Diffusion Approximation for a Heavily Loaded Multi-User Wireless Communication System with Cooperation

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Joint work with Sumit Bhardwaj

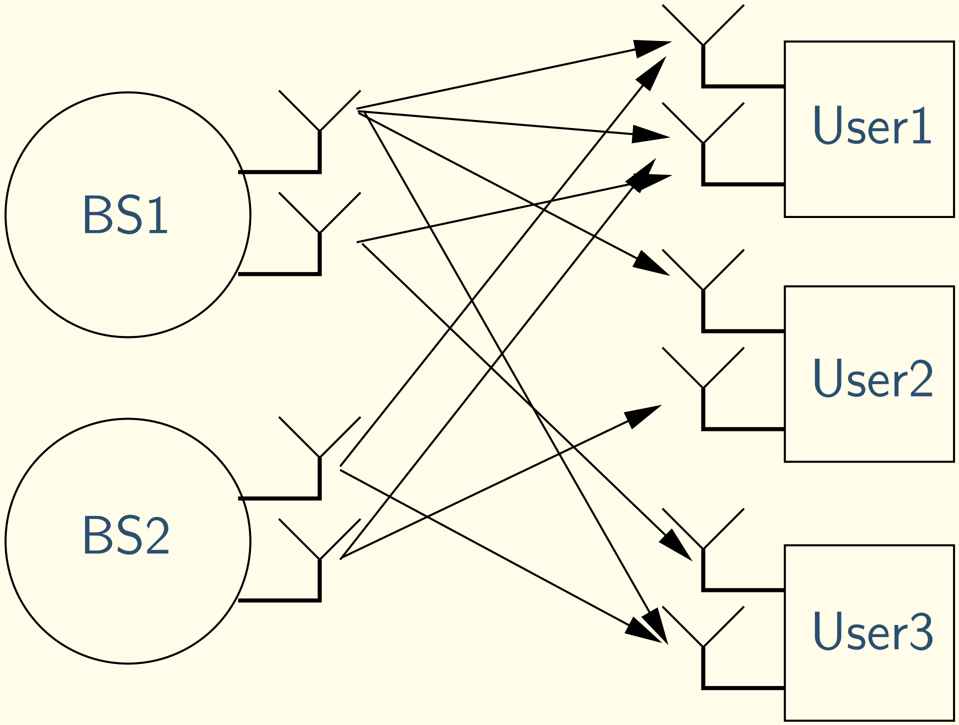
Communication System

Queueing Model

Heavy Traffic

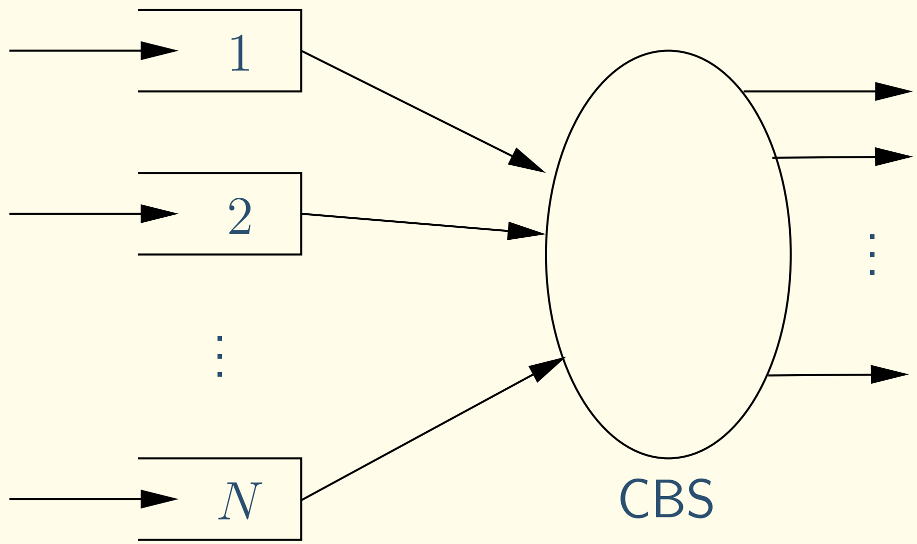
Diffusion Approximation

Communication System

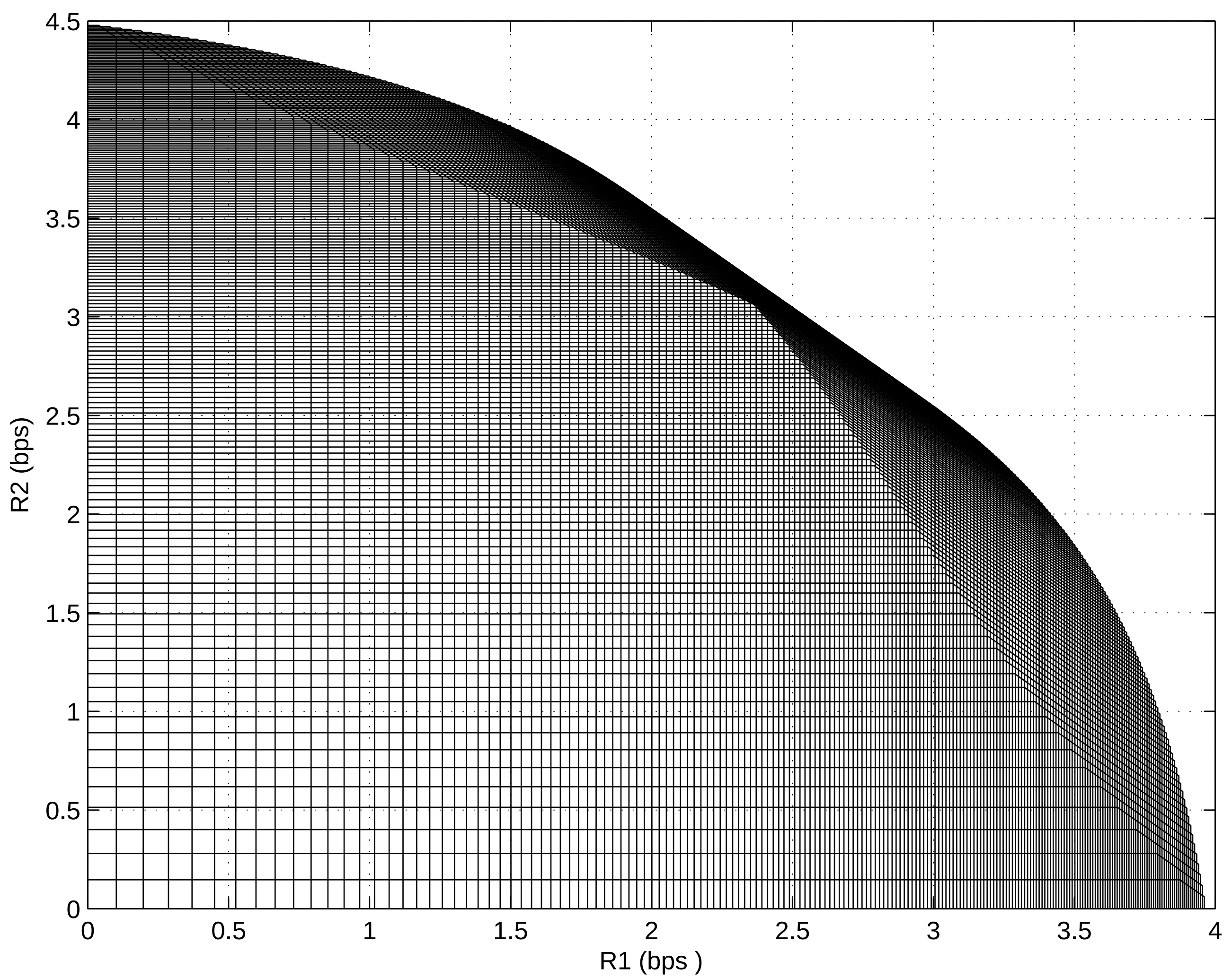

BS1
BS2
User1
User2
User3

- Downlink (base station to mobiles) modeled as multi-user MIMO broadcast channel
- Multiple cooperating base stations
- Total channel capacity is enhanced by cooperation (can be achieved by dirty paper coding)
- Packet-based traffic
- Channel fixed over the period of interest

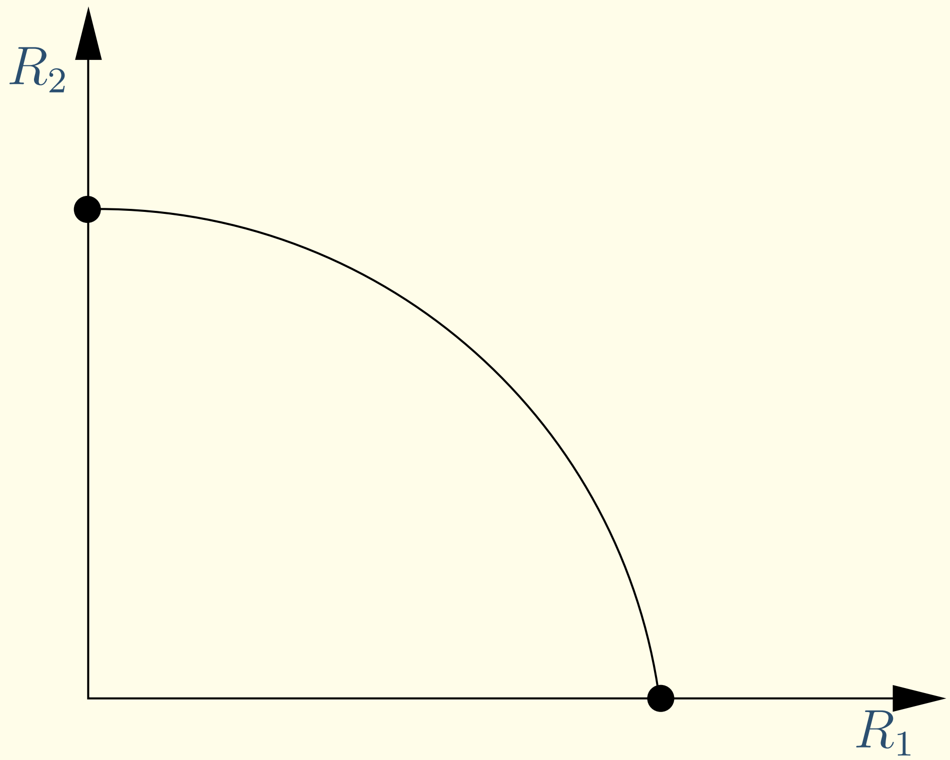
Queueing Schematic



Capacity Region for 2-Users

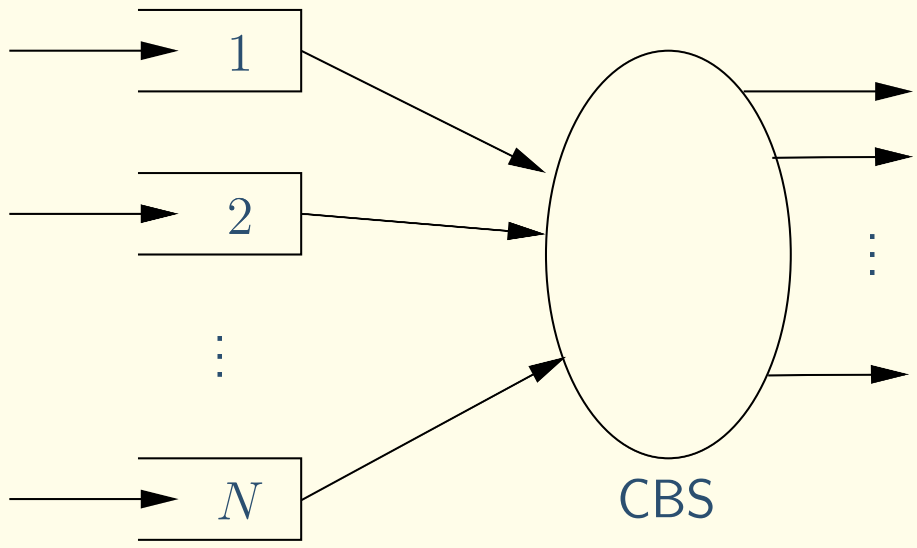


Assume that cooperation is expressed in terms of sums of rates



Queueing Model

Queueing Schematic



Primitives

- N users
- $E_i(t)$ number of packet arrivals for user i up to t
  - renewal process with rate λ_i
- $V_i(n)$ sum of first n packet lengths (in bits) for user i
  - i.i.d. packet lengths with mean m_i
- $E_i, V_i,\ i = 1, \ldots, N$ are all mutually independent
- Nominal bit arrival rate vector $b = (b_1, \ldots, b_N)$ is known
- System starts empty

Workload Process

- Workload (number of bits waiting to be transmitted) for the i-th queue at time t:

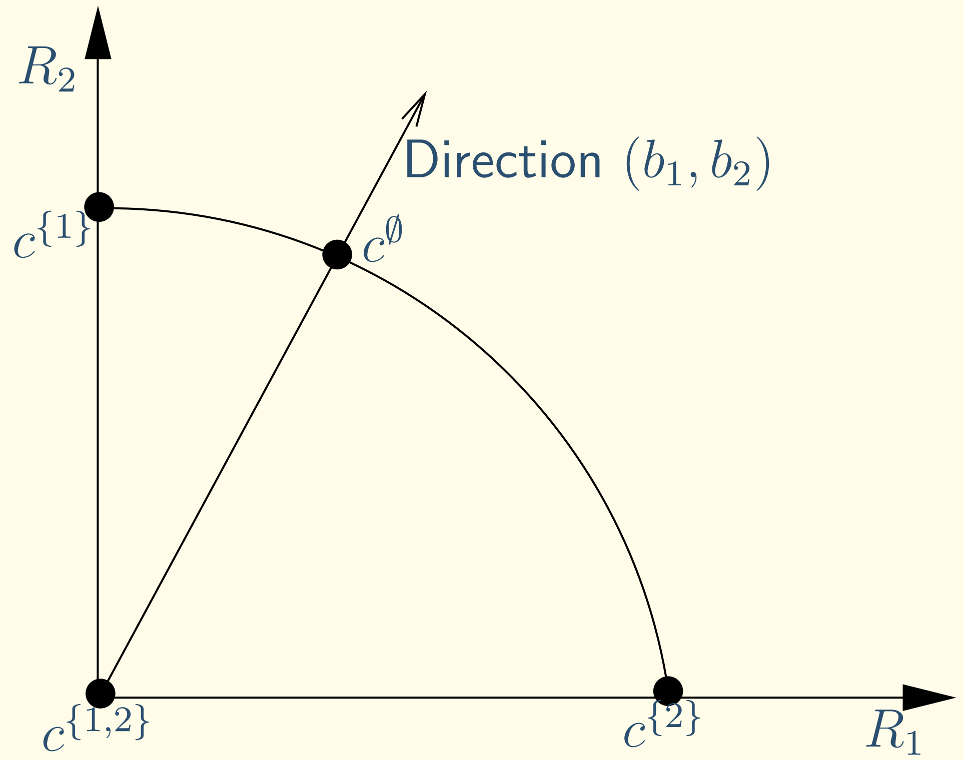
$$W_i(t) = V_i(E_i(t)) - T_i(t)$$

- Total amount of service (measured in bits) given to the i-th queue up to time t:

$$T_i(t) = \int_0^t \Lambda_i(W(s))\, ds$$

where $\Lambda : \mathbb{R}_+^N \to \mathbb{R}_+^N$ specifies the service rate for each queue as a function of the workload

Example



Define

$$
\begin{aligned}
\mathcal{K}(w) &= \{i : w_i = 0\} \text{ for } w \in \mathbb{R}_+^N \\
\Lambda(w) &= c^{\mathcal{K}(w)}
\end{aligned}
$$

where for each $\mathcal{K} \subseteq \{1, \ldots, N\}$,

$$
\begin{aligned}
c_i^{\mathcal{K}} &= 0 \quad \text{if } i \in \mathcal{K} \\
c_i^{\mathcal{K}} &= \delta_{\mathcal{K}} b_i \quad \text{if } i \notin \mathcal{K}
\end{aligned}
$$

where $\delta_{\mathcal{K}} > 0$ is as large as possible while respecting the constraint that $c^{\mathcal{K}}$ is in the capacity region

Cooperation Assumptions

- The vector $c^{\emptyset}$ has the (strict) maximum sum of rates (corresponding to maximum cooperation)

$$\sum_i c_i^{\emptyset} > \sum_i c_i^{\mathcal{K}} \quad \text{for all } \mathcal{K} \neq \emptyset$$

- The service rate for a fixed non-empty queue is least when all queues are non-empty

$$c_i^{\emptyset} \leq c_i^{\mathcal{K}} \quad \text{for all } i \notin \mathcal{K} \text{ and } \mathcal{K} \neq \emptyset$$

Performance?

- No closed-form expression for the performance of the policy
- Here we consider performance when the system is heavily loaded
- Seek a diffusion approximation for the workload process

Heavy Traffic

- Heavy traffic (simplest case):

$$b_i = \lambda_i m_i = c_i^{\emptyset} \quad \text{for } i = 1, \ldots, N$$

- Finite second moments for i.i.d. interarrival times and packet lengths:

$$\begin{aligned} \alpha_i^2 &= \text{SCV for interarrivals times for queue } i \\ \beta_i^2 &= \text{SCV for packet lengths for queue } i \end{aligned}$$

- Diffusion scaling: let $r \to \infty$ through a sequence and

$$\begin{aligned}\hat{W}^r(t) &= \frac{W(r^2 t)}{r}\\ \hat{E}^r(t) &= \frac{1}{r}\left(E(r^2 t) - \lambda r^2 t\right)\\ \hat{V}^r(t) &= \frac{1}{r}\left(V(r^2 t) - m r^2 t\right)\end{aligned}$$

- Standard FCLT for stochastic primitives:

$$(\hat{E}^r, \hat{V}^r) \Rightarrow (\tilde{E}, \tilde{V}) \quad \text{as } r \to \infty$$

where $\tilde{E}$ and $\tilde{V}$ are independent N-dimensional Brownian motions with non-degenerate diagonal covariance matrices $\Gamma_E = \text{diag}(\lambda_i \alpha_i^2)$ and $\Gamma_V = \text{diag}(m_i^2 \beta_i^2)$

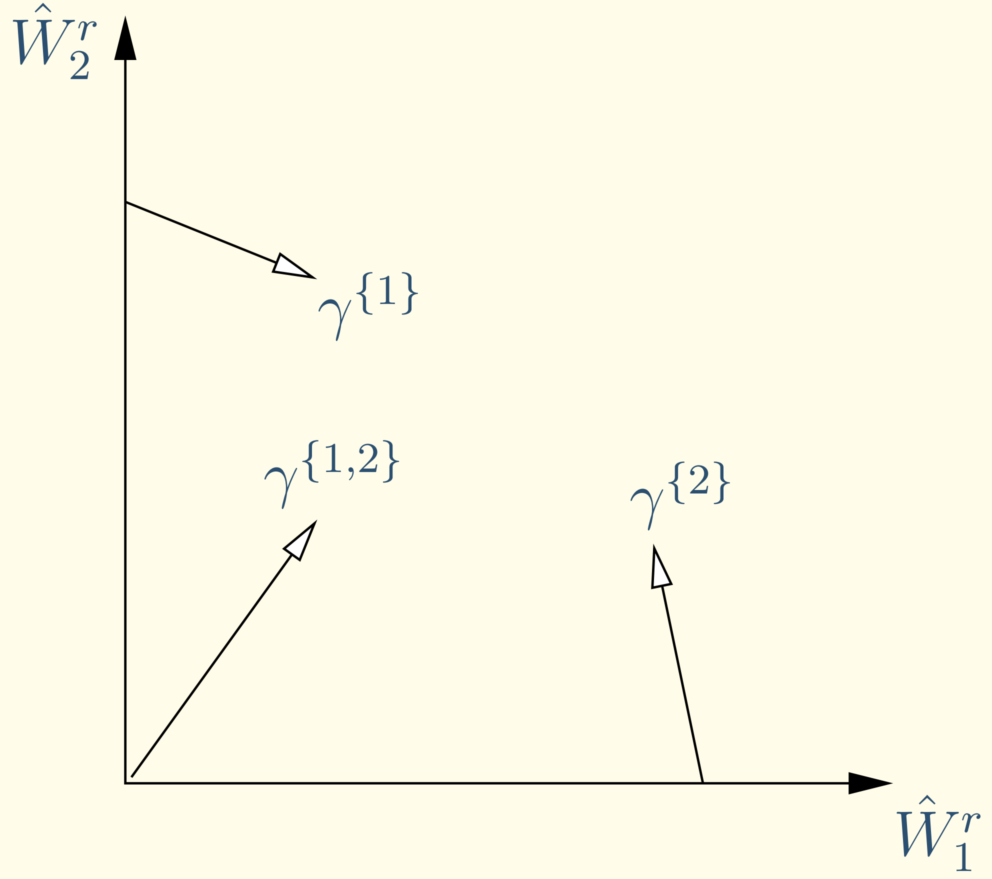
Diffusion Scaled Workload

$$\hat{W}_i^r(t) = \frac{1}{r} V_i(E_i(r^2 t)) - \frac{1}{r} T_i(r^2 t)$$
$$= \hat{X}_i^r(t) + \sum_{\mathcal{K} \neq \emptyset} \gamma^{\mathcal{K}} \hat{U}^{r,\mathcal{K}}(t)$$

where

$$\hat{X}_i^r(t) = \hat{V}_i^r(\bar{E}_i^r(t)) + m_i \hat{E}_i^r(t)$$
$$\hat{U}^{r,\mathcal{K}}(t) = \frac{1}{r} \int_0^{r^2 t} 1_{\{\mathcal{K}(W(s)) = \mathcal{K}\}} \, ds$$
$$\gamma^{\mathcal{K}} = c^{\emptyset} - c^{\mathcal{K}} \quad \text{for } \mathcal{K} \neq \emptyset$$

Note, $\gamma_i^{\mathcal{K}} > 0$ if $i \in \mathcal{K}$ and $\gamma_i^{\mathcal{K}} \leq 0$ if $i \notin \mathcal{K}$



$\hat{W}_2^r$
$\gamma^{\{1\}}$
$\gamma^{\{1,2\}}$
$\gamma^{\{2\}}$
$\hat{W}_1^r$

Diffusion Approximation

Theorem 1. *The sequence of diffusion scaled processes* $\{(\hat{W}^r(\cdot), \hat{X}^r(\cdot), \hat{U}^r(\cdot))\}$ *is C-tight and any weak limit point* $(\tilde{W}, \tilde{X}, \tilde{U})$ *defines a semimartingale reflecting Brownian motion in* $\mathbb{R}_+^N$ *of the form*

$$\tilde{W}(t) = \tilde{X}(t) + \sum_{\mathcal{K} \neq \emptyset} \gamma^{\mathcal{K}} \tilde{U}^{\mathcal{K}}(t), \quad t \geq 0,$$

where $\tilde{X}$ *is an* N*-dimensional driftless Brownian motion with covariance matrix* $\Gamma = \text{diag}(\lambda_i m_i^2(\alpha_i^2 + \beta_i^2))$ *and each* $\tilde{U}^{\mathcal{K}}$ *is a continuous, non-decreasing one-dimensional process that can increase only when* $\tilde{W}$ *is on the face*

$$F_{\mathcal{K}} = \{w \in \mathbb{R}_+^N : w_i = 0 \text{ for all } i \in \mathcal{K}\}.$$

(Proof uses invariance principle of Kang-W '07)

For $\emptyset \neq \mathcal{L} \subseteq \mathcal{K}$,

$$\begin{aligned} n^{\mathcal{K}} \cdot \gamma^{\mathcal{L}} &= \sum_{i \in \mathcal{K}} \gamma_i^{\mathcal{L}} \\ &= \sum_{i \in \mathcal{K}} (c_i^{\emptyset} - c_i^{\mathcal{L}}) \\ &= \sum_{i} (c_i^{\emptyset} - c_i^{\mathcal{L}}) - \sum_{i \notin \mathcal{K}} (c_i^{\emptyset} - c_i^{\mathcal{L}}) \\ &\geq \sum_{i} (c_i^{\emptyset} - c_i^{\mathcal{L}}) \\ &> 0 \end{aligned}$$

because of cooperation

Theorem 2. *For each $\mathcal{K}$ such that $|\mathcal{K}| \geq 2$ and each $\emptyset \neq \mathcal{L} \subseteq \mathcal{K}$, we have*

$$\int_0^\infty 1_{F_\mathcal{K}}(\tilde{W}(s))d\tilde{U}^\mathcal{L}(s) = 0 \quad \text{almost surely.}$$

Consequently, almost surely,

$$\tilde{W}(t) = \tilde{X}(t) + \sum_{i=1}^N \gamma^{\{i\}} \tilde{U}^{\{i\}}(t), \quad t \geq 0.$$

(Proof uses an extension of an argument given by Reiman-W '88)

Theorem 3. *As $r \to \infty$, the diffusion scaled process $\hat{W}^r$ converges in distribution to a semimartingale reflecting Brownian motion $\tilde{W}$ in $\mathbb{R}^N_+$ of the form*

$$\tilde{W} = \tilde{X} + \sum_{i=1}^{N} \gamma^{\{i\}} \tilde{U}^{\{i\}}$$

where the directions of reflection $\{\gamma^{\{i\}}\}_{i=1}^N$ determine a reflection matrix of Harrison-Reiman type $(I - P')$

- The policy has $2^N - 1$ points of operation
- Heavy traffic diffusion approximation
  - N-dimensional SRBM
  - Key result: Only reflections on $N-1$-dimensional faces matter

Thank You!

Abstract

We consider a model for a cellular wireless communication system in which data is transmitted to multiple users over a common channel. For information theoretic reasons, the rate of transmission over this channel can be enhanced by cooperation. Assuming a fixed channel and that the average arrival rate of data for each user is known, we consider a simple scheduling policy which exploits cooperation and which has been shown to be throughput-optimal under Markovian assumptions. As a measure of performance under this policy, we establish a heavy traffic diffusion approximation for the workload process. This diffusion process is a semimartingale reflecting Brownian motion (SRBM) living in the positive orthant of N-dimensional space (where N is the number of users). Nominally, this SRBM has one direction of reflection associated with each of the $2^N - 1$ boundary faces. However, we show that in fact only those directions associated with the (N-1)-dimensional boundary faces matter in the heavy traffic limit.