57th ANNUAL
HIGH SCHOOL HONORS MATHEMATICS CONTEST

April 19, 2014
on the campus of the
University of California, San Diego

PART II
4 Questions

Welcome to Part II of the contest!

Please print your Name, School, and Contest ID number:

Name ____
First   Last

School ____

3-digit Contest ID number ________________________________

Please do not open the exam until told do so by the proctor.

EXAMINATION DIRECTIONS:

1. Part II consists of 4 problems, each worth 25 points. These problems are "essay" style questions. You should put all work towards a solution in the space following the problem statement. You should show all work and justify your responses as best you can.

2. The score on Part II is added to the score on Part I to determine the highest individual scores in the contest.

3. Scoring is based on the progress you have made in understanding and solving the problem. The clarity and elegance of your response is an important part of the scoring. You may use the back side of each sheet to continue your solution, but be sure to call the reader's attention to the back side if you use it.

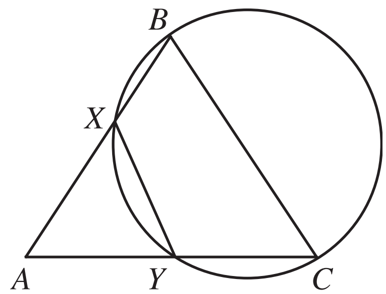
4. Give this entire exam to a proctor when you have completed the test to your satisfaction.

Please let your coach know if you plan to attend the Awards Dinner on Wednesday, April 30, 6:00–8:15pm in the UCSD Faculty Club.

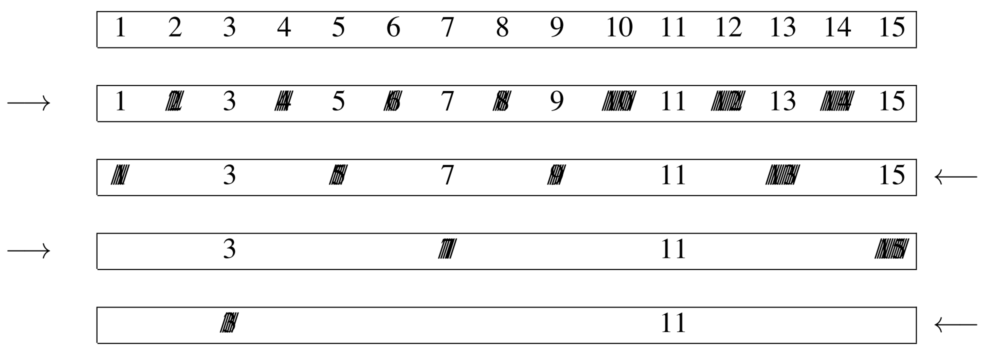
1. Alice and Bob each have a bag of 9 balls. The balls in each bag are numbered from 1 to 9. Alice and Bob each remove one ball uniformly at random from their own bag. Let a be the sum of the numbers on the balls remaining in Alice's bag. Let b be the sum of the numbers on the balls remaining in Bob's bag. Determine the probability that a and b differ by a multiple of 4.

2. Suppose that $x \in \mathbb{Q}$ is a rational number with the property that $x^2 - x \in \mathbb{Z}$ is an integer. Prove that, in fact, $x \in \mathbb{Z}$ is an integer.

3. In triangle ABC, $AB = BC = 25$ and $AC = 30$. The circle with diameter BC intersects AB at X and AC at Y. Determine the length of XY.



4. A school has a row of n open lockers, numbered 1 through n. Starting at the beginning of the row, you walk past and close every second locker until reaching the end of the row, as shown in the example below. Then you turn around, walk back, and close every second locker that is still open. You continue in this manner back and forth along the row, until only one locker remains open. Define $f(n)$ to be the number of the last open locker. For example, if there are 15 lockers, then $f(15) = 11$ as shown below.



Calculate $f(2014)$.