda_0)$.*

Remark: In particular, if $z < A(\lambda_0)$ then $N(\lambda) = L(\lambda)$. The fact that we shall need is that since $A(\lambda_0) > 0$, we have that if $z < 0$ then $N(\lambda) = L(\lambda)$.

From the Howe duality theory we have that the modules V_μ and E_μ are irreducible modules. It is the case that they are in fact unitary highest weight modules (see [2]). The question answered by the above theorem is: *When are the modules generalized Verma Modules?* We consider the two cases $\mathfrak{g} = so(2m)$ and $\mathfrak{g} = sp(m)$ and give a sufficient (but not necessary) condition for when this happens.

3.3.1 The $so^*(2m)$ case.

$$\begin{aligned}
\mathfrak{g}_0 &= so^*(2m)\\
\mathfrak{g} &= so(2m,\mathbb{C})\\
\mathfrak{k} &= gl(m,\mathbb{C})\\
\mathfrak{p}^+ &= \wedge^2(\mathbb{C}^m)\\
\mathfrak{p}^- &= \wedge^2(\mathbb{C}^m)^*\\
\Phi &= D_m\\
\Phi_c^+ &= \{e_i - e_j | 1 \le i < j \le m\}\\
\Phi_n^+ &= \{e_i + e_j | 1 \le i < j \le m\}\\
\beta &= e_1 + e_2\\
\rho &= (m-1, m-2, m-3, \ldots, 2, 1, 0)\\
\zeta &= \frac{1}{2}(1, 1, \ldots, 1)
\end{aligned}$$

Proposition 25 *For the $Sp(k) \times so(2m)$-decomposition,*

$$\mathcal{P}(M^*_{2k,m}) = \bigoplus V^\mu \otimes V_\mu$$

where the direct sum runs over non-negative integer partitions,

$$\mu = (\mu_1, \mu_2, \ldots, \mu_k)$$

with $l = min(k,m)$ non-zero parts, the highest weight of V_μ is the m tuple:

$$\tilde{\mu} = (-k, \ldots, -k, -k-\mu_l, \ldots, -k-\mu_1)$$

The highest weight of the $Sp(k)$ representation is the k tuple:

$$\mu = (\mu_1, \ldots, \mu_l, 0, \ldots, 0)$$

Proof:

This proposition is a form of the well known Howe duality theorem mentioned earlier. We indicate below enough detail to determine the explicit parameters.

For $1 \leq i \leq min(2k, m)$, define:

$$\omega_i = Det\left(\begin{bmatrix} x_{11} & x_{12} & \dots & x_{1i} \\ x_{21} & x_{22} & \dots & x_{2i} \\ \dots & \dots & & \dots \\ x_{i1} & x_{i2} & \dots & x_{ii} \end{bmatrix}\right)$$

Let

$$v_\mu = \omega_1^{\mu_1-\mu_2}\omega_2^{\mu_2-\mu_3}\dots\omega_{m-1}^{\mu_{m-1}-\mu_m}\omega_m^{\mu_m}$$

In [2] p. 124 the vector v_μ is a highest weight vector relative to the chosen positive system with highest weight $\tilde{\mu}$. Furthermore, let T be the usual diagonal torus for the groups $Sp(k, \mathbb{C})$. For $\mathbf{t} = (t_1, \dots, t_k)$ we have $\mathbf{t}\dot{v}_\mu = t_1^{\mu_1} \dots t_k^{\mu_k} v_\mu$. And it can be checked that v_μ is killed by the positive nilpotent part of the Lie algebra of $Sp(k, \mathbb{C})$. This means that v_μ is a highest weight vector for both actions. **Q.E.D.**

Let:

$$\begin{aligned} \eta &= 2k\zeta \\ &= \underbrace{(k, k, \dots, k)}_{m\ k\text{'s}} \end{aligned}$$

Proposition 26 *As a $gl(m, \mathbb{C})$ representation under restriction, the $so^*(2m)$ generalized Verma module,*

$$N(-k, \dots, -k, -k - \mu_l, \dots, -k - \mu_1)$$

is equivalent to $\mathcal{P}(\wedge^2(\mathbb{C}^m)^*) \otimes F^{-\mu} \otimes F^{-\eta}$

Proof: This follows from the definition of a generalized Verma modules and

$$\mathfrak{p}^- = \wedge^2(\mathbb{C}^m)^*$$

Q.E.D.

Remark: In the above, $F^{-\eta}$ is the one dimensional representation of the *complex Lie algebra $gl_m(\mathbb{C})$*. This representation is equivalent to $-k\,Tr(X)$. In the case that $N(\tilde{\mu}) \cong E_\mu$ then it is a representation under the *right action* of $GL(m)$. Note that the differential of the right action of $GL(m, \mathbb{C})$ on E_μ is not equivalent to the restriction of $sp(m, \mathbb{C})$ to the Lie subalgebra $gl_m(\mathbb{C})$. However, we do have the following,

Proposition 27 *If V_μ has highest weight,*

$$\tilde{\mu} = (-k, \dots, -k, -k - \mu_l, \dots, -k - \mu_1)$$

and $V_\mu = N(\tilde{\mu})$ then under the right action of $GL(m)$ we have,

$$V_\mu \cong S(\wedge^2(\mathbb{C}^m)^*) \otimes F^\mu$$

Proof: The action of the Cartan subalgebra,

$$\mathfrak{h} \subset gl(m, \mathbb{C}) \subset so(2m, \mathbb{C})$$

defined by the differential operators $E_{ij} + k\delta_{i,j}$ differs by $k\, Id_m$ from the action of the Cartan subalgebra,

$$\mathfrak{h}' \subset Lie(GL(m, \mathbb{C}))$$

defined by the differential of the right action of $GL(m, \mathbb{C})$. **Q.E.D.**

Proposition 28 *In the decomposition, $\mathcal{P}(M^*_{2k,k}) = \bigoplus V^\mu \otimes V_\mu$*

$$\begin{aligned} V_\mu &= N(-k - \mu_k, \dots, -k - \mu_1) \\ &= L(-k - \mu_k, \dots, -k - \mu_1) \end{aligned}$$

Proof: Recall first that in this example, $\rho = (k-1, k-2, \dots, 0)$. So,

$$\begin{aligned} \lambda + \rho = (-k - \mu_k, -k - \mu_{k-1}, \dots) \\ +(k-1, k-2, \dots) \\ = (-\mu_k - 1, -\mu_{k-1} - 2, \dots) \end{aligned}$$

For this example, $\beta = e_1 + e_2$. So,

$$\begin{aligned} z &= \frac{2\langle \lambda + \rho, \beta \rangle}{(\beta, \beta)} \\ &= \langle \lambda + \rho, \beta \rangle \\ &= -\mu_k - \mu_{k-1} - 2 \\ &< 0 \end{aligned}$$

The result follows from Theorem 24. **Q.E.D.**

Proposition 29 ([8] chapter 5) *As a $GL(k)$ representation $S(\wedge^2(\mathbb{C}^k))$ is a multiplicity free space in which an irreducible representation F^ν occurs if and only if ν has each part repeated an even number of times. That is, the Young diagram has even columns.*

Corollary 30 *The multiplicity of the $GL(m)$ representation F^λ, in $S(\wedge^2(\mathbb{C}^m)) \otimes F^\mu$ is,*

$$\sum_{\nu : l((2\nu)^c) \leq m} N^\lambda_{(2\nu)^c,\mu}$$

Proof: Tensor every irreducible component of $S(\wedge^2(\mathbb{C}^m))$ with F^μ. **Q.E.D.**

A proof of Littlewood's restriction rules for $Sp(k)$

Let

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0)$$

$$\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k \geq 0)$$

be non-negative integer partitions with at most k parts and set $\tilde{\mu}$ to be the following vector in $\mathbb{C}^k$,

$$\tilde{\mu} = (-k - \mu_k, \ldots, -k - \mu_1)$$

- By definition c^λ_μ is the multiplicity of the irreducible representation V^μ in the $GL(k)$ representation F^λ.
- c^λ_μ is the same as the multiplicity of the $GL(k)$-representation F^λ in the $so(2k)$-representation V_μ by Proposition 23
- V_μ is an irreducible unitary highest weight representation with highest weight $\tilde{\mu}$ by Proposition 25. Hence, $V_\mu \cong L(\tilde{\mu})$.
- V^μ occurs in the space $\mathcal{P}(M^*_{2k,k})$ by Corollary 19.
- V_μ occurs in the space $\mathcal{P}(M^*_{2k,k})$ by Theorem 21
- If $L(\tilde{\mu})$ occurs in $\mathcal{P}(M^*_{2k,k})$ then $L(\tilde{\mu}) = N(\tilde{\mu})$ by Proposition 28
- As a right $GL(k)$ representation, $V_\mu \cong S(\wedge^2(\mathbb{C}^k)) \otimes F^\mu$ by Proposition 27

- The multiplicity of the irreducible representation F^λ of $GL(m)$ in

$$S(\wedge^2(\mathbb{C}^k)) \otimes F^\mu$$

is c^λ_μ and by Proposition 30 this number is:

$$\sum_{\nu : l((2\nu)^c) \leq m} N^\lambda_{(2\nu)^c,\mu}$$

Q.E.D.

3.3.2 The $sp(m)$ case.

$$\begin{aligned}
\mathfrak{g}_0 &= sp(2m) \\
\mathfrak{g} &= sp(m,\mathbb{C}) \\
\mathfrak{k} &= gl(m,\mathbb{C}) \\
\mathfrak{p}^+ &= S^2(\mathbb{C}^m) \\
\mathfrak{p}^- &= S^2(\mathbb{C}^m)^* \\
\Phi &= C_m \\
\Phi_c^+ &= \{e_i - e_j | 1 \leq i < j \leq m\} \\
\Phi_n^+ &= \{e_i + e_j | 1 \leq i \leq j \leq m\} \\
\beta &= 2e_1 \\
\rho &= (m, m-1, m-2, \ldots, 3, 2, 1) \\
\zeta &= (1, 1, \ldots, 1)
\end{aligned}$$

For this section let $r = \lfloor \frac{n}{2} \rfloor$ and $l = min(r, m)$.

Proposition 31 *For the*
$O(n) \times sp(m)$-decomposition,

$$\mathcal{P}(M^*_{n,m}) = \bigoplus E^\mu \otimes E_\mu$$

where the direct sum runs over non-negative integer partitions μ with l non-zero parts, the highest weight of E_μ is:

$$\tilde{\mu} = (-\frac{n}{2}, \ldots, -\frac{n}{2}, -\frac{n}{2} - \mu_l, \ldots, -\frac{n}{2} - \mu_1)$$

Proof: For $1 \leq i \leq min(n, m)$,

$$\omega_i = Det\left(\begin{bmatrix} x_{11} & x_{12} & \ldots & x_{1i} \\ x_{21} & x_{22} & \ldots & x_{2i} \\ \ldots & \ldots & & \ldots \\ x_{i1} & x_{i2} & \ldots & x_{ii} \end{bmatrix}\right)$$

Let

$$v_\lambda = \omega_1^{\lambda_1 - \lambda_2} \omega_2^{\lambda_2 - \lambda_3} \ldots \omega_{m-1}^{\lambda_{m-1} - \lambda_m} \omega_m^{\lambda_m}$$

In [12], p.23 prop 6.6 we have that the vector $v_{\tilde{\mu}}$ is a highest weight vector which under the action of the $O(n, \mathbb{C})$ generates an irreducible representation of $O(n, \mathbb{C}$ isomorphic to E^μ. **Q.E.D.**

In the next proposition we will denote by $\tilde{\eta}$ the partition in which each part is equal to $\frac{n}{2}$. That is,

$$\begin{aligned} \tilde{\eta} &= \tfrac{n}{2}\zeta \\ &= \underbrace{(\frac{n}{2}, \frac{n}{2}, \ldots, \frac{n}{2})}_{m\ \frac{n}{2}\text{'s}} \end{aligned}$$

Proposition 32 ($(O(n), sp(m))$ case) *As a $gl(m, \mathbb{C})$ representation under restriction, the generalized Verma module,*

$$N(-\frac{n}{2}, \ldots, -\frac{n}{2}, -\frac{n}{2} - \mu_l, \ldots, -\frac{n}{2} - \mu_1)$$

is equivalent to

$$\mathcal{P}(S^2(\mathbb{C}^m)^*) \otimes F^\mu \otimes F^{\tilde{\eta}}$$

Proof: This follows from the definition of a generalized Verma modules and

$$\mathfrak{p}^- = S^2(\mathbb{C}^m)^*$$

Q.E.D.

Remark: In the above, $F^{\tilde{\eta}}$ is the one dimensional representation of the *complex Lie algebra* representation equivalent to $\frac{n}{2}Tr(X)$. This is not the differential of a *group* representation of the $GL(m,\mathbb{C})$. However, in the case that $N(\tilde{\mu}) \cong E_\mu$ then it is a representation under the *right action* of $GL(m)$. Note that the differential of the right action of $GL(m,\mathbb{C})$ on E_μ is not equivalent to the restriction of $sp(m,\mathbb{C})$ to the Lie subalgebra $gl_m(\mathbb{C})$. Although we do have the following,

Proposition 33 *In the case that, E_μ has highest weight*

$$\tilde{\mu} = (-\frac{n}{2}, \ldots, -\frac{n}{2}, -\frac{n}{2} - \mu_l, \ldots, -\frac{n}{2} - \mu_1)$$

and $E_\mu \cong N(\tilde{\mu})$ then under the right action of $GL(m)$ we have

$$E_\mu \cong S(S^2(\mathbb{C}^m)) \otimes F^\mu$$

Proof: The action of the Cartan subalgebra,

$$\mathfrak{h} \subset gl(m,\mathbb{C}) \subset sp(m,\mathbb{C})$$

defined by the differential operators $E_{ij} + \frac{n}{2}\delta_{i,j}$ differs by $\frac{n}{2}I_m$ from the action of the Cartan subalgebra,

$$\mathfrak{h}' \subset Lie(GL(m,\mathbb{C}))$$

defined by the differential of the right action of $GL(m,\mathbb{C})$. **Q.E.D.**

Proposition 34 *Let $r = \lfloor \frac{n}{2} \rfloor$. In the decomposition,*

$$\mathcal{P}(M^*_{n,r}) = \bigoplus E^\mu \times E_\mu$$

$$\begin{aligned} E_\mu &= N(-\frac{n}{2} - \mu_r, \ldots, -\frac{n}{2} - \mu_1) \\ &= L(-\frac{n}{2} - \mu_r, \ldots, -\frac{n}{2} - \mu_1) \end{aligned}$$

Proof: Recall first that in this example, $\rho = (r, r-1, \ldots, 1)$. So,

$$\begin{aligned} \lambda + \rho = (-\frac{n}{2} - \mu_r, \ldots) \\ +(r, r-1, \ldots) \\ = (r - \frac{n}{2} - \mu_r, \ldots) \end{aligned}$$

For this example, $\beta = 2e_1$. So,

$$\begin{aligned} z &= \frac{2\langle \lambda + \rho, \beta \rangle}{(\beta, \beta)} \\ &= \frac{\langle \lambda + \rho, \beta \rangle}{2} \\ &= \frac{2(r - \frac{n}{2} - \mu_r)}{2} \\ &= \frac{2(\lfloor \frac{n}{2} \rfloor - \frac{n}{2} - \mu_r)}{2} \\ &= \lfloor \frac{n}{2} \rfloor - \frac{n}{2} - \mu_r \\ &\leq 0 \end{aligned}$$

The result follows from Theorem 24. **Q.E.D.**

Proposition 35 (see [8] chapter 5) *As a $GL(m)$ representation $S(S^2(\mathbb{C}^m))$ is a multiplicity free space in which an irreducible representation F^ν occurs if and only if ν has even parts. That is, the Young diagram has even rows.*

Corollary 36 *The multiplicity of the $GL(m)$ representation F^λ in*

$$S(S^2(\mathbb{C}^m)) \otimes F^\mu$$

is,

$$\sum_{\nu : l(\nu) \leq n} N^\lambda_{2\nu,\mu}$$

Proof: Tensor every irreducible component of $S(S^2(\mathbb{C}^m))$ with F^μ. **Q.E.D.**

A proof of Littlewood's restriction rules for $O(n)$

Let $r = \lfloor \frac{n}{2} \rfloor$

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 0)$$

$$\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_r \geq 0)$$

be non-negative integer partitions with at most r parts and set $\tilde{\mu}$ to be the following vector in $\mathbb{C}^r$,

$$\tilde{\mu} = (-\frac{n}{2} - \mu_r, \ldots, -\frac{n}{2} - \mu_1)$$

- By definition b^λ_μ is the multiplicity of the irreducible representation E^μ in the $GL(r)$ representation F^λ.
- b^λ_μ is the same as the multiplicity of the $GL(r)$-representation F^λ in the $sp(r)$-representation E_μ by Proposition 22
- E_μ is an irreducible unitary highest weight representation with highest weight $\tilde{\mu}$ by Proposition 31. Hence, $E_\mu \cong L(\tilde{\mu})$.
- E^μ occurs in the space $\mathcal{P}(M^*_{n,r})$ by Corollary 18.
- E_μ occurs in the space $\mathcal{P}(M^*_{n,r})$ by Theorem 20
- If $L(\tilde{\mu})$ occurs in $\mathcal{P}(M^*_{n,r})$ then $L(\tilde{\mu}) = N(\tilde{\mu})$ by Proposition 34
- As a right $GL(r)$ representation, $E_\mu \cong S(S^2(\mathbb{C}^r)) \otimes F^\mu$ by Proposition 33
- The multiplicity of the irreducible representation F^λ of $GL(r)$ in

$$S(S^2(\mathbb{C}^r)) \otimes F^\mu$$

is b^λ_μ and by Corollary 36 is also:

$$\sum_{\nu:l(\nu)\leq n} N^\lambda_{2\nu,\mu}$$

Q.E.D.

In conclusion, we point out that the techniques employed in this chapter are somewhat more general than what we have actually used. In the research following this thesis we will give stronger branching rules as further applications of this same theory.

Chapter 4

Spherical Harmonics

4.1 Spherical Harmonics

4.1.1 Classical Setup

Let $\{x_i\}_{i=1}^{n}$ be the coordinate basis of the vector space $(\mathbb{C}^n)^*$ and $(*,*)$ the standard non-degenerate bilinear form on $\mathbb{C}^n$ with respect to this basis, ie:

$$(x_i, x_j) = \delta_{i,j}$$

Using this form we will identify $\mathbb{C}^n$ with $(\mathbb{C}^n)^*$. $\mathcal{P}(\mathbb{C}^n)$ will denote the ring of polynomial functions on $\mathbb{C}^n$. This ring is graded by degree, so take $\mathcal{P}^d(\mathbb{C}^n)$ to be the degree d homogeneous polynomials. Hence,

$$\mathcal{P}(\mathbb{C}^n) = \bigoplus_{d \geq 0} \mathcal{P}^d(\mathbb{C}^n)$$

For $x \in \mathbb{C}^n$, set $Q(x) = (x, x)$. Q is a quadratic polynomial function on $\mathbb{C}^n$. Let the subring of $\mathcal{P}(\mathbb{C}^n)$ generated by Q be denoted by $\mathbb{C}[Q]$. As usual, let the Laplacian,

$$\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial {x_i}^2}$$

Define the degree d spherical harmonics,

$$H_d(\mathbb{C}^n) = \left\{ f \in P^d(\mathbb{C}) | \Delta(f) = 0 \right\}$$

We will abbreviate $H_d = H_d(\mathbb{C}^n)$, and set

$$\mathcal{H}(\mathbb{C}^n) = \bigoplus_{d \geq 0} H_d$$

$O(n)$ will (specifically) be the subgroup of $GL(n)$ consisting of orthogonal matrices with respect to Q. $GL(n)$ acts linearly on $\mathbb{C}^n$ by the standard representation, so let $\mathcal{P}(\mathbb{C}^n)^{O(n)}$ denote the subring of $O(n)$-invariant polynomial functions. For the classical theory of spherical harmonics refer to [8]. The part of the theory we require is summarized by, $\mathcal{P}(\mathbb{C}^n) = \mathcal{P}(\mathbb{C}^n)^{O(n)} \otimes \mathcal{H}$, and $\mathcal{P}(\mathbb{C}^n)^{O(n)} = \mathbb{C}[Q]$. As mentioned in the last chapter a list of operators which generate the commutant of this action are,

$$\begin{aligned} \Delta &= \sum_{i=1}^{n} \frac{\partial^2}{\partial {x_i}^2} \\ E &= \frac{n}{2} + \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} \\ M &= \sum_{i=1}^{n} x_i^2 \end{aligned}$$

The above operators span a vector space which under the usual Lie bracket is a Lie algebra isomorphic to $sl_2(\mathbb{C})$. In fact, if $X = \frac{1}{2}\Delta$, $Y = -\frac{1}{2}M$, and $H = -E$ then $\{X, Y, H\}$ is a standard TDS, see [8].

4.1.2 A formal expression

Let the symbols $t_1, \ldots, t_n$ and h be indeterminates and let $\mathbf{t} = \mathbf{t}_{(n)}$ denote the list $t_1, \ldots, t_n$. Let $\mathbb{N}$ denote the set of non-negative integers. For $\mathbf{a} \in \mathbb{N}^n$, let $\lambda(\mathbf{a})$ be a partition with its i^{th} part,

$$\lambda(\mathbf{a})_i = \sum_{j=i}^{n} a_j$$

Also, for a non-negative integer partition λ with at most n parts, let

$$A(\lambda)_i = \begin{cases} \lambda_i - \lambda_{i+1} & \text{For } 1 \leq i \leq n-1 \\ \lambda_n & \text{For } i = n \end{cases}$$

Observe that for all partitions, λ with $l(\lambda) \leq n$, and for all, $\mathbf{a} \in \mathbb{N}^n$,

$$\lambda(A(\mu)) = \mu$$

$$A(\lambda(\mathbf{a})) = \mathbf{a}$$

Recall the notation from chapter 3 where we indexed the irreducible representations of $O(n, \mathbb{C})$ by non-negative integer partitions. Note that the spherical harmonics correspond to the partitions with exactly one non-zero part, that is $H_d \cong E^\mu$, where $\mu = (d) = (d, 0, \ldots, 0)$. As in chapter 3, set:

$$b^\lambda_{(d)} := \dim Hom_{O(n)}(H_d, F^\lambda)$$

In the above, the subscript (d) is intended to denote a partition with one part equal to d. Consider the following formal sum,

$$R_{O(n)}(\mathbf{t}_{(n)}, h) := \sum_{\mathbf{a} \in \mathbb{N}^n,\, d \geq 0} b^{\lambda(a)}_{(d)}\ \mathbf{t}^{\mathbf{a}}\ h^d$$

The idea in the above is that the coefficient of $t_1^{a_1} \ldots t_n^{a_n} h^d$ is the multiplicity of the spherical harmonics of degree d in $F^{\lambda(\mathbf{a})}$, where the highest weight λ is indexed by its non-negative integer combination of fundamental weights for $GL(n)$.

Theorem 37 *For $n \geq 2$,*

$$R_{O(n)} = \frac{\prod_{i=1}^{n-1}(1 + t_i t_{i+1} h)}{(1 - t_1 h) \prod_{i=1}^{n}(1 - t_i^2) \prod_{i=2}^{n-1}(1 - t_i^2 h^2)}$$

Remarks: In the above an empty product has the value 1, as usual.

By expanding the above expression into a formal power series we obtain an effective method to compute the multiplicity of the space of spherical harmonics of degree d in a irreducible $GL(n)$ representation with highest weight λ.

In the next section we will give a somewhat technical proof of theorem 37. It is instructive to point out that what we will prove is the following inductive approach.

Proposition 38

$$\begin{aligned} R_{O(n+1)}(\mathbf{t}_{(n+1)}, h) &= \frac{1 + t_n t_{n+1} h}{(1 - t_{n+1}^2)(1 - t_n^2 h^2)} R_{O(n)}(\mathbf{t}_{(n)}, h) \quad \textit{for } n \geq 1 \\ R_{O(1)}(t, h) &= \frac{1 + th}{1 - t^2} \end{aligned}$$

4.2 Proof of theorem 37

Define:

$$W_n(\mathbf{t}_{(n)}, h) = \sum_{\mathbf{a}\in\mathbb{N}^n,\, d\geq 0} \dim\left(F^{\lambda(\mathbf{a})} \otimes S^d(\mathbb{C}^n)\right)^{O(n)} \mathbf{t}^{\mathbf{a}} h^d$$

Recall the isomorphism,

$$\begin{aligned} Hom_{O(n)}(H_d, F^\lambda) &\cong ((H_d)^* \otimes F^\lambda)^{O(n)} \\ &\cong (H_d \otimes F^\lambda)^{O(n)} \end{aligned}$$

As an application of this observe,

$$\begin{aligned} \sum_{d\geq 0} \dim\left(F^\lambda \otimes S^d(\mathbb{C}^n)\right)^{O(n)} h^d &= \sum_{d\geq 0} \dim\left(F^\lambda \otimes (S^d(\mathbb{C}^n))^*\right)^{O(n)} h^d \\ &= \sum_{d\geq 0} \dim\left(F^\lambda \otimes \mathcal{P}^d(\mathbb{C}^n)\right)^{O(n)} h^d \\ &= \frac{1}{1-h^2} \sum_{d\geq 0} \dim\left(F^\lambda \otimes H_d\right)^{O(n)} h^d \\ &= \frac{1}{1-h^2} \sum_{d\geq 0} b^\lambda_{(d)}\, h^d \end{aligned}$$

We then obtain,

$$R_{O(n)} = (1-h^2) W_n(\mathbf{t}_{(n)}, h)$$

Given partitions λ and μ with at most n parts, λ is said to *interlace* μ if $\lambda_i \geq \mu_i \geq \lambda_{i+1}$ for all $1 \leq i \leq n-1$. Another way to state this condition is that the corresponding Young diagrams differ by a horizontal strip. With this in mind we shall compute $W_n(\mathbf{t}, h)$ another way, first observe the classical interlacing result.

Theorem 39 (Interlacing, [8] Chapter 9)

$$F^\mu \bigotimes S^d(\mathbb{C}^n) = \bigoplus_\lambda F^\lambda$$

Where the sum is over all partitions λ with at most n parts and size $|\mu| + d$ that interlace μ.

In order to find the $O(n)$ invariants we will again (see chapter 2) use the following special case of Cartan and Helgason, (see [8], chapter 10).

Theorem 40

$$\dim \left(F^{\lambda}\right)^{O(n)} = \begin{cases} 1 \textit{ if } \lambda \textit{ has all even parts} \\ 0 \textit{ otherwise} \end{cases}$$

To obtain our objective we will give a weighted enumeration of even partitions which interlace a given partition. We define the following set,

$$L_n = \{(\lambda, \mu)|\ 2\lambda_i \geq \mu_i \geq 2\lambda_{i+1} \geq \mu_{i+1} \text{ for all } i,\ 1 \leq i \leq n-1\}$$

Let $\mathbb{C}[\mathbf{t}_{(n)}, h]$ denote the ring of polynomials in $x_1, \ldots, x_n$ and h. Consider the weight, $w_n : L_n \to \mathbb{C}[\mathbf{t}_{(n)}, h]$, by,

$$\begin{aligned} w_n(\lambda, \mu) &= t_n^{\mu_n} \left[\prod_{i=1}^{n-1} t_i^{\mu_i - \mu_{i+1}}\right] h^{\sum_{i=1}^{n}(2\lambda_i - \mu_i)} \\ &= \mathbf{t}^{A(\mu)}\ h^{2|\lambda|-|\mu|} \end{aligned}$$

Proposition 41

$$W_n(\mathbf{t}_{(n)}, h) = \sum_{(\lambda,\mu)\in L_n} w_n(\lambda, \mu)$$

Proof:

$$\begin{aligned} W_n(\mathbf{t}_{(n)}, h) &= \sum_{\mathbf{a}\in\mathbb{N}^n,\, d\geq 0} \dim \left(F^{\lambda(\mathbf{a})} \otimes S^d(\mathbb{C}^n)\right)^{O(n)} t^{\mathbf{a}} h^d \\ &= \sum_{\substack{\mu_1 \geq \cdots \geq \mu_n \geq 0 \\ d \geq 0}} \left(\sum_{\substack{\lambda\,:\, 2\lambda \, interlaces \, \mu \\ |2\lambda| - |\mu| = d}} 1 \right) t^{A(\mu)}\, h^d \\ &= \sum_{(\lambda,\mu)\in L_n} t^{A(\mu)}\, h^{|2\lambda|-|\mu|} \end{aligned}$$

Q.E.D.

We will express $W_n(\mathbf{t}_{(\mathbf{n})}, h)$ as a rational function. We will proceed inductively as follows. Set,

$$\mathbb{N}_e^2 = \{(a, b) \in \mathbb{N}^2 | a, b \geq 0,\ a + b \in 2\mathbb{Z}\}$$

and define another weight,

$$\widetilde{w}_n : \mathbb{N}_e^2 \times L_n \to \mathbb{C}[\ \mathbf{t}_{(n+1)},\ h]$$

$$\widetilde{w}_n\left((a,b),(\lambda,\mu)\right) = t_{n+1}^a\ t_m^b\ h^b\ w_n(\lambda,\mu)$$

Theorem 42 *Given a quadruple* $(a, b, \widetilde{\lambda}, \widetilde{\mu}) \in \mathbb{N}_e^2 \times L_n$*, let*

$$\begin{aligned} \lambda_i &= \widetilde{\lambda}_i + \frac{a+b}{2} \qquad \textit{for } 1 \le i \le n \\ \lambda_{n+1} &= \frac{a+b}{2} \\ \mu_i &= \widetilde{\mu}_i + a + b \qquad \textit{for } 1 \le i \le n \\ \mu_{n+1} &= a \end{aligned}$$

Let: $\Psi\left((a,b),(\widetilde{\lambda},\widetilde{\mu})\right) = (\lambda,\mu)$*. It is the case that* Ψ *defines a function,*

$$\Psi : \mathbb{N}_e^2 \times L_n \to L_{n+1}$$

Furthermore, Ψ *is bijective and* $w_{n+1} \circ \Psi = \widetilde{w}_n$

Proof: The fact that Ψ is a function is a just a check that the values are in the set L_{n+1}. It can be easily checked that the following is also a function, $\Psi^* : L_{n+1} \to \mathbb{N}_e^2 \times L_n$ Given (λ,μ) set,

$$\begin{aligned} a &= \mu_{n+1} \\ b &= 2\lambda_{n+1} - \mu_{n+1} \\ \widetilde{\lambda}_i &= \lambda_i - \lambda_{n+1} \qquad \text{for } 1 \le i \le n \\ \widetilde{\mu}_i &= \mu_i - 2\lambda_{n+1} \qquad \text{for } 1 \le i \le n \end{aligned}$$

$$\Psi^*(\lambda,\mu) = \left((a,b),(\widetilde{\lambda},\widetilde{\mu}\right)$$

It is an easy exercise to check that, $\Psi \circ \Psi^* = I_{L_{n+1}}$ and, $\Psi^* \circ \Psi = I_{\mathbb{N}_e^2 \times L_n}$. To verify the second part of the theorem observe the following computation,

$$w_{n+1} \circ \Psi\left((a,b),(\widetilde{\lambda},\widetilde{\mu})\right) = w_{n+1}(\lambda,\mu)$$

$$
\begin{aligned}
w_{n+1}(\lambda,\mu) &= t_{n+1}^{\mu_{n+1}} \left[\prod_{i=1}^{n-1} t_i^{\mu_i-\mu_{i+1}} \; (t_n^{\mu_n-\mu_{n+1}})\right] h^{\left[\sum_{i=1}^{n}(2\lambda_i-\mu_i)+(2\lambda_{n+1}-\mu_{n+1})\right]} \\
&= t_{n+1}^{a} \left[t_n^{\tilde{\mu}_n+b} \prod_{i=1}^{n-1} t_i^{\tilde{\mu}_i-\tilde{\mu}_{i+1}}\right] h^{\left[b+\sum_{i=1}^{n}(2\tilde{\lambda}_i-\tilde{\mu}_i)\right]} \\
&= (t_{n+1}^{a}\, t_n^{b}\, h^{b})\, t_n^{\tilde{\mu}_n} \left[\prod_{i=1}^{n-1} t_i^{\tilde{\mu}_i-\tilde{\mu}_{i+1}}\right] h^{2|\lambda|-|\mu|} \\
&= t_{n+1}^{a}\, t_n^{b}\, h^{b}\, w_n(\tilde{\lambda},\tilde{\mu}) \\
&= \widetilde{w}_n((a,b),(\tilde{\lambda},\tilde{\mu}))
\end{aligned}
$$

Q.E.D.

Before proceeding with our computation we will need the following lemma:

Lemma 2

$$
\sum_{(i,j)\in N_e^2} x^i\, y^j = \frac{1+xy}{(1-x^2)(1-y^2)}
$$

Proof:

$$
\begin{aligned}
\sum_{(i,j)\in N_e^2} x^i\, y^j &= \sum_{i,j\geq 0} (x^{2i}\, y^{2j} + x^{2i+1}\, y^{2j+1}) \\
&= \sum_{i,j\geq 0} x^{2i}\, y^{2j}\, (1+xy) \\
&= \frac{1+xy}{(1-x^2)(1-y^2)}
\end{aligned}
$$

Q.E.D.

Now we will set up the induction step for our computation starting with Proposition 41 and then applying Theorem 42.

$$
\begin{aligned}
W_{n+1}(\mathbf{t}_{(n+1)}, h) &= \sum_{(\lambda,\mu)\in L_{n+1}} w_{n+1}(\lambda,\mu) &(4.1)\\
&= \sum_{((a,b),(\tilde{\lambda},\tilde{\mu}))\in N_e^2\times L_n} w_{n+1}\circ\Psi((a,b),(\tilde{\lambda},\tilde{\mu})) &(4.2)\\
&= \sum_{((a,b),(\widetilde{\lambda},\widetilde{\mu}))\in \mathbb{N}_e^2\times L_n} \widetilde{w}_n((a,b),(\widetilde{\lambda},\widetilde{\mu})) &(4.3)\\
&= \sum_{((a,b),(\widetilde{\lambda},\widetilde{\mu}))\in \mathbb{N}_e^2\times L_n} t_{n+1}^a t_n^b h^b w_n(\widetilde{\lambda},\widetilde{\mu}) &(4.4)\\
&= \left(\sum_{(a,b)\in N_e^2} t_{n+1}^a t_n^b\right)\left(\sum_{(\tilde{\lambda},\tilde{\mu})\in L_n} w_n(\tilde{\lambda},\tilde{\mu})\right) &(4.5)\\
&= \left(\frac{1+t_nt_{n+1}h}{(1-t_{n+1}^2)(1-t_n^2h^2)}\right) W_n(\mathbf{t}_{(n)}, h) &(4.6)
\end{aligned}
$$

The last line follows from the definition of $W_n(\mathbf{t_{(n)}}, h)$ and Lemma 2. This sets up an inductive procedure for the computation. The basis case will be provided by,

Proposition 43

$$
\begin{aligned}
W_1(t_1, h) &= \sum_{\substack{\lambda_1,\,\mu_1\,\in\,\mathbb{N}\\ 2\lambda_1\,-\,\mu_1\,\geq\,0}} t^{\mu_1}h^{2\lambda-\mu}\\
&= \frac{1+t_1h}{(1-t_1^2)(1-h^2)}
\end{aligned}
$$

Proof: Use Lemma 2 **Q.E.D.**

Now we will express $W_n(\mathbf{t}_{(n)}, h)$ as a rational function of the variables $t_1, \ldots, t_n$ and h.

Proposition 44

$$
W_n(\mathbf{t}_{(n)}, h) = \frac{\prod_{i=1}^{n-1}(1+t_it_{i+1}h)}{(1-t_1h)(1-h^2)\prod_{i=1}^{n}(1-t_i^2)\prod_{i=2}^{n-1}(1-t_i^2h^2)}
$$

Proof: This has been proved above by the induction argument with basis case from equation (4.7) and induction step supplied by equation (4.6). **Q.E.D.** This completes the inductive approach to the computation. We now push back the recursion to yield the following statement for $n \geq 2$,

$$R_{O(n)} = \frac{\prod_{i=1}^{n-1}(1 + t_i t_{i+1} h)}{(1 - t_1 h)\prod_{i=1}^{n}(1 - t_i^2)\prod_{i=2}^{n-1}(1 - t_i^2 h^2)} \tag{4.7}$$

4.3 Graded multiplicity formulae for the spherical harmonics

Here we give an application of section 4.1 by computing the graded multiplicity formula for the spherical harmonics of the groups, $O(n)$ and $SO(n)$ in the spaces $\mathcal{P}(\wedge^2(\mathbb{C}^n)^*)$, and $\mathcal{P}(S^2(\mathbb{C}^n)^*)$. Here we will assume $n \geq 3$. Recall the classical $GL(n)$ decompositions (see [8], chapter 5),

$$\begin{aligned}
\mathcal{P}(\wedge^2(\mathbb{C}^n)^*) &= \bigoplus_{d\geq 0} \mathcal{P}^d(\wedge^2(\mathbb{C}^n)^*) \\
&= \bigoplus_{d\geq 0} \bigoplus_{\substack{\lambda\,:\,l((2\lambda)^c)\,\leq\, n \\ |\lambda|\,=\,d}} F^{(2\lambda)^c} \\
\mathcal{P}(S^2(\mathbb{C}^n)^*) &= \bigoplus_{d\geq 0} \mathcal{P}^d(S^2(\mathbb{C}^n)^*) \\
&= \bigoplus_{d\geq 0} \bigoplus_{\lambda\,:\,l(\lambda)\,\leq\, n|\lambda|\,=\,d} F^{2\lambda}
\end{aligned}$$

Both of the spaces above are graded by degree. We will now apply the results of section 4.1 to the problem of finding the graded multiplicity of the $O(n)$ representations, H_d in the above spaces.

4.3.1 Graded Multiplicity in $\mathcal{P}(S^2(\mathbb{C}^n))$

It is interesting to point out that if we restrict the sum in equation (4.7) to only the partitions with even parts the numerator cleans up. This restriction can be done algebraically by applying the operator $\mathcal{E}$ as follows:

Define the following operator on the ring of formal power series in the variables $t_1 \dots t_n$ and h,

$$\mathcal{E}_n : \mathbb{C}[[\mathbf{t}_{(n)}, h]] \to \mathbb{C}[[\mathbf{t}_{(n)}, h]]$$

$$\mathcal{E}_n(p) = \frac{1}{2^n} \sum_{\mathbf{e} \in \{1,-1\}^n} p(e_1 t_1, \ldots, e_n t_n, h)$$

$$\begin{aligned}
\sum_{\mathbf{a} \in \mathbb{N}^n, d \geq 0} b_{(d)}^{2\lambda(a)} \mathbf{t}^{\mathbf{2a}} h^d \\
&= \mathcal{E}_n \left(\sum_{\mathbf{a} \in \mathbb{N}^n, d \geq 0} b_{(d)}^{\lambda(a)} \mathbf{t}^{\mathbf{a}} h^d \right) \\
&= \frac{1}{2^n} \sum_{\mathbf{e} \in \{1,-1\}^n} R_{O(n)}(e_1 t_1, \ldots, e_n t_n, h)
\end{aligned}$$

Using the recursion for $R_{O(n)}$ we obtain,

$$\begin{aligned}
&= \frac{1}{2^n} \sum_{\mathbf{e} \in \{1,-1\}^n} \frac{1 + e_{n-1} e_n t_{n-1} t_n h}{(1-t_n^2)(1-t_{n-1}^2 h^2)} R_{O(n-1)}(e_1 t_1, \ldots, e_{n-1} t_{n-1}, h) \\
&= \frac{1}{(1-t_n^2)(1-t_{n-1}^2 h^2)} \frac{1}{2^{n-1}} \sum_{\mathbf{e} \in \{1,-1\}^{n-1}} R_{O(n-1)}(e_1 t_1, \ldots, e_{n-1} t_{n-1}, h) \\
&= \frac{1}{(1-t_n^2)(1-t_{n-1}^2 h^2)} \mathcal{E}_{n-1}(R_{O(n-1)}(\mathbf{t}_{(n-1)}, h)) \\
&= \frac{1}{\prod_{i=1}^n (1-t_i^2) \prod_{i=1}^{n-1} (1-t_i^2 h^2)}
\end{aligned}$$

The specialization of $t_i = q^{i/2}$, for $1 \leq i \leq n$ gives the generating function for the graded multiplicity of the degree d spherical harmonics in the space $S(S^2(\mathbb{C}^n))$.

$$\sum_{k,d \geq 0} \dim Hom_{O(n)}(H_d, \mathcal{P}^k(S^2(\mathbb{C}^n)^*))\, q^k\, t^d = \prod_{i=1}^{n} \left(\frac{1}{1-q^i} \right) \prod_{i=1}^{n-1} \left(\frac{1}{1-q^i\, t^2} \right)$$

Recall that as an $O(n)$ representation, $S^2(\mathbb{C}^n)$ is reducible, specifically it has a one dimensional invariant subspace which as an irreducible complement which we call $\mathfrak{p}$. After taking into account this invariant occurring in degree 1 we obtain,

$$\sum_{k,d \geq 0} \dim Hom_{O(n)}(H_d, \mathcal{P}^k(\mathfrak{p}^*))\, q^k\, t^d = \prod_{i=2}^{n} \left(\frac{1}{1-q^i} \right) \prod_{i=1}^{n-1} \left(\frac{1}{1-q^i\, t^2} \right)$$

Recall that as with any representation of the group $O(n)$, $\mathfrak{p}^* \cong \mathfrak{p}$. From the Kostant-Rallis theorem (see chapter 1) for the pair $(GL(n), O(n))$, the polynomial functions on $\mathfrak{p}$ are a free module of the ring of invariant polynomials. The ring of invariant polynomials is a polynomial ring by the Chevalley restriction theorem (see [1], or [8]). Viewing the harmonic polynomials as a quotient by the invariant polynomials one obtains,

$$\sum_{k,d\geq 0} \dim Hom_{O(n)}(H_d, \mathcal{H}^k(\mathfrak{p}))\, q^k\, t^d = \prod_{i=1}^{n-1} \left(\frac{1}{1-q^i\, t^2} \right)$$

4.3.2 Graded multiplicity for in $\mathcal{P}(S^2(\mathbb{C}^n)^*)$ for $SO(n)$

In the same spirit that the above computation is carried out, we can obtain the generating function for the graded multiplicity of the spherical harmonics (viewed this time as irreducible representations of $SO(n)$) in the space $\mathcal{H}(\mathfrak{p})$ for the pair $(SL(n), SO(n))$. This is done by first observing that,

Proposition 45 ($SO(n)$ Invariants)

$$\dim \left(F^\lambda\right)^{SO(n)} = \begin{cases} 1 \text{ \textit{if} } \lambda \text{ \textit{has all even or all odd parts}} \\ 0 \text{ \textit{otherwise}} \end{cases}$$

Proof: Use theorem 40.

Define,

$$R_{SO(n)} = \sum_{\mathbf{a}\in\mathbb{N}^n,\, d\geq 0} \dim Hom_{SO(n)}(H_d, F^{\lambda\mathbf{a}})\mathbf{t}^{\mathbf{a}}\, h$$

In a sense, theorem 45 states that we have found one half the the induced representation. We will use a process of induction in stages to find the other half. Define:

$$M = R_{SO(n)} - R_{O(n)} \tag{4.8}$$

The problem is reduced to the calculation of M. The essential part of the calculation is provided below.

$$
\begin{aligned}
M(\mathbf{t}_{(n)}, h) = & \sum_{\substack{\mathbf{a} \in \mathbb{N}^n,\ d \geq 0 \\ a_n > 0}} b_{(d)}^{\lambda(\mathbf{a})} t_1^{a_1} \dots t_{n-1}^{a_{(n-1)}}\, t_n^{(a_n)-1}\, h^d \\
= & \frac{1}{t_n}\Big(\sum_{\mathbf{a}\in\mathbb{N}^n,\, d\geq 0} b_{(d)}^{\lambda(\mathbf{a})} t_1^{a_1} \dots t_{n-1}^{a_{n-1}}\, t_n^{a_n}\, h^d - \\
& \sum_{\substack{\mathbf{a} \in \mathbb{N}^n,\ d \geq 0 \\ a_n = 0}} b_{(d)}^{\lambda(\mathbf{a})} t_1^{a_1} \dots t_{n-1}^{a_{(n-1)}}\, h^d \Big) \\
= & \frac{1}{t_n}\left(R_{O(n)}(\mathbf{t}_{(n)}, h) - R_{O(n)}(\mathbf{t}_{(n)}, h)|_{t_n^0}\right)
\end{aligned}
$$

View the above as a computation of formal power series in t_n with coefficient ring $\mathbb{C}[t_1, \dots, t_n,\, h]$. The coefficient of ${t_n}^0$ in $R_{O(n)}$ is,

$$
\frac{\prod_{i=1}^{n-2}(1 + t_i t_{i+1}\, h)}{(1 - t_1\, h)\prod_{i=2}^{n-1}(1 - t_i^2\, h^2)\prod_{i=1}^{n-1}(1 - {t_i}^2)}
$$

Subtracting,

$$
t_n M(\mathbf{t}_{(n)}, h) = \frac{\prod_{i=1}^{n-2}(1 + t_i t_{i+1}\, h)\ \ (1 + t_{n-1} t_n\, h - (1 - {t_n}^2))}{(1 - t_1\, h)\prod_{i=2}^{n-1}(1 - {t_i}^2\, h^2)\prod_{i=1}^{n-1}(1 - {t_i}^2)\ \ (1 - {t_n}^2)}
$$

Finally we are lead to,

$$
M(\mathbf{t}_{(n)}, h) = \frac{\prod_{i=1}^{n-2}(1 + t_i t_{i+1}\, h)\ \ (t_{n-1}\, h + t_n)}{(1 - t_1\, h)\prod_{i=2}^{n-1}(1 - {t_i}^2\, h^2)\prod_{i=1}^{n}(1 - {t_i}^2)}
$$

Combining this result with equation (4.8) proves,

Proposition 46

$$
R_{SO(n)} = \frac{(1 + t_n)(1 + h t_{n-1})\prod_{i=1}^{n-2}(1 + t_i t_{i+1} h)}{(1 - t_1 h)\prod_{i=1}^{n}(1 - t_i^2)\prod_{i=2}^{n-1}(1 - t_i^2 h^2)}
$$

Returning now to our application, we seek to find the graded multiplicity of a spherical harmonic viewed as an $SO(n)$ representation. As before we average over the group Z_2^n.

$$\begin{aligned}&\tfrac{1}{2^n}\sum_{\mathbf{e}\in\{1,-1\}^n} M(e_1t_1,\ldots,e_nt_n,h)\\ =\;&\frac{t_1^2t_2^2\ldots t_{n-1}^2\,h^n}{\prod_{i=1}^n(1-t_i^2)\prod_{i=1}^{n-1}(1-t_i^2h^2)}\end{aligned}$$

Now we can easily find the graded multiplicity, Take $t_i = q^{i/2}$ for $1 \leq i \leq n$ and obtain,

$$\begin{aligned}&\textstyle\sum_{k,d\geq 0}\ \dim Hom_{SO(n)}(H_d, S^k(S^2(\mathbb{C}^n)))\ q^k\ t^d\\ =\;&\frac{1+t^n\,q^{\binom{n}{2}}}{\prod_{i=1}^{n-1}(1-q^it^2)\prod_{i=1}^{n}(1-q^i)}\end{aligned}$$

Removing the Hilbert series for the $SO(n)$-invariants we are left with,

Corollary 47

$$\begin{aligned}&\textstyle\sum_{k,d\geq 0}\ \dim Hom_{SO(n)}(H_d, \mathcal{H}^k(\mathfrak{p}))\ q^k\ t^d\\ =\;&\frac{1+t^n\,q^{\binom{n}{2}}}{\prod_{i=1}^{n-1}(1-q^it^2)}\end{aligned}$$

4.3.3 Graded Multiplicity in $\mathcal{P}(\wedge^2(\mathbb{C}^n))$

CASE 1: $n = 2k$. It is enough to extract the coefficient of $t_1^0\,t_3^0\,t_5^0\ldots t_{2k-1}^0$ in equation (4.7). That is sum over only the monomials with variables indexed by even numbers. An easy induction argument on k gives that the appropriate coefficient is,

$$\frac{1}{\prod_{i=1}^{k-1}(1-t_{2i}^2h^2)\prod_{i=1}^{k}(1-t_{2i}^2)}$$

CASE 2: $n = 2k+1$. This time, it is enough to extract the coefficient of $t_1^0\,t_3^0\,t_5^0\ldots t_{2k+1}^0$ in equation (4.7). That is sum over only the monomials with variables indexed by even numbers.

$$\frac{1}{\prod_{i=1}^{k}(1-t_{2i}^2h^2)(1-t_{2i}^2)}$$

Chapter 5

An Identity

5.1 Schur polynomials

As before, by a non-negative integer partition we mean a list of non-negative integers, $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n \geq 0)$. The λ_i will be called the parts of the partition. The number of non-zero λ_i will be called the length of the partition, denoted $l(\lambda)$. The sum of the λ_i will be called the size of the partition and will be denoted by $|\lambda|$.

In [17], the Schur polynomial, $s_\lambda(x_1, \dots, x_n)$ indexed by λ is defined by,

$$\frac{\sum_{w \in S_n} sgn(w)\, x_{w(1)}^{\lambda_1+n-1} x_{w(2)}^{\lambda_2+n-2} \dots x_{w(n)}^{\lambda_n}}{\prod_{1 \leq i < j \leq n}(x_i - x_j)}$$

In this chapter we consider the above polynomials from two points of view. On one hand, the Weyl character formula asserts that the above formulae are in fact the characters of the irreducible representations of $GL(n, \mathbb{C})$ with highest weight $\lambda_1\epsilon_1 + \cdots + \lambda_n\epsilon_n$ evaluated on the maximal torus parameterized by the diagonal matrix entries $(x_1 = x_{11}, \dots, x_n = x_{nn})$.

In this sense, we have $x_i \in \mathbb{C}^\times$, for each i. This defines a homogeneous polynomial function on a dense subset of $\mathbb{C}^n$. Therefore, we can view these "characters" as polynomials in the indeterminates $x_1, \dots, x_n$. These polynomials will have degree $|\lambda| = \lambda_1 + \cdots + \lambda_n$. The coefficients of the monomials are weight multiplicities, which are invariant under the Weyl group, S_n. Hence, these polynomials are in fact symmetric polynomials.

These indeed form a vector space basis of the ring of symmetric polynomials in n indeterminates. This is the other role of these polynomials. In [17], the theory of symmetric polynomials is highly developed. We recall some facts and formalisms here in order to state and prove theorem 51. Later we will interpret this formal identity in terms of representation theoretic ideas via the results of chapter 3.

For non-negative integer partitions (with at most n nonzero parts) define the non-negative integers $N^{\lambda}_{\mu,\nu}$

$$s_{\mu}(\mathbf{x})s_{\nu}(\mathbf{x}) = \sum_{\lambda} N^{\lambda}_{\mu\nu} s_{\lambda}(\mathbf{x})$$

The above numbers are called the Littlewood-Richardson coefficients. Also, define the skew Schur function, $s_{\lambda/\mu}$ by:

$$s_{\lambda/\mu}(\mathbf{x}) = \sum_{\nu} N^{\lambda}_{\mu\nu}\, s_{\nu}(\mathbf{x})$$

From the fact that the degree of a product of polynomials is the sum of the degrees of the factors, we see that if $N^{\lambda}_{\mu,\nu} \neq 0$, then $|\lambda| = |\mu| + |\nu|$. Also, the degree of $s_{\lambda/\mu}(\mathbf{x})$ is $|\lambda| - |\mu|$. A consequence of this is that the above sum is finite.

Next, we will consider the case where we have two sets of variables:

$$\mathbf{x} = (x_1, \cdots, x_n)$$

$$\mathbf{y} = (y_1, \cdots, y_m)$$

Proposition 48

$$s_{\lambda}(\mathbf{x}, \mathbf{y}) = \sum_{\lambda,\mu} s_{\lambda/\mu}(\mathbf{x}) s_{\mu}(\mathbf{y})$$

Equivalently,

$$(s_{\lambda}, s_{\mu}s_{\nu}) = (s_{\lambda/\mu}, s_{\nu})$$

Where $(,)$ is the Hall scalar product (under which the Schur functions are an orthonormal basis of the symmetric polynomials, see [17]. In the next section we will give a proof of Proposition 48.

Two additional facts that we will use in this chapter are:

Proposition 49

$$\prod_{\substack{1 \le i \le n \\ 1 \le j \le m}} \frac{1}{1 - x_i y_j} = \sum_{\lambda : l(\lambda) \le min(n,m)} s_\lambda(\mathbf{x}) s_\lambda(\mathbf{y})$$

This identity follows from the fact that each side is a graded character of the $GL(n, \mathbb{C})$ representation, $\mathcal{P}(M_{n,m})$, which is an instance of Howe duality, see [10]. For a proof from a symmetric function point of view see [17] or [4].

Proposition 50

$$\prod_{1 \le i \le j \le n} \frac{1}{1 - x_i x_j} = \sum_{\lambda : l(\lambda) \le n} s_{2\lambda}(\mathbf{x})$$

Again, this proposition can be found in many places such as [17] or [4]. It also results from the fact that each side is the graded character of the $GL(n, \mathbb{C})$ representation, $\mathcal{P}(S^2(\mathbb{C}^n) * n)$. For a complete account of this point of view see [8].

5.2 A symmetric function identity

We now state the main identity we need for the stability result. I have not seen this identity in the literature, but have seen variations of it.

Proposition 51

$$\sum_{\lambda,\mu} s_{2\lambda/2\mu}(\mathbf{x}) q^{|\mu|} = \prod_{k=1}^{\infty} \left(\frac{1}{1 - q^k} \prod_{1 \le i \le j \le n} \frac{1}{1 - q^{k-1} x_i x_j} \right)$$

Remark: Observe that on the left side of the formula in Proposition 51 the coefficient of q^d is a formal series in $x_1, \ldots, x_n$ for all d.

Corollary 52

$$\sum_{\lambda,\mu} s_{2\lambda/2\mu}(\mathbf{x}) q^{|\lambda|} = \prod_{k=1}^{\infty} \left(\frac{1}{1 - q^k} \prod_{1 \le i \le j \le n} \frac{1}{1 - q^k x_i x_j} \right)$$

Proof: Follows from the homogeneity of $s_{2\lambda/2\mu}(\mathbf{x})$ and Proposition 51 by substituting $q^{\frac{1}{2}}x_i$ for x_i. **Q.E.D.**

The proof of Proposition 48 is standard and is included here:

Proof of Proposition 48:

Consider another set of indeterminates, $z_1, \ldots, z_{n+m}$. We will expand the following product using Proposition 49 and 48.

$$\prod_{\substack{1 \le i \le n \\ 1 \le j \le n+m}} \frac{1}{1 - x_i z_j} \prod_{\substack{1 \le i \le m \\ 1 \le j \le n+m}} \frac{1}{1 - y_i z_j} \tag{5.1}$$

$$= \sum_{\mu,\nu} s_\mu(\mathbf{x}) s_\mu(\mathbf{z}) s_\nu(\mathbf{y}) s_\nu(\mathbf{z}) \tag{5.2}$$

$$= \sum_{\lambda} s_\lambda(\mathbf{x}, \mathbf{y}) s_\lambda(\mathbf{z}) \tag{5.3}$$

$$= \sum_{\tau,\nu} N^{\tau}_{\mu\nu} s_\tau(\mathbf{z}) s_\mu(\mathbf{x}) s_\nu(\mathbf{y}) \tag{5.4}$$

$$= \sum_{\tau,\nu} s_{\tau/\nu}(\mathbf{x}) s_\nu(\mathbf{y}) s_\tau(\mathbf{z}) \tag{5.5}$$

Taking the coefficient of $s_\lambda(\mathbf{z})$ in 5.3 and 5.5 gives the result. **Q.E.D.**

Proposition 53

$$\sum_{\lambda} s_{2\lambda/\mu}(\mathbf{x}) = \sum_{\sigma} s_{\mu/2\sigma}(\mathbf{x}) \prod_{1 \le i \le j \le n} \frac{1}{1 - x_i x_j}$$

Remark: If $N^{\lambda}_{\mu\nu} \neq 0$ then the Young diagrams of μ and ν must fit inside the Young diagram of λ. This is a consequence of the fact that weights of the tensor product are the sums of the weights of the tensor factors. We mention it here to observe the consequence that if $N^{\lambda}_{\mu,\nu} \neq 0$ then $l(\lambda) \le l(\mu) + l(\nu)$. Recall also that if $l(\nu) > n$ then we take $s_\nu(\mathbf{x}) = 0$. Hence, $s_{\lambda/\mu}(\mathbf{x}) = 0$ if $l(\lambda) > l(\mu) + n$. Therefore, for a fixed μ,

$$\sum_{\lambda} s_{2\lambda/2\mu}(\mathbf{x}) = \sum_{\lambda : l(\lambda) \le n+m} s_{2\lambda/2\mu}(\mathbf{x})$$

$$\sum_{\sigma} s_{\mu/2\sigma}(\mathbf{x}) = \sum_{\sigma:l(\sigma)\leq m} s_{\mu/2\sigma}(\mathbf{x})$$

The proof of Proposition 53 can be found in [17], but we will include it here.

Proof of Proposition 53: Let $z_i = x_i$ for $1 \leq i \leq n$ and $z_{i+n} = y_i$ for $1 \leq i \leq m$.

$$\prod_{1\leq i\leq j\leq n+m} \frac{1}{1-z_iz_j} = \sum_{\lambda:l(\lambda)\leq n+m} s_{2\lambda}(\mathbf{x},\mathbf{y}) \tag{5.6}$$

$$= \sum_{\substack{\lambda\,:\,l(\lambda)\,\leq\,n+m \\ \nu\,:\,l(\nu)\,\leq\,m}} s_{2\lambda/\nu}(\mathbf{x})s_{\nu}(\mathbf{y}) \tag{5.7}$$

For a partition μ with at most m parts, the coefficient of $s_\mu(\mathbf{y})$ in (5.7) is,

$$\sum_{\lambda:l(\lambda)\leq n+m} s_{2\lambda/\mu}(\mathbf{x})$$

Now reconsider the expression,

$$\prod_{1\leq i\leq j\leq n+m} \frac{1}{1-z_iz_j}$$

and factor as follows,

$$\prod_{1\leq i\leq j\leq n} \frac{1}{1-x_ix_j} \prod_{\substack{1\,\leq\,i\,\leq\,n \\ 1\,\leq\,j\,\leq\,m}} \frac{1}{1-x_iy_j} \prod_{1\leq i\leq j\leq m} \frac{1}{1-y_iy_j} \tag{5.8}$$

We will develop the last two products in the above as follows:

$$\prod_{\substack{1 \le i \le n \\ 1 \le j \le m}} \frac{1}{1 - x_i y_j} \prod_{1 \le i \le j \le m} \frac{1}{1 - y_i y_j} \tag{5.9}$$

$$= \sum_{\alpha : l(\alpha) \le min(n,m)} s_\alpha(\mathbf{x}) s_\alpha(\mathbf{y}) \sum_{\sigma : l(\sigma) \le m} s_{2\sigma}(\mathbf{y}) \tag{5.10}$$

$$= \sum_{\substack{\alpha \,:\, l(\alpha) \,\le\, min(n,\, m) \\ \sigma \,:\, l(\sigma) \,\le\, m}} s_\alpha(\mathbf{x}) s_\alpha(\mathbf{y}) s_{2\sigma}(\mathbf{y}) \tag{5.11}$$

$$= \sum_{\substack{\alpha \,:\, l(\alpha) \,\le\, min(n,\, m) \\ \sigma \,:\, l(\sigma) \,\le\, m \\ \gamma \,:\, l(\gamma) \,\le\, m}} s_\alpha(\mathbf{x}) \; N^{\gamma}_{\alpha, 2\sigma} s_\gamma(\mathbf{y}) \tag{5.12}$$

$$= \sum_{\substack{\alpha \,:\, l(\alpha) \,\le\, min(n,\, m) \\ \sigma \,:\, l(\sigma) \,\le\, m \\ \gamma \,:\, l(\gamma) \,\le\, m}} N^{\gamma}_{2\sigma, \alpha} s_\alpha(\mathbf{x}) s_\gamma(\mathbf{y}) \tag{5.13}$$

$$= \sum_{\substack{\sigma \,:\, l(\sigma) \,\le\, m \\ \gamma \,:\, l(\gamma) \,\le\, m}} s_{\gamma/2\sigma}(\mathbf{x}) s_\gamma(\mathbf{y}) \tag{5.14}$$

We see that the coefficient of $s_\mu(\mathbf{y})$ in equation (5.8) is,

$$\prod_{1 \le i \le j \le n} \frac{1}{1 - x_i x_j} \sum_{\sigma : l(\sigma) \le m} s_{\mu/2\sigma}(\mathbf{x})$$

The coefficients of equations (5.7) and (5.8) are the same so we obtain:

$$\sum_{\lambda : l(\lambda) \le n+m} s_{2\lambda/\mu}(\mathbf{x}) = \prod_{1 \le i \le j \le n} \frac{1}{1 - x_i x_j} \sum_{\sigma : l(\sigma) \le m} s_{\mu/2\sigma}(\mathbf{x})$$

Q.E.D.

5.2.1 Proof of Proposition 51

Define:

$$\tilde{A}_q(\mathbf{x}) = \prod_{k=1}^{\infty} \left(\frac{1}{1 - q^k} \prod_{1 \le i \le j \le n} \frac{1}{1 - q^{k-1} x_i x_j} \right)$$

and,

$$A_q(\mathbf{x}) = \sum_{\lambda, \mu} s_{2\lambda/2\mu}(\mathbf{x}) q^{|\mu|}$$

By Proposition 53 we have:

$$\begin{aligned}
A_q(\mathbf{x}) &= \sum_{\mu} q^{|\mu|}\left(\sum_{\lambda} s_{2\lambda/2\mu}(\mathbf{x})\right) \\
&= \sum_{\mu} q^{|\mu|}\left(\sum_{\sigma} s_{2\mu/2\sigma}(\mathbf{x})\right)\prod_{1\le i\le j\le n}\frac{1}{1-x_ix_j} \\
&= \sum_{\mu,\sigma} s_{2\mu/2\sigma}(\mathbf{x})q^{|\mu|}\prod_{1\le i\le j\le n}\frac{1}{1-x_ix_j} \\
&= \sum_{\mu,\sigma} s_{2\mu/2\sigma}(q^{\frac{1}{2}}\mathbf{x})q^{|\sigma|}\prod_{1\le i\le j\le n}\frac{1}{1-x_ix_j} \\
&= A_q(q^{\frac{1}{2}}\mathbf{x})\prod_{1\le i\le j\le n}\frac{1}{1-x_ix_j}
\end{aligned}$$

Iterating the above argument we obtain,

$$A_q(\mathbf{x}) = \frac{A_q(q^{\frac{d}{2}}\mathbf{x})}{\prod_{k=0}^{d}\prod_{1\le i\le j\le n}(1-q^kx_ix_j)} \tag{5.15}$$

This holds for all $d \ge 0$. View $A_q(q^{\frac{d}{2}}\mathbf{x})$ as a formal power series in q with coefficients from the ring of formal power series in the n indeterminates, $x_1,\ldots,x_n$.

$$\begin{aligned}
A_q(q^{\frac{d}{2}}\mathbf{x}) &= \sum_{\lambda,\mu} s_{2\lambda/2\mu}(q^{\frac{d}{2}}\mathbf{x})q^{|\mu|} \\
&= \sum_{\lambda,\mu} s_{2\lambda/2\mu}(\mathbf{x})q^{d(|\lambda|-|\mu|)+|\mu|}
\end{aligned}$$

In the above,

$$\begin{aligned}
d\left(|\lambda|-|\mu|\right)+|\mu| &= |\mu| \text{ if } |\lambda| = |\mu| \\
&\ge d \text{ otherwise}
\end{aligned}$$

Hence, up to degree $d-1$ the series $A_q(q^{\frac{d}{2}}\mathbf{x})$ has only terms where $|\lambda| = |\mu|$. For such terms $s_{2\lambda/2\mu}(\mathbf{x}) = 1$. This implies that up to degree $d-1$, $A_q(q^{\frac{d}{2}}\mathbf{x})$ agrees with the expansion of,

$$\prod_{k=1}^{\infty}\frac{1}{1-q^k}$$

For $0 \leq N < d$, observe that the coefficient of q^N in,

$$\prod_{d \geq k \geq 0} \left(\prod_{1 \leq i \leq j \leq n} \frac{1}{1 - q^k x_i x_j} \right)$$

is the same as the coefficient of q^N in,

$$\prod_{k=0}^{\infty} \left(\prod_{1 \leq i \leq j \leq n} \frac{1}{1 - q^k x_i x_j} \right)$$

We have shown that as a series in q, up to degree d, $A_q(\mathbf{x})$ agrees with $\tilde{A}_q(\mathbf{x})$. In light of the fact that both $A_q(\mathbf{x})$ and $\tilde{A}_q(\mathbf{x})$ are independent of d we are finished.

5.3 A stable range of the multiplicities in the space of Harmonic polynomials

Consider the symmetric pair $(GL(n, \mathbb{C}), O(n, \mathbb{C}))$. When the adjoint representation, $\mathfrak{g} = gl(n, \mathbb{C})$ of $GL(n, \mathbb{C})$ is restricted to $O(n, \mathbb{C})$ we have the decomposition,

$$gl(n, \mathbb{C}) = S^2(\mathbb{C}^n) \oplus \wedge^2(\mathbb{C}^n)$$

Note that $S^2(\mathbb{C}^n)$ contains a unique invariant vector (up to scalar multiple) whose complement, denoted $\mathfrak{p}_n$, is irreducible. $\wedge^2(\mathbb{C}^n)$ is irreducible for all $n \geq 2$. We have a natural grading by degree on the polynomial functions on $S^2(\mathbb{C}^n)^*$,

$$\mathcal{P}(S^2(\mathbb{C}^n)^*) = \bigoplus_d \mathcal{P}^d(S^2(\mathbb{C}^n)^*)$$

Let $\mathcal{I}(\mathfrak{p}_n)$ be the ideal generated by the $O(n, \mathbb{C})$ invariant polynomials on $\mathfrak{p}_n$ which vanish at the origin. Set:

$$\mathcal{I}^d(\mathfrak{p}_n) = \mathcal{P}^d(\mathfrak{p}_n) \cap \mathcal{I}(\mathfrak{p}_n)$$

Let $\mathcal{H}^d(\mathfrak{p}_n)$ be the $O(n)$ invariant complement of $\mathcal{I}^d(\mathfrak{p}_n)$ in the representation $\mathcal{P}^($\mathfrak{p}_n$)$. Set:

$$\mathcal{H}(\mathfrak{p}_n) = \bigoplus_d \mathcal{H}^d(\mathfrak{p}_n)$$

The Kostant-Rallis theorem asserts that as a representation of $O(n,\mathbb{C})$,

$$\mathcal{P}(S^2(\mathbb{C}^n)^*) \cong \mathcal{P}(S^2(\mathbb{C}^n)^*)^{O(n,\mathbb{C})} \otimes \mathcal{H}(\mathfrak{p}_n)$$

Each irreducible representation of $O(n)$ occurs with finite multiplicity in the space $\mathcal{H}(\mathfrak{p}_n)$. The question arises: what is the distribution of a given irreducible $O(n,\mathbb{C})$ representation in the graded components of $\mathcal{H}(\mathfrak{p}_n)$. We can address this question at the character level by introducing an indeterminate q to keep track of the grading on $\mathcal{H}(\mathfrak{p}_n)$.

$$char_q\ \mathcal{P}(S^2(\mathbb{C}^n)^*) = char_q\ \left(\mathcal{P}(S^2(\mathbb{C}^n)^*)^{O(n,\mathbb{C})}\right)\quad char_q\ (\mathcal{H}(\mathfrak{p}_n))$$

From the Chevalley restriction theorem (see [8], [1]), we can write out the q character for the invariants.

$$char_q\ \mathcal{P}(S^2(\mathbb{C}^n)^*)^{O(n,\mathbb{C})} = \prod_{k=1}^{n} \frac{1}{1-q^k}$$

Because we have a finite multiplicity space, we define the *polynomials* $p_\nu(q)$ to be the graded multiplicity of an irreducible representation, E^ν indexed by ν as in chapter 3 of this work or as in chapter 10 of [8]. Let the character of E^ν be denoted by χ_ν.

$$char_q\ \mathcal{H}(\mathfrak{p}_n) = \sum_\nu p_\nu(q)\chi_\nu$$

$$\sum_\nu p_\nu(q)\chi_\nu = char_q \mathcal{P}(S^2(\mathbb{C}^n)^*) \prod_{k=1}^{n}(1-q^k) \tag{5.16}$$

As a representation of the larger group $GL(n,\mathbb{C})$ we have,

$$char_q \mathcal{P}(S^2(\mathbb{C}^n)^*) = \sum_{\lambda: l(\lambda)\leq n} s_{2\lambda}\ q^{|\lambda|}$$

In chapter 3 we have that if we address the restriction of representations of $GL(n,\mathbb{C})$ to $O(n,\mathbb{C})$. At the character level, restrict s_λ to the group $O(n,\mathbb{C})$ and expand in terms of the irreducible $O(n)$ characters,

$$s_\lambda = \sum_{\nu: l(\nu)\leq n} b_\nu^\lambda\ \chi_\nu$$

Recall the definition, $b_\nu^\lambda = \dim \mathrm{Hom}_{O(n)}(E^\nu, F^\lambda)$. From equation 5.16 we obtain,

$$\sum_{\nu:l(\nu)\leq n} p_\nu(q)\chi_\nu = \sum_{\nu:l(\nu)\leq n} b_\nu^{2\lambda}\ \chi_\nu\ q^{|\lambda|}\ \prod_{k=1}^{n}(1-q^k)$$

Hence, for each ν we have,

$$\begin{aligned} p_\nu(q) &= \left(\sum_{\lambda:\lambda\leq n} b_\nu^{2\lambda} q^{|\lambda|}\right)\prod_{k=1}^{n}(1-q^k) \\ &= \sum_{i\geq 0}\ \sum_{\substack{\lambda\,:\,l(\lambda)\,\leq\,n \\ |\lambda|\,=\,i}} b_\nu^{2\lambda} q^i \prod_{k=1}^{n}(1-q^k) \end{aligned}$$

Let: $r = \lfloor \frac{n}{2} \rfloor$. From the Littlewood restriction formula proven in chapter 3, if $l(\lambda) \leq r$ then we have that,

$$b_\nu^\lambda = \sum_{\mu:l(\mu)\leq r} N_{2\mu,\nu}^\lambda$$

Observe that if $|\lambda| \leq i$ then $l(\lambda) \leq i$, so up to degree r,

$$p_\nu(q) = \sum_{0\leq i\leq r}\ \sum_{\substack{\lambda\,:\,l(\lambda)\,\leq\,n \\ |\lambda|\,=\,i}}\ \sum_{\mu:l(\mu)\leq r} N_{2\mu,\nu}^{2\lambda} q^i \prod_{k=1}^{n}(1-q^k) \tag{5.17}$$

$$+ \sum_{i>r}\ \sum_{\substack{\lambda\,:\,l(\lambda)\,\leq\,n \\ |\lambda|\,=\,i}} b_\nu^{2\lambda} q^i \prod_{k=1}^{n}(1-q^k) \tag{5.18}$$

We will define the formal power series $\widetilde{p}_\nu(q)$ by,

$$\sum_\nu \widetilde{p}_\nu(q) s_\nu(\mathbf{x}) = \prod_{k=1}^{\infty}\prod_{1\leq i\leq j\leq n}\frac{1}{1-q^k x_i x_j}$$

Or, by expanding the individual factors indexed by k into Schur functions we obtain,

$$\sum_\nu \widetilde{p}_\nu(q) s_\nu(\mathbf{x}) = \prod_{k=1}^{\infty}\left(\sum_\lambda s_{2\lambda}(\mathbf{x}) q^{k|\lambda|}\right)$$

Theorem 54 *Up to degree $r = \lfloor \frac{n}{2} \rfloor$ in q, $p_\nu(q)$ agrees with the $\widetilde{p}_\mu(q)$.*

Proof: From corollary 52 we have,

$$\begin{aligned}
\sum_{\nu} \widetilde{p}_{\nu}(q) s_{\nu}(\mathbf{x}) &= \frac{\prod_{k=1}^{\infty}(1-q^k)}{\prod_{k=1}^{\infty}(1-q^k)} \prod_{k=1}^{\infty} \prod_{1\le i\le j\le n} \frac{1}{1-q^k x_i x_j} \\
&= \sum_{\lambda,\mu} s_{2\lambda/2\mu}(\mathbf{x}) q^{|\lambda|} \prod_{k=1}^{\infty}(1-q^k) \\
&= \sum_{\lambda,\mu,\nu} N^{2\lambda}_{2\mu,\nu} s_{\nu}(\mathbf{x}) q^{|\lambda|} \prod_{k=1}^{\infty}(1-q^k)
\end{aligned}$$

Up to degree r in q, $\widetilde{p}_{\nu}(q)$ is the same as,

$$\sum_{\lambda,\mu} N^{2\lambda}_{2\mu,\nu} q^{|\lambda|} \prod_{k=1}^{n}(1-q^k) \tag{5.19}$$

$$= \sum_{0\le i\le r} \sum_{\substack{\lambda\,:\, l(\lambda)\,\le\, n \\ |\lambda|\,=\,i}} \sum_{\mu: l(\mu)\le r} N^{2\lambda}_{2\mu,\nu} q^{i} \prod_{k=1}^{n}(1-q^k) \tag{5.20}$$

$$+ \sum_{i>r} \sum_{\substack{\lambda\,:\, l(\lambda)\,\le\, n \\ |\lambda|\,=\,i}} \sum_{\mu} N^{2\lambda}_{2\mu,\nu} q^{i} \prod_{k=1}^{n}(1-q^k) \tag{5.21}$$

Comparing 5.20 with equation 5.17 we obtain the result. **Q.E.D.**

Corollary 55 *For $n \ge 2d$, and $\nu = (\nu_1, \ldots, \nu_{\lfloor \frac{n}{2} \rfloor})$,
the multiplicity of the irreducible $O(n)$ representation, E^{ν} in $\mathcal{H}^d(\mathfrak{p}_n)$ is the coefficient of q^d in the formal series $\widetilde{p}_{\nu}(q)$.*

Proof: Immediate from the definition of $p_{\nu}(q)$.

Chapter 6

Low rank examples.

6.1 Background

In the general situation, G will denote a semi-simple linear algebraic group with Lie algebra $\mathfrak{g}$. Θ will denote a regular involution with differential at the identity $\theta : \mathfrak{g} \to \mathfrak{g}$. Let K be the set of fix points of Θ in G while the Lie algebra of K will be denoted by $\mathfrak{k}$. It is implied that $\mathfrak{k}$ is the fixed point set of θ. We take $\mathfrak{p}$ to be the -1 eigenspace of θ, ie:

$$\mathfrak{p} = \{X \in \mathfrak{g} : \theta(X) = -X\}$$

The example which will be addressed in this chapter is $G = SL(n, \mathbb{C})$ with $\Theta(g) = (X^{-1})^T$. So, $K = SO(n, \mathbb{C})$.

$$\mathfrak{k} = \left\{X \in M_n(\mathbb{C}) | X^T + X = 0\right\}$$

$$\mathfrak{p} = sl_n(\mathbb{C}) \cap SM_n$$

Where: $SM_n = \{X \in M_n(\mathbb{C}) | X^T = X\}$.

For any vector space V, let $\mathcal{P}(V)$ denote the complex valued polynomial functions on V. This ring is graded by degree, so let $\mathcal{P}^d(V)$ denote the subspace of homogeneous polynomials of degree d. And we have,

$$\mathcal{P}(V) = \bigoplus_{d \geq 0} \mathcal{P}^d(V)$$

As usual, denote the subring of K-invariant polynomial functions on $\mathfrak{p}$ by $\mathcal{P}(\mathfrak{p})^K$. For the case that we are interested in $K = SO_n(\mathbb{C})$ and $\mathfrak{p} = sl_n(\mathbb{C}) \cap SM_n$. The Chevalley restriction theorem (see [1]) implies that $\mathcal{P}(\mathfrak{p})^K$ is a polynomial ring and in particular,

$$\mathcal{P}(\mathfrak{p})^K = \mathbb{C}[Tr\, X^2, Tr\, X^3, \ldots, Tr\, X^n]$$

We set $\mathcal{I} = \mathcal{P}(\mathfrak{p})\mathcal{P}(\mathfrak{p})_+{}^K$ and $\mathcal{I}^d = \mathcal{I} \cap \mathcal{P}^d(\mathfrak{p})$. So $\mathcal{I}$ is the ideal of $\mathcal{P}(\mathfrak{p})$ generated by the K-invariant functions which vanish at the origin and this ideal is graded by degree. $\mathcal{I}^d$ is stable under the action of K so it has a unique K-invariant complement in $\mathcal{P}^d(\mathfrak{p})$ as K is a reductive group. Let $\mathcal{H}(\mathfrak{p})^d$ denote this complement and set $\mathcal{H}(\mathfrak{p}) = \bigoplus_{d \geq 0} \mathcal{H}(\mathfrak{p})^d$. In analogy with the theory of spherical harmonics we will call $\mathcal{H}(\mathfrak{p})$ the harmonics. From a general theorem proven by Kostant and Rallis we have,

Theorem 56 (Kostant-Rallis)

$$\mathcal{P}(\mathfrak{p}) \cong \mathcal{P}(\mathfrak{p})^K \otimes \mathcal{H}(\mathfrak{p})$$

and furthermore as a representation of K, $\mathcal{H}(\mathfrak{p}) \cong Ind_M^K\, 1$.
Where: M is the centralizer in K of an Abelian subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ maximal with respect to the condition of being contained in $\mathfrak{p}$ and consisting of only semi-simple elements.

Proof (See [8] or [13])

In the case where $K = SO_n(\mathbb{C})$. one can show that $\mathfrak{a} = \mathfrak{p} \cap D_n$ and $M = K \cap D_n$ where D_n denotes the set of diagonal $n \times n$ matrices.

The fact that $\mathcal{H}(\mathfrak{p})$ is an induced representation and Frobenius reciprocity allows us to determine the multiplicity of an irreducible representation of K in the space of harmonics. Indeed,

$$\dim Hom_K(V, \mathcal{H}(\mathfrak{p})) = \dim V^{\rho(M)}$$

where (ρ, V) is an irreducible regular representation of K. The question which we will answer concerns the distribution of the above multiplicity in the graded components of the harmonics. In particular, if V^λ is an irreducible representation of K with highest weight λ then we will find $\dim Hom_K(V^\lambda, \mathcal{H}(\mathfrak{p}))$ in terms of λ. Again, for this chapter we will only consider the case when $K = SO_4(\mathbb{C})$. Before proceeding, we will describe an isomorphism between the Lie algebra $so_4(\mathbb{C})$ and $sl_2(\mathbb{C}) \oplus sl_2(\mathbb{C})$.

6.2 Parameterization of $SO_4(\mathbb{C})$ representations

Proposition 57 *The groups $SL_2(\mathbb{C})\times SL_2(\mathbb{C})$ and $SO_4(\mathbb{C})$ are locally isomorphic linear algebraic groups.*

Proof First, let $\widetilde{K} = SL_2 \times SL_2$. Let $V = M_2(\mathbb{C})$ and consider the map $\widetilde{K} \times V \to V$ defined by $(g,h), X \mapsto gXh^{-1}$ for all $(g,h) \in \widetilde{K}, X \in V$. This defines a regular representation of $\widetilde{K}$ which we will denote by $\alpha : \widetilde{K} \to GL(V)$. The image of this map is contained in a subgroup of $GL(V)$ isomorphic to $SO_4(\mathbb{C})$. This can be seen by observing that the determinant is a quadratic, irreducible polynomial on the two by two matrices and $Det(\alpha(g,h)(X)) = Det(X)$, hence $\alpha(\widetilde{K}) \subseteq O(V, Det)$. α is continuous and $\widetilde{K}$ is connected so the image is connected and we have $\alpha(\widetilde{K}) \subseteq SO(V, Det)$. Because α is a morphism of algebraic groups, the image is closed. After observing that $\widetilde{K}$ and $SO_4(\mathbb{C})$ are six dimensional, we see, that α is surjective. A direct computation will show that $\ker \alpha = \{\pm 1\}$. **Q.E.D.**

Corollary 58

$$sl_2(\mathbb{C}) \oplus sl_2(\mathbb{C}) \cong so_4(\mathbb{C})$$

Proof The differential of α (from the proof of the last proposition) at the identity is an explicit isomorphism between the Lie algebras $sl_2(\mathbb{C}) \oplus sl_2(\mathbb{C})$ and $so_4(\mathbb{C})$. **Q.E.D.**

Using the fact that these two algebras are isomorphic will allow us to parameterize the irreducible regular representations of $SO_4(\mathbb{C})$.

Proposition 59 *In general, if G_1 and G_2 are reductive groups and $\{V^\lambda\}$ and $\{W^\mu\}$ are distinct representations of equivalence classes of irreducible regular representations of G_1 and G_2 respectively, then $\{V^\lambda \otimes W^\mu\}$ are the irreducible regular representations of $G_1 \times G_2$. Furthermore, the same statement is true if we replace the reductive groups with reductive Lie algebras and the representations of the groups with representations of Lie algebras.*

Proof: See [8].

We will apply this fact when $G_1 = G_2 = SL_2(\mathbb{C})$ and then obtain a concrete parameterization of the irreducible representations of $SO_4(\mathbb{C})$.

Proposition 60 *Let $V = \mathbb{C}^2$ be the standard representation of $SL_2(\mathbb{C})$ and set $F^d = \mathcal{P}^d(V)$. Define: $\pi_d(g)(f)(v) = f(g^{-1}v)$ for $f \in F^d, g \in SL_2(\mathbb{C}), v \in V$. Then,*

1. *(π_d, F^d) is an irreducible regular representation of $SL_2(\mathbb{C})$.*
2. *If (ρ, W) is an irreducible regular representation of $SL_2(\mathbb{C})$ then $W \cong F^d$, for some $d = 0, 1, 2, \ldots$.*
3. *For $s \in \mathbb{C}^\times$ let $\chi^d(s)$ denote the character of the representation (ρ, W) evaluated at the matrix $\begin{bmatrix} s & 0 \\ 0 & s^{-1} \end{bmatrix}$. Then,*

$$\chi^d(s) = \frac{s^{d+1} - s^{-d-1}}{s - s^{-1}}$$

Proof: see [8]

$SO(4, \mathbb{C})$ **representations.**

So the irreducible representations of $SO_4(\mathbb{C})$ will be taken to be

$$F^{k,l} \equiv F^k \otimes F^l$$

for all $k, l = 0, 1, 2, \ldots$. Observe that, F^{kl} has highest weight $k\omega_1 + l\omega_2$ where ω_1 and ω_2 are the two fundamental weights of $SO_4(\mathbb{C})$. Also note that $\dim F^{k,l} = (k+1)(l+1)$ and in fact we have,

$$char\ F^{k,l}\left(\begin{bmatrix} s & 0 \\ 0 & s^{-1} \end{bmatrix}, \begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix}\right) = \chi^k(s)\chi^l(t)$$
$$= \frac{(s^{k+1} - s^{-k-1})(t^{l+1} - t^{-l-1})}{(s - s^{-1})(t - t^{-1})}$$

6.3 Application of the Kostant-Rallis theorem

Any character of a regular representation of $SO_4(\mathbb{C})$ can be written as a non-negative integer linear combination of the above formulae. We will now return to the theorem of Kostant and Rallis where $k = sl_2 \oplus sl_2 \cong F^{2,0} \oplus F^{0,2}$ and $\mathfrak{p} = SM_n \cap sl_4 \cong F^{2,2}$ as an SO_4 representations. From the Kostant-Rallis theorem we have,

$$\mathcal{P}(F^{2,2}) \cong \mathbb{C}[u_1, u_2, u_3] \otimes \mathcal{H}(\mathfrak{p})$$

Where: $u_i = Tr(X^{i+1})$ for $i = 1, 2, 3$. $\{u_1, u_2, u_3\}$ are algebraically independent polynomials generating the invariants. Also, we have the additional structure that $\mathcal{H}(\mathfrak{p})$ is an induced representation, which allows us to compute the multiplicity of irreducible representations. Define $m(k, l)$ by:

$$\mathcal{H}(\mathfrak{p}) = \bigoplus_{k,l \geq 0} \underbrace{F^{k,l} \oplus \cdots \oplus F^{k,l}}_{m(k,l) \text{ copies}}$$

and as stated before we see from Frobenius reciprocity and the Kostant-Rallis result that $m_{k,l} = \dim(F^{k,l})^M$. The goal of this section is to prove a formula for the computation of $m_d(k, l)$ which are defined by,

$$\mathcal{H}(\mathfrak{p})^d = \bigoplus_{k,l \geq 0} \underbrace{F^{k,l} \oplus \cdots \oplus F^{k,l}}_{m_d(k,l) \text{ copies}}$$

In order to apply the Kostant-Rallis theorem to compute the non-graded multiplicity of $F^{k,l}$ in $\mathcal{H}(\mathfrak{p})$ we must find $\widetilde{M}$ such that,

$$\begin{array}{ccc} \alpha : SL_2 \times SL_2 & \rightarrow & SO_4(\mathbb{C}) \\ \cup & & \cup \\ \widetilde{M} = \alpha^{-1}(M) & \rightarrow & M \end{array}$$

$\widetilde{M}$ consists of 16 pairs of SL_2 matrices. We then arrive at,

$$m(2k, 2l) = \begin{cases} \frac{(2k+1)(2l+1)-3}{4} & \text{for } k-l \text{ odd,} \\ \frac{(2k+1)(2l+1)+3}{4} & \text{for } k-l \text{ even.} \end{cases}$$

while, $m(k, l) = 0$ if either k or l are odd.

6.4 Graded Multiplicity

Let $V = \bigoplus_{i\geq 0} V^i$ be a graded K-representation such that $\dim V^i < \infty$ and each V^i is K-invariant. Define:

$$char_q\, V = \sum_{i\geq 0} q^i\, char\, V^i$$

Let $\widehat{K}$ be a complete index set of distinct irreducible regular representations of K. Define:

$$char_q\, V = \sum_{\lambda\in\widehat{K}} p_\lambda(q)\, char\, V^\lambda$$

So the multiplicity of V^λ in V^d is $p_\lambda(q)|_{q^d}$ and furthermore the nongraded multiplicity of V^λ in V is $p_\lambda(1)$. From the Kostant-Rallis theorem we have,

$$\begin{aligned} char_q\, \mathcal{P}(F^{2,2}) &= char_q\, \mathcal{P}(F^{2,2})^{SO(4,\mathbb{C})}\, char_q\, \mathcal{H}(\mathfrak{p}) \\ &= \frac{char_q\, \mathcal{H}(\mathfrak{p})}{(1-q^2)(1-q^3)(1-q^4)} \end{aligned}$$

In order to compute $char_q\, \mathcal{P}(F^{2,2})$ we need only observe that,

$$\chi^{2\,2}(s,t) = \begin{matrix} s^2t^2 & + & t^2 & + & s^{-2}t^2 & + \\ s^2 & + & 1 & + & s^{-2} & + \\ s^2t^{-2} & + & t^{-2} & + & s^{-2}t^{-2} & \end{matrix}$$

Which leads to the formal expression for the character of the polynomial functions on $F^{2,2}$,

$$char_q\, \mathcal{P}(F^{2,2}) = \frac{1}{\prod_{i,j=-2,0,2}(1-qs^it^j)}$$

Hence, finding the graded multiplicity is equivalent to finding the polynomials $p_{k\,l}(q)$ such that,

$$\frac{(1-q^2)(1-q^3)(1-q^4)}{\prod_{i,j=-2,0,2}(1-qs^it^j)} = \sum_{k,l\geq 0} p_{k\,l}(q)\chi^{2k,2l}(s,t) \tag{6.1}$$

Before proceeding we should point out that one can prove the following facts directly, yet they will come out automatically from the formula for the graded multiplicity to follow this section.

1. $p_{k\,l}(q) = p_{l\,k}(q)$ for all $k, l \geq 0$

2. degree $p_{k\,l}(q) \leq 2k + l$ for all $k, l \geq 0$ with $k \geq l$

3. The order of zero in $p_{k\,l}(q) \geq \max(k, l)$

The initial values of $p_{k\,l}(q)$ can be easily be computed by expansion of the above into power series. Below are some initial data.

	0	1	2	3
0	1	0	$q^2 + q^4$	q^6
1	0	$q^3 + q^2 + q$	$q^5 + q^4 + q^3$	$q^7 + q^6 + 2q^5 + q^4 + q^3$
2	$q^4 + q^2$	$q^5 + q^4 + q^3$	$2q^6 + q^5 + 2q^4 + q^3 + q^2$	$q^8 + 2q^7 + 2q^6 + 2q^5 + q^4$
3	q^6	$q^7 + q^6 + 2q^5 + q^4 + q^3$	$q^8 + 2q^7 + 2q^6 + 2q^5 + q^4$	$2q^9 + 2q^8 + 3q^7 + 2q^6 + 2q^5 + q^4 + q^3$

Several properties of $p_{k\,l}(q)$ were found and these are now proven by a computation later in this chapter, some of the most important are:

Proposition 61 *For all, $j \geq 0$*

$$p_{j\,j}(q) = \frac{q^j(1 - q^{j+2})(1 - q^{j+1}) + q^{j+4}(1 - q^j)(1 - q^{j-1})}{(1 - q)^2(1 + q)}$$

Proposition 62 *For all, $j \geq 0$*

$$p_{j+1\,j}(q) = \frac{q^{j+2}(1 - q^{j+2})(1 - q^j)}{(1 - q)^2}$$

But most importantly,

Proposition 63 (Shift formula) *For all, $k, l \geq 0$ with $k \geq l$,*

$$p_{k+2\,l}(q) - q^2 p_{k\,l}(q) = q^{2k-l+4}\left(\frac{1 - q^{2l+1}}{1 - q}\right)$$

The above three statements provide a formula for $p_{k\,l}(q)$. In the next section we will prove a stronger form of the shift formula which gives a more interesting algorithm for computing the values of $p_{k,l}(q)$. For now we will describe how the above facts give us the graded multiplicity. The following theorem is a statement about formal power series and is intentionally stated without reference to any representation theory.

Proposition 64 *For all integers $k, l \geq 0$ let $p_{k\,l}(q)$ be a formal power series in q which satisfies propositions 1, 2 and 3. Then, for each $k, l \geq 0$ $p_{k\,l}(q)$ is a polynomials in q. In particular, for all $k \geq l, k - l \in 2\mathbb{Z}$,*

$$p_{k\,l}(q) = \frac{q^k(1 + q^2 + q^4) - q^{k+l}(q + q^2 + q^3 + q^4) + q^{2k}(q^{l+3} - q^{-l+2}) + q^{k+2l+3}}{(1 - q)(1 - q^2)} \tag{6.2}$$

For all $k \geq l, k-l \in 2\mathbb{Z}+1$,

$$p_{k\,l}(q) = \frac{(q^{2k+l}+q^{k+2l})(q^3-q^4)+q^{2k-l}(q^3-q^2)+q^k(q-q^4)+q^{k+l}(q^5-q)}{(1-q^2)(1-q)^2} \tag{6.3}$$

Proof The proof is an easy induction argument on $k-l$ in which the base case is stated in propositions 1 and 2 and the induction step is propositions 3.

Theorem 65 *If* $p_{k\,l}(q)$ *are given by (6.2) and (6.3) then,*

$$\frac{(1-q^2)(1-q^3)(1-q^4)}{\prod_{i,j=-2,0,2}(1-qs^it^j)} = \sum_{k,l\geq 0} p_{k\,l}(q)\chi^{2k,2l}(s,t)$$

Proof: This was done by formal methods using the computer package MAPLE. Because of the parity condition on $p_{k\,l}(q)$ we will define $E_{r\,l}(q)$ and $O_{r\,l}(q)$ by re-indexing the parameters,

$$E_{r\,l}(q) := p_{l+2r\,l}$$

and

$$O_{r\,l}(q) := p_{l+2r+1\,l}$$

It is enough to show,

$$\sum_{r,l\geq 0}\Big(E_{r\,l}(q)\chi^{2(l+2r)\,2l}(s,t)+O_{r\,l}(q)\chi^{2(l+2r+1)\,2l}(s,t)\Big) \tag{6.4}$$

$$+\sum_{r,l\geq 0}\Big(E_{r\,l}(q)\chi^{2l\,2(l+2r)}(s,t)+O_{r\,l}(q)\chi^{2l\,2(l+2r+1)}(s,t)\Big) \tag{6.5}$$

$$-\sum_{l\geq 0}E_{r\,l}(q)\chi^{2l\,2l}(s,t) \tag{6.6}$$

$$= \frac{(1-q^2)(1-q^3)(1-q^4)}{\prod_{i,j=-2,0,2}(1-qs^it^j)} \tag{6.7}$$

Here (6.4) is the part of the sum where $k \geq l$ and (6.5) is where $k \leq l$ and then we subtract (6.6) because it was counted twice. By substituting the conjectured formulae of $p_{k\,l}(q)$ into (6.4),(6.5), and (6.6), and evaluating all summations which will result in a rational expression in q, s, and t. This expression can be checked against (6.7) with MAPLE. Let $c_{n\,m}(s,t) = s^{n+1}t^{m+1}$ and observe that,

$$(s-s^{-1})(t-t^{-1})\chi^{m,n} = c_{n\,m}(s,t)-c_{n\,m}(s^{-1},t)-c_{n\,m}(s,t^{-1})+c_{n\,m}(s^{-1},t^{-1})$$

Define:

$$\begin{aligned} A(s,t) &:= \sum_{r,l\geq 0} E_{r\,l}(q)c_{l+2r\;2l}(s,t) + O_{r\,l}(q)c_{2(l+2r+1)\;2l}(s,t) \\ &+ \sum_{r,l\geq 0} E_{r\,l}(q)c_{2l\;2(l+2r)}(s,t) + O_{r\,l}(q)c_{2l\;2(l+2r+1)}(s,t) \\ &- \sum_{l\geq 0} E_{r\,l}(q)c_{2l\;2l}(s,t) \end{aligned}$$

To be shown that,

$$A(s,t) - A(s^{-1},t) - A(s,t^{-1}) + A(s^{-1},t^{-1}) =$$
$$\frac{(1-q^2)(1-q^3)(1-q^4)(s-s^{-1})(t-t^{-1})}{\prod_{i,j=-2,0,2}(1-qs^it^j)}$$

Evaluating the geometric series which appear in $A(s,t)$ and then obtaining a simplified rational expression in q, s and t the left side of this equation can be found using MAPLE. Again with MAPLE we verify that this symbolic expression is equal to,

$$\frac{(1-q^2)(1-q^3)(1-q^4)(s-s^{-1})(t-t^{-1})}{\prod_{i,j=-2,0,2}(1-qs^it^j)}$$

It is the case that the decomposition of a representation is unique up to equivalence so the above gives us,

Corollary 66 *The graded multiplicity of $F^{2k,2l}$ in $\mathcal{H}(\mathfrak{p})$ is given by the equations (6.2) and (6.3).*

6.5 $SO_4(\mathbb{C})$ invariants

The last result is the key to solving the problem of computing the Hilbert series of $\mathcal{P}(M_4(\mathbb{C}))^{SO_4(\mathbb{C})}$. This is done by observing that as an $SO_4(\mathbb{C})$-representation we have:

$$M_4(\mathbb{C}) \cong \mathfrak{p} \oplus \mathfrak{k} \oplus \mathbb{C}I$$

Recall that $\mathfrak{k}$ is the Lie algebra $so_4(\mathbb{C})$. If we have a graded decomposition of the polynomial functions on each of these irreducible components of $M_4(\mathbb{C})$, then a graded decomposition of $\mathcal{P}(M_4(\mathbb{C}))$ can in principle be calculated for the graded multiplicity of all representations that arise from tensoring irreducibles occurring in the space of polynomial functions on each of the components of $M_4(\mathbb{C})$. The Hilbert series for the invariants will then be the graded multiplicity of the trivial representation which can be computed with the aid of MAPLE. Before pushing this through we observe the following decomposition,

Proposition 67 *As a K-representation,*

$$char_q\,\mathcal{P}(\mathfrak{k}) = \frac{\sum_{k,l\geq 0} q^{k+l}\,\chi^{2k,2l}(s,t)}{(1-q^2)^2}$$

Proof As an $SL_2(\mathbb{C}) \times SL_2(\mathbb{C})$-representation, $\mathfrak{k} \cong F^{2,0} \oplus F^{0,2}$ and as an $SL_2(\mathbb{C})$-representation,

$$char_q\,\mathcal{P}(F^2) = \frac{\sum_{k\geq 0} q^k \chi^{2k}(s)}{1-q^2}$$

This follows from the fact that locally $SL_2(\mathbb{C})$ is isomorphic to $SO_3(\mathbb{C})$. and the above decomposition follows from spherical harmonics. The result can be verified by evaluating the sum to a rational expression which is equal to,

$$\frac{1}{(1-qs^2)(1-qs^0)(1-qs^{-2})}$$

Using this fact and that $\mathcal{P}(F^{2,0} \oplus F^{0,2}) \cong \mathcal{P}(F^{2,0}) \otimes \mathcal{P}(F^{0,2})$

$$char_q\,\mathcal{P}(F^{2,0} \oplus F^{0,2}) = \frac{\sum_{k\geq 0} q^k \chi^{2k,0} \sum_{l\geq 0} q^l \chi^{0,2l}}{(1-q^2)^2}$$

Next observe that as $SO_4(\mathbb{C})$-representations,

$$F^{n,0} \otimes F^{0,m} \cong F^{n,m}$$

Q.E.D.

Theorem 68

$$char_q\,\mathcal{P}(M_4(\mathbb{C}))^{SO_4(\mathbb{C})} = \frac{q^{15}+q^{11}+q^{10}+3q^9+2q^8+2q^7+3q^6+q^5+q^4+1}{(1-q^6)(1-q^4)^3(1-q^3)^2(1-q^2)^3(1-q)}$$

Proof. The result is only a computation using the Clebsh-Gordan formula for $SL_2(\mathbb{C})$ representations. The computation reduces to,

$$char_q\,\mathcal{P}(M_4(\mathbb{C}))^{SO_4(\mathbb{C})} = \frac{\sum_{k,l\geq 0} q^{k+l} p_{k\,l}(q)}{(1-q)(1-q^2)^3)(1-q^4)}$$

Which, using MAPLE can be summed to a rational expression in q that is the above result.

6.6 A proof of the shift.

In this section we give a full decomposition for the space $\mathcal{H}_{\mathfrak{p}}$ as an $SO(4,\mathbb{C})$-representation. We will accomplish this by proving the following:

Theorem 69 (The Shift Formula) *For $k \geq l \geq 0$,*

$$p_{k,l}(q) - q^2\, p_{k-2,\,l}(q) = q^{2k-l}\left(\frac{1-q^{2l+1}}{1-q}\right)$$

Proposition 70 *For all $k, l \geq 0$,*

$$p_{k,l}(q) = p_{l,k}(q)$$

Lemma 3

$$p_{0,0}(q) = 1 \tag{6.8}$$

$$p_{1,0}(q) = 0 \tag{6.9}$$

$$p_{1,1}(q) = q + q + q^2 \tag{6.10}$$

First we recall some basic facts which follow from the Clebsh-Gordan formula for tensoring a $sl_2(\mathbb{C}) \oplus sl_2(\mathbb{C})$ representation with itself and then projecting onto the symmetric and skew-symmetric tensors.

Lemma 4 *For $k \geq 0$,*

$$S^2\left(F^{k,k}\right) \cong \bigoplus_{\substack{2k \geq r,\, s \geq 0 \\ r+s \in 2\mathbb{Z}}} F^{2r,2s}$$

Lemma 5 *For $k \geq 0$,*

$$\wedge^2\left(F^{k,k}\right) \cong \bigoplus_{\substack{2k \geq r,\, s \geq 0 \\ r+s \in 2\mathbb{Z}+1}} F^{2r,2s}$$

As before we denote the standard representation of $SO_4(\mathbb{C})$ by V and we recall that as an $K = SL_2 \times SL_2$ representation $V \cong F^{1,1}$. In the following is a full decomposition of the polynomial functions on $S^2(V)$ as a graded K representation.

Theorem 71 *Define:*

$$\begin{aligned} \mathcal{S}^k &= S^2(S^kV) \\ \mathcal{E}^k &= \wedge^2(S^kV) \end{aligned}$$

and set,

$$\begin{aligned} \mathcal{S} &= \bigoplus_{k\geq 0} \mathcal{S}^k \\ \mathcal{E} &= \bigoplus_{k\geq 0} \mathcal{E}^k \end{aligned}$$

Then,

$$char_q\mathcal{P}(S^2V) = char_q\mathcal{P}(\wedge^2V)\,(char_q\mathcal{S} - char_q\mathcal{E})$$

Proof: From lemma 4 we have $S^2V \cong F^{0,0} \oplus F^{2,2}$. The weights occurring in this representation are:

$$\{(0,0)\} \cup \{(2i,2j) : i,j = \text{-1, 0 or 1}\}$$

From this fact we obtain that,

$$\begin{aligned} char_q\mathcal{P}(S^2V) &= \frac{1}{(1-q)\prod_{i,j=-1,0,1}(1-qu^{2i}v^{2j})} \\ &= \left(\frac{1}{\prod_{i=-1,0,1}(1-qu^{2i}v^0)\prod_{j=-1,0,1}(1-qu^0v^{2j})}\right) \\ &\quad \left(\frac{1}{\prod_{i,j=-1,1}(1-qu^{2i}v^{2j})}\right) \end{aligned}$$

From Lemma 5 we obtain,

$$char_q\mathcal{P}(\wedge^2V) = \frac{1}{\prod_{i=-1,0,1}(1-qu^{2i}v^0)\prod_{j=-1,0,1}(1-qu^0v^{2j})}$$

The set of weights W of S^kV is,

$$W = \left\{ \begin{array}{lcl} a_1(1,1)+a_2(1,-1)+ & \text{such that} & a_1,a_2,a_3,a_4 \geq 0 \\ a_3(-1,1)+a_4(-1,-1) & & a_1+a_2+a_3+a_4 = k \end{array} \right\}$$

The set of weights in $\wedge^2(S^kV)$ consists of all two element subsets of W, while the set of weights in $S^2(S^kV)$ consists of all two element multi-subsets of W. Hence, we obtain:

$$char_q\mathcal{S} - char_q\mathcal{E} = \sum_{k\geq 0} q^k \left(char\ S^2(S^kV) - char\ \wedge^2(S^kV)\right)$$

$$= \sum_{k\geq 0} q^k \left(\sum_{\substack{a_1,a_2,a_3,a_4\geq 0 \\ a_1+a_2+a_3+a_4=k}} \begin{array}{c} ((u^1v^1)^{a_1})^2((u^1v^{-1})^{a_2})^2 \\ ((u^{-1}v^1)^{a_3})^2((u^{-1}v^{-1})^{a_4})^2 \end{array} \right)$$

$$= \frac{1}{\prod_{i,j=-1,1}(1-qu^{2i}v^{2j})}$$

Q.E.D.

Theorem 72 (Spherical Harmonics for $n = 3$) *Let $SO(3,\mathbb{C})$ act on $\mathbb{C}^3$ via the standard representation. Then, as an $SO(3,\mathbb{C})$-representation we have,*

$$char_q\mathcal{P}(\mathbb{C}^2) = \frac{\sum_{k\geq 0} q^k\chi^{2k}}{1-q^2}$$

Next we observe that as a K-representation, the Lie algebra $\mathfrak{k}$ is isomorphic to $\wedge^2 V$. Furthermore, we have,

Corollary 73

$$char_q\mathcal{P}(\mathfrak{k}) = \frac{\sum_{i,j\geq 0} q^{i+j}\chi^{2i,2j}}{(1-q^2)^2}$$

Theorem 74 (Spherical Harmonics for n=4) *Let $SO(4,\mathbb{C})$ act on $\mathbb{C}^4$ via the standard representation. Then, as an $SO(4,\mathbb{C})$-representation we have,*

$$char_q\mathcal{P}(\mathbb{C}^4) = \frac{\sum_{k\geq 0} q^k\chi^{k,k}}{(1-q^2)^2}$$

Lemma 6

$$char_q\mathcal{S} - char_q\mathcal{E} = \frac{\sum_{k\geq 0}\left(char\ S^2(F^{k,k}) - char\ \wedge^2(F^{k,k})\right)}{(1-q^2)}$$

Proof: Define:

$$char_q\mathcal{S} - char_q\mathcal{E} = \sum_{k\geq 0} D_k q^k$$

For $k \geq 0$

$$\begin{aligned}
D_{k+2} - D_k = \quad & char\ S^2(S^{k+2}V) - char\ \wedge^2 (S^{k+2}V) - char\ S^2(S^kV) \\
& + char\ \wedge^2 (S^kV) \\
= \quad & char\ S^2 \left(S^kV \oplus F^{k+2,k+2}\right) - char\ S^2(S^kV) \\
& - char\ \wedge^2 \left(S^kV \oplus F^{k+2,k+2}\right) + char\ \wedge^2 (S^kV) \\
= \quad & char\ S^2(S^kV) + char\ S^2(F^{k+2,k+2}) \\
& + char S^{k+2}V \otimes F^{k+2,k+2} - char\ S^2(S^kV) \\
& - char\ \wedge^2 (S^kV) - char\ \wedge^2 (F^{k+2,k+2}) \\
& - char S^{k+2}V \otimes F^{k+2,k+2} + char\ \wedge^2 (S^kV) \\
= \quad & char\ S^2(F^{k+2,k+2}) - char\ \wedge^2 (F^{k+2,k+2})
\end{aligned}$$

Observe that,

$$D_0 = 1 = char\ S^2(F^{0,0}) - char\ \wedge^2 (F^{0,0})$$

and since $V \cong F^{1,1}$,

$$D_1 = char\ S^2(V) - char\ \wedge^2 (V) = char\ S^2(F^{1,1}) - char\ \wedge^2 (F^{1,1})$$

We obtain,

$$\begin{aligned}
(1 - q^2)\,(char_q\mathcal{S} - char_q\mathcal{E}) \quad & = \sum_{k \geq 0} (1 - q^2) D_k q^k \\
& = \sum_{k \geq 0} D_k q^k - \sum_{k \geq 0} D_k q^{k+2} \\
& = D_0 + D_1 q + \sum_{k \geq 2} (D_k - D_{k-2})\, q^k \\
& = \sum_{k \geq 0} \left(char\ S^2(F^{k,k}) - char\ \wedge^2 (F^{k,k})\right)
\end{aligned}$$

Q.E.D.

Theorem 75

$$char_q\mathcal{P}(S^2V) =$$

$$\frac{1}{(1-q)(1-q^2)^3}\sum_{n,m\geq 0}\left(\sum_{\substack{i,j,k,l\geq 0\\ |i-k|\leq n\leq i+k\\ |j-l|\leq m\leq j+l}}(-1)^{k+l}q^{i+j+max(k,l)}\chi^{2n,2m}\right)$$

Proof: From Lemma 4 and Lemma 5 we obtain,

$$\begin{aligned}
char_q\mathcal{S} - char_q\mathcal{E} &= \frac{1}{1-q^2}\sum_{d\geq 0}q^d\left(charS^2(F^{d,d}) - char\wedge^2(F^{d,d})\right)\\
&= \frac{1}{1-q^2}\sum_{d\geq 0}q^d\left(\sum_{d\geq k,l\geq 0}(-1)^{k+l}\chi^{2k,2l}\right)\\
&= \frac{1}{1-q^2}\sum_{\substack{r,k,l\geq 0\\ d=r+max(k,l)}}(-1)^{k+l}q^{r+max(k,l)}\chi^{2k,2l}\\
&= \frac{1}{1-q^2}\left(\sum_{r\geq 0}q^r\right)\left(\sum_{k,l\geq 0}(-1)^{k+l}q^{max(k,l)}\chi^{2k,2l}\right)\\
&= \frac{1}{(1-q)(1-q^2)}\sum_{k,l\geq 0}(-1)^{k+l}q^{max(k,l)}\chi^{2k,2l}
\end{aligned}$$

And from Corollary 73 we have an expression for $char_q\mathcal{P}(\wedge^2(V))$. Next, recall from the Clebsh-Gordan formula that for $i,j,k,l\geq 0$

$$F^{2i,2j}\otimes F^{2k,2l}\cong\bigoplus_{\substack{|i-k|\leq r\leq i+k\\ |j-l|\leq s\leq j+l}}F^{2r,2s}$$

Q.E.D.

If we multiply the formula in theorem 75 by $(1-q)(1-q^2)(1-q^3)(1-q^4)$ then we will have obtained the following expression for the graded multiplicity of the irreducible representation $F^{n,m}$ in $\mathcal{H}_{\mathfrak{p}}$.

$$p_{n,m}(q) = \frac{(1-q^3)(1-q^4)}{(1-q^2)^2}\sum_{\substack{i,j,k,l\geq 0\\ |i-k|\leq n\leq i+k\\ |j-l|\leq m\leq j+l}}(-1)^{k+l}q^{i+j+max(k,l)}$$

This expression is not a finite sum, however we shall see that $p_{0,0}(q) = 1$, $p_{1,0}(q) = 0$, $p_{1,1}(q) = q + q^2 + q^3$, $p_{n,m}(q) = p_{m,n}(q)$ for $n, m \geq 0$ and the following result. These facts will set up our inductive procedure for computing $p_{n,m}(q)$.

Theorem 76 *For $n \geq m$ and $n \geq 2$ and $m \geq 0$*

$$p_{n,m}(q) - q^2 p_{n-2,m}(q) = q^{2n-m}(1 + q + q^2 + \cdots + q^{2m})$$

Proof: Let $R(q) = \frac{(1-q^3)(1-q^4)}{(1-q^2)^2}$. We will write $p_{n,m}(q)$ in the following way,

$$p_{n,m}(q) = R(q) \sum_{\substack{j,\, l \,\geq\, 0 \\ |j-l| \,\leq\, m \,\leq\, j+l}} (-1)^l q^j \widetilde{p}_{n,l}(q)$$

Where: For $n, l \geq 0$,

$$\widetilde{p}_{n,l}(q) = \sum_{\substack{i,\, k \,\geq\, 0 \\ |i-k| \,\leq\, n \,\leq\, i+k}} (-1)^k q^{i+max(k,l)}$$

Claim: For $n \geq 2$ and $l \geq 0$,

$$\widetilde{p}_{n,l}(q) - q^2 \widetilde{p}_{n-2,l}(q) = (1-q^2)(-1)^n q^l \sum_{r=0}^{l-n} (-q)^r$$

We will establish this claim by a straightforward computation. Recall that if $l - n < 0$ then by convention the empty sum is taken to be zero. In order to see the idea of the computation we have drawn the sets we will be summing over for n=5.

7			x	x	x	x	x	x
6		x	x	x	x	x	x	x
5	x	x	x	x	x	x	x	x
4		x	x	x	x	x	x	x
3			x	x	x	x	x	x
2				x	x	x	x	x
1					x	x	x	
0						x		
	0	1	2	3	4	5	6	7

Below is the set we would sum over for n=3.

7					x	x	x	x
6				x	x	x	x	x
5			x	x	x	x	x	x
4		x	x	x	x	x	x	x
3	x	x	x	x	x	x	x	
2		x	x	x	x	x		
1			x	x	x			
0				x				
	0	1	2	3	4	5	6	7

Upon substituting $n-2$ for n we obtain,

$$\widetilde{p}_{n-2,l}(q) = \sum_{\substack{i,\,k\,\geq\,0 \\ |i-k|\,\leq\,n-2\,\leq\,i+k}} (-1)^k q^{i+max(k,l)}$$

Next is the last picture with the i,k-axis shifted to the left by two.

7							x	x
6						x	x	x
5					x	x	x	x
4				x	x	x	x	x
3			x	x	x	x	x	x
2				x	x	x	x	x
1					x	x	x	
0						x		
	0	1	2	3	4	5	6	7

Next, we will multiply by q^2 and shift the i index by 2.

$$\begin{aligned} q^2\widetilde{p}_{n-2,l}(q) &= \sum_{\substack{i,\,k\,\geq\,0 \\ |i-k|\,\leq\,n-2\,\leq\,i+k}} (-1)^k q^{i+2+max(k,l)} \\ &= \sum_{\substack{(i-2),\,k\,\geq\,0 \\ |(i-2)-k|\,\leq\,n-2\,\leq\,(i-2)+k}} (-1)^k q^{i+max(k,l)} \end{aligned}$$

Because the second is a subset of the first we will display the difference of the sets.

7			x	x	x	x		
6		x	x	x	x			
5	X	X	x	x				
4		X	X					
3								
2								
1								
0								
	0	1	2	3	4	5	6	7

In the above expression for $\widetilde{p}_{n-2,l}(q)$ the sum is over a set of indices which contains the summation indices in the shifted expression for $q^2\widetilde{p}_{n,l}(q)$. As a consequence of this, there is a significant amount of cancellation in the difference of the sums. The difference in the sets of indices is to be thought of as four diagonal lines parameterized below with $r \geq 0$.

$$\begin{array}{rcl}
(i,k) & = & (0,n) + (r,r) \\
(i,k) & = & (1,n) + (r,r) \\
(i,k) & = & (1,n-1) + (r,r) \\
(i,k) & = & (1,n-1) + (r,r)
\end{array}$$

The computation of the $\widetilde{p}$-shift follows,

$$\begin{array}{rcl}
\widetilde{p}_{n,l}(q) - q^2\widetilde{p}_{n-2,l} & = & \displaystyle\sum_{r\geq 0}\left[\begin{array}{l}
(-1)^{n+r}q^{r+max(n+r,l)}+ \\
(-1)^{n+r}q^{r+1+max(n+r,l)}+ \\
(-1)^{r-1+r}q^{r+1+max(n-1+r,l)}+ \\
(-1)^{r-1+r}q^{r+2+max(n-1+r,l)}
\end{array}\right] \\
 & = & \displaystyle\sum_{r\geq 0}(1+q)\left[\begin{array}{l}
(-1)^{n+r}q^{r+max(n+r,l)}+ \\
(-1)^{n+1+r}q^{r+1+max(n-1+r,l)}
\end{array}\right] \\
 & = & (-1)^n(1+q)\displaystyle\sum_{r\geq 0}(-1)^r q^r[q^{max(n+r,l)} - q^{1+max(n-1+r,l)}]
\end{array}$$

Observe that,

$$q^{max(n+r,l)} - q^{1+max(n-1+r,l)} = \begin{cases} (1-q)q^l & \text{if } l > n-1+r \\ 0 & \text{otherwise} \end{cases}$$

Substituting this into our $\widetilde{p}$-shift formula we obtain the claim,

$$\widetilde{p}_{n,l}(q) - q^2\widetilde{p}_{n-2,l}(q) = (1-q^2)(-1)^{n-2}q^l \sum_{\substack{r \geq 0 \\ l > n-1+r}} (-q)^r$$

Next, we return to the p-shift and substitute our result for the $\widetilde{p}$-shift.

$$\begin{aligned}
&p_{n,l}(q) - q^2 p_{n-2,l}(q) \\
&= R(q) \sum_{\substack{j,l \geq 0 \\ |j-l| \leq m \leq j+l}} (-1)^l q^j (\widetilde{p}_{n,l}(q) - q^2\widetilde{p}_{n,l}(q)) \\
&= R(q) \sum_{\substack{j,l \geq 0 \\ |j-l| \leq m \leq j+l \\ l \geq n}} (-1)^l q^j \left[(1-q^2)(-1)^n q^l \sum_{r \geq 0}^{l-n} (-q)^r \right]
\end{aligned}$$

To proceed further we note the set equality resulting from the fact that $n \geq m$,

$$\begin{aligned}
&\{(j,l) \geq 0 : |j-l| \leq m \leq j+l,\, l \geq n\} = \\
&\{(j,l) \geq 0 : l \geq n,\, l-m \leq j \leq l+m\}
\end{aligned}$$

To simplify the notation we will introduce the following notation for integers a and b with $a \leq b$:

$$[a,b]_q = q^a + q^{a+1} + q^{a+2} ... + q^b$$

We will now write the p-shift as,

$$
\begin{aligned}
&p_{n,l}(q) - q^2 p_{n-2,l}(q)\\
&= R(q) \sum_{l \geq n} \left[\sum_{l-m \leq j \leq l+m} (-1)^l q^j \left((1-q^2)(-1)^n q^l \sum_{r=0}^{l-n} (-q)^r \right) \right]\\
&= R(q)(1-q^2)(-1)^n \sum_{l \geq n} \left[(-1)^l q^l \left(\sum_{r=0}^{l-n} (-q)^r \right) \left(\sum_{l-m \leq j \leq l+m} q^j \right) \right]\\
&= R(q)(1-q^2)(-1)^n [-m,m]_q \left(\sum_{l \geq n} (-1)^l q^{2l} \left(\sum_{r=0}^{l-n} (-q)^r \right) \right)\\
&= R(q)(1-q^2)(-1)^n [-m,m]_q (-1)^n q^{2n} \left(\sum_{l \geq 0} (-1)^l q^2 l \left(\sum_{r=0}^{l} (-q)^r \right) \right)\\
&= R(q)(1-q^2) \left(\sum_{\substack{r,\, l \geq 0 \\ l \geq r \geq 0}} (-1)^{l+r} q^{r+2l} \right) [-m,m]_q q^{2n}
\end{aligned}
$$

Observe that we have the identity of formal power series,

$$
\begin{aligned}
&\sum_{\substack{r,\, l \geq 0 \\ l \geq r \geq 0}} (-1)^{l+r} q^{r+2l} = \sum_{\substack{r,\, s \geq 0 \\ l = r+s}} (-1)^{2r+s} q^{3r+2s}\\
&= \left(\frac{1}{1-q^3} \right) \left(\frac{1}{1+q^2} \right)
\end{aligned}
$$

This identity allows us to cancel the $R(q)$ in the p-shift formula.

$$
\begin{aligned}
&p_{n,m}(q) - q^2 p_{n-2,m}(q)\\
&= R(q)(1-q^2) \frac{1}{(1-q^3)(1+q^2)} [-m,m]_q q^{2n}\\
&= \frac{(1-q^3)(1-q^4)}{(1-q^2)^2} (1-q^2) \frac{1}{(1-q^3)(1+q^2)} [-m,m]_q q^{2n}\\
&= [-m,m]_q q^{2n}\\
&= q^{2n-m} [0,2m]_q\\
&= q^{2n-m} \left(1 + q + q^2 + \cdots + q^{2m} \right)
\end{aligned}
$$

Q.E.D.

The text of this chapter, is in part a reprint of material as it appears in the paper *On Some q-Analogs of a Theorem of Kostant-Rallis*, in the Canadian Journal of Mathematics, Vol. 52(2), 2000, pp. 438-448; Canadian Mathematical Society, Ottawa Ontario, Canada; co-authored with Nolan R. Wallach. I was the secondary author of this paper and made substantial contributions to the research as did my co-author.

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