57th ANNUAL
HIGH SCHOOL HONORS MATHEMATICS CONTEST

April 19, 2014
on the campus of the
University of California, San Diego

PART I
25 Questions

Welcome to the contest! Please do not open the exam until told do so by the proctor.

EXAMINATION DIRECTIONS:

1. Print your name on the lower right side of the answer sheet.
2. Print and bubble in your 3-digit contest ID number under "EXAM NUMBER". **Keep your ID number for Part II**.
   Do not make extraneous marks on the answer sheet. Your answers must be indicated by neatly blackening the box with a #2 pencil. If you erase, do so thoroughly!
3. Calculators may **not** be used. You should use the exam paper for scratchwork.
4. There are 25 multiple-choice questions on Part I. You have 90 minutes for Part I.
5. At the end of Part I, hand in the answer sheet — you may keep the examination questions. Make sure your name and ID number are on the answer sheet.
6. Your score on Part I will be used to calculate your team score. The sum of scores from Part I and Part II determine the individual awards.

Scoring
$+4$ points for a correct answer.
0 points for no answer.
-1 points for an incorrect answer.

Good Luck!

There will be a 15 minute break after Part I before proceeding to Part II.

Please let your coach know if you plan to attend the Awards Dinner on Wednesday, April 30, 6:00–8:15pm in the UCSD Faculty Club.

You may take these exam questions with you after you are done. You may write on this exam and use it to discuss your results outside the room after completion of the exam.

1. A rectangular box is 2 units wide, 3 units deep and 6 units long. How long is the longest diagonal of the box?

   (A) $\sqrt{65}$ (B) 9 (C) $\sqrt{74}$ (D) 8 (E) 7

2. The number 2014 is the product of three prime numbers. What is their sum?

   (A) 74 (B) 114 (C) 87 (D) 53 (E) 61

3. What is the exact value of $\dfrac{1}{\log_2(100!)} + \dfrac{1}{\log_3(100!)} + ... + \dfrac{1}{\log_{100}(100!)}$?

   (A) $\log_2(100!)$ (B) $\log_{10}(100!)$ (C) 1 (D) 2 (E) $\log_{100}(100!)$

4. In a certain state at a certain time, license plates used three letters followed by three digits. Letters could be used multiple times. Digits could be used multiple times. What number of license plates could be made?

   (A) $117,576,000$ (B) $7,576,000$ (C) $57,600$ (D) 760 (E) $17,576,000$

5. A certain university wants to be sure to enroll enough students from the fifty states so that there is a state from which at least 100 students come. The selection of students is not to be made by their home state. What is the minimum number of students that must be enrolled to guarantee that there is at least one state from which 100 students come?

   (A) $2,501$ (B) $4,951$ (C) $36,979$ (D) $8,951$ (E) $9,149$

6. Consider the set of numbers $S = \{1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23\}$. What is the minimum number of numbers selected from S that will guarantee that at least one pair of these numbers adds up to 24?

(A) 10 (B) 9 (C) 8 (D) 7 (E) 6

7. What is an equivalent expression for $\dfrac{x^2-3x-18}{x^2-8x+12} \cdot \dfrac{x^2-4x+3}{x^2+x-2} \cdot \dfrac{x^2-4}{x^2-9}$?

(A) $\dfrac{x^2+6x+5}{x^2-6x+5}$ (B) $\dfrac{x^2+9x+20}{x^2+3x-4}$ (C) $\dfrac{x^2+3x-4}{x^2+9x+20}$ (D) $\dfrac{x^2-5x+20}{x^2+6x+9}$ (E) 1

8. Consider two fair six-sided dice. The sides of each die are numbered 1 through 6. The dice are to be thrown onto a level table. Let $p(n)$ be the probability that on one throw the numbers showing sum to n. What is the ratio $\dfrac{p(2)}{p(7)}$?

(A) $\dfrac{1}{6}$ (B) 1 (C) $\dfrac{1}{3}$ (D) $\dfrac{1}{216}$ (E) $\dfrac{1}{36}$

9. The sum of a positive number and twenty times its reciprocal is equal to the cube of the number. What is the number?

(A) 2 (B) -2 (C) $\sqrt{2}$ (D) 5 (E) $\sqrt{5}$

10. What is the smallest integer M such that for every set S of 9 distinct odd integers that sum to at least M, the sum of the 3 largest integers in S is greater than or equal to 75?

(A) 111 (B) 1005 (C) 171 (D) 51 (E) 97

11. Let f be a function of a real variable with the properties that $f(x) + f(1-x) = 11$ and $f(1+x) = 3 + f(x)$ for all real x. What is the value of $f(x) + f(-x)$?

(A) 8 (B) 9 (C) 10 (D) 11 (E) depends on x

12. What is the smallest number B such that $\dfrac{xy}{x^2+y^2} \leq B$ for all positive numbers x and y?

(A) 2 (B) $\sqrt{2}$ (C) 1 (D) $\dfrac{1}{2}$ (E) 0

13. Simplify the following expression.

$$\frac{1}{\sqrt{5}-\sqrt{3}} + \frac{1}{\sqrt{5}+\sqrt{3}}.$$

(A) $\sqrt{3}$ (B) 2 (C) $\sqrt{5}$ (D) 3 (E) 4

14. If a regular hexagon is inscribed in a circle of radius 1, the enclosed area of the hexagon is $\frac{3}{2}\sqrt{3}$. What is the enclosed area of a regular 12-sided polygon inscribed in a circle of radius 1?

(A) $\sqrt{10}$ (B) 3 (C) $\frac{25}{8}$ (D) $\frac{22}{7}$ (E) $\frac{19}{6}$

15. An estate wills a substantial amount of wealth to four charities A, B, C, and D. Certain interested parties find out that

$$A + B + C = \$475,000, \quad B + C + D = \$550,000,$$
$$C + D + A = \$400,000, \quad D + A + B = \$525,000.$$

What is the amount left to charity C?

(A) \$100,000 (B) \$125,000 (C) \$150,000 (D) \$175,000

(E) there is not sufficient information to answer the question

16. If the expression $(x^3 - x^2y + xy^2 + y^3)^3$ is expanded and simplified, what is the sum of all the coefficients of the resulting polynomial?

(A) 1 (B) 2 (C) 4 (D) 8 (E) 16

17. John visits the San Diego Zoo with his 7 children. Entering the gift shop at closing time, the store has only one of each toy remaining. John asks each child what toys they prefer. Here are their answers.

James	wombat	panda	gorilla	meerkat	giraffe
Josh	panda	donkey	cheetah		
Jezebel	panda	donkey	meerkat	cheetah	
Jamelia	panda	donkey	cheetah		
Janine	wombat	gorilla	donkey	meerkat	giraffe
Jason	panda	cheetah			
Julia	wombat	gorilla	cheetah	giraffe	meerkat

John decides to buy each child one of their preferred toys. What animal does he give to Jezebel?

(A) meerkat (B) gorilla (C) giraffe (D) wombat (E) donkey

18. In a conventional clock, how many times between noon on Saturday and noon on Sunday does the second hand pass the minute hand?

(A) 720 (B) 1415 (C) 1440 (D) 1521 (E) 1620

19. A dime and a foreign coin exactly twice the diameter of the dime are laid on a flat table. The large coin is held fixed on the table, while the dime is rolled without slipping around the edge of the large coin. When the dime has completed one full rotation around the large coin, how many times has the face on the dime rotated?

(A) 1 (B) 2 (C) 3 (D) 4 (E) 4.5

20. Let $a, b, c > 0$ with $a+b+c=1$, and set $P = \left(\frac{1}{a}-1\right)\left(\frac{1}{b}-1\right)\left(\frac{1}{c}-1\right)$. What is the largest integer less than or equal to P?

(A) 4 (B) 5 (C) 6 (D) 7 (E) 8

21. How many subsets of $\{1, 2, \ldots, 10\}$ contain no two successive numbers?

(A) 55 (B) 89 (C) 144 (D) 233 (E) 512

22. What is the maximum area of a quadrilateral with side lengths 1, 4, 7, and 8?

(A) 4 (B) $\sqrt{65}$ (C) 14 (D) 18 (E) 28

23. Simplify the following sum:

$$\frac{1}{1\cdot 4}+\frac{1}{4\cdot 7}+\frac{1}{7\cdot 10}+\cdots+\frac{1}{2011\cdot 2014}.$$

(A) $\frac{671}{2014}$ (B) 1 (C) $\frac{2011}{2014}$ (D) $\frac{1}{3}$ (E) $\frac{2013}{2014}$

24. Let $x_1, x_2, x_3, \ldots$ list the numbers that can be written as a sum of one or more distinct powers of 3, in increasing order ($x_1 < x_2 < x_3 < \cdots$). For example, $x_1 = 3^0 = 1$, $x_2 = 3^1 = 3$, $x_3 = 3^1 + 3^0 = 4$. What is the value of x_{100}?

(A) 972 (B) 981 (C) 999 (D) 1002 (E) 1053

25. Let $f_1(x) = (x+53)(x+106)(x+159)\cdots(x+2014)$. Define $f_2(x) = f_1(x+1) - f_1(x)$, $f_3(x) = f_2(x+1) - f_2(x)$, and in general $f_n(x) = f_{n-1}(x+1) - f_{n-1}(x)$. What is the smallest value of n for which $f_n(x)$ is a constant polynomial?

(A) 1 (B) 19 (C) 38 (D) 39 (E) 2014