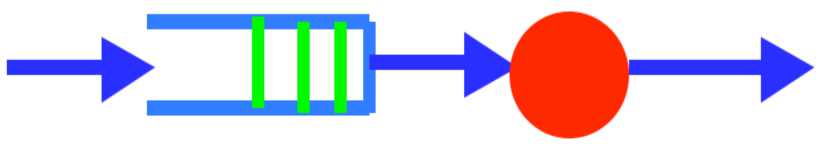
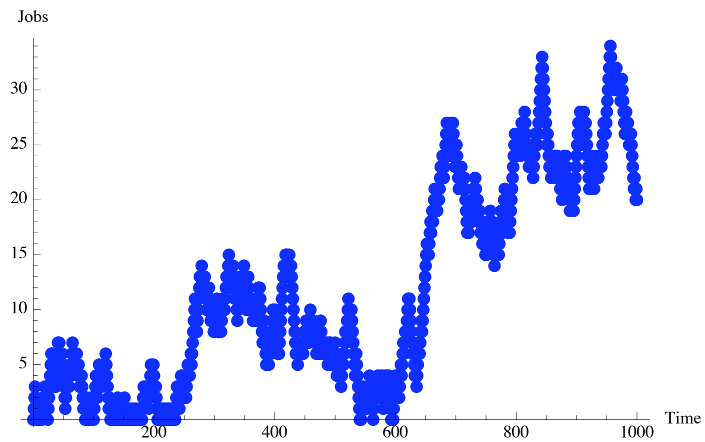
THE MATHEMATICS OF QUEUEING





Professor Ruth J. Williams

University of California, San Diego

April 27, 2011

QUEUES

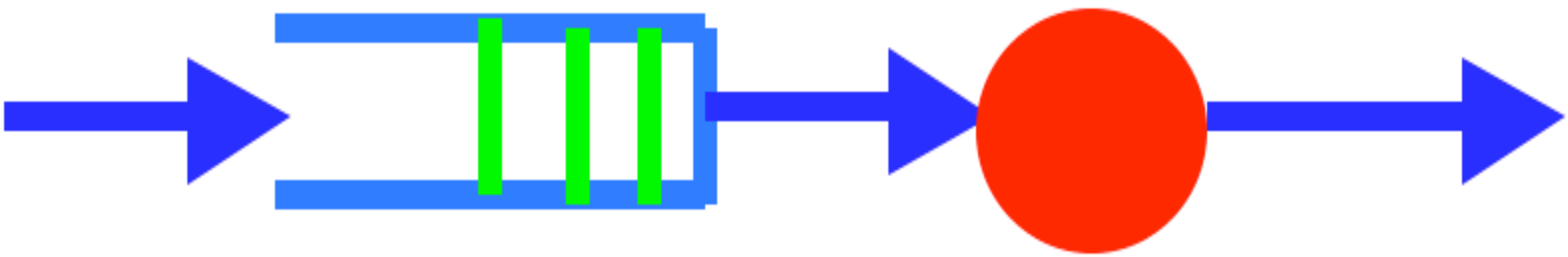
Photograph courtesy of Ilze Ziedins

EXAMPLES OF QUEUES

- Customers waiting in supermarket lines
- Cars waiting at a traffic light
- Patients waiting in an emergency department
- Clients waiting for service by a telephone agent
- Parts waiting to be assembled into finished products
- Packets waiting to be transmitted through a router in the Internet

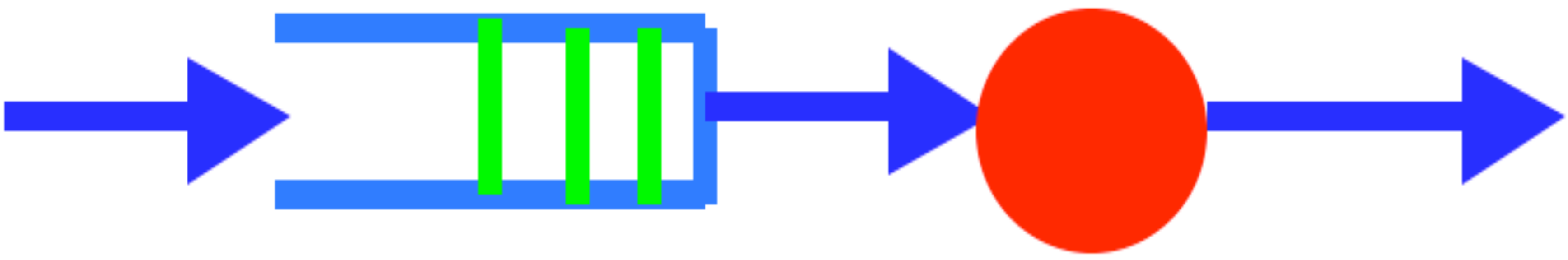
A SIMPLE QUEUEING EXAMPLE

A SINGLE SERVER QUEUE



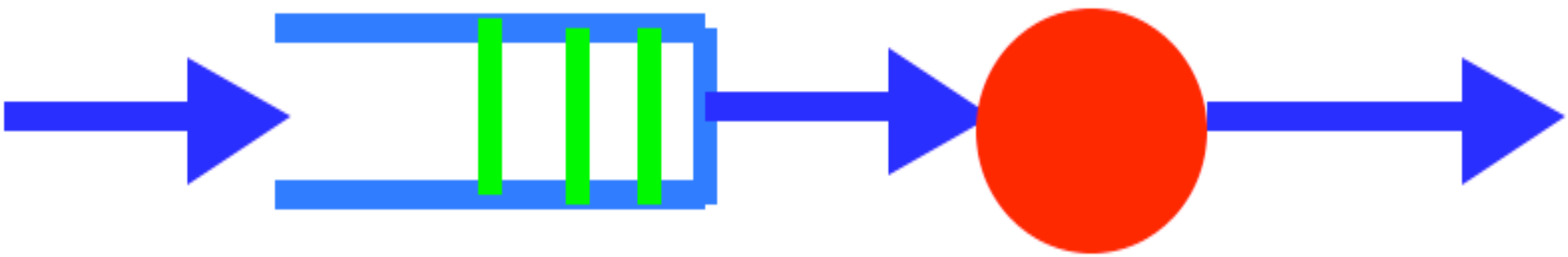
- Green bars: “jobs”
- Red disc: server

RELIABLE SERVER ASSUMPTION



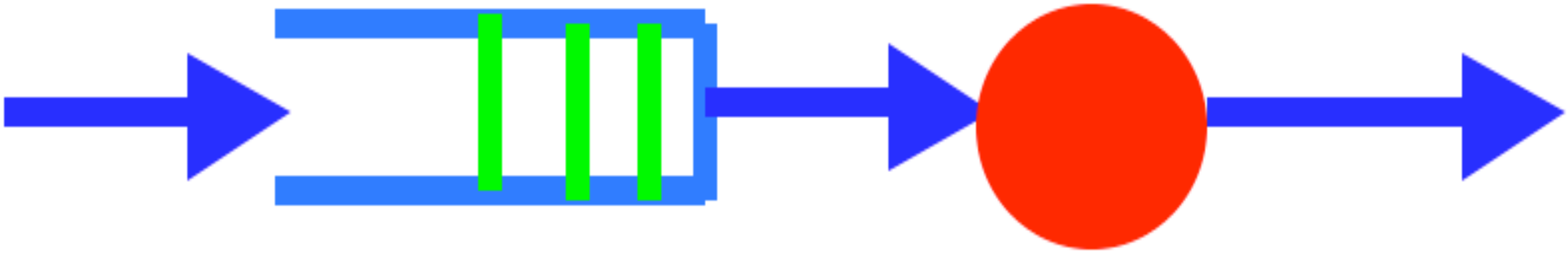
- The server can process one “job” per minute

ARRIVALS: SCENARIO 1



- One job arrives at the beginning of each minute
- Balanced deterministic system: no queue will build up

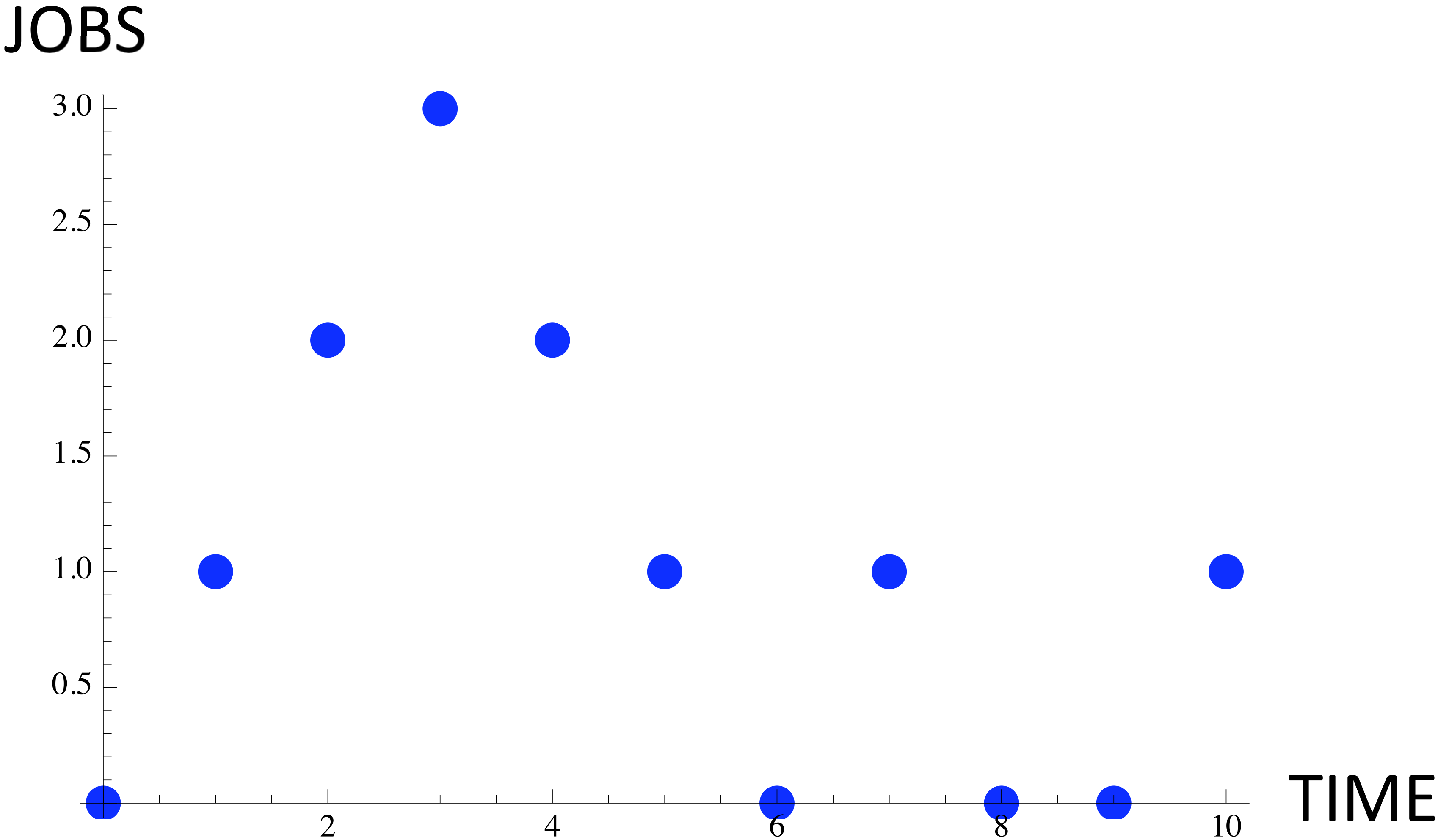
ARRIVALS: SCENARIO 2



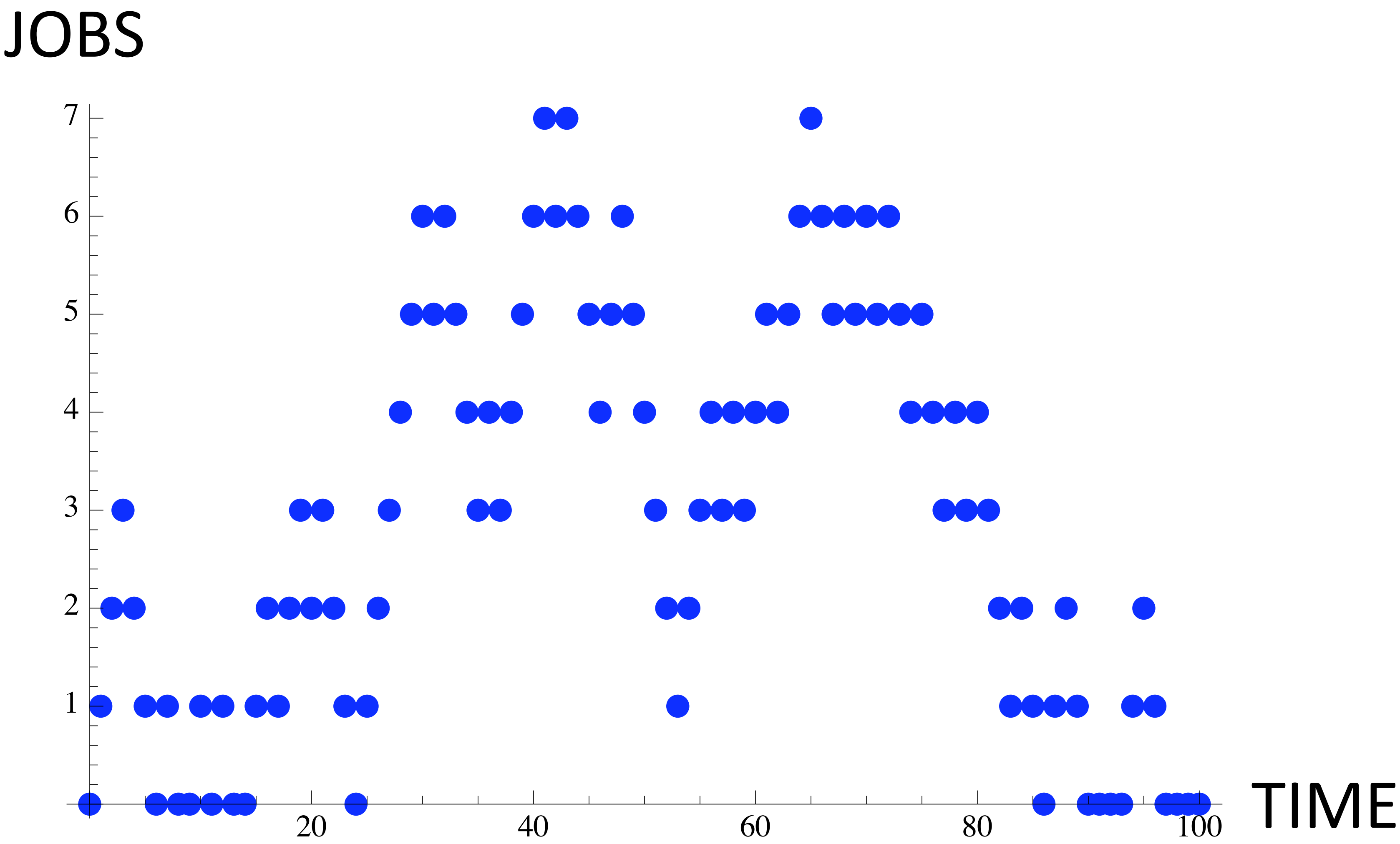
- At the beginning of each minute, either zero or two jobs arrive (equal probability for each)

- Random arrivals: simulate using coin flips
- Average # of arrivals per min = 1

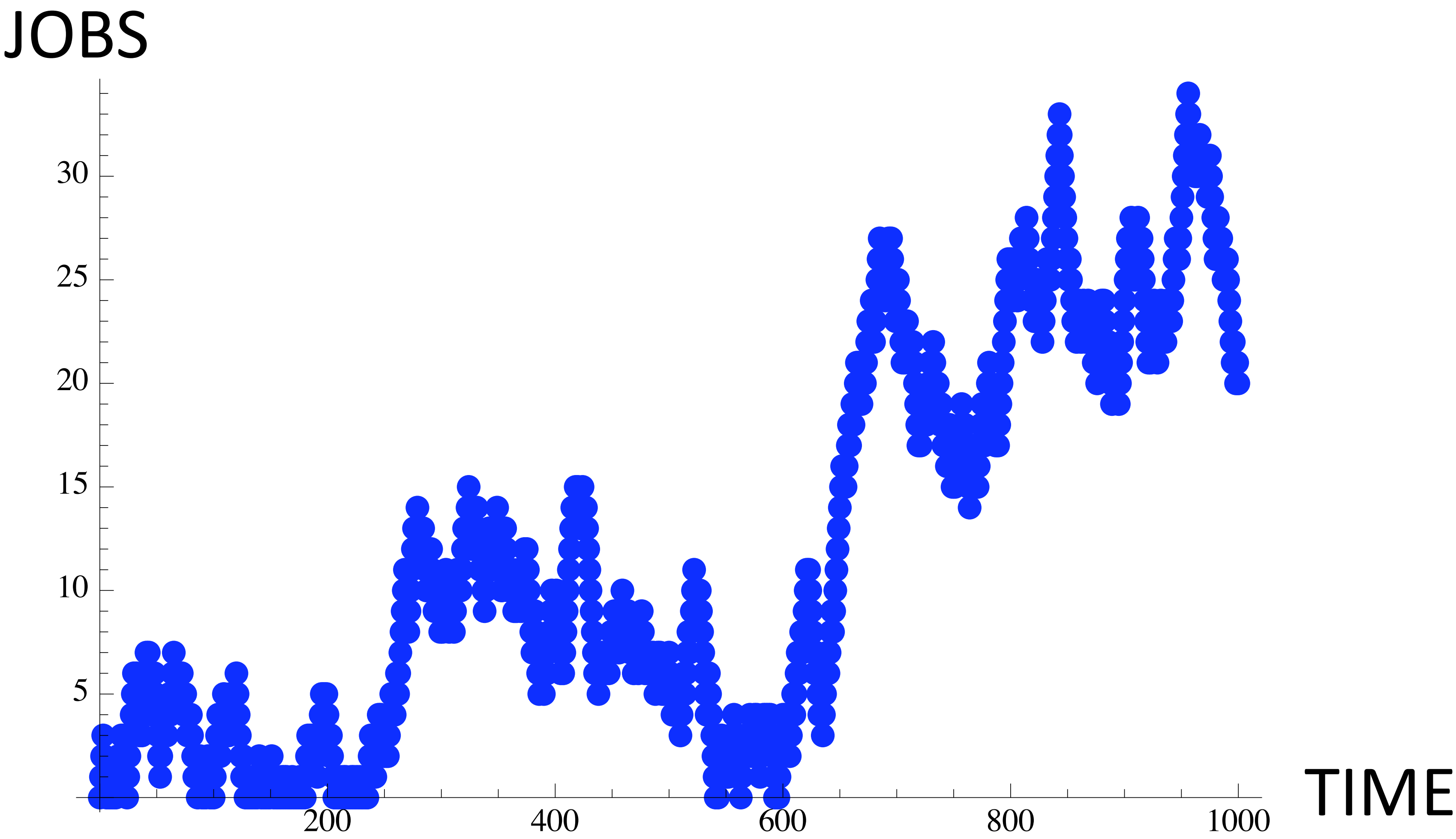
A SAMPLE REALIZATON AFTER 10 MINUTES



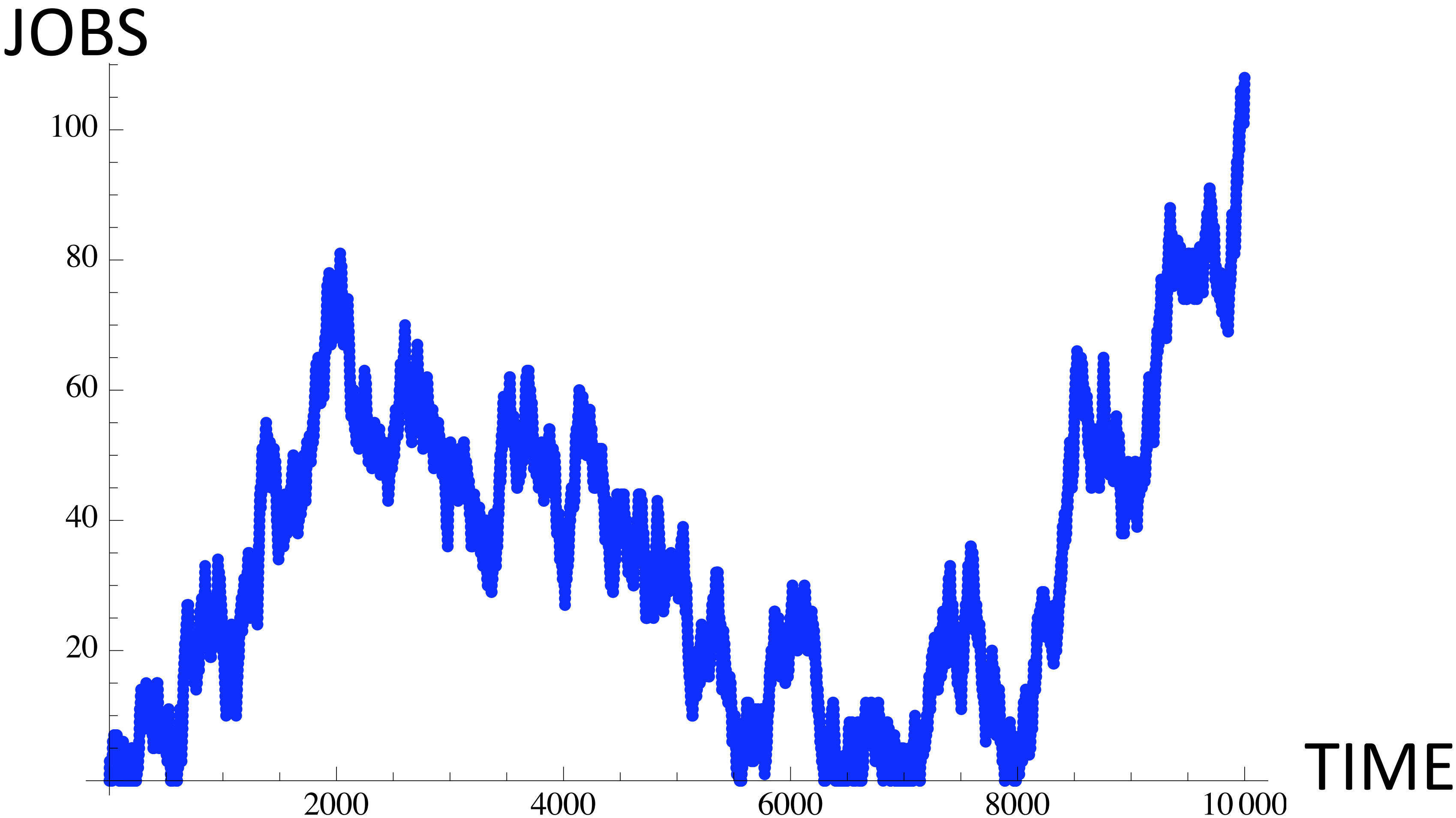
AFTER 100 MINUTES



AFTER 1,000 MINUTES



AFTER 10,000 MINUTES



MATHEMATICAL ANALYSIS

Queuelength

$Q(n)$=number of jobs in the system after n mins

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$$A(n) = \frac{Q(1)+Q(2)+\cdots+Q(n)}{n}$$

MATHEMATICAL ANALYSIS

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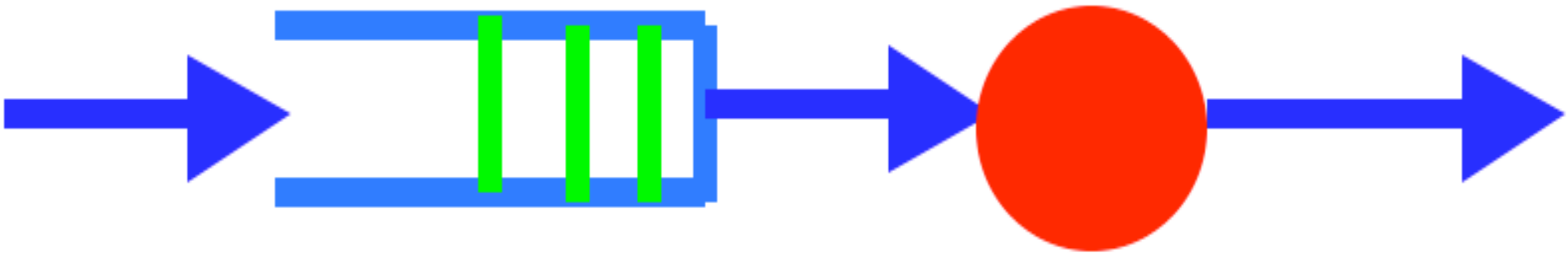
THEOREM: $A(n) \to \infty$ as $n \to \infty$

SUBCRITICAL ARRIVALS: SCENARIO 3



- Fix $0 < \varepsilon < 0.5$
- At the beginning of each minute,

 - two jobs arrive with probability $\frac{1}{2} - \varepsilon$ and

 - zero jobs arrive with probability $\frac{1}{2} + \varepsilon$

- Average # of arrivals per min = $1 - 2\varepsilon$

SUBCRITICAL ARRIVALS

Average after n mins: $A(n) = \dfrac{Q(1)+Q(2)+\cdots+Q(n)}{n}$

THEOREM

$$A(n) \rightarrow \tilde{A} = \frac{1}{4\varepsilon} - \frac{1}{2} \qquad \text{as } n \rightarrow \infty$$

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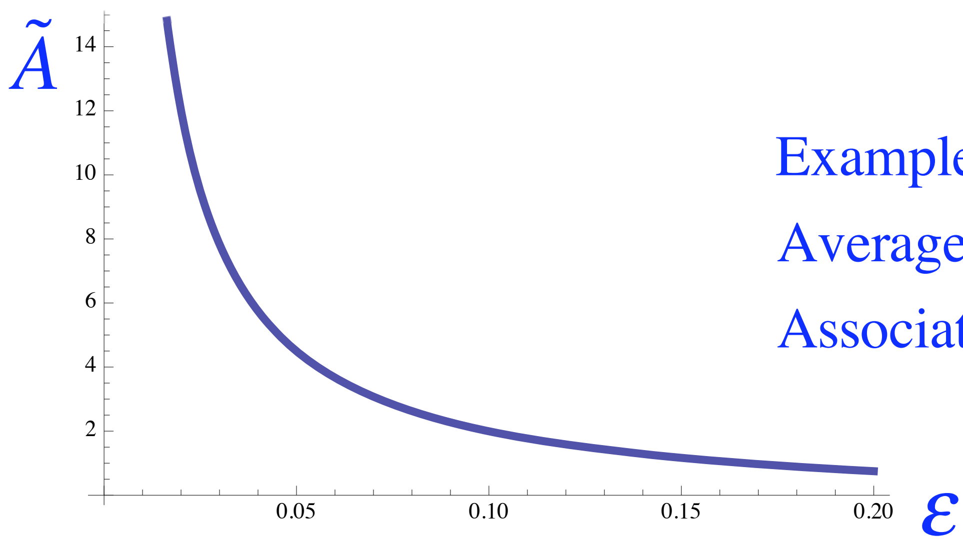
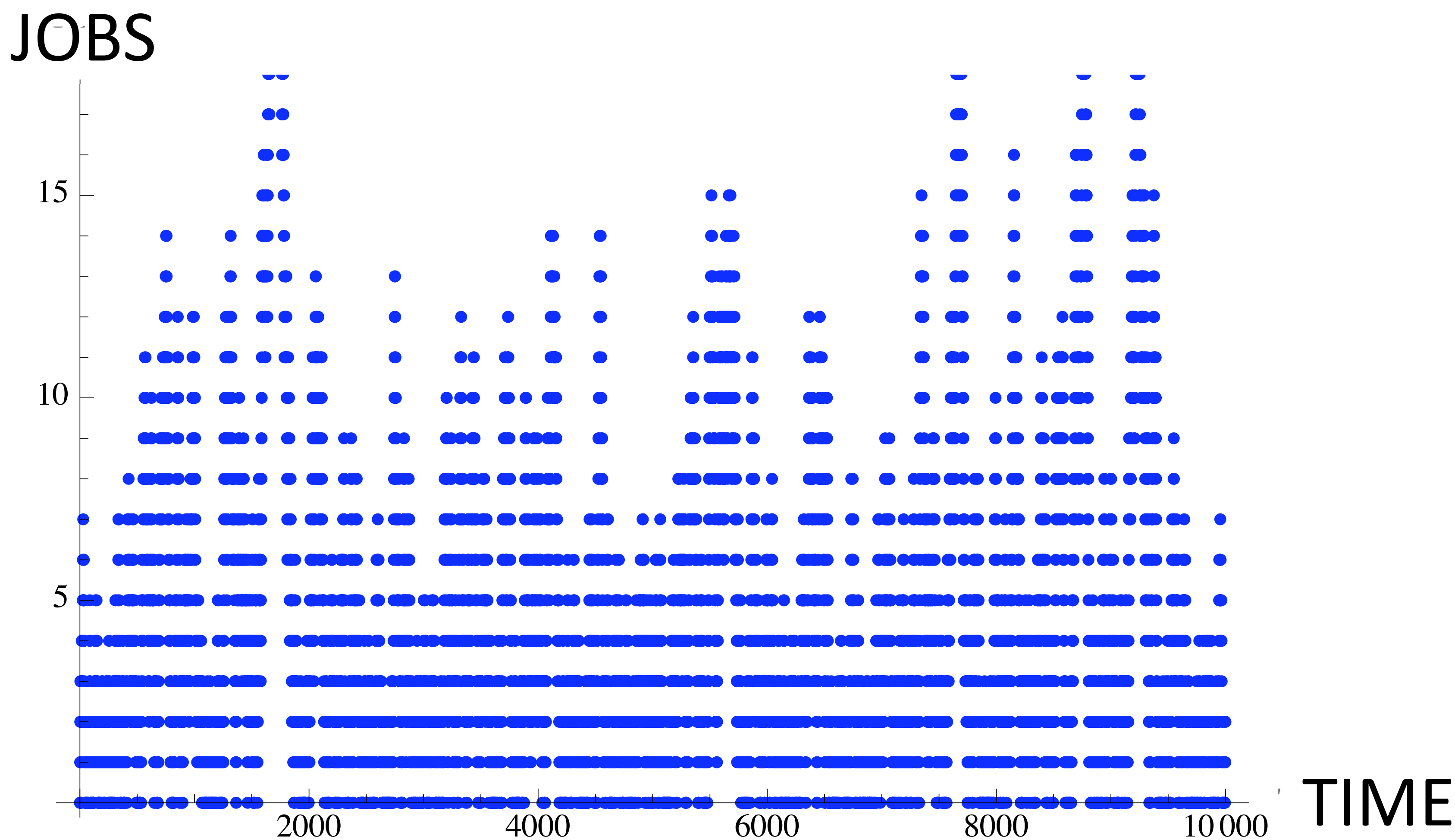


Example: $\varepsilon = 0.05$

Average queuelength (in long run) = 4.5

Associated average wait = 10 minutes

SIMULATION FOR 10,000 MINUTES



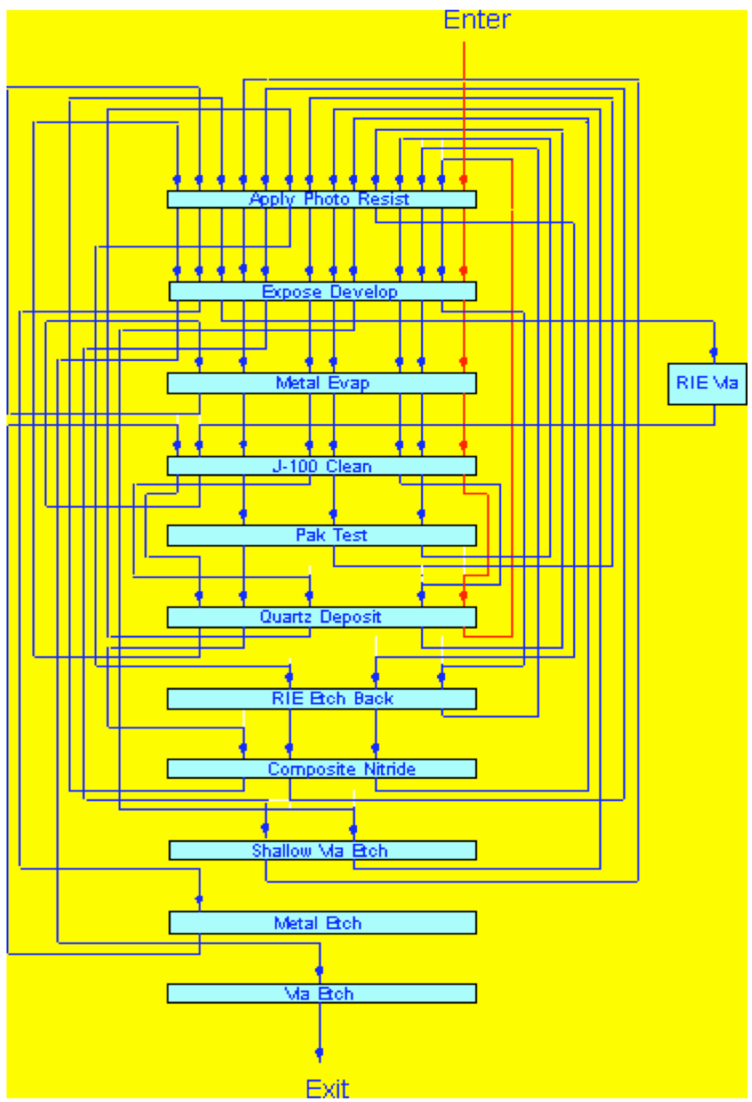
$\varepsilon = 0.05$ Average queuelength from simulation = 4.13

NETWORKS OF QUEUES

Analysis and control of networks of queues is a challenging mathematical problem with applications to

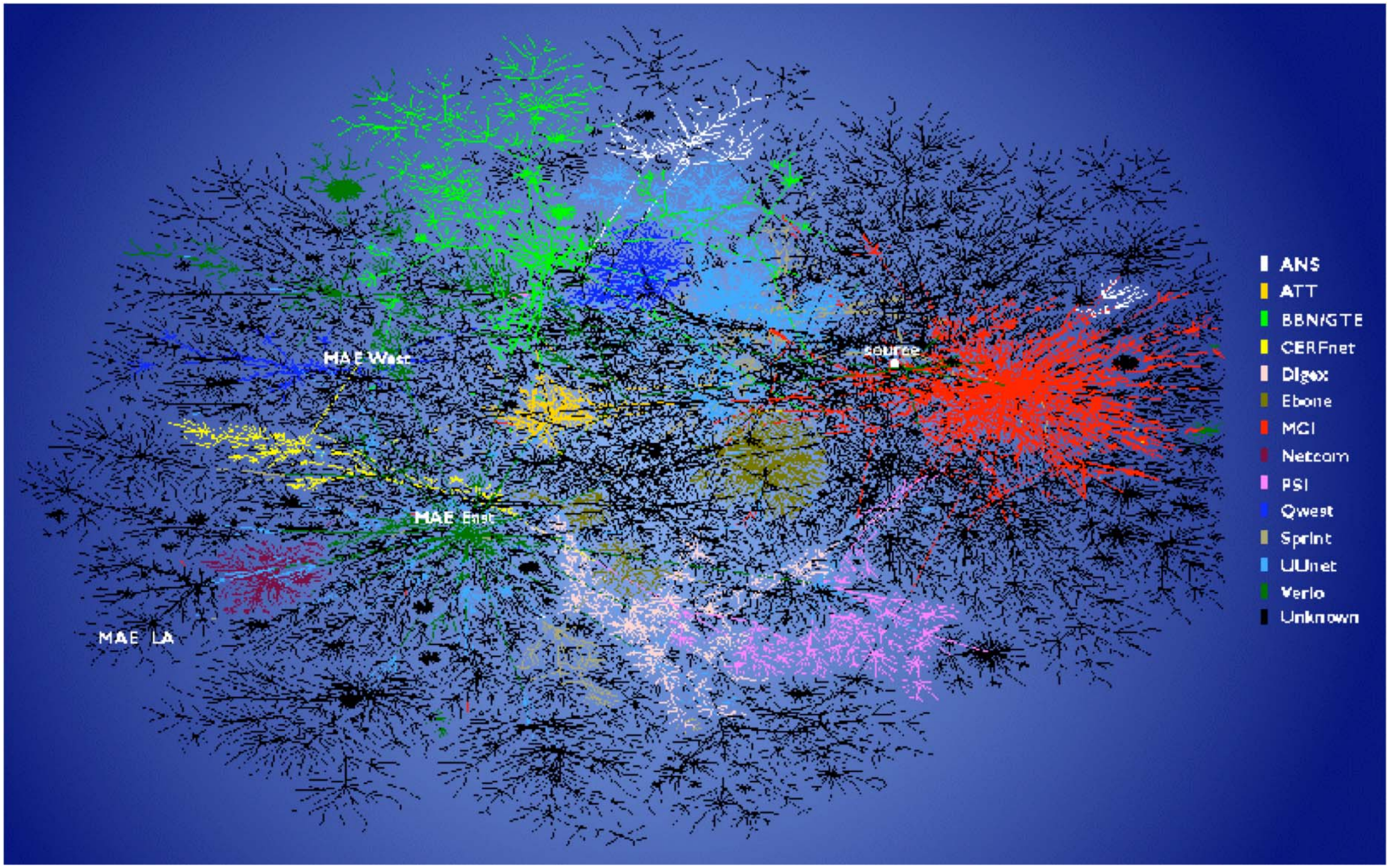
- -manufacturing systems
- -telecommunication networks
- -transportation networks
- -service networks
- -biological networks

SEMICONDUCTOR MANUFACTURING



Courtesy of P. R. Kumar

INTERNET



K C Claffy   http://www.caida.org/publications/

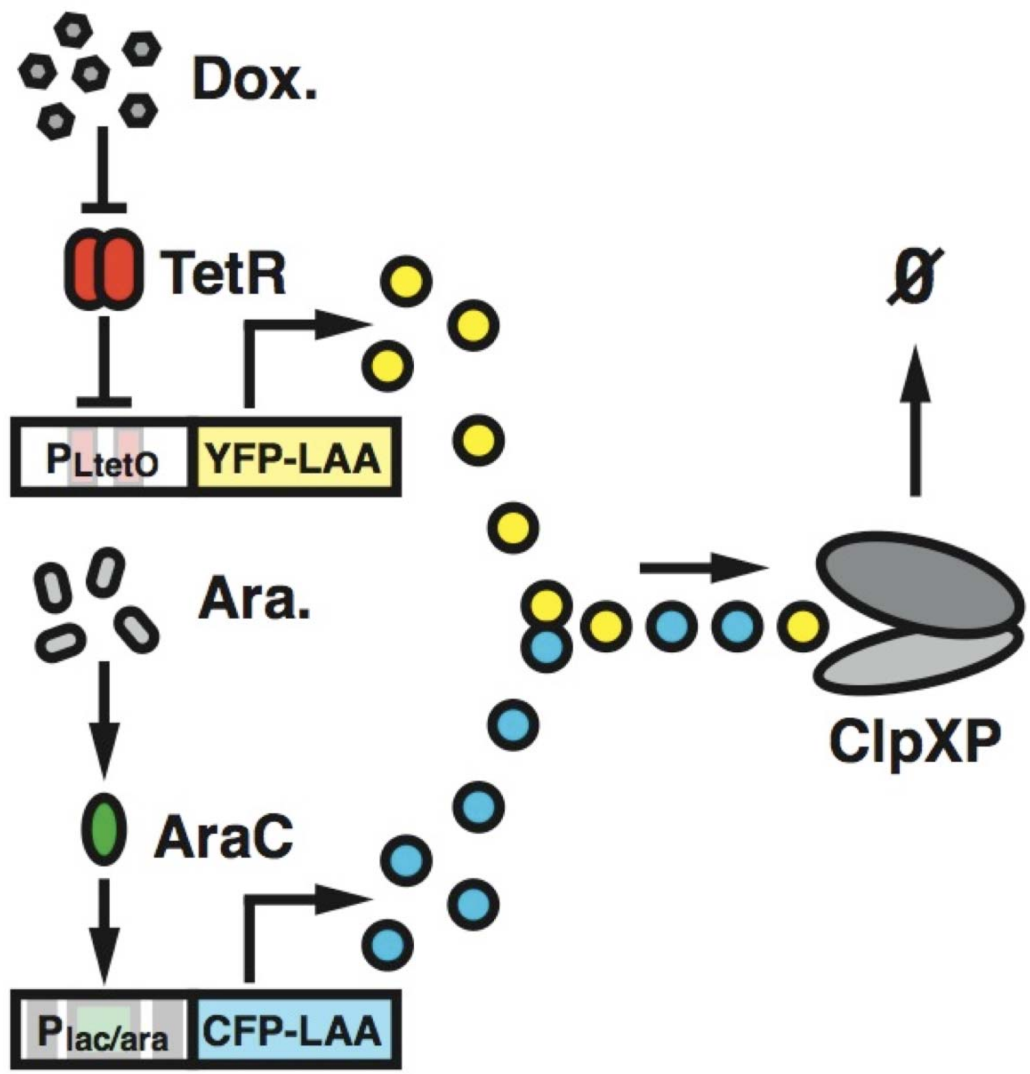
TRANSPORTATION NETWORKS

http://www.highways.gov.uk/

CALL CENTERS

First Direct (branchless retail banking) Larreche et al., INSEAD

BIOLOGICAL NETWORKS



THANK YOU