The Hard Edge of Unitary Brownian Motion

Abstract:

Random matrix theory is filled with laws of large numbers and central limit theorems, all specialized to different statistics and scaling regimes. The two most famous laws of large numbers are for Gaussian Wigner (Hermitian) matrices: the bulk (empirical distribution) of eigenvalues converge a.s. to the semicircle law, and the largest eigenvalue converges to twice the common variance of the entries. The corresponding central limit theorems give Gaussian fluctuations in the bulk, but a completely different distribution (the Tracy-Widom Law) for the largest eigenvalue.

One can view a Gaussian Wigner matrix as (a marginal of) Brownian motion on the Lie algebra of Hermitian matrices. It is then natural to study the analogous questions for the eigenvalues of the Brownian motion on the associated Lie group: the unitary group. For the bulk, the a.s. limit empirical eigenvalue distribution was discovered by Biane and Rains independently in the late 1990s; the corresponding bulk central limit result was largely found by Lévy and Maïda in 2010, and completed by Cébron and me last year. This left the corresponding questions about the largest eigenvalue.

This talk will focus on my recent joint work with Collins and Dahlqvist, where we prove that the largest and smallest (in polar angle) eigenvalues converge a.s. to their expected limits. Interestingly, the rate of convergence is much faster than in the Hermitian setting. The main technical result is a growth bound on moments that is a vast improvement over previously known bounds. The techniques involve some stochastic calculus, and a healthy dose of representation theory, which I will briefly describe.

Time permitting, I will also discuss our extension of these results to the multivariate (fully noncommutative) case, with applications to the Jacobi process. And show lots of pictures.